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THE SIZE OF (L^2, L^p) MULTIPLIERS

BY

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0. Introduction. A complex valued function φ defined on the dual Γ of an infinite compact abelian group G is called an (L^p, L^q) *multiplier* if for all $f \in L^p(G)$, $M_\varphi f \in L^q(G)$ where by $M_\varphi f$ we mean the function whose Fourier transform is given by $\widehat{M_\varphi f}(\chi) = \varphi(\chi)\widehat{f}(\chi)$ for $\chi \in \Gamma$. The space of (L^p, L^q) multipliers will be denoted by $M(p,q)$. When μ is a bounded Borel measure on G, then $M_{\hat\mu} \in M(p,p)$ (we will write $\mu \in M(p,p)$). If a multiplier $\varphi \in M(2,p)$ for some $p > 2$ then φ is called L^p-*improving*. For basic properties, and background information on L^p-improving multipliers we refer the reader to [5] and [8].

In this paper we investigate the relationship between the size of the function φ and membership in $M(2,p)$ for certain types of multipliers, furthering the work of [2] and [5] in particular.

By a *one-sided Riesz product* we mean a multiplier φ given by

$$\varphi(\chi) = \begin{cases} \prod a_i^{\varepsilon_i} & \text{if } \chi = \prod \chi_i^{\varepsilon_i},\ \varepsilon_i = 0, 1, \\ 0 & \text{otherwise} \end{cases}$$

where $\{\chi_i\}$ is a dissociate subset of Γ and $\{a_i\}$ is a bounded sequence of complex numbers. We will write $\varphi = \prod(1 + a_i\chi_i)$ for short. When $\chi_i^2 = 1$ for all χ_i then a one-sided Riesz product is actually the Fourier transform of a Riesz product; and like Riesz products, one-sided Riesz products exhibit interesting phenomena. Extending work of Bonami [2], in Section 2 we characterize certain (one-sided) Riesz products on T^∞, D^∞ and T which belong to $M(2,p)$. This characterization shows that the necessary conditions on the size of (L^2, L^p) multipliers which we obtain in Section 1 are best possible, but are not sufficient even for (one-sided) Riesz products, answering an open problem in [5].

In [8] (L^2, L^p) multipliers are "almost" characterized. The necessary conditions we establish are combined with this result to sharpen the known estimates of the $\Lambda(p)$ constants of sums of dissociate sets. The previously known best estimates were developed in [2] by mainly combinatorial methods.

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1. Necessary conditions. As a preliminary result we obtain lower bounds for L^p norms of Riesz products and one-sided Riesz products.

LEMMA 1.1. *Let* $\{\chi_i\}_1^\infty$ *be a dissociate subset of* Γ *such that* $\chi_i^2 \neq 1$. *For each* $p > 0$ *there are positive constants* $k = k_p$ *and* $c = c_p < 1/2$ *such that*

$$\text{(a)}\qquad \prod_{i=1}^N (1 + (p-1)|c_i|^2 - k|c_i|^3) \le \Big\| \prod_{i=1}^N (1 + c_i\chi_i + \overline{c_i\chi_i}) \Big\|_p \le \prod_{i=1}^N (1 + (p-1)|c_i|^2 + k|c_i|^3)\,,$$

and

$$\text{(b)}\qquad \prod_{i=1}^N (1 + |c_i|^2 p/4 - k|c_i|^3) \le \Big\| \prod_{i=1}^N (1 + c_i\chi_i) \Big\|_p \le \prod_{i=1}^N (1 + |c_i|^2 p/4 + k|c_i|^3)$$

whenever $N \in \mathbb{N}$ *and* $\{c_i\}$ *is a sequence of complex numbers with* $|c_i| \le c$ *for all* i.

Proof. In what follows the constant $k = k_p$ may vary from one line to another.

(a) The Taylor series expansion of $(1+x)^p$ for $|x|$ small yields that

$$\Big\| \prod_{i=1}^N (1 + c_i\chi_i + \overline{c_i\chi_i}) \Big\|_p^p \ge \int \prod_{i=1}^N \bigg(1 + p(c_i\chi_i + \overline{c_i\chi_i}) + \frac{p(p-1)}{2}(c_i\chi_i + \overline{c_i\chi_i})^2 - k|c_i|^3 \bigg).$$

As $\{\chi_i\}$ is a dissociate set this integral equals $\prod_{i=1}^N (1 + p(p-1)|c_i|^2 - k|c_i|^3)$. By taking pth roots and another application of Taylor series we obtain the first inequality in (a). The other is similar.

For (b) first we observe that

$$\Big\| \prod_{i=1}^N (1 + c_i\chi_i) \Big\|_p = \Big[\int \prod_{i=1}^N ((1 + c_i\chi_i)(1 + \overline{c}_i\overline{\chi}_i))^{p/2} \Big]^{1/p} = \prod_{i=1}^N (1 + |c_i|^2)^{1/2} \Big[\int \prod_{i=1}^N \Big(1 + \frac{c_i\chi_i + \overline{c_i\chi_i}}{1 + |c_i|^2} \Big)^{p/2} \Big]^{1/p}.$$

Using part (a) it follows that if constants $|c_i|$ are sufficiently small, than the integral in the line above dominates

$$\prod_{i=1}^{N}\left(1+\left(\frac{p}{2}-1\right)\frac{|c_i|^2}{(1+|c_i|^2)^2}-k|c_i|^3\right)^{p/2}.$$

This estimate together with another application of Taylor series establishes the lower bound for $||\prod_{i=1}^{N}(1+c_i\chi_i)||_p$, and similar arguments give the upper bound. ■

Remark. Of course, for any sequence $\{c_i\}$, the L^2 norms of $\prod_{i=1}^{N}(1+c_i\chi_i+\overline{c_i\chi_i})$ and $\prod_{i=1}^{N}(1+c_i\chi_i)$ are $\prod_{i=1}^{N}(1+2|c_i|^2)^{1/2}$ and $\prod_{i=1}^{N}(1+|c_i|^2)^{1/2}$ respectively.

