Reza Sarhangi

The Sky Within: Mathematical Aesthetics of Persian Dome Interiors

The decorations of domes represent the sky, heaven and what lies beyond the "seven skies". Reza Sarhangi examines the spatial effects of domes interiors and the geometric construction of the designs.

Introduction

The decoration of dome interiors, in some cases similar to the decoration of pavement, windows, and walls, is closely related to geometrical properties of shapes, in both two- and three-dimensional space, known to artists several centuries ago. Based on existing domes, we cannot trace this art back beyond ten centuries, for there were many natural and social disasters that destroyed them. However, we may be reasonably certain of a much earlier existence of such sophisticated designs of dome interiors based on references in earlier literature as well as the level of geometry available in those times.

Mathematics had its crude beginnings perhaps fifty centuries ago, in the civilizations of the Middle East. For the Babylonians and the Egyptians it was a practical tool, essential in day-to-day living. Greeks, beginning with Thales of Miletus, established mathematics based on deductive reasoning rather than by trial and error. Pythagoras and his disciples continued the systematization effort initiated by Thales over the next two centuries. Euclid, a disciple of the Platonic school, was the last in the chain of great mathematicians of classical Greeks that brought earlier efforts to axiomatize the geometry to conclusion in his 13-volume book, *The Elements*. Liking the challenge, the Greeks set very tight limits on which tools were permissible for construction, essentially utilizing the compass and straight edge. With a few notable exceptions, almost all of the figures that were dealt with could be constructed using these two tools.

The works on geometry and geometrical constructions were translated and then collected later on in the Middle East somewhere between the seventh to the fourteenth century. Scholars such as Persian mathematician al-Khwarizmi, who was a member of the "House of Wisdom" in Baghdad in the early part of the eighth century, produced several manuscripts on arithmetic, algebra, and the use of Hindu numerals. Their translations of Greek geometry and collections of the methods from Hindus and Hebrews helped preserve important knowledge for later study in the West. The word *algorithm* is from the Latin translation of al-Khwarizmi and *algebra* comes from the title of one of his books. Alhazen, another Middle-Eastern mathematician, was the first to correct the understanding of seeing an object. Greeks thought that to see an object, light from the eye goes to the object. He correctly reversed this concept, providing the basis for the science of perspective. His treatise on optics was translated into Latin in 1270 as *Opticae thesaurus Alhazeni libri vii*.

The flourishing of geometry and geometrical designs and the challenge of using only compass and straight edge for creating intricate structures were in harmony with the beliefs of religious scholars of the Islamic empire that included North Africa, Spain, and a part of Eastern Europe and the Middle East. Artists were forbidden to represent people and living objects in their works, for these representations were perceived as idols replacing God. During a trip to Spain in 1936, Escher visited the Alhambra, a structure by Moors from North Africa that had first intrigued him in 1922. Afterwards he remarked, "This is the richest source of inspiration that I have ever struck… What a pity it is that the religion of the Moors forbade them to make graven images!"[1]

Two points must be emphasized here. First, even though the flourishing of tiling designs occurred during the Islamic Empire, this art is much older, as it was used in Babylonian constructions and handcrafts of the early Central Asian civilizations. Second, the practice of using geometrical designs rather than idols could possibly be traced to a much earlier period. Herodotus, in the fifth century BC, wrote: "It is not customary amongst Persians to have idols

Figure 1.
Stucco dome interior with another dome at the center.

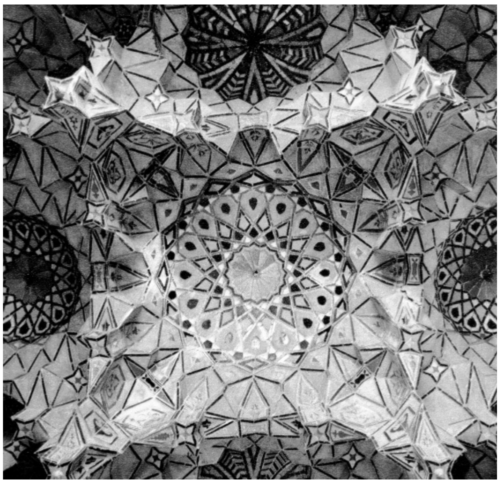

Figure. 2.
Stucco dome interior in a private house in Kashan.

made and temples built and alters erected, they even consider the use of them a sign of folly."[2] Herodotus' quote referred to temples for idol worship. Zoroastrian temples of ancient Persian tradition have been uncovered and dated as early as 2000 BC by archeologists.

