

# The Small Maturity Implied Volatility Slope for Lévy Models

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# Introduction and motivation

- ▶ There are many results on asymptotics of implied volatility
- ▶ Our focus:
  - ▶ Lévy models
  - ▶ Slope of implied vol (strike derivative)
  - ▶ Short maturity
- ▶ Lee's moment formula + refinements: steepness of wings
- ▶ We focus on the ATM slope
- ▶ Diffusion models: limit smile as  $T \rightarrow 0$
- ▶ Lévy models: Implied vol explodes off-the-money, no limit smile

# Introduction and Motivation

- ▶ Good slope approximation for very short maturities
- ▶ Identify the sign of the slope for short maturities
- ▶ → simple parameter constraint for market calibration
- ▶ Qualitative influence of parameters on model behavior
- ▶ Our asymptotic method should be of general applicability
- ▶ Method extends to other time-asymptotics problems for Lévy processes (densities, option prices ...)

## Implied volatility slope and digitals

- ▶ For simplicity:  $S_0 = 1, r = 0$
- ▶ Implied volatility  $\sigma_{\text{imp}} = \sigma_{\text{imp}}(K, T)$ :

$$C_{\text{BS}}(K, \sigma_{\text{imp}}, T) = C(K, T) := \mathbb{E}[(S_T - K)^+]$$

- ▶ Implied volatility slope:

$$\partial_K \sigma_{\text{imp}} = -\frac{\partial_K C_{\text{BS}} - \partial_K C}{\partial_\sigma C_{\text{BS}}}$$

- ▶ If law of  $S_T$  is continuous:

$$\partial_K \sigma_{\text{imp}} = -\frac{\partial_K C_{\text{BS}} + \mathbb{P}[S_T \geq K]}{\partial_\sigma C_{\text{BS}}}$$

## Implied volatility slope and digitals

- ▶ More explicitly:

$$\partial_K \sigma_{\text{imp}} = \frac{\Phi(-\sigma_{\text{imp}}\sqrt{T}/2) - \mathbb{P}[S_T \geq K]}{K\sqrt{T} n(\sigma_{\text{imp}}\sqrt{T}/2)}$$

- ▶ Under mild assumptions we have  $\sigma_{\text{imp}}\sqrt{T} = o(1)$ ,  $T \rightarrow 0$ , and so

$$\partial_K \sigma_{\text{imp}} \sim \frac{\sqrt{2\pi}}{K\sqrt{T}} \left( \frac{1}{2} - \mathbb{P}[S_T \geq K] - \frac{\sigma_{\text{imp}}\sqrt{T}}{2\sqrt{2\pi}} + O((\sigma_{\text{imp}}\sqrt{T})^3) \right)$$

- ▶ Two cases for ATM slope asymptotics:
  - ▶  $\mathbb{P}[S_T \geq S_0] \not\rightarrow \frac{1}{2}$ : Need first order term of  $\mathbb{P}[S_T \geq S_0]$
  - ▶  $\mathbb{P}[S_T \geq S_0] \rightarrow \frac{1}{2}$ : Need second order term of  $\mathbb{P}[S_T \geq S_0]$  and first term of  $\sigma_{\text{imp}}\sqrt{T}$

## ATM digital calls: Small-time expansions for Lévy models

- ▶  $S_t = \exp(X_t)$ ,  $X$  Lévy process with characteristic triplet  $(\sigma, \nu, b)$

$$\log E[e^{zX_1}] = \frac{\sigma^2 z^2}{2} + bz + \int_{-\infty}^{\infty} (e^{zx} - 1 - zx\mathbf{1}_{\{|x|\leq 1\}}) \nu(dx).$$

- ▶ If  $\sigma > 0$ , then

$$\lim_{T \rightarrow 0} \mathbb{P}[S_T > S_0] = \lim_{T \rightarrow 0} \mathbb{P}[X_T > 0] = \frac{1}{2}$$

- ▶ If  $\sigma = 0$ , then the limit can attain all values from  $[0, 1]$ .

