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The Smallest C^*-Algebra for Canonical Commutations Relations

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Abstract. We consider the C^*-algebras which contain the Weyl operators when the symplectic form which defines the C.C.R. is possibly degenerate. We prove that the C.C.R. are all obtained as a quotient of a universal C^*-algebra by some of its ideals, and we characterize all these ideals.

I. Introduction

In a recent paper [1] Slawny derived the following very interesting result.

There exists a C^*-algebra $\mathfrak{A}$ which is such that to every representation (not necessarily continuous) of the C.C.R. there corresponds a representation of $\mathfrak{A}$; moreover, $\mathfrak{A}$ is simple and minimal.

Non degeneracy of the symplectic form which defines the C.C.R. (see below) seems essential to his derivation; in this paper, we shall not make this assumption and through a quite different approach we shall be able to give a description of the C^*-algebras which contain the Weyl operators.

We define a universal C^*-algebra which coincides with the one defined by Manuceau [2] and Slawny [1] in the case where the symplectic form of the C.C.R. is non degenerate; the definition is specially simple to handle and in particular we prove that any positive linear form on finite combinations of Weyl operators extends to a state of this algebra.

Moreover any C^*-algebra containing the Weyl operators is the quotient of the universal one by an ideal which is in some sense characterized by its intersection with the center of the algebra.

A section is devoted to the study of central states, and our results are close of Slawny in the case of non degeneracy.

Finally we make the following remark: Degeneracy of the symplectic form which defines the C.C.R. has been encountered already in the study of quasi-free bose gas below the critical temperature [3] and is possibly interesting to consider in the study of field theory with massless particles.

Another possible application is the C.C.R. representations occurring in solid state physics where even the vector space structure of the one particle state space is replaced by an abelian group structure [4].

II. Mathematical Preleminaries

In this paragraph we collect most of the material we shall need; we generalize a construction which can be found, e.g. in [2].

First let us define a symplectic group (H, σ) where H is an abelian group and σ is an application from $H \times H$ into $\boldsymbol{R}$ such that:

$$\sigma(x, y) = -\sigma(y, x), \qquad \forall x, y \in H, \tag{2.1}$$

$$\sigma(x, y+z) = \sigma(x, y) + \sigma(x, z), \qquad \forall x, y, z \in H. \tag{2.2}$$

Notice that it implies that:

$$\sigma(x, 0) = 0, \qquad \forall x \in H \tag{2.3}$$

and

$$\sigma(x, -y) = -\sigma(x, y) = \sigma(-x, y), \qquad \forall x, y \in H. \tag{2.4}$$

We shall not assume that σ is non degenerate and we denote by H_0 the subgroup of H

$$H_0 = \{x \in H \mid \forall y \in H, \ \exp(2i\sigma(x, y)) = 1\}. \tag{2.5}$$

Let $\Delta(H, \sigma)$ be the complex vector space generated by the functions δ_x, $(x \in H)$ from H to $\boldsymbol{C}$ defined by

$$\delta_x(y) = \begin{cases} 0 & \text{if} \quad x \neq y \\ 1 & \text{if} \quad x = y. \end{cases} \tag{2.6}$$

$\Delta(H, \sigma)$ is an algebra with unit δ_0 with respect to the product which satisfies

$$\delta_x \cdot \delta_y = e^{-i\sigma(x,y)} \delta_{x+y}, \qquad \forall x, y \in H. \tag{2.7}$$

Moreover it is a $*$-algebra with respect to the involution such that

$$(\delta_x)^* = \delta_{-x}, \qquad \forall x \in H. \tag{2.8}$$

Note that the δ_x's are unitaries.

The δ_x's previously defined are linearly independent and they form a basis of $\Delta(H, \sigma)$; any element $\boldsymbol{a}$ of $\Delta(H, \sigma)$ can be written as

$$\boldsymbol{a} = \sum_{i=1}^{N} \alpha_i \delta_{x_i}, \qquad \alpha_i \in \boldsymbol{C}, \qquad x_i \in H \tag{2.9}$$

and we shall always assume if not stated otherwise that in the previous expression the x_i's are all different.