The art of stucco

Stucco is a typically Persian art form for the decoration of dome interiors. In most cases it has accompanied the art of tessellation, ceramics, and mirror works. The fine work of elaborately carved stucco has survived for centuries and can be found in cities including Esfahan, Mashhad, and Kashan in Iran. Examples of stucco dome interiors similar to the Persian style are also located in the western hemi-sphere in the Alhambra in Granada, Spain.

A circular dome was generally supported on a square base with various corner squinch designs. The transition of interior designs from the building below to the dome above is achieved by constructions of several three dimensional shapes, such as wings of a star having the center as the center of the dome. In a stucco dome interior, the three-dimensional cuts as triangles, diamonds, or stars are designed in such a way that they have some corners toward the center. This gives a feeling of attraction of all other points on the dome toward the center. This reminds us of an attractor in a dynamical system -- a stable point that all states near it are attracted to. Figure 1 is an example of a stucco dome interior. The attractor of the design is inside a second dome. This second dome is above several windows that are constructed

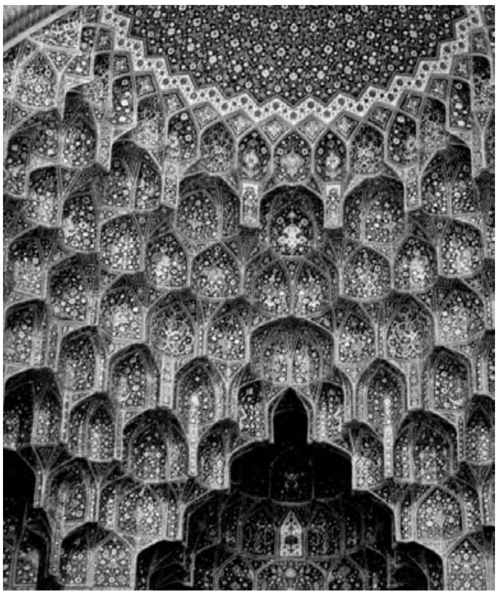

Figure 3.
Entrance portal of the Shah Mosque, Esfahan.

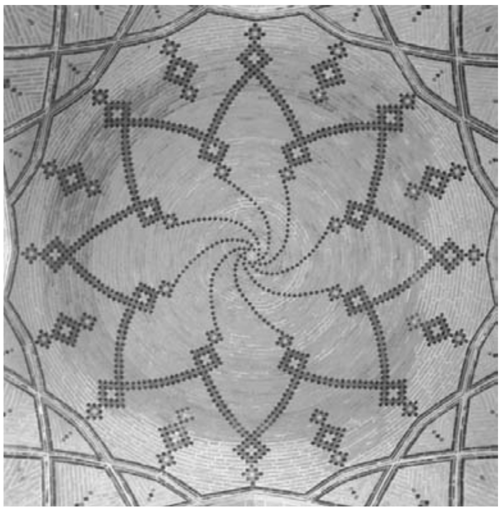

Figure 4.
Brick pattern construction interior dome in Kashan.

around a cylinder. In the summer, these windows allow a flow of air. People of the past used to cover these windows with bushes similar to tumbleweeds, compacted and drenched in water. The hot and strong summer wind of cities close to deserts would remove the drops of water from the bushes. This action takes energy and, as a result, a flow of cold air would come inside the building. It is worth mentioning that such a dome with a cooler system was usually built over the main room at the center, surrounded by smaller rooms containing smaller domes. In some cases, the pattern in the dome interior reveals more than one attractor. For instance, Figure 2 shows a dome with four attractors surrounding the main one at the center. Besides these five attractors, the dome has another twelve attractors in twelve holes with centers on the circumference of a circle larger than the circle that contains the center of four attractors. Figure 3 is a work of stucco that has been combined with the art of tiling. It is the entrance portal of the Shah Mosque, Esfahan, built by Shah Abbas the Great between 1611 and 1629. Figure 4 illustrates an example of a dome interior that uses the idea of attractors in combination with a brick pattern construction.

