The size of a hypergraph and its matching number

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Abstract

More than forty years ago, Erdős conjectured that for any $t \leq \frac{n}{k}$, every k-uniform hypergraph on n vertices without t disjoint edges has at most $\max\left\{\binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k}\right\}$ edges. Although this appears to be a basic instance of the hypergraph Turán problem (with a t-edge matching as the excluded hypergraph), progress on this question has remained elusive. In this paper, we verify this conjecture for all $t < \frac{n}{3k^2}$. This improves upon the best previously known range $t = O\left(\frac{n}{k^3}\right)$, which dates back to the 1970's.

1 Introduction

A k-uniform hypergraph is a pair $H = (V, E)$, where $V = V(H)$ is a finite set of vertices, and $E = E(H) \subseteq \binom{V}{k}$ is a family of k-element subsets of V called edges. A matching in H is a set of disjoint edges in $E(H)$. We denote by $\nu(H)$ the size of the largest matching, i.e., the maximum number of disjoint edges in H. The problem of finding the maximum matching in a hypergraph has many applications in various different areas of mathematics, computer science, and even computational chemistry. Yet although the graph matching problem is fairly well-understood, and solvable in polynomial time, most of the problems related to hypergraph matching tend to be very difficult and remain unsolved. Indeed, the hypergraph matching problem is known to be NP-hard even for 3-uniform hypergraphs, without any good approximation algorithm.

One of the most basic open questions in this area was raised in 1965 by Erdős [5], who asked to determine the maximum possible number of edges that can appear in any k-uniform hypergraph with matching number $\nu(H) < t \leq \frac{n}{k}$ (equivalently, without any t pairwise disjoint edges). He conjectured that this problem has only two extremal constructions. The first one is a clique consisting of all the k-subsets on $kt-1$ vertices, which obviously has matching number $t-1$. The second example is a k-uniform hypergraph on n vertices containing all the edges intersecting a fixed set of $t-1$ vertices, which also forces the matching number to be at most $t-1$. Neither construction is uniformly better than the other across the entire parameter space, so the conjectured bound is the maximum of these two possibilities. Note that in the second case, the complement of this hypergraph is a clique on $n-t+1$ vertices together with $t-1$ isolated vertices, and thus the original hypergraph has $\binom{n}{k} - \binom{n-t+1}{k}$ edges.

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Conjecture 1.1 *Every k-uniform hypergraph H on n vertices with matching number $\nu(H) < t \le \frac{n}{k}$ satisfies*

$$e(H) \le \max\left\{\binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k}\right\}. \tag{1}$$

In addition to being important in its own right, this Erdős conjecture has several interesting applications, which we discuss in the concluding remarks. Yet although it is more than forty years old, only partial results have been discovered so far. In the case $t = 2$, the condition simplifies to the requirement that every pair of edges intersects, so Conjecture 1.1 is thus equivalent to a classical theorem of Erdős, Ko, and Rado [7]: that any intersecting family of k-subsets on $n \ge 2k$ elements has size at most $\binom{n-1}{k-1}$. The graph case ($k = 2$) was separately verified in [6] by Erdős and Gallai. For general fixed t and k, Erdős [5] proved his conjecture for sufficiently large n. Frankl [8] showed that Conjecture 1.1 was asymptotically true for all n by proving the weaker bound $e(H) \le (t-1)\binom{n-1}{k-1}$.