The application $\|\cdot\|_1$ from $\Delta(H,\sigma)$ to $\boldsymbol{R}^+$

$$\left\|\sum_{i=1}^{N} \alpha_i \delta_{x_i}\right\|_1 = \sum_{i=1}^{N} |\alpha_i| \tag{2.10}$$

is a $*$-norm, and the completion $\overline{\Delta(H,\sigma)}^1$ is a $*$-Banach algebra with unit as one can easily verify [1].

For an arbitrary character χ of the abelian group H we shall define τ_χ which is a $*$-automorphism of $\Delta(H,\sigma)$ by

$$\tau_\chi(\delta_x) = \chi(x)\,\delta_x\,. \tag{2.12}$$

τ_χ is isometric hence it extends to $\overline{\Delta(H,\sigma)}^1$; amongst the previous characters we can consider those of the form

$$\chi_y(x) = \exp(2i\sigma(y,x)), \qquad \forall x \in H\,. \tag{2.13}$$

Let K be the set of $\chi_y (y \in H)$ and $\bar{K}$ the closure of this set in the dual group $\hat{H}$ of H with the discrete topology. Notice that $\bar{K}$ can be identified to the dual group of H/H_0. Moreover if $\chi_y \in K$, then the corresponding $*$-automorphism is inner:

$$\tau_{\chi_y}(\boldsymbol{a}) = \delta_y \cdot \boldsymbol{a} \cdot \delta_{-y}, \qquad \forall y \in H, \qquad \forall \boldsymbol{a} \in \overline{\Delta(H,\sigma)}^1\,. \tag{2.14}$$

Later on we shall need the whole set of states of $\overline{\Delta(H,\sigma)}^1$ but for the moment we specialize ourself to a special class, the central states which satisfy

$$\omega(\boldsymbol{a}\cdot\boldsymbol{b}) = \omega(\boldsymbol{b}\cdot\boldsymbol{a})\,. \tag{2.15}$$

They can also be defined by the fact that they are invariant by the $*$-automorphism τ_χ, $\chi \in \bar{K}$. One of these states is specially interesting; it is defined by

$$\omega_0(\delta_x) = 0, \quad \text{if} \quad x \in H \quad \text{and} \quad x \neq 0\,. \tag{2.16}$$

Actually we have defined the previous state over $\Delta(H,\sigma)$ but the following proposition allows to identify the state of $\Delta(H,\sigma)$ with those of $\overline{\Delta(H,\sigma)}^1$.

Proposition (2.17). *Any positive linear form on $\Delta(H,\sigma)$ extends to a positive linear form on $\overline{\Delta(H,\sigma)}^1$.*

Proof. Indeed, from the Cauchy-Schwartz inequality, one has

$$|f(\delta_x)| \leqq f(\delta_0)$$

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[1] Notice that the most general element $\boldsymbol{a}$ of $\overline{\Delta(H,\sigma)}^1$ is of the form

$$\boldsymbol{a} = \sum_{i=1}^{\infty} \alpha_i \delta_{x_i} \quad \text{with} \quad \sum_{i=1}^{\infty} |\alpha_i| < \infty\,. \tag{2.11}$$

for any linear positive form f on $\Delta(H, \sigma)$ and every $x \in H$ hence for $a = \sum_{i=1}^{N} \alpha_i \delta_{x_i}$

$$|f(a)| \leqq f(\delta_0)\left(\sum_{i=1}^{N} |\alpha_i|\right) = f(\delta_0)\, \|a\|_1 \tag{2.18}$$

and f extends by continuity. Q.E.D.

As a consequence one can equivalently define the central state ω_0 by the fact that it is invariant by any $*$-automorphism τ_χ.

The norm $\|\cdot\|_1$ is not a C^*-algebra norm[2]; hence is not isometrically isomorphic to an algebra of bounded operators. We have to find a C^*-algebra norm and the aim of the next section is to solve this problem.