With the estimates of this lemma we can now obtain necessary quantitative estimates for certain (L^2, L^p) multipliers. First we consider the case when the multiplier arises from a measure. Recall that a measure μ is *tame* if for each $\varphi \in \Delta M(G)$ there exists $a \in \mathbb{C}$ and $\gamma \in \Gamma$ such that $\varphi_\mu = a\gamma$ a.e. $d\mu$ ([6, 6.1]). A Riesz product is an example of a tame measure.

THEOREM 1.2. *Let μ be a tame measure on a compact abelian group G and assume $\mu \in M(2,p)$ for some $p > 2$. Suppose that Γ has no elements of order 2. Then $|\varphi_\mu|^2 \le 1/(p-1)$ for all $\varphi \in \overline{\Gamma}\setminus\Gamma \subset \Delta M(G)$.*

Before proving this we state an immediate corollary and make some initial remarks.

COROLLARY 1.3. *If tame $\mu \in M(2,p)$ for $p > 2$ then*

$$\limsup_{\chi\in\Gamma}|\widehat{\mu}(\chi)|^2 \le \frac{1}{p-1}||\mu||^2_{M(G)}.$$

Remarks. (1) For background information on $\Delta M(G)$ see [6].

(2) This result improves the estimate in [5] and [7] for tame measures, and was shown by Bonami to be both necessary and sufficient for certain Riesz products ([2, p. 376, 385]).

Proof of Theorem. Let $\varphi \in \overline{\Gamma}\setminus\Gamma$ and suppose $\varphi_\mu = z\chi d\mu$ a.e. where, without loss of generality, we may assume $z \ne 0$. Replacing μ by $\gamma\mu$ if necessary we may assume $\widehat{\mu}(1) \ne 0$. Fix $0 < \delta < |z|$. Observe that $|\widehat{\mu}((\varphi\overline{\chi})^k)| = |\widehat{\mu}((\overline{\varphi}\chi)^k)| = |z^k\widehat{\mu}(1)|$ for all non-negative integers k, thus we may choose a dissociate set $\{\chi_i\}_{i=1}^{\infty}$ such that

$$\left|\widehat{\mu}\Big(\prod \chi_i^{\varepsilon_i}\Big)\right| \ge (|z|-\delta)^{\sum|\varepsilon_i|}|\widehat{\mu}(1)| \quad \text{whenever } \varepsilon_i = 0, \pm 1.$$

For $\varepsilon > 0$ (and small), define the trigonometric polynomial $f_{N,\varepsilon}$ by

$$\widehat{f}_{N,\varepsilon}(\chi) = \begin{cases} \dfrac{(\varepsilon(|z|-\delta))^k}{\widehat{\mu}(\chi)} & \text{if } \chi = \displaystyle\prod_{j=1}^{N}\chi_j^{\varepsilon_j},\ \varepsilon_j = 0, \pm 1 \text{ and } \displaystyle\sum_{j=1}^{N}|\varepsilon_j| = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mu * f_{N,\varepsilon} = \prod_{j=1}^{N}(1 + \varepsilon(|z| - \delta)(\chi_j + \overline{\chi}_j)) .$$

Thus

$$|\widehat{f}_{N,\varepsilon}(\chi)| \le \begin{cases} \dfrac{\varepsilon^k}{|z| - \delta} & \text{if } \chi = \prod_{j=1}^{N} \chi_j^{\varepsilon_j},\ \varepsilon_j = 0, \pm 1 \text{ and } \sum_{j=1}^{N} |\varepsilon_j| = k, \\ 0 & \text{otherwise} \end{cases}$$

so

$$||f_{N,\varepsilon}||_2 \le \frac{1}{|z| - \delta}(1 + 2\varepsilon^2)^{N/2} .$$

An application of the closed graph theorem shows that there is a constant C such that $||\mu * f||_p \le C||f||_2$ for all $f \in L^2$. Together with Lemma 1.1 this shows that for all N and for all sufficiently small ε,

$$C \ge \frac{||\mu * f_{N,\varepsilon}||_p}{||f_{N,\varepsilon}||_2} \ge \frac{1}{|z| - \delta}\left[\frac{1 + (p-1)\varepsilon^2(|z| - \delta)^2 - k\varepsilon^3(|z| - \delta)^3}{(1 + 2\varepsilon^2)^{1/2}}\right]^N .$$

Hence for all small ε,

$$1 + (p-1)\varepsilon^2(|z| - \delta)^2 - k\varepsilon^3(|z| - \delta)^3 \le (1 + 2\varepsilon^2)^{1/2} .$$

Letting $\varepsilon \to 0$ we see that this can occur only if $(p-1)(|z| - \delta)^2 \le 1$, but as $\delta > 0$ was arbitrary this implies that $|z|^2 \le 1/(p-1)$ as desired. ∎

Unlike measures, for general (L^2, L^p) multipliers φ it is not necessary that $\limsup|\varphi(\chi)| < ||\varphi||_{l^\infty}$. Indeed, it is easy to see that the characteristic function of a Sidon set is an (L^2, L^p) multiplier for all $p > 2$ (cf. [8] or [12]). However, in the next proposition we will prove that for one-sided Riesz products a better estimate can be obtained, and we will prove an estimate sharper than Corollary 1.3 for Riesz products.