A short calculation shows that when $t \le \frac{n}{k+1}$, we always have $\binom{n}{k} - \binom{n-t+1}{k} > \binom{kt-1}{k}$, so the potential extremal example in this case has all edges intersecting a fixed set of $t-1$ vertices. One natural question is then to determine the range of t (with respect to n and $k \ge 3$) for which the maximum is indeed equal to $\binom{n}{k} - \binom{n-t+1}{k}$, i.e., where the second case is optimal. Recently, Frankl, Rödl, and Ruciński [9] studied 3-uniform hypergraphs ($k = 3$), and proved that for $t \le n/4$, the maximum was indeed $\binom{n}{3} - \binom{n-t+1}{3}$, establishing the conjecture in that range. For general $k \ge 4$, Bollobás, Daykin, and Erdős [4] explicitly computed the bounds achieved by the proof in [5], showing that the conjecture holds for $t < \frac{n}{2k^3}$. Frankl and Furëdi [8] established the result in a different range $t < \left(\frac{n}{100k}\right)^{1/2}$, which improves the original bound when k is large relative to n. In this paper, we extend the range in which the Erdős conjecture holds to all $t < \frac{n}{3k^2}$.

Theorem 1.2 *For any integers n, k, t satisfying $t < \frac{n}{3k^2}$, every k-uniform hypergraph on n vertices without t disjoint edges contains at most $\binom{n}{k} - \binom{n-t+1}{k}$ edges.*

To describe the idea of our proof, we first outline Erdős's original approach for the case $t < \frac{n}{2k^3}$. Let v be a vertex of maximum degree. By induction on t we find $t-1$ disjoint edges $F_1, \ldots, F_{t-1}$, none of which contain v. If $\deg(v)$ exceeds $k(t-1)\binom{n-2}{k-2}$, which is the maximum possible number of edges containing v which also meet a vertex in $\bigcup_{i=1}^{t-1} F_i$, then we can find t disjoint edges. Otherwise, the number of edges meeting any of F_i is at most $|\bigcup_{i=1}^{t-1} F_i| \cdot k(t-1)\binom{n-2}{k-2} = k(t-1) \cdot k(t-1)\binom{n-2}{k-2}$, which turns out to be less than the total number of edges when $n \ge 2k^3 t$. Any other edge will serve as the t-th edge in the matching.

To improve Erdős's bound, we show that in the first part of the argument, we are already done if the t-th largest degree exceeds $2t\binom{n-2}{k-2}$. This puts a tighter constraint on the sum of the degrees of the $k(t-1)$ vertices in $\bigcup_{i=1}^{t-1} F_i$, allowing the second stage to proceed under the relaxed assumption $n \ge 3k^2 t$. The fact that t vertices of degree at least $2t\binom{n-2}{k-2}$ are enough to find t disjoint edges leads naturally to the following multicolored version of the Erdős conjecture, which was also considered independently by Aharoni and Howard in [1].

Conjecture 1.3 *Let $\mathcal{F}_1, \ldots, \mathcal{F}_t$ be families of subsets in $\binom{[n]}{k}$. If $|\mathcal{F}_i| > \max\left\{\binom{n}{k} - \binom{n-t+1}{k}, \binom{kt-1}{k}\right\}$ for all $1 \le i \le t$, then there is a "rainbow" matching of size t: one that contains exactly one edge from each family.*

The $k = 2$ case of this conjecture was established by Meshulam (see [1]). To obtain Theorem 1.2, we prove an asymptotic version of Conjecture 1.3, by showing that a rainbow matching exists whenever $|\mathcal{F}_i| > (t-1)\binom{n-1}{k-1}$ for every $1 \le i \le t$.

The rest of this paper is organized as follows. In the next section, we describe the so-called *shifting* method, which is a well known technique in extremal set theory, and use it to prove some preliminary results. In Section 3 we first prove the multicolored Erdős conjecture asymptotically, and then use it to prove Theorem 1.2. There, we also use the same argument to show that Conjecture 1.3 holds for all $t < \frac{n}{3k^2}$. The last section contains some concluding remarks and open problems.

2 Shifting

In extremal set theory, one of the most important and widely-used tools is the technique of shifting, which allows us to limit our attention to sets with certain structure. In this section we will only state and prove the relevant results for Section 3. For more background on the applications of shifting in extremal set theory, we refer the reader to the survey [8] by Frankl.