Self-similarity in stucco

Let us study the stucco dome interior design in Figure 5 in detail. The entire dome is an eight-winged star that has one attractor. Its symmetries comprise the dihedral group of order 16, D_8. This star has been divided into a second group of stars. These are stars with 4, 5, 6, and 7 wings made from mirrors. The 7-winged stars, heptagrams, are irregular, not all sides and angles are congruent. The stars are connected with geometrical cuts that are surrounded with stars with sharper wings that are pointed toward the center. In some cases, when the artist uses tiles instead of mirrors in order to cover the surface of each five-winged star, we have another sequence of stars that resembles the entire dome as we can see in Figure 6 and Figure 7. In this stage, because of limitation on the size of each tile, the artist does not continue to produce the next series of smaller stars. However, the idea of self-similarity is evident. We observe that the attractor of the five-winged design is at the center of a ten-winged star and it has the dihedral group of symmetries of order 10, D5. Figure 8 shows the design behind the work in the previous figure. The design can be constructed using only a compass and straight edge. To do this, the first step is to construct the surrounding five-winged star in Figure 8. This star can be constructed by dividing a circle into ten equal arcs. To accomplish this, we divide a circle into five arcs as illustrated in Figure 9 and then divide each arc in half using a compass and straight edge. We also may directly divide it into ten arcs as in Figure 10.[3] It is worth mentioning that the larger part of the golden cut of the radius of a circle divides the circle into 10 equal arcs.[4] We connect the endpoint of one of these ten arcs in Figure 10, calling the first point A, to the fourth endpoint D, clockwise, and continue. We can then construct a 3/10 star polygon (Figure 11). Number 3 indicates the number of vertices skipped between each pair of connected vertices, while 10 is the total number of vertices. Using this star polygon we can construct the five-winged star that we needed. This star polygon may be seen to be composed of five rhombi, one of which is rhombus AKFL in Figure 11. In Figure 12 we consider one of the five rhombi that construct this star (in fact, only one fourth of this rhombus can create the entire star design by using

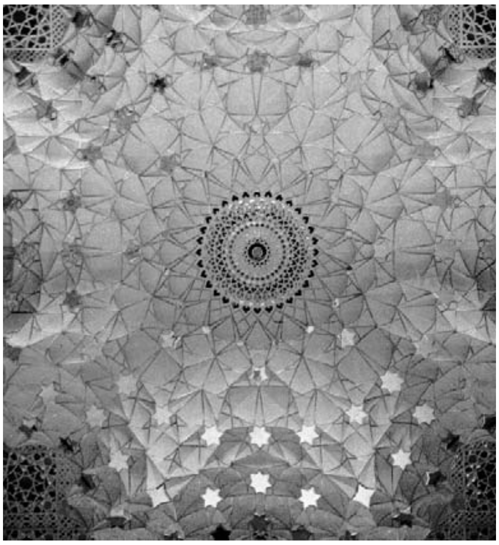

Figure 5.
Mirror work stucco dome interior.

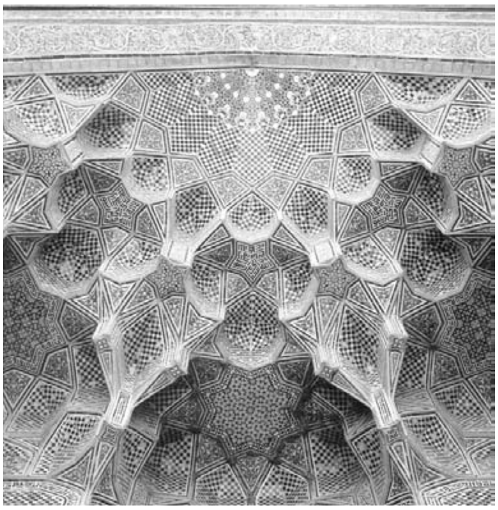

Figure 6.
Ali-Gholi Agha Mosque, Esfahan.

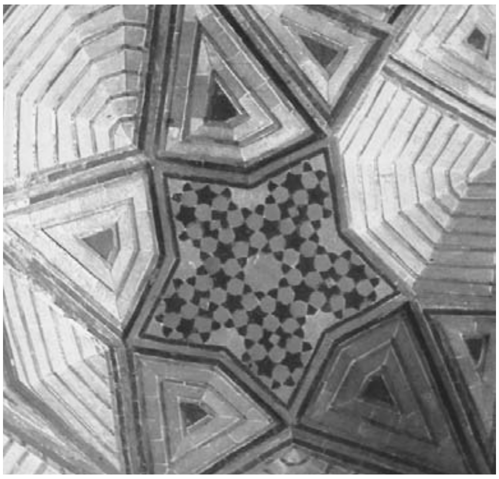

Figure 7.
Kaseh Garan School, Esfahan.