PROPOSITION 1.4. *Let $\{\chi_n\}$ be a dissociate set in Γ with $\chi_n^2 \ne 1$ and let $1 < p < q < \infty$. Suppose $\{r_n\}$ and $\{t_n\}$ are sets of complex numbers and let*

$$\varepsilon_n^{(1)} = \max\left(|r_n|^2 - \frac{p-1}{q-1}, 0\right), \qquad \varepsilon_n^{(2)} = \max\left(|t_n|^2 - \frac{p}{q}, 0\right).$$

If either $\varphi_1 = \prod(1 + r_n\chi_n + \overline{r_n\chi_n})$ or $\varphi_2 = \prod(1 + t_n\chi_n)$ belong to $M(p,q)$, then $\sum_n(\varepsilon_n^{(i)})^3 < \infty$ for $i = 1, 2$.

If, in addition, $\{\chi_n\}$ satisfies the further independence condition

$$\prod \chi_n^{\delta_n} = 0 \text{ for } \delta_n = 0, \pm 1, \pm 2, \pm 3 \quad \text{implies} \quad \delta_n = 0 ,$$

then $\sum(\varepsilon_n^{(i)})^2 < \infty$ is a necessary condition.

Remark. If $|r_n| \le 1/2$ then φ_1 is a measure, otherwise by φ_1 we simply mean the obvious multiplier.

Proof. Note that a necessary condition for φ_1 or φ_2 to be an element of $M(p,q)$ is that $\{\varepsilon_n^{(i)}\}$ is a bounded sequence for $i=1,2$. Define trigonometric polynomials $f_N^{(1)} = \prod_{n=1}^N (1 + c\varepsilon_n^{(1)}(\chi_n + \overline{\chi}_n))$ and $f_N^{(2)} = \prod_{n=1}^N (1 + c\varepsilon_n^{(2)}\chi_n)$ where $c \ge 0$ is a small constant.

As $\varphi_1, \varphi_2 \in M(p,q)$ the usual closed graph theorem argument shows that for $i=1,2$, $\sup_N \|M_{\varphi_i} f_N^{(i)}\|_q / \|f_N^{(i)}\|_p < \infty$. Thus for c chosen sufficiently small, Lemma 1.1 implies that

$$\begin{aligned}\infty > \sup_N \frac{\|M_{\varphi_1} f_N^{(1)}\|_q}{\|f_N^{(1)}\|_p} &\ge \sup_N \prod_{n=1}^N \left(\frac{1 + (q-1)|c\varepsilon_n^{(1)} r_n|^2 - k|c\varepsilon_n^{(1)} r_n|^3}{1 + (p-1)|c\varepsilon_n^{(1)}|^2 + k|c\varepsilon_n^{(1)}|^3} \right) \\ &= \sup_N \prod_{n=1}^N \left(1 + \frac{(\varepsilon_n^{(1)})^3((q-1)c^2 - kc^3|r_n|^3)}{(p-1)|c\varepsilon_n^{(1)}|^2 + k|c\varepsilon_n^{(1)}|^3} \right),\end{aligned}$$

which forces $\sum (\varepsilon_n^{(1)})^3 < \infty$. Similar arguments apply to $\sum (\varepsilon_n^{(2)})^3$.

If $\{\chi_i\}$ satisfies the stronger independence property, then Lemma 1.1 can be improved. The stronger property implies that for every i,

$$\int \chi_i^\delta \overline{\chi}_i^{3-\delta} \prod_{j\ne i} \chi_j^{\delta_j} = 0 \quad \text{for } \delta = 0,1,2,3 \text{ and } \delta_j = 0, \pm 1, \pm 2, \pm 3,$$

thus arguments similar to Lemma 1.1, but taking the first four terms of the Taylor series expansion, show that for c_i sufficiently small

$$\begin{aligned}\prod_{i=1}^N (1 + (p-1)|c_i|^2 - k|c_i|^4) &\le \Big\| \prod_{i=1}^N (1 + c_i\chi_i + \overline{c_i\chi_i}) \Big\|_p \\ &\le \prod_{i=1}^N (1 + (p-1)|c_i|^2 + k|c_i|^4)\end{aligned}$$

and

$$\prod_{i=1}^N (1 + |c_i|^2 p/4 - k|c_i|^4) \le \Big\| \prod_{i=1}^N (1 + c_i\chi_i) \Big\|_p \le \prod_{i=1}^N (1 + |c_i|^2 p/4 + k|c_i|^4).$$

If we take $g_N^{(1)} = \prod_{n=1}^N (1 + \sqrt{c\varepsilon_n^{(1)}}(\chi_n + \overline{\chi}_n))$ and $g_N^{(2)} = \prod_{n=1}^N (1 + \sqrt{c\varepsilon_n^{(2)}}\chi_n)$, then by estimating $\|M_{\varphi_i} g_N^{(i)}\|_q / \|g_N^{(i)}\|_p$ with these sharper estimates we get the necessary condition $\sum (\varepsilon_n^{(i)})^2 < \infty$ for $i = 1,2$. ∎

COROLLARY 1.5. *Let $\{\chi_i\}$ be a dissociate subset of Γ and let $\varphi = \prod (1 + a_i\chi_i)$ be a one-sided Riesz product. If $\varphi \in M(2,p)$ then* $\limsup |a_i|^2 \le 2/p$.

Remark. This condition is both necessary and sufficient for certain one-sided Riesz products (see [2, p. 389] and §2).