Given a family $\mathcal{F}$ of equal-size subsets of $[n]$, for integers $1 \le j < i \le n$, we define the (i, j)-shift map S_{ij} as follows: for any set $F \in \mathcal{F}$,

$$S_{ij}(F) = \begin{cases} F \setminus \{i\} \cup \{j\}\,, & \text{iff } i \in F,\ j \notin F \text{ and } F \setminus \{i\} \cup \{j\} \notin \mathcal{F}\,; \\ F\,, & \text{otherwise.} \end{cases}$$

Also, we denote the family after shifting as

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F) : F \in \mathcal{F}\}\ .$$

Lemma 2.1 *The shift map S_{ij} satisfies the following properties.*

(i) $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$.

(ii) *If $\mathcal{F}$ is k-uniform, then so is $S_{ij}(\mathcal{F})$.*

(iii) *If the families $\mathcal{F}_1, \ldots, \mathcal{F}_t$ have the property that no subsets $F_1 \in \mathcal{F}_1, \ldots, F_t \in \mathcal{F}_t$ are pairwise disjoint, then the shifted families $S_{ij}(\mathcal{F}_1), \ldots, S_{ij}(\mathcal{F}_t)$ still preserve this property.*

Proof. Claims (i) and (ii) are obvious. For (iii), assume that the statement is false, i.e., we have $F_i \in \mathcal{F}_i$ such that $S_{ij}(F_1), \ldots, S_{ij}(F_t)$ are pairwise disjoint, while $F_1, \ldots, F_t$ are not. Without loss of generality, $F_1 \cap F_2 \neq \emptyset$. Next, observe that whenever $S_{ij}(F_k) \neq F_k$, we also have $j \in S_{ij}(F_k)$, so the pairwise disjointness of the $S_{ij}(F_k)$ implies that the only possible case (re-indexing if necessary) is for $S_{ij}(F_1) = F_1 \setminus \{i\} \cup \{j\}$, and $S_{ij}(F_k) = F_k$ for every $k \ge 2$. Note also that since F_1 and F_2 intersect while $S_{ij}(F_1)$ and $S_{ij}(F_2)$ do not, we must have $i \in F_2$ and $j \notin F_2$.

Therefore the only reason that $S_{ij}(F_2) = F_2$ is because $F_2' = F_2 \setminus \{i\} \cup \{j\}$ is already in $\mathcal{F}_2$. The pair of disjoint sets $S_{ij}(F_1)$ and $S_{ij}(F_2) = F_2$ have the same union as the pair of disjoint sets F_1 and F_2'. Using the pairwise disjointness of the $S_{ij}(F_k)$, we conclude that the sets $F_1, F_2', F_3, \ldots, F_t$ are pairwise disjoint as well, contradicting our initial assumption. □

In practice, we often combine the shifting technique with induction on the number of elements in the underlying set. Indeed, let us apply the shifts $\{S_{ni}\}_{1\le i\le n-1}$ successively, and with slight abuse of notation, let us again call the resulting families $\mathcal{F}_1, \ldots, \mathcal{F}_t$. Create from each $\mathcal{F}_i$ two sub-families based on containment of the final element n:

$$\begin{aligned} \mathcal{F}_i(n) &= \{F \setminus \{n\} : F \in \mathcal{F}_i, n \in F\} \ , \\ \mathcal{F}_i(\bar{n}) &= \{F \qquad\quad : F \in \mathcal{F}_i, n \notin F\} \ . \end{aligned}$$

It turns out that the rainbow matching number does not increase by this decomposition.