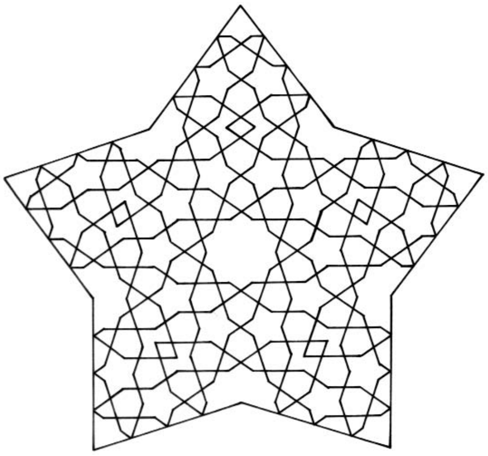

Figure 8.
"Skeletal" design of the five-winged star.

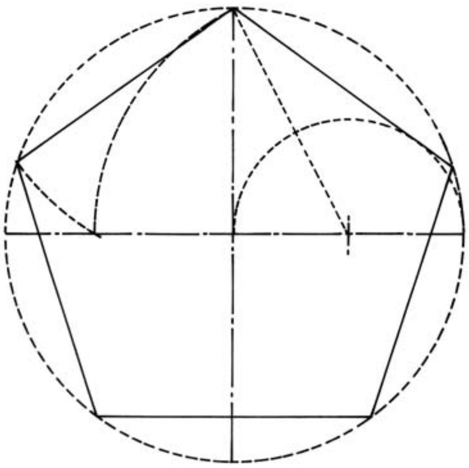

Figure 9.
The division of a circle into five equal arcs.

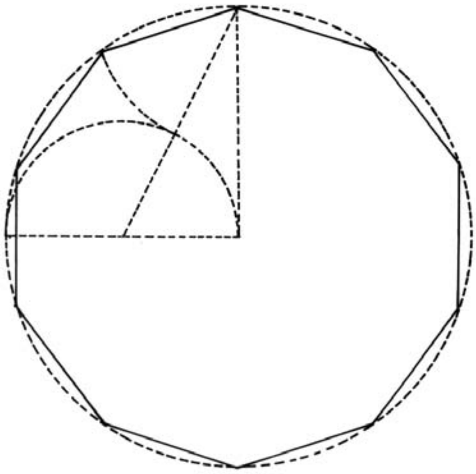

Figure 10.
The division of a circle into ten equal arcs.

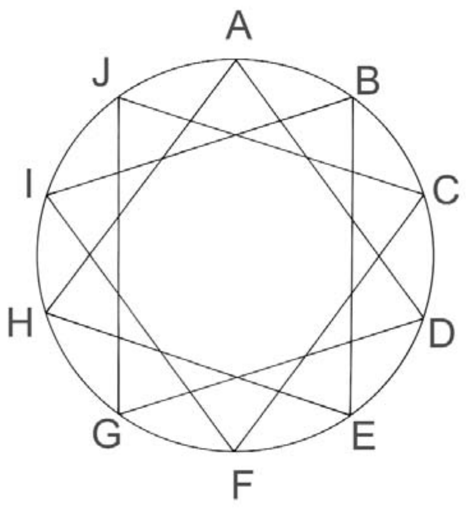

Figure 11.
Construction of a five-winged star with acute angle of 72 degrees. The star polygon also comprises five rhombi such as ALFK.

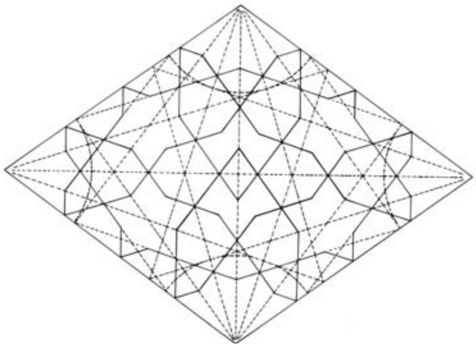

Figure 12.
Division of the rhombus using a compass and straight edge.