Proof. Assume $\{\chi_i\} = \{\chi_i\}_{i\in J} \cup \{\chi_i\}_{i\in K}$ where $\chi_i^2 = 1$ for $i \in J$ and $\chi_i^2 \neq 1$ for $i \in K$. Let $\alpha = \prod_{i\in J}(1 + a_i\chi_i)$ and $\beta = \prod_{i\in K}(1 + a_i\chi_i)$. By duality $M(2,p) = M(p',2)$ and as $|\alpha(\chi)|$ and $|\beta(\chi)|$ are both dominated by $|\varphi(\chi)|$ for all $\chi \in \Gamma$ it follows that α and β belong to $M(2,p)$. But α is actually a Riesz product so, by [5] or [7], $\limsup_{i\in J}|a_i|^2 \leq 2/p$. By the previous proposition $\limsup_{i\in K}|a_i|^2 \leq 2/p$. ∎

COROLLARY 1.6. *A one-sided Riesz product φ maps L^2 to L^p for some $p > 2$ if and only if* $\limsup|\varphi(\chi)| < 1$.

Proof. Necessity has already been established. For sufficiency, assume $|\varphi(\chi_i)| \leq 1 - \delta < 1$ for all $i \geq k$ and let $\varphi_1 = \prod_{i=k}^{\infty}(1 + \varphi(\chi_i)\chi_i)$. Let φ_1^N denote the composition of φ_1 with itself N times. If N is chosen sufficiently large, and μ is the Riesz product $\mu = \prod_{i=k}^{\infty}(1 + (\chi_i + \overline{\chi}_i)/4)$ then $|\varphi_1^N(\chi)| \leq |\widehat{\mu}(\chi)|$ for all $\chi \in \Gamma$. By [13], $\mu \in M(2,p)$ for some $p > 2$, hence $\varphi_1^N \in M(2,p)$. An interpolation argument ([8, 1.3]) shows that $\varphi_1 \in M(2,q)$ for some $2 < q \leq p$. As φ is a finite linear combination of translates of φ_1, the multiplier $\varphi \in M(2,q)$. ∎

2. L^p-Improving Riesz products and one-sided Riesz products. Perhaps the most difficult problem in the study of (L^2, L^p) multipliers, and the one with the least satisfactory solutions, is of finding good (and practical) sufficient conditions to describe the $p > 2$ for which a multiplier φ maps L^2 to L^p. Other than for monotonic functions ([5, 2.2]), optimal sufficient conditions are known only for certain (one-sided) Riesz products.

In Chapter 3 of [2], Bonami showed that the Riesz products $\prod(1 + re_n(\chi))$ on D^∞ and $\prod(1 + 2r\cos x_j)$ on T^∞ belong to $M(p,q)$ if and only if $r^2 \leq (p-1)/(q-1)$, and for even integers p the one-sided Riesz products $\prod(1 + re^{ix_j})$ on T^∞ belong to $M(2,p)$ if and only if $r^2 \leq 2/p$. In contrast, our Proposition 1.4 shows that there are Riesz products μ satisfying $\limsup|\widehat{\mu}|^2 \leq (p-1)/(q-1)$ but with $\mu \notin M(p,q)$, answering [5, 3.2(vi)], and similarly that there are one-sided Riesz products φ with $\limsup|\varphi|^2 \leq 2/p$ but with $\varphi \notin M(2,p)$. In this section we characterize a more general class of L^p-improving (one-sided) Riesz products and as a corollary extend Bonami's result on one-sided Riesz products to all $p > 2$.

THEOREM 2.1. *Let $p > 2$ and let $\{r_j\}$ and $\{t_j\}$ be sequences of complex numbers such that $|t_j|^2 \geq 2/p$ and $|r_j|^2 \geq 1/(p-1)$. Let $\varphi = \prod(1 + t_je^{ix_j})$ be a one-sided Riesz product on T^∞, and $\mu = \prod(1 + r_je^{ix_j} + \overline{r}_je^{-ix_j})$ be a Riesz product on T^∞. Then $\varphi \in M(2,p)$ if and only if $\sum(|t_j|^2 - 2/p)^2 < \infty$, and $\mu \in M(2,p)$ if and only if $\sum(|r_j|^2 - 1/(p-1))^2 < \infty$.*

Proof. Notice that the characters defined on T^∞ by $(x_k) \mapsto e^{ix_j}$ satisfy the "further independence condition" of Proposition 1.4, thus necessity is clear in both cases.

To prove sufficiency we need the following lemma which is a straightforward modification of [2, p. 374].

LEMMA 2.2. *Let $\varphi = \prod(1 + a_j e^{ix_j} + b_j e^{-ix_j})$ be a multiplier on T^∞. For each n let $\varphi_n = 1 + a_n e^{ix_n} + b_n e^{-ix_n}$ and let $||\varphi_n||_{p,q}$ denote the norm of φ_n as an operator from L^p to L^q. Then $\varphi \in M(p,q)$ if and only if $\prod ||\varphi_n||_{p,q} < \infty$ and in this case $||\varphi||_{p,q} \le \prod ||\varphi_n||_{p,q}$.*

Proof of Theorem 2.1 (ctd.). *Sufficiency for one-sided Riesz products.* Let $p = 2s$ (so $s > 1$) and set $\varepsilon_n = |t_n|^2 - 2/p$. Let $s_0 = 1$ and let

$$s_k = \frac{s(s-1)\ldots(s-k+1)}{k!} \quad \text{if } k \neq 0\,.$$

Thus $s_k = \binom{s}{k}$ if s is an integer (where $\binom{s}{k} = 0$ if $k > s$). One can easily check that $0 \le s_k \le s^k/k!$ if $k \le [s] + 1$ and $|s_k| \le s^{[s]+1}/(k(k-1))$ if $k > [s] + 1$.