Lemma 2.2 *Let $\mathcal{F}_1, \ldots, \mathcal{F}_t$ be the shifted families, where each $\mathcal{F}_i$ is k_i-uniform and $\sum_{i=1}^t k_i \le n$. Suppose that no subsets $F_1 \in \mathcal{F}_1, \ldots, F_t \in \mathcal{F}_t$ are pairwise disjoint. Then, for any $0 \le s \le t$, the families $\mathcal{F}_1(n), \ldots, \mathcal{F}_s(n), \mathcal{F}_{s+1}(\bar{n}), \ldots, \mathcal{F}_t(\bar{n})$ still have the same property.*

Proof. Assume for the sake of contradiction that there exist pairwise disjoint sets $F_1 \in \mathcal{F}_1(n)$, $\ldots$, $F_s \in \mathcal{F}_s(n)$, $F_{s+1} \in \mathcal{F}_{s+1}(\bar{n})$, $\ldots$, $F_t \in \mathcal{F}_t(\bar{n})$. By definition of $\mathcal{F}_i(n)$ and $\mathcal{F}_i(\bar{n})$, we know that $F_i \cup \{n\} \in \mathcal{F}_i$ for $1 \le i \le s$, and $F_i \in \mathcal{F}_i$ for $s+1 \le i \le t$. The size of $\bigcup_{i=1}^t F_i$ is equal to

$$\sum_{i=1}^t |F_i| = \sum_{i=1}^s (k_i - 1) + \sum_{i=s+1}^t k_i = \sum_{i=1}^t k_i - s \le n - s \, ,$$

so there exist distinct elements $x_1, \ldots, x_s \notin \bigcup_{i=1}^t F_i$. Since $F_i \cup \{n\}$ is invariant under the shift S_{nx_i}, the set $F_i \cup \{x_i\} = (F_i \cup \{n\}) \setminus \{n\} \cup \{x_i\}$ must also be in the family $\mathcal{F}_i$. Taking $F_i' = F_i \cup \{x_i\}$ for $1 \le i \le s$, together with F_i for $s+1 \le i \le t$, it is clear that we have found pairwise disjoint sets from $\mathcal{F}_i$, contradiction. $\Box$

3 Main result

In this section, we discuss the Erdős conjecture and its multicolored generalizations, and prove the original conjecture for the range $t < \frac{n}{3k^2}$. The colored interpretation arises from considering the collection of families $\mathcal{F}_i$ as a single uniform hypergraph (possibly with repeated edges) on the vertex set $[n]$, where each set in $\mathcal{F}_i$ introduces a hyperedge colored in the i-th color. The following lemma is a multicolored generalization of Theorem 10.3 in [8], and provides a sufficient condition for a multicolored hypergraph to contain a rainbow matching of size t.

Lemma 3.1 *Let $\mathcal{F}_1, \ldots, \mathcal{F}_t$ be families of subsets of $[n]$ such that for each i, $\mathcal{F}_i$ only contains sets of size k_i, $|\mathcal{F}_i| > (t-1)\binom{n-1}{k_i-1}$, and $n \ge \sum_{i=1}^t k_i$. Then there exist t pairwise disjoint sets $F_1 \in \mathcal{F}_1$, $\ldots$, $F_t \in \mathcal{F}_t$.*

Proof. We proceed by induction on t and n. The case $t = 1$ is trivial. For general t, we can also handle all minimal cases of the form $n = \sum_{i=1}^t k_i$. Indeed, consider a uniformly random permutation π of $[n]$, and define a series of indicator random variables $\{X_i\}$ as follows: $X_1 = 1$ iff $\{\pi(1), \ldots, \pi(k_1)\}$ is a set in $\mathcal{F}_1$ and $X_1 = 0$ otherwise, and in general, $X_j = 1$ iff $\{\pi(k_1+\cdots+k_{j-1}+1), \ldots, \pi(k_1+\cdots+k_j)\}$ is a set in $\mathcal{F}_j$. We assume that there are no t disjoint sets from different families, so we deterministically have:

$$X_1 + \cdots + X_t \leq t - 1 \,. \tag{2}$$

On the other hand, it is easy to see that the expectation of X_i is the probability that a random k_i-set is in $\mathcal{F}_i$, so

$$\mathbb{E}X_i = \frac{|\mathcal{F}_i|}{\binom{n}{k_i}} \,.$$

Yet we know that for every i, we have $|\mathcal{F}_i| > (t-1)\binom{n-1}{k_i-1}$, so

$$\mathbb{E}X_i > \frac{(t-1)\binom{n-1}{k_i-1}}{\binom{n}{k_i}} = (t-1)\frac{k_i}{n} \,.$$

Summing these inequalities over $1 \leq i \leq t$, we obtain that $\sum_{i=1}^{t} \mathbb{E}X_i > t-1$, a contradiction to (2).