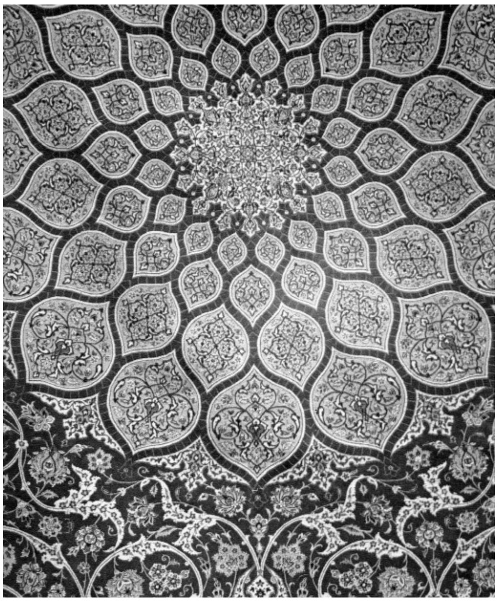

Figure 13.
Dome interior of the Marble Place, Tehran.

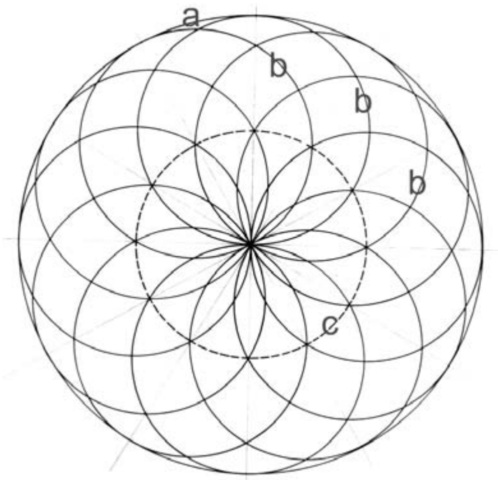

Figure 14.
Rosette design of 12 circles. a) *reference circle*; b) *radial circles*; c) *centrum ring*.

reflective and rotational symmetries). We divide the obtuse angle into six equal angles and the acute angle into four. To do this, if we look at Figure 11, we notice that the acute angle, such as <A, is an inscribed angle opposite to arc DH, clockwise. This arc has been divided into four equal arcs DE, EF, FG, and GH. Join A to these points and divide the acute angle to four equal angles. The obtuse angle is equal to angle <IAC in figure 11. This angle is opposite to arc CI, clockwise, which is divided into six equal arcs. With the same procedure as for the acute angle, we can divide this angle into six equal angles.

Let O be the intersection of two diagonals. Two lines C-4 and B-3 meet at E. We make an arc with the center of C and radius CE to find point F on DC. From F, we draw a line that is parallel to C-5. This line and D-1 meet at G. From G we make a parallel to D-3 to meet CD on H and C-5 on Z. We find L on CD such that DH = LC. From H we make a line parallel to AC and from the intersection of this line we make a parallel line to A-1. This gives us the quadrilateral with side HG. K is the midpoint of DC. From K we make two parallels, one with AC and the other with B-3. From L we make a line parallel to AC to meet C-5 on M and D-1 on N. R is the intersection of LR that is parallel to B-3 and AC. T is the intersection of C-4 and a line from N parallel to C-1. The intersection of a line from L parallel to B-3 and the line C-5 gives us a point. S is the intersection of AC and the parallel line from this point to C-1.

Rosette dome interiors

Figure 13 shows the interior of the dome of the Marble Palace in Tehran. The dome is closely modeled on the Sheik Lotfolah mosque in Esfahan, built by Shah Abbas early in the seventeenth century. The circular rosette pattern is created based on the arrangement of overlapping circles (Figure 14). Sixteen congruent circles, called *radial circles*, are arranged so that they have a point in common. Therefore, the centers of each radial circle lie on a circle, called the *centrum ring*, which is itself congruent with any of the radial circles. The common point is the center of the centrum ring. The outer circle, which is concentric with the centrum ring and whose radius is equal to the diameter of the radial circles, is the *reference circle*. Changing the number of radial circles or increasing the diameter of the radial circles with respect to the radius of the reference circle, produces different rosettes.[5] Figures 15 and Figure 16 present the design of another dome interior based on the division of a circle into 16.

The designs for dome interiors, and other designs for walls and pavements, were constructed by artist-geometers. They were very familiar with the Euclidean geometry theorems and properties. These designs were normally gathered by stucco makers and other artist-constructors, who would pass them along to the next generation. The designs were graphed on a scroll. Ink pens were used for major lines. However, all circles were sketched with a compass without lead. Both end points of the compass were sharp metal. The metal etched barely visible grids onto the scroll. Then using straight edge, they drew the design with ink. Today, these scrolls have been disappearing.