Certainly the assumption that $\sum \varepsilon_n^2 < \infty$ implies that $\varepsilon_n \to 0$ so we may choose N so that for all $n > N$ we have $|t_n| < 1$, $s_k(1/s + \varepsilon_n)^k < 3/4$ if $k = 2, 3, \ldots, [s]$, and $\varepsilon_n < \varepsilon = \varepsilon(s)$ where $0 < \varepsilon \le 1 - 1/s$ will be specified later.

It is easy to see that if $\varphi_n = 1 + t_n e^{ix}$ then

$$||\varphi_n||_{2,p} = \sup_b \frac{||1 + bt_n e^{ix}||_p}{||1 + be^{ix}||_2}\,.$$

CLAIM. *For $|r| \le 1$ and any complex number b with $|b| > 1$, $|1 + bre^{ix}| \le |\bar{b} + re^{ix}|$.*

To prove this observe that

$$|\bar{b} + re^{ix}|^2 - |1 + bre^{ix}|^2 = |b|^2 - 1 + |r|^2 - |br|^2\,.$$

The latter expression is a decreasing function of $|r|^2$, whose value at $|r|^2 = 1$ is zero. This proves the claim.

From this inequality we see that if $|b| > 1$ and $|t_n| \le 1$ then

$$||1 + bt_n e^{ix}||_p \le ||\bar{b} + t_n e^{ix}||_p = |b|\,||1 + \bar{b}^{-1} t_n e^{ix}||_p\,.$$

As $||1 + be^{ix}||_2 = |b|\,||1 + \bar{b}^{-1} e^{ix}||_2$ and $|\bar{b}^{-1}| < 1$ it follows that in computing $||\varphi_n||_{2,p}$, for $n \ge N$, we need only take the supremum over $b \in \mathbb{C}$ with $|b| \le 1$. By taking limits we may further reduce to

$$||\varphi_n||_{2,p} = \sup_{|b|<1} \frac{||1 + bt_n e^{ix}||_p}{||1 + be^{ix}||_2}\,.$$

Thus we now assume $|b| < 1$ and $n \geq N$. Compute the Taylor series expansion for

$$(1 + bt_n e^{ix})^s = \sum_{k=0}^{\infty} s_k (bt_n)^k e^{ikx}$$

(of course the sum terminates at $k = s$ if s is an integer). Since $|bt_n e^{ix}| \leq |b| < 1$ this series converges uniformly so

$$||1 + bt_n e^{ix}||_p^p = ||(1 + bt_n e^{ix})^s||_2^2 = \sum_{k=0}^{\infty} s_k^2 |bt_n|^{2k} ,$$

and this series converges absolutely. Also,

$$||1 + be^{ix}||_2^p = (1 + |b|^2)^s = \sum_{k=0}^{\infty} s_k |b|^{2k} ,$$

and this series converges absolutely as well.

We must estimate

$$\begin{aligned} \frac{||1 + bt_n e^{ix}||_p^p}{||1 + be^{ix}||_2^p} &= \frac{\sum_{k=0}^{\infty} s_k^2 |bt_n|^{2k}}{\sum_{k=0}^{\infty} s_k |b|^{2k}} \\ &= 1 + \frac{s^2|b|^2\varepsilon_n + |b|^4 \sum_{k=2}^{\infty} s_k |b|^{2(k-2)} (s_k(1/s + \varepsilon_n)^k - 1)}{(1 + |b|^2)^s} . \end{aligned}$$

We break the infinite sum into two terms:

(i) $$\sum_{k=2}^{[s]} s_k |b|^{2(k-2)} (s_k(1/s + \varepsilon_n)^k - 1)$$

(If $[s] = 1$ this term is not present.)

(ii) $$\sum_{k=[s]+1}^{\infty} s_k |b|^{2(k-2)} (s_k(1/s + \varepsilon_n)^k - 1)$$

(If s is an integer this term is not present.)

In (i) the choice of $n \geq N$ ensures that $s_k(1/s + \varepsilon_n)^k - 1 < -1/4$, and as $s_k > 0$ for $k = 2, \ldots, [s]$ the first sum is at most $-s_2/4$ if $[s] \neq 1$.

Sum (ii) we further break down as

$$\sum_{k=[s]+1}^{\infty} s_k |b|^{2(k-2)} (s_k s^{-1} - 1) + \sum_{k=[s]+1}^{\infty} s_k^2 |b|^{2(k-2)} ((1/s + \varepsilon_n)^k - s^{-k}) .$$

By the mean-value theorem and the assumption that $\varepsilon_n \leq \varepsilon \leq 1 - 1/s$,

$$(1/s + \varepsilon_n)^k - s^{-k} \leq \varepsilon_n k (1/s + \varepsilon_n)^{k-1} \leq \varepsilon k .$$

Thus for some constant $C_1(s)$,

$$\sum_{k=[s]+1}^{\infty} s_k^2|b|^{2(k-2)}((1/s+\varepsilon_n)^k - s^{-k}) \le \sum_{k=[s]+1}^{\infty} \left(\frac{s^{[s]+1}}{k(k-1)}\right)^2 |b|^{2(k-2)}\varepsilon k$$
$$\le |b|^{2([s]-1)} \le C_1(s)\,.$$

Clearly $\{s_k(s_k s^{-k}-1)\}_{k=[s]+1}^{\infty}$ is an alternating sequence tending to zero, with first term negative. We claim that it is a (strictly) decreasing sequence (in absolute value). To prove this we first remark that as $s_{k+1}/s_k = (s-k)/(k+1)$ it suffices to show that for $k \ge [s]+1$,

$$\frac{s_k}{s^k}\left(k+1+\frac{(k-s)^2}{(k+1)s}\right) < s+1\,.$$

Since $|s_k s^{-k}| \le 1/(k(k-1))$ and $k^2+1+s+k^2/s \le 2k(k+1)$,

$$\frac{s_k}{s^k}\left(k+1+\frac{(k-s)^2}{(k+1)s}\right) \le \frac{1}{k(k-1)}\left(\frac{2k(k+1)}{k+1}\right) = \frac{2}{k-1} < s+1\,,$$

as desired. Hence the first sum in (ii) is at most the sum of its first two terms, which is at most $|b|^{2([s]-1)}C_2(s)$ where $C_2(s) < 0$. If $\varepsilon > 0$ is chosen so that $\varepsilon C_1(s) < |C_2(s)|/2$ then sum (ii) is negative, and more specifically, if $[s] = 1$ then (ii) is at most $C_2(s)/2$.