## Warm-up: Jump diffusions

- ▶ Log-underlying

$$X_T = \sigma W_T + bT + \text{compound Poisson}$$

- ▶ Let  $\tau$  be the first jump time. Then

$$\begin{aligned}\mathbb{P}[X_T > 0] &= \mathbb{P}[X_T > 0 | \tau \leq T] \cdot \mathbb{P}[\tau \leq T] \\ &\quad + \mathbb{P}[X_T > 0 | \tau > T] \cdot \mathbb{P}[\tau > T] \\ &= O(T) + \mathbb{P}[\sigma W_T + bT > 0](1 + O(T)) \\ &= \mathbb{P}[\sigma W_T + bT > 0] + O(T)\end{aligned}$$

- ▶ Hence

$$\mathbb{P}[X_T > 0] = \frac{1}{2} + \frac{b}{\sigma\sqrt{2\pi}}\sqrt{T} + O(T), \quad T \rightarrow 0.$$

## Warm-up: Jump diffusions

- ▶ Recall

$$\partial_K \sigma_{\text{imp}} \sim \frac{\sqrt{2\pi}}{K\sqrt{T}} \left( \frac{1}{2} - \mathbb{P}[S_T \geq K] - \frac{\sigma_{\text{imp}}\sqrt{T}}{2\sqrt{2\pi}} + O((\sigma_{\text{imp}}\sqrt{T})^3) \right)$$

- ▶ From  $\sigma_{\text{imp}} \rightarrow \sigma$  and

$$\mathbb{P}[X_T > 0] = \frac{1}{2} + \frac{b}{\sigma\sqrt{2\pi}}\sqrt{T} + O(T)$$

we get the ATM slope

$$\lim_{T \rightarrow 0} \partial_K \sigma_{\text{imp}}|_{K=1} = -\frac{b}{\sigma} - \frac{\sigma}{2}$$

- ▶ What about infinite activity?



## Case Study: The Normal Inverse Gaussian Model

- ▶ Moment generating function

$$\mathbb{E}[e^{zX_T}] = \exp\left(T\left(\frac{1}{2}\sigma^2 z^2 + bz + \delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}\right)\right)\right)$$

- ▶ First suppose  $\sigma = 0$ . Thus we need the first term of  $\mathbb{P}[X_T > 0]$ .
- ▶ Special case of thm by Rosenbaum, Tankov (2011):  $\varepsilon^{-1}X_{\varepsilon t}$  converges in law to a 1-stable Lévy process  $X^*$ .
- ▶ Hence

$$\begin{aligned}\lim_{T \rightarrow 0} \mathbb{P}[X_T > 0] &= \lim_{\varepsilon \rightarrow 0} \mathbb{P}[\varepsilon^{-1}X_\varepsilon > 0] \\ &= \mathbb{P}[X_1^* > 0] = \frac{1}{2} + \frac{1}{\pi} \arctan(b/\delta).\end{aligned}$$

## Case Study: The Normal Inverse Gaussian Model

- ▶ Recall

$$\partial_K \sigma_{\text{imp}} \sim \frac{\sqrt{2\pi}}{K\sqrt{T}} \left( \frac{1}{2} - \mathbb{P}[S_T \geq K] - \frac{\sigma_{\text{imp}}\sqrt{T}}{2\sqrt{2\pi}} + O((\sigma_{\text{imp}}\sqrt{T})^3) \right)$$

- ▶ From

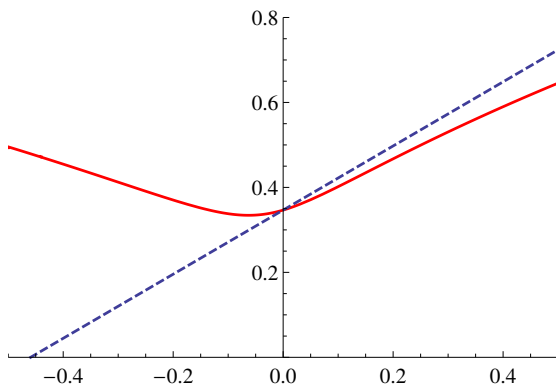
$$\lim_{T \rightarrow 0} \mathbb{P}[X_T > 0] = \frac{1}{2} + \frac{1}{\pi} \arctan(b/\delta)$$

we thus get

$$\partial_K \sigma_{\text{imp}}|_{K=1} \sim -\sqrt{2/\pi} \arctan(b/\delta) \frac{1}{\sqrt{T}}$$