Now we consider a generic instance with $n > \sum_{i=1}^{t} k_i$, and inductively assume that all instances with smaller n are known. By Lemma 2.1, after applying all shifts $\{S_{ni}\}_{1 \leq i \leq n-1}$, we obtain families in which any rainbow t-matching can be pulled back to a rainbow t-matching in $\{\mathcal{F}_i\}$. For convenience we still call the shifted families $\{\mathcal{F}_i\}$. Our next step is to partition each $\mathcal{F}_i$ into $\mathcal{F}_i(n) \cup \mathcal{F}_i(\bar{n})$, but in order to avoid empty sets, we first dispose of the case when there is some $k_i = 1$ with $\{n\} \in \mathcal{F}_i$. After re-indexing, we may assume that this is $\mathcal{F}_1$. Since $|\mathcal{F}_i| > (t-1)\binom{n-1}{k_i-1}$ and there are at most $\binom{n-1}{k_i-1}$ sets containing n, every other $\mathcal{F}_i$ has more than $(t-2)\binom{n-1}{k_i-1}$ sets which in fact lie in $[n-1]$. By induction on the $t-1$ sizes $k_2, \ldots, k_t$, we find $t-1$ such disjoint sets from $\mathcal{F}_2$, $\ldots$, $\mathcal{F}_t$ which, together with $\{n\} \in \mathcal{F}_1$, establish the claim.

Returning to the general case, since $|\mathcal{F}_i| = |\mathcal{F}_i(n)| + |\mathcal{F}_i(\bar{n})|$ and our size condition is

$$|\mathcal{F}_i| > (t-1)\binom{n-1}{k_i-1} = (t-1)\binom{n-2}{k_i-2} + (t-1)\binom{n-2}{k_i-1} \,,$$

we conclude that for each i, either $|\mathcal{F}_i(n)| > (t-1)\binom{n-2}{k_i-2}$ or $|\mathcal{F}_i(\bar{n})| > (t-1)\binom{n-2}{k_i-1}$. Without loss of generality, we may assume that $|\mathcal{F}_i(n)| > (t-1)\binom{n-2}{k_i-2}$ for $1 \leq i \leq s$, and $|\mathcal{F}_i(\bar{n})| > (t-1)\binom{n-2}{k_i-1}$ for $s+1 \leq i \leq t$. Note that $\mathcal{F}_i$ is (k_i-1)-uniform for $1 \leq i \leq s$ and k_i-uniform for $s+1 \leq i \leq t$, and the base set now has $n-1$ elements. Induction on n and Lemma 2.2 then produce t disjoint sets from different families. $\Box$

As mentioned in the introduction, the conjectured extremal hypergraph when $t \leq \frac{n}{k+1}$ is the hypergraph consisting of all edges intersecting a fixed set of size $t-1$. If we inspect the vertex degree sequence of this hypergraph, we observe that although there are $t-1$ vertices with high degree $\binom{n-1}{k-1}$, the remaining vertices only have degree $\binom{n-1}{k-1} - \binom{n-t}{k-1}$. For small t, this is asymptotically about $(t-1)\binom{n-2}{k-2}$, which is much smaller than $\binom{n-1}{k-1} = \frac{n-1}{k-1}\binom{n-2}{k-2}$. The following corollary of Lemma 3.1 shows that this sort of phenomenon generally occurs when hypergraphs satisfy the conditions in the Erdős conjecture.