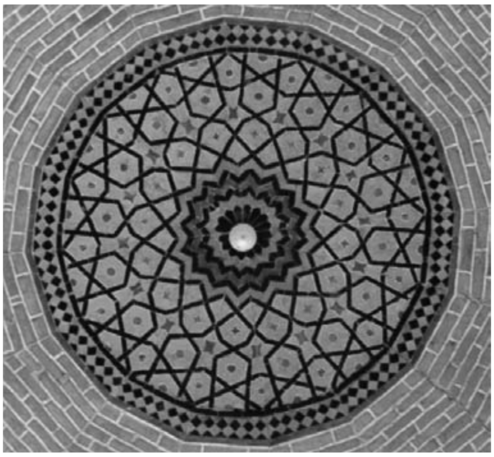

Figure 15.
A dome interior design and its constuction based on the division of the circle into sixteen parts.

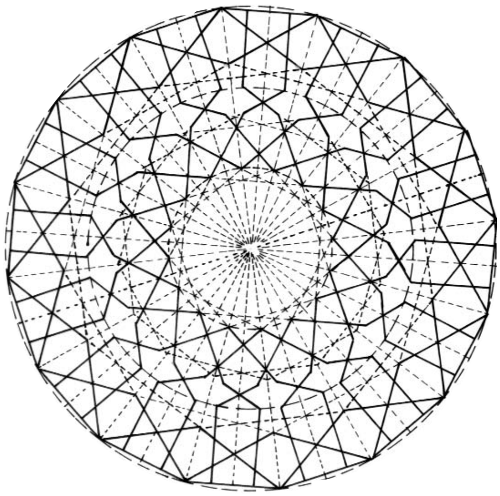

Figure 16.
The construction for the dome interior in Figure 15.

The missing regular heptagon

Even though a number such as seven has mystical and religious importance in eastern cultures, we don't observe any dome design or any other wall or floor designs that incorporate the regular heptagon or regular seven-winged stars. In fact, the geometrical structures of designs introduced in previous sections include regular 3, 4, 5, 6, 8, and 10 but misses 7-and 9-gons. The reason for it may very well be related to the idea of the constructable regular polygons.

The ancient mathematicians discovered how to construct regular polygons of 3, 4, 5, 6, 8 and 10 sides using a compass and straight-edge alone. The list of other constructable regular polygons known to them included 15-gons and any polygon with twice sides as a given constructable polygon. No matter how much effort was expended in the exercise, mathematicians were not successful in constructing a regular heptagon by compass and straight-edge or in proving that the construction is impossible until 1796 when Gauss, then a 19-year old student, proved the impossibility of its construction.

In fact, he proved that, in general, construction of a regular polygon having an odd number of sides is possible when, and only when, that number is either a prime Fermat number (a prime of the form 2k + 1, where k=2n) or is made up by multiplying together different Fermat primes. Such a construction is not possible for 7 nor 9. Gauss first showed that a regular 17-gon is constructible, and after a short period he completely solved the problem. It was this discovery, announced on June 1, 1796 but made on March 30th, that induced the young man

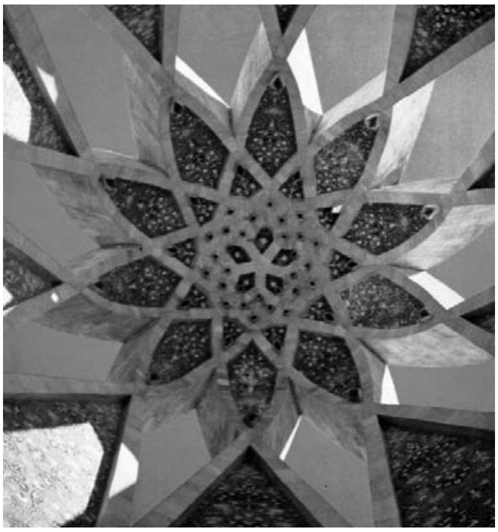

Figure 17.
Dome Interior of Kahayyam's Mausoleum.