Combining (i) and (ii) we get

$$\sum_{k=2}^{\infty} s_k|b|^{2(k-2)}(s_k(1/s+\varepsilon_n)^k - 1) \le C_3(s) \equiv \begin{cases} -s_2/4 & \text{if } [s] \ne 1, \\ C_2(s)/2 & \text{if } [s] = 1. \end{cases}$$

Thus for $|b| < 1$ and $n \ge N$,

$$\frac{||1+bt_ne^{ix}||_p^p}{||1+be^{ix}||_2^p} \le 1 + \frac{s^2|b|^2\varepsilon_n + |b|^4C_3(s)}{(1+|b|^2)^s}\,.$$

If $|b|^2 \le 2s^2\varepsilon_n/|C_3(s)|$ then clearly

$$\frac{||1+bt_ne^{ix}||_p^p}{||1+be^{ix}||_2^p} \le 1 + \varepsilon_n^2 C_4(s)$$

for $C_4(s) = 2s^4/|C_3(s)|$, while if $2s^2\varepsilon_n/|C_3(s)| \le |b|^2 < 1$,

$$\frac{||1+bt_ne^{ix}||_p^p}{||1+be^{ix}||_2^p} \le 1 + \frac{s^2|b|^2(\varepsilon_n - 2\varepsilon_n)}{(1+|b|^2)^s} \le 1\,.$$

Thus $||\varphi_n||_{2,p} \le (1+\varepsilon_n^2 C_4(s))^{1/p}$ whenever $n \ge N$. As $||\varphi_n||_{2,p} < \infty$ for all n, $\prod ||\varphi_n||_{2,p} < \infty$ when $\sum \varepsilon_n^2 < \infty$. By Lemma 2.2, $\varphi \in M(2,p)$.

Sufficiency for Riesz products. The proof is similar to that for one-sided Riesz products so only the main ideas will be sketched.

Let $\varphi_n = 1 + r_n e^{ix} + \bar{r}_n e^{-ix}$. We need to bound $||\varphi_n||_{2,p}$. Since $\mu \in M(2,p)$ if and only if $\prod(1 + |r_n|(e^{ix_n} + e^{-ix_n})) \in M(2,p)$, without loss of generality we may assume $r_n \geq 0$. Since this operator maps real-valued functions to real-valued functions, Bonami [2, p. 377] has shown that

$$||\varphi_n||_{2,p} = \sup_{b\in\mathbb{R}} \frac{||1 + br_n \cos x||_p}{||1 + b\cos x||_2}\,.$$

For $0 \leq r \leq 1$ and $|b| > 1$

$$|1 + br\cos x| \leq |b + r\cos x| = |b|\,|1 + rb^{-1}\cos x|\,.$$

This simple inequality shows that whenever $r_n \leq 1$ then in computing $||\varphi_n||_{2,p}$ we may restrict ourselves to $|b| \leq 1$. Choose N so that $r_n < 1$ for $n \geq N$ and let $p = 2s$.

The power series expansion of $(1+x)^{2s}$ converges uniformly on $[-\alpha, \alpha]$ for any $\alpha < 1$, thus for $n \geq N$ and $|b| \leq 1$

$$\begin{aligned} ||1 + br_n \cos x||_p^p &= \sum_{k=0}^{\infty} \int_0^{2\pi} (2s)_k (br_n)^k \cos^k x\, dx \\ &= 1 + \sum_{k=1}^{\infty} (2s)_{2k} (br_n)^{2k} \frac{(2k-1)(2k-3)\ldots 1}{2k(2k-2)\ldots 2}\,. \end{aligned}$$

and the latter series converges absolutely. (Of course, this is a finite sum if $2s$ is an integer.) It follows that

$$\frac{||1 + br_n\cos x||_p^p}{||1 + b\cos x||_2^p} = 1 + \frac{\sum_{k=1}^{\infty} \frac{s_k}{2^k} b^{2k}\left[\left(\frac{1}{2s-1} + \varepsilon_n\right)^k \frac{(2s-1)(2s-3)\ldots(2s-2k+1)}{k!} - 1\right]}{(1 + b^2/2)^s}\,.$$

Let

$$a_k(s) \equiv a_k \equiv \frac{1}{(2s-1)^k} \frac{(2s-1)\ldots(2s-2k+1)}{k!}\,.$$

When $s \geq 3/2$ then $(2s-1) \geq 2$ and with this observation it is not hard to show that $|a_k| \leq 1/k$. (It is helpful to consider the cases $[2s]$ an even or odd integer separately.) Also, $\{s_k(a_k - 1)/2^k\}_{k=[s]+1}^{\infty}$ is an alternating sequence which is decreasing (in absolute value) to zero and with first term negative. Thus arguments similar to those used for the one-sided Riesz products show that $||\varphi_n||_{2,p} \leq (1 + C(s)\varepsilon_n^2)^{1/p}$ for $n \geq N$.