Corollary 3.2 *If a k-uniform hypergraph H on n vertices has t distinct vertices $v_1, \ldots, v_t$ with degrees $d(v_i) > 2(t-1)\binom{n-2}{k-2}$, and $kt \leq n$, then H contains t disjoint edges.*

Proof. Let H_i be a $(k-1)$-uniform hypergraph containing all the subsets of $V(H) \setminus \{v_1, \ldots, v_t\}$ of size $k-1$ which together with v_i form an edge of H. For any fixed $1 \le i \le t$ and $j \ne i$, there are at most $\binom{n-2}{k-2}$ edges of H containing both vertices v_i and v_j. Therefore for every hypergraph H_i,

$$e(H_i) \ge d(v_i) - (t-1)\binom{n-2}{k-2} > (t-1)\binom{n-2}{k-2} \ge (t-1)\binom{n-t-1}{k-2}.$$

Since every hypergraph H_i is $(k-1)$-uniform and has $n-t$ vertices, we can use Lemma 3.1 with $\mathcal{F}_i = E(H_i)$, $k_i = k-1$ and n replaced by $n-t$, to find t disjoint edges $e_1 \in E(H_1), \ldots, e_t \in E(H_t)$. Taking the edges $e_i \cup \{v_i\} \in E(H)$, we obtain t disjoint edges in the original hypergraph H. □

Now we are ready to prove our main result, Theorem 1.2, which states that for $t < \frac{n}{3k^2}$, every k-uniform hypergraph on n vertices without t disjoint edges contains at most $\binom{n}{k} - \binom{n-t+1}{k}$ edges.

Proof of Theorem 1.2. We proceed by induction on t. The base case $t = 1$ is trivial, so we consider the general case, assuming that the $t-1$ case is known. Suppose $e(H) > \binom{n}{k} - \binom{n-t+1}{k}$, and let us seek t disjoint edges in H. We first consider the situation when there is a vertex v of degree $d(v) > k(t-1)\binom{n-2}{k-2}$. Let H_v be the sub-hypergraph induced by the vertex set $V(H) \setminus \{v\}$. Since there are at most $\binom{n-1}{k-1}$ edges containing v,

$$\begin{aligned} e(H_v) \ge e(H) - \binom{n-1}{k-1} &> \binom{n}{k} - \binom{n-t+1}{k} - \binom{n-1}{k-1} \\ &= \binom{n-1}{k} - \binom{(n-1)-(t-1)+1}{k}. \end{aligned}$$

By induction, there are $t-1$ disjoint edges $e_1, \ldots, e_{t-1}$ in H_v, spanning $(t-1)k$ distinct vertices $u_1, \ldots, u_{(t-1)k}$. Note that the number of edges containing v and any vertex u_j is at most $\binom{n-2}{k-2}$. Therefore since we assumed that $d(v) > k(t-1)\binom{n-2}{k-2}$, there must be another edge e_t which contains v but avoids $u_1, \ldots, u_{(t-1)k}$. We then have t disjoint edges $e_1, \ldots, e_t$ in H.

Now suppose that the maximum vertex degree in H is at most $k(t-1)\binom{n-2}{k-2}$. After re-indexing the vertices, we may assume that $k(t-1)\binom{n-2}{k-2} \ge d(v_1) \ge \cdots \ge d(v_n)$. If the t-th largest degree satisfies $d(v_t) > 2(t-1)\binom{n-2}{k-2}$, then Corollary 3.2 immediately produces t disjoint edges in H, so we may also assume for the remainder that $d(v_t) \le 2(t-1)\binom{n-2}{k-2}$.