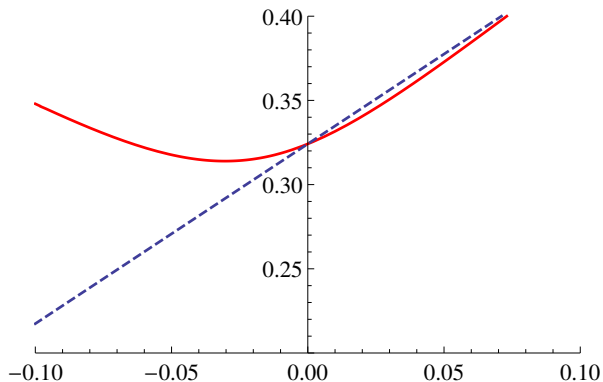
- ▶ ATM slope explodes

## NIG implied vol, with $\sigma = 0$



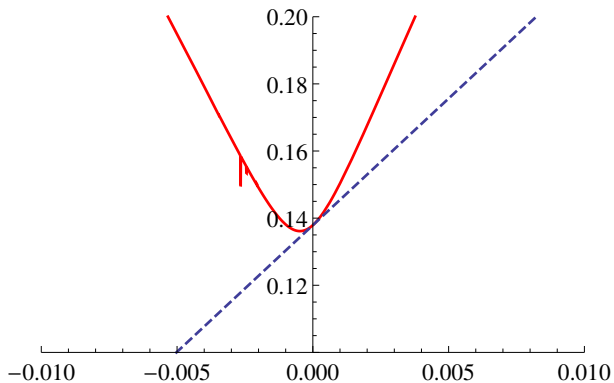
Maturity  $T = 0.1$

## NIG implied vol, with $\sigma = 0$



Maturity  $T = 0.05$

NIG implied vol, with  $\sigma = 0$



Maturity  $T = 0.001$

## Robustness of Lee's moment formula for $\sigma = 0$

- ▶ ATM slope  $\sim \text{const} \cdot T^{-1/2}$  extends to other models (Variance gamma, CGMY, Meixner)
- ▶ Lee's moment formula (Lee 2004;  $K = \log k$ ):

$$\sigma_{\text{imp}}(K, T) \sim \text{const} \cdot \sqrt{k/T}, \quad k \rightarrow \infty$$

$$\sigma_{\text{imp}}(K, T) \sim \text{const} \cdot \sqrt{-k/T}, \quad k \rightarrow -\infty$$

- ▶ Fact: For these models, the ATM slope is positive for small  $T$  if and only if the right smile wing is steeper

## Case Study: The Normal Inverse Gaussian Model. Now $\sigma > 0$ .

- ▶ What about  $\sigma > 0$ ?
- ▶ Econometric studies suggest the necessity of a diffusion component (Ait-Sahalia, Jacod 2009, 2010)
- ▶ We need the second order term in

$$\mathbb{P}[X_T > 0] = \frac{1}{2} + ?, \quad T \rightarrow 0$$

- ▶ The methods we've mentioned so far do not apply

## Mellin transform asymptotics

- ▶ Goal: Asymptotics of a function  $H(T)$ , as  $T \rightarrow 0$ . (We:  $H(T) = \mathbb{P}[X_T > 0]$ .)
- ▶ Mellin transform

$$\mathcal{M}(s) = \int_0^\infty T^{s-1} H(T) dT$$

- ▶ Inversion formula

$$H(T) = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} T^{-s} \mathcal{M}(s) ds$$

- ▶ Move contour to the left. Suppose  $\mathcal{M}$  has poles at  $0 \geq s_0 > s_1 > \dots$
- ▶ Wrap contour around poles; residue theorem  $\implies$

$$H(T) = \text{res}_{s=s_0} T^{-s} \mathcal{M}(s) + \text{res}_{s=s_1} T^{-s} \mathcal{M}(s) + \dots$$



## Mellin transform asymptotics: Lévy processes

- ▶ ATM digital:

$$H(T) = \mathbb{P}[X_T > 0] = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \frac{1}{z} e^{-T\psi(z)} dz$$