When $1 < s < 3/2$, the factors $(2s)_{2k}$ are negative for $k \geq 2$. Thus

$$||1 + br_n \cos x||_p^p \leq 1 + (2s)_2 (br_n)^2/2\,.$$

Hence

$$\frac{||1+br_n\cos x||_p^p}{||1+b\cos x||_2^p} \le 1 + \frac{\frac{1}{4}2s(2s-1)b^2\varepsilon_n - \sum_{k=2}^{\infty} s_k(b^2/2)^k}{(1+b^2/2)^s}\,.$$

Since $\{s_k/2^k\}$ is an alternating sequence which is decreasing (in absolute value) to zero and with first term positive, the same sort of arguments as before again prove that $||\varphi_n||_{2,p} \le (1+C(s)\varepsilon_n^2)^{1/p}$ for $n \ge N$.

Since $\varphi_n \in M(2,p)$ for all n we can conclude (in either case) that $\varphi \in M(2,p)$ when $\sum \varepsilon_n^2 < \infty$. ■

An obvious corollary to this theorem is

COROLLARY 2.3. *The one-sided Riesz product* $\varphi = \prod(1+re^{ix_j})$ *belongs to* $M(2,p)$ *if and only if* $|r| \le \sqrt{2/p}$. ■

The next corollary is in the same spirit as [2, p. 387].

COROLLARY 2.4. *Let* $1 < p \le 2 < q < \infty$ *and*

$$|r_n|^2 = \frac{p-1}{q-1} + \varepsilon_n$$

where $\varepsilon_n \ge 0$. *Then the Riesz product* μ *on* T^∞ *given by* $\mu = \prod(1+2r_j\cos x_j)$ *belongs to* $M(p,q)$ *if and only if* $\sum \varepsilon_n^2 < \infty$.

Proof. First we prove sufficiency. Let $t_n = r_n/\sqrt{p-1}$, $\nu_1 = \prod(1+2\sqrt{p-1}\cos x_j)$ and $\nu_2 = \prod(1+2t_n\cos x_j)$. Clearly μ is the composition of the multipliers ν_1 and ν_2. Since $|t_n|^2 = r_n^2/(p-1) \ge 1/(q-1)$ and

$$\sum\left(|t_n|^2 - \frac{1}{q-1}\right)^2 = \frac{1}{(p-1)^2}\sum\left(|r_n|^2 - \frac{p-1}{q-1}\right)^2 < \infty\,,$$

by the theorem $\nu_2 \in M(2,q)$. If $1/p + 1/p' = 1$ then $p-1 = 1/(p'-1)$, so $\nu_1 \in M(2,p') = M(p,2)$. Therefore $\mu \in M(p,q)$.

Necessity follows from Proposition 1.4. ■

EXAMPLE 2.5. Let $1 < p \le 2 < q < \infty$. The multiplier on T^∞ given by

$$\varphi = \prod(1+2\sqrt{a_n}\cos x_n) \quad \text{where} \quad a_n = \frac{p-1}{q-1} + \frac{1}{\sqrt{n}}$$

does not belong to $M(p,q)$ but does belong to $M(s,t)$ for all $1 < s \le 2 < t < \infty$ satisfying $(p-1)/(q-1) < (s-1)/(t-1)$.

Proof. By the previous corollary $\varphi \notin M(p,q)$. Suppose $(s-1)/(t-1) > (p-1)/(q-1)$. Let $\varphi_1 = \prod(1+2\sqrt{s-1}\cos x_n)$ and $\varphi_2 = \prod(1+2\sqrt{a_n/(s-1)}\cos x_n)$. Clearly $\varphi_1 \in M(s,2)$ and as $a_n/(s-1) < 1/(t-1)$ for n sufficiently large, $\varphi_2 \in M(2,t)$. Since φ is the composition of φ_1 and φ_2, we see that $\varphi \in M(s,t)$. ■

Just as in [2, pp. 392–393] the following is another consequence of Theorem 2.1:

COROLLARY 2.6. *Let* $p > 2$ *and let* $\{n_i\}$ *be a lacunary sequence of positive integers satisfying* $n_{i+1}/n_i \geq 3$. *Then* $\varphi = \prod(1 + re^{in_jx}) \in M(2,p)$ *if* $|r| \leq \sqrt{1/2p}$, *and if in addition* $\sum n_i/n_{i+1} < \infty$, *then* $\varphi \in M(2,p)$ *if and only if* $|r| \leq \sqrt{2/p}$.

We will omit the proofs as they are similar to the corresponding results in [2].

Let e_n be the character on D^∞ given by $e_n((x_j)) = x_n$. Similar arguments to those used in Theorem 2.1 enable one to prove

PROPOSITION 2.7. *Let* $1 < p \leq 2 < q < \infty$ *and let* $|r_n|^2 = (p-1)/(q-1) + \varepsilon_n$ *with* $\varepsilon_n \geq 0$. *Then the Riesz product* $\mu = \prod(1 + r_n e_n(x))$ *on* D^∞ *belongs to* $M(p,q)$ *if and only if* $\sum \varepsilon_n^2 < \infty$.

We leave the details to the reader.

3. Computation of $\Lambda(p)$ constants. Let $p > 2$. A subset E of Γ is called a $\Lambda(p)$ *set* if there is a constant C_p such that $\|f\|_p \leq C_p\|f\|_2$ for all $f \in \{g \in L^2 : \operatorname{supp}\widehat{g} \subseteq E\}$. The least such constant C_p is called the $\Lambda(p)$ *constant* of E and is denoted by $\Lambda(E,p)$. For standard results on $\Lambda(p)$ sets see [10] or [15].