By induction (with room to spare), we also know that there are $t-1$ disjoint edges in H, spanning $(t-1)k$ vertices. Among these vertices, the $t-1$ largest degrees are at most $k(t-1)\binom{n-2}{k-2}$ by our maximum degree assumption, while the remaining $(t-1)(k-1)$ vertices cannot have degrees exceeding $d(v_t) \le 2(t-1)\binom{n-2}{k-2}$. Therefore the sum of degrees of these $(t-1)k$ vertices is at most

$$(t-1) \cdot k(t-1)\binom{n-2}{k-2} + (t-1)(k-1) \cdot 2(t-1)\binom{n-2}{k-2} = (t-1)^2(3k-2)\binom{n-2}{k-2}.$$

However, we know that the total number of edges exceeds

$$\begin{aligned} e(H) &> \binom{n}{k} - \binom{n-t+1}{k} \\ &= \left[1 - \left(1 - \frac{t-1}{n}\right) \cdots \left(1 - \frac{t-1}{n-k+1}\right)\right] \binom{n}{k} \\ &\geq \left[1 - \left(1 - \frac{t-1}{n}\right)^k\right] \binom{n}{k} \\ &\geq \left[k \cdot \frac{t-1}{n} - \binom{k}{2} \left(\frac{t-1}{n}\right)^2\right] \frac{n(n-1)}{k(k-1)} \binom{n-2}{k-2} \\ &\geq \left(\frac{(n-1)(t-1)}{k-1} - \frac{(t-1)^2}{2}\right) \binom{n-2}{k-2}, \end{aligned}$$

where we used that $(1-x)^k \leq 1 - kx + \binom{k}{2}x^2$ when $0 \leq kx \leq 1$. Since $n > 3k^2t$, we also have $n - 1 > 3k(k-1)(t-1)$. Therefore,

$$e(H) > (t-1)^2 \left(3k - \frac{1}{2}\right) \binom{n-2}{k-2},$$

and so there is another edge in H disjoint from the previous $t-1$ edges, again producing t disjoint edges in H. □

Based on the same idea and technique, we can also obtain a multicolored version of the Erdős conjecture, which is an analogue of a theorem of Kleitman [10] for matching number greater than one. Note that Theorem 1.2 is the $\mathcal{F}_1 = \cdots = \mathcal{F}_t$ case of the following result.

Theorem 3.3 *Let $\mathcal{F}_1, \ldots, \mathcal{F}_t$ be k-uniform families of subsets of $[n]$, where $t < \frac{n}{3k^2}$, and every $|\mathcal{F}_i| > \binom{n}{k} - \binom{n-t+1}{k}$. Then there exist pairwise disjoint sets $F_1 \in \mathcal{F}_1, \ldots, F_t \in \mathcal{F}_t$.*

Proof. For any vertex $v \in \mathcal{F}_i$, let H_v^j be the sub-hypergraph of $\mathcal{F}_j$ induced by the vertex set $[n] \backslash \{v\}$. Then as in the previous proof,

$$e(H_v^j) \geq |\mathcal{F}_i| - \binom{n-1}{k-1} > \binom{n-1}{k} - \binom{(n-1)-(t-1)+1}{k}.$$

By induction on t, for every i there exist $t-1$ disjoint edges $\{e_j\}_{j \neq i}$ such that $e_j \in H_v^j$. So as before, if some $\mathcal{F}_i$ has a vertex with degree $d(v) > k(t-1)\binom{n-2}{k-2}$, then there is an edge in $\mathcal{F}_i$ which contains v and is disjoint from $\{e_j\}_{j \neq i}$. Hence we may assume the maximum degree in each hypergraph $\mathcal{F}_i$ is at most $k(t-1)\binom{n-2}{k-2}$.

On the other hand, by induction on t we also know that for every i there exist $t-1$ disjoint edges from the families $\{\mathcal{F}_j\}_{j \neq i}$, spanning $(t-1)k$ vertices. If some $\mathcal{F}_i$ has t-th largest degree at most $2(t-1)\binom{n-2}{k-2}$, then the sum of degrees of these $(t-1)k$ vertices in $\mathcal{F}_i$ is again at most

$$(t-1)^2(3k-2)\binom{n-2}{k-2} \leq \binom{n}{k} - \binom{n-t+1}{k} < e(\mathcal{F}_i),$$

which guarantees the existence of an edge in $\mathcal{F}_i$ disjoint from the previous $t-1$ edges from $\{\mathcal{F}_j\}_{j\neq i}$. So, we may assume that each $\mathcal{F}_i$ contains at least t vertices with degree above $2(t-1)\binom{n-2}{k-2}$.