- ▶ Mellin transform (using Fubini)

$$\begin{aligned} \mathcal{M}(s) &= \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \frac{1}{z} \left( \int_0^\infty T^{s-1} e^{-T\psi(z)} dT \right) dz \\ &= \frac{\Gamma(s)}{2i\pi} \int_{a-i\infty}^{a+i\infty} \frac{1}{z} \psi^{-s}(z) dz =: \Gamma(s)F(s) \end{aligned}$$

- ▶  $\Gamma(s)$  has poles at  $s = 0, -1, -2, \dots$
- ▶ If  $F(s)$  has meromorphic continuation, we get asymptotics of  $H(T)$

## Mellin transform asymptotics

- ▶ Poles of  $\mathcal{M}$  correspond to terms in the asymptotic expansion of  $H$
- ▶ Simple poles yield powers of  $T$ :

$$\operatorname{res}_{s=s_0} T^{-s} \left( \frac{c}{s-s_0} + \dots \right) = cT^{-s_0}$$

- ▶ Double poles yield logarithmic terms:

$$\begin{aligned} & \operatorname{res}_{s=s_0} T^{-s} \left( \frac{c_2}{(s-s_0)^2} + \frac{c_1}{s-s_0} + \dots \right) \\ &= T^{-s_0} \operatorname{res}_{s=s_0} (1 - (\log T)(s-s_0) + \dots) \left( \frac{c_2}{(s-s_0)^2} + \frac{c_1}{s-s_0} + \dots \right) \end{aligned}$$

## Case study: NIG with $\sigma > 0$

- ▶ Lévy exponent

$$\psi(z) = -\left(\frac{1}{2}\sigma^2 z^2 + bz + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2})\right)$$

- ▶ Fourier representation of ATM digital:

$$\mathbb{P}[X_T > 0] = \frac{1}{\pi} \operatorname{Re} \int_0^\infty \frac{e^{-T\psi(a+iy)}}{a+iy} dy$$

- ▶ Mellin transform:

$$\int_0^\infty \frac{e^{-T\psi(a+iy)}}{a+iy} dy \longrightarrow \Gamma(s) \int_0^\infty \frac{\psi^{-s}(a+iy)}{a+iy} dy$$

## Case study: NIG with $\sigma > 0$

- ▶ Mellin transform:

$$\Gamma(s) \int_0^\infty \frac{\psi^{-s}(a + iy)}{a + iy} dy$$

- ▶ Analytic for  $\Re(s) > 0$
- ▶ We need meromorphic continuation to  $\Re(s) \leq 0$
- ▶ Goal: “analytic function” + “explicit meromorphic function”
- ▶  $y$  large:

$$\psi^{-s}(a + iy) = (\sigma^2 y^2 / 2)^{-s} + O(y^{-2s-1})$$

## Case study: NIG with $\sigma > 0$

- ▶  $y$  large:

$$\psi^{-s}(a + iy) = (\sigma^2 y^2 / 2)^{-s} + O(y^{-2s-1})$$

- ▶ Add and subtract convergence-inducing summand

$$\int_0^\infty \frac{(\sigma^2 y^2 / 2)^{-s}}{a + iy} dy = \frac{\textit{analytic}}{\sin 2\pi s}$$

$\implies$  meromorphic continuation to  $-\frac{1}{2} < \Re s$

- ▶ Second convergence-inducing summand  $\implies$  continuation to  $-1 < \Re s$
- ▶  $\text{res}_{s=0} T^{-s} \mathcal{M}(s) = \textit{const} = \frac{\pi}{2} + \textit{imaginary}$
- ▶  $\text{res}_{s=-1/2} T^{-s} \mathcal{M}(s) = \textit{const} \cdot \sqrt{T}$

## Case study: NIG with $\sigma > 0$

- ▶ Result of Mellin transform asymptotics:

$$\mathbb{P}[X_T > 0] = \frac{1}{2} + \frac{b}{\sigma\sqrt{2\pi}}\sqrt{T} + O(T \log T)$$