Let $\{\chi_i\} \subseteq \Gamma$ be a dissociate set. Sets of the form

$$\Big\{\prod \chi_i^{\varepsilon_i} : \sum |\varepsilon_i| \leq n,\ \varepsilon_i = 0, \pm 1 \text{ (or } \varepsilon_i = 0,1)\Big\}$$

are well known examples of $\Lambda(p)$ sets for all $2 < p < \infty$, but are not Sidon sets. Using mainly combinatorial methods Bonami found estimates for the $\Lambda(p)$ constants of such sets [2, Ch. 2]. She then used her estimates in the proof of her result for (L^2, L^p) one-sided Riesz products. Here we take the opposite approach and use the earlier results of this paper to improve upon Bonami's estimates of $\Lambda(p)$ constants (when they are not already optimal). The connection between the two subjects is due to the following theorem which almost characterizes (L^2, L^p) multipliers.

THEOREM 3.1 ([8]). *Let* φ *be a bounded function on* Γ *and for each* $\varphi > 0$ *let* $E(\varphi) = \{\chi : |\varphi(\chi)| \geq \varepsilon\}$. *If* $\varphi \in M(2,p)$ *for some* $p > 2$, *then for each* $\varepsilon > 0$, $E(\varepsilon)$ *is a* $\Lambda(p)$ *set and* $\Lambda(E(\varepsilon),p) \leq \|\varphi\|_{2,p}\varepsilon^{-1}$. *If* $E(\varepsilon)$ *is a* $\Lambda(p)$ *set for every* $\varepsilon > 0$ *and* $\Lambda(E(\varepsilon),p) = O(\varepsilon^{-1})$, *then* $\varphi \in M(2,r)$ *for all* $r < p$.

Before applying this theorem it is convenient to establish some notation.

Notation. Let

$$T_k = \Big\{(n_i) \in \sum \mathbb{Z} : n_i = 0, \pm 1,\ \sum |n_i| = k\Big\},$$

$$T_k^+ = \Big\{(n_i) \in \sum \mathbb{Z} : n_i = 0, 1, \ \sum |n_i| = k\Big\},$$
$$\Gamma_k = \Big\{(\varepsilon_i) \in \sum \mathbb{Z}(2) : \sum \varepsilon_i = k\Big\}.$$

Given $E \subseteq \mathbb{Z}$ let

$$E_k = \Big\{\sum \varepsilon_i n_i : \varepsilon_i = 0, \pm 1, \ n_i \in E, \ \sum |\varepsilon_i| = k\Big\},$$
$$E_k^+ = \Big\{\sum \varepsilon_i n_i : \varepsilon_i = 0, 1, \ n_i \in E, \ \sum |\varepsilon_i| = k\Big\}.$$

Given two real-valued functions, F and G, defined on $\mathbb{N} \times (2, \infty)$, we will say that F is *exactly dominated* by G if for every $2 < p < \infty$, $F(k,p) \le G(k,p)$ for all $k \in \mathbb{N}$, and for every $2 < q < p$, $\limsup_k F(k,p)/G(k,q) = \infty$.

PROPOSITION 3.2. *Let $p > 2$. Then both $\Lambda(T_k, p)$ and $\Lambda(\Gamma_k, p)$ are exactly dominated by $(p-1)^{k/2}$, and $\Lambda(T_k^+, p)$ is exactly dominated by $(p/2)^{k/2}$.*

Proof. Let $\varphi = \prod (1 + \sqrt{2/p} e^{ix_j})$ be a one-sided Riesz product on T^∞. Then φ is an (L^2, L^p) multiplier and

$$E((2/p)^{k/2}, \varphi) \equiv \Big\{(n_i) \in \sum \mathbb{Z} : |\varphi((n_i))| \ge (2/p)^{k/2}\Big\} = \bigcup_{j=1}^{k} T_j^+ .$$

The proof of Theorem 2.1 shows that $||\varphi||_{2,p} = 1$, thus Theorem 3.1 gives $\Lambda(T_k^+, p) \le (p/2)^{k/2}$. Suppose $\limsup_k \Lambda(T_k^+, p) \le C(q/2)^{k/2}$ for some $2 < q < p$. As T_k^+ is a $\Lambda(p)$ set for every k there is a constant C_1 such that

$$\Lambda\Big(\bigcup_{j=1}^{k} T_k^+, p\Big) \le k \sup_{1 \le j \le k} \Lambda(T_j^+, p) \le C_1 (q/2)^{k/2},$$

Let $\varphi_1 = \prod (1 + \sqrt{2/q} e^{ix_j})$. Since $E((2/q)^{k/2}, \varphi_1) \subseteq \bigcup_{j=1}^k T_k^+$ the converse direction of Theorem 3.1 tells us $\varphi_1 \in M(2, r)$ for every $r < p$. But this is false for $r > q$.

The estimates of the $\Lambda(p)$ constants for the sets T_k and Γ_k follow similarly from Theorem 3.1 and [2, p. 376, 385]. ∎

PROPOSITION 3.3. *Let $E = \{n_i\}$ be a lacunary set of positive integers satisfying $n_{i+1}/n_i \ge 3$ for all i.*

(a) *$\Lambda(E_k^+, p) \le (2p)^{k/2}$ and $\Lambda(E_k, p) \le (4(p-1))^{k/2}$ for all $k \in \mathbb{N}$.*

(b) *If $\sum n_i/n_{i+1} < \infty$, then for some constant C, $\Lambda(E_k^+, p)$ and $\Lambda(E_k, p)$ are exactly dominated by $C(p/2)^{k/2}$ respectively.*

Proof. The proof is similar using Corollary 2.6 and [2, pp. 392–393]. We remark that in (a) the (L^2, L^p) operator norm of the appropriate multiplier can be shown to be 1. ∎

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