Now select distinct vertices v_i, such that for each $1 \leq i \leq t$, the degree of v_i in $\mathcal{F}_i$ exceeds $2(t-1)\binom{n-2}{k-2}$. Consider all the subsets of $[n] \setminus \{v_1, \ldots, v_t\}$ which together with v_i form an edge of $\mathcal{F}_i$. Denote this $(k-1)$-uniform hypergraph by T^i. The same calculation as in Corollary 3.2 gives

$$e(T^i) > (t-1)\binom{n-t-1}{k-2}.$$

Applying Lemma 3.1 to $\{T^i\}$, we again find t disjoint edges from different families, as desired. □

4 Concluding Remarks

- In this paper, we proved that for $t < \frac{n}{3k^2}$, every k-uniform hypergraph on n vertices with matching number less than t has at most $\binom{n}{k} - \binom{n-t+1}{k}$ edges. This verifies the conjecture of Erdős in this range of t, and improves upon the previously best known range by a factor of k. As we discussed in the introduction, if the Erdős conjecture is true in general, then for $t < \frac{n}{k+1}$, the maximum number of edges cannot exceed $\binom{n}{k} - \binom{n-t+1}{k}$. It would be very interesting to tighten the range to $t < O\left(\frac{n}{k}\right)$.

- *A fractional matching* in a k-uniform hypergraph $H = (V, E)$ is a function $w : E \to [0, 1]$ such that for each $v \in V$ we have $\sum_{e \ni v} w(e) \leq 1$. The *size* of w is the sum $\sum_{e \in E} w(e)$, and the size of the largest fractional matching in H is denoted by $\nu^*(H)$. The fractional version of the Erdős conjecture states that among k-uniform hypergraphs H on n vertices with fractional matching number $\nu^*(H) < xn$, the maximum number of edges is asymptotically $(1 + o(1)) \max\left\{(kx)^k, 1 - (1-x)^k\right\}\binom{n}{k}$.

  It appears that these conjectures are closely related to several other interesting problems. For example, it was shown in [3] that the integral version can be used to determine the minimum degree condition which ensures the existence of perfect matchings in uniform hypergraphs. Furthermore, it turns out that the fractional version is closely related to an old probability conjecture of Samuels [12] and in computer science, it has applications to finding optimal data allocations in distributed storage systems (see [3] for more details). In [2], the fractional Erdős conjecture was used to attack an old problem of Manickam-Miklós-Singhi, which states that for $n \geq 4k$, every set of n real numbers with nonnegative sum has at least $\binom{n-1}{k-1}$ k-element subsets whose sums are also nonnegative.

- Pyber [11] proved the following product-type generalization of the Erdős-Ko-Rado theorem. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be families of k_1- and k_2-element subsets of $[n]$. If every pair of sets $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ intersects, then $|\mathcal{F}_1||\mathcal{F}_2| \leq \binom{n-1}{k_1-1}\binom{n-1}{k_2-1}$ for sufficiently large n. The special case when $k_1 = k_2$ and $\mathcal{F}_1 = \mathcal{F}_2$ corresponds to the Erdős-Ko-Rado theorem. Our Theorem 3.3 is a minimum-type result of similar flavor. Hence, it would be interesting to study the following multicolor analogue of Pyber's result.

  Question 4.1 *What is the maximum of $\prod_{i=1}^{t} |\mathcal{F}_i|$ among families $\mathcal{F}_1, \ldots, \mathcal{F}_t$ of subsets of $[n]$, where each $\mathcal{F}_i$ is k_i-uniform, and there are no t pairwise disjoint subsets $F_1 \in \mathcal{F}_1, \ldots, F_t \in \mathcal{F}_t$?*

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