Figure 18.
General view of Khayyam's Mausoleum.

to choose mathematics instead of philology as his life work. He requested that a regular 17-sided polygon to be engraved on his tombstone. The Cyclotomic Extensions is a topic that ties together results from modern algebra and ancient geometric construction problems. In this topic, Gauss' claim can be proved in a fairly short argument using Galois Theory. Of course, Gauss did not use Galois theory in his proof because of the simple reason that the proof occurred 15 years before Galois was born![6]

A jug of wine, a loaf of bread – and thou

Omar Khayyam, born in 1048 in Neyshabur, a city in Persia, was a mathematician and an astronomer. Nonetheless, his fame in the western hemisphere mainly comes from a paraphrase version of his *Rubaiyat* by Edward Fitzgerald, a collection of his quatrains, pieces of verse complete in four rhymed lines. He is chiefly responsible for revising the Jalali Solar Calendar which is still in official use in Iran. In his native home and the northern neighboring countries, which in a time constituted the Soviet Union, he is regarded as "the proof of Truth", the highest praise for a scientist.

In Khayyam's time, universities flourished in the Islamic world and many observatories were built. He was a professor at the Neyshabur Nazamieh, one of a series of university colleges founded by his contemporary Nezam Ol-Molk, a celebrated vice-minister. Khayyam studied and obtained original results in algebra. His work continued many of the main lines of development in 19th-century mathematics. Not only did he discover a general method of extracting roots of arbitrary high degree, but also his *Algebra* contains the first complete treatment of the solution of cubic equation.

Much of Khayyam's work in geometry centered around Euclid's fifth postulate, the parallel postulate. He contributed the idea of a quadrilateral with two congruent sides perpendicular to the base. The parallel postulate would be proved, he recognized, if he could show that the remaining two angles were right angles. In this he failed, but his question about the quadrilateral became the standard way of discussing the parallel postulate.

Although the question of Khayyam's personal religious beliefs remains a vexed one, the balance of scholarly opinion is that he was an orthodox theologian who wrote his quatrains as a private exercise in skepticism. He died in 1122. His mausoleum, built in recent years, is in Neyshabur. The entire mausoleum consists of a dome, open from every direction (Figures 17 and 18). Its design reflects a combination of traditional patterns and contemporary construction.

Conclusion

In Persian architecture, it was geometry that provided diverse stylistic developments for constructions and designs; not only to serve a function, but also to evoke an emotional response by harmonization of the constructional elements, such as domes and columns and decorative elements. The artists and architects of those times transferred the geometry into the art of harmonization, engaging feelings and emotions. The sophisticated geometry involved in dome interiors shows how artists try to express their feelings and emotions, as well as their beliefs and philosophy, through complex geometrical designs involving repetition, rhythm, pacing, scale, and color combination. The construction of stucco domes shows that they also were aware of the geometry of 3-dimensional Euclidean space. The designs reveal, through self-similarity, that the artists had a sense of fractal geometry.

Notes

1. D. Seymour and J. Britton, *Introduction to Tesselations*, p. 183.
2. F. Mehr, *The Zoroastrian Tradition*, p. 15.
3. Figure 10 is after a figure from *De re Aedificatoria* by Renaissance architect Leon Battista Alberti, first published in 1485. Cf. Leon Battista Alberti, *The Ten Books of Architecture*, Plate 21.
4. For examples in art illustrating the influence of the division of a circle into ten and five arcs, cf. Kappraff, *Connections: The Geometric Bridge Between Art and Science*, pp. 90 - 91.
5. For more about the rosette, see Williams, "Spirals and the Rosette in Architectural Ornament"; Williams, *Italian Pavements: Patterns in Space*, pp. 123-129.
6. A proof appropriate for an amateur mathematician can be found in Kazarinoff, *Ruler and the Round or Angle Trisection and Circle Division*. Gauss' approach can be found in *Disquisitiones Arithmeticae* and in Dickson, "Constructions with Ruler and Compasses; Regular Polygons".

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Reza Sarhangi was a high school math teacher, teacher trainer, drama teacher, play writer, play director and scene designer before coming to the US in 1986. After obtaining a Ph.D. in applied mathematics (Control Theory-Distributed Parameter Systems) from Wichita State University, Kansas, he joined the faculty of Southwestern College. Besides teaching mathematics courses, he teaches courses such as Math and Art, Chaos and Fractals, and Mathematical Bridges to Science. These courses are offered through the Integrative Studies Program -- the general studies component -- at Southwestern College. Recently he and a group of his students wrote and directed a play. He is the conference director and the proceedings editor of *Bridges: Mathematical Connections an Art, Music, and Science*.