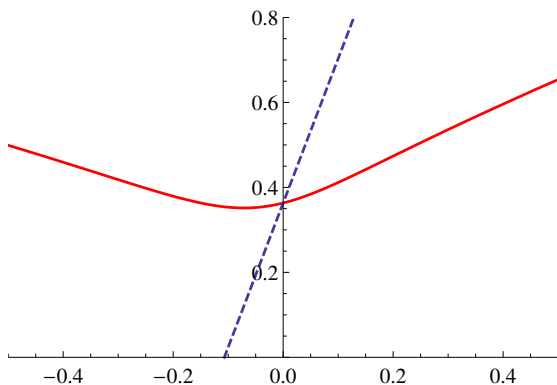
- ▶ ATM slope

$$\lim_{T \rightarrow 0} \partial_K \sigma_{\text{imp}}|_{K=1} = -\frac{b}{\sigma} - \frac{\sigma}{2}$$

with

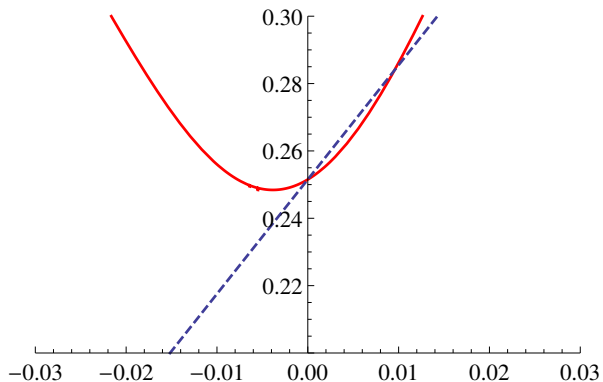
$$b = -\frac{1}{2}\sigma^2 - \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2})$$

## NIG implied vol, with $\sigma > 0$



$\sigma = .1$ , maturity  $T = 0.1$

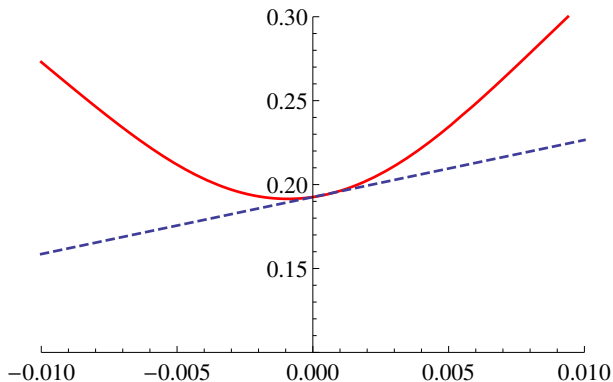
## NIG implied vol, with $\sigma > 0$



$\sigma = .1$ , maturity  $T = 0.05$



## NIG implied vol, with $\sigma > 0$



$\sigma = .1$ , maturity  $T = 0.001$

## Further results, outlook

- ▶ Method works for CGMY, variance gamma, Meixner,... with  $\sigma > 0$ :

$$\lim_{T \rightarrow 0} \partial_K \sigma_{\text{imp}}|_{K=1} = -\frac{b}{\sigma} - \frac{\sigma}{2}$$

- ▶  $b$  such that  $S = e^X$  martingale
- ▶  $b$  depends on all model parameters
- ▶ Work in progress: general theorem (with conditions on Lévy triplet)
  
- ▶ Refined expansion for NIG,  $\sigma = 0$ :

$$\partial_K \sigma_{\text{imp}}|_{K=1} = -\sqrt{2/\pi} \arctan(b/\delta) \frac{1}{\sqrt{T}} + \frac{\beta\delta}{\pi} T \log T + O(T)$$

## Summary

- ▶ Implied vol slope of Lévy models for small maturity
- ▶ Related to transition probabilities (ATM digital calls)
- ▶ Slope sign gives useful constraint for calibration
- ▶ Mellin transform technique extends to other problems (time asymptotics of Lévy processes)

## References

- Y. Ait-Sahalia, J. Jacod, *Is Brownian motion necessary to model high frequency data?* Annals of Statistics 2010.
- L. Andersen, A. Lipton, *Asymptotics for exponential Lévy processes and their volatility smile: survey and new results*, IJTAF 2013.
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- P. Tankov, *Pricing and hedging in exponential Lévy models: review of recent results*, Paris-Princeton Lecture Notes in Mathematical Finance, 2010.