III. Minimal Regular Norm

We shall follow a standard procedure to define a C^*-algebra norm on $\overline{\Delta(H, \sigma)}^1$, (see e.g. [5], p. 260) but we need the following lemma, which tells us that $\overline{\Delta(H, \sigma)}^1$ is reduced.

Lemma (3.1). *Let $a \in \overline{\Delta(H, \sigma)}^1$, $a \neq 0$, then*

$$\omega_0(a^* a) > 0$$

where ω_0 is the central state defined previously.

The proof is obvious according (2.11) and (2.16).

The minimal regular norm on $\overline{\Delta(H, \sigma)}^1$ is then defined as follows:

$$\|a\| = \operatorname*{Sup}_{\varrho \in \mathscr{F}} \sqrt{\varrho(a^* a)}, \qquad a \in \overline{\Delta(H, \sigma)}^1 \tag{3.2}$$

where $\mathscr{F}$ stands for the set of states of $\overline{\Delta(H, \sigma)}^1$: from Lemma (3.1) $\|\cdot\|$ is a norm. Moreover it is a C^*-algebra norm (cf. [5], p. 261) and one has

$$\|a\| \leqq \|a\|_1 . \tag{3.3}$$

We shall denote by $\overline{\Delta(H, \sigma)}$ the completion of $\overline{\Delta(H, \sigma)}^1$ or equivalently of $\Delta(H, \sigma)$ with respect to this norm. By Prop. (2.17), any linear positive form on $\Delta(H, \sigma)$ extends to a positive linear form on $\overline{\Delta(H, \sigma)}$.

The next proposition characterizes the representations of $\overline{\Delta(H, \sigma)}$ and will be needed to describe the different C^*-algebra norm on $\Delta(H, \sigma)$.

Proposition (3.4). *Let U be a Weyl system, viz U is a mapping of H into the group of unitaries of an Hilbert space $\mathscr{H}$ such that:*

$$U(x)\, U(y) = \exp(-i\sigma(x, y))\, U(x+y) \tag{3.5}$$

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[2] Choose e.g. $a = \delta_0 + \delta_y - \delta_{-y}$ $y \in H$ and $y \neq 0$.

then

$$\pi\left(\sum_{i=1}^{N} \alpha_i \delta_{x_i}\right) = \sum_{i=1}^{N} \alpha_i U(x_i) \tag{3.6}$$

extends to a representation of $\overline{\Delta(H, \sigma)}$

Indeed if $\boldsymbol{a} = \sum_{i=1}^{N} \alpha_i \delta_{x_i} \in \Delta(H, \sigma)$

$$\begin{aligned} \|\pi(\boldsymbol{a})\| &= \operatorname*{Sup}_{\substack{\psi \in \mathscr{H} \\ \|\psi\| = 1}} \sqrt{(\psi \mid \pi(\boldsymbol{a}^* \boldsymbol{a}) \psi)} \\ &\leqq \operatorname*{Sup}_{\varrho \in \mathscr{F}} \sqrt{\varrho(\boldsymbol{a}^* \boldsymbol{a})} = \|\boldsymbol{a}\| . \end{aligned} \tag{3.7}$$

We are now in a position to prove that the norm $\|\cdot\|$ is maximal, more precisely

Corollary (3.8). *Let* $\| \ \|_0$ *be a* C^**-algebra norm on* $\Delta(H, \sigma)$ *then for any element* $\boldsymbol{a} \in \overline{\Delta(H, \sigma)}$

$$\|\boldsymbol{a}\|_0 \leqq \|\boldsymbol{a}\| .$$

Indeed let $\overline{\Delta(H, \sigma)}^0$ be the closure of $\Delta(H, \sigma)$ with respect to $\|\cdot\|_0$; it is a C^*-algebra which has an isometric representation π and $x \in H \to \pi(\delta_x)$ is a Weyl system.

Corollary (3.9). *Let* $\|\cdot\|_0$ *be a* C^**-algebra norm on* $\Delta(H, \sigma)$ *then there exists an ideal* $\mathscr{I}$ *in* $\overline{\Delta(H, \sigma)}$ *such that*

$$\overline{\Delta(H, \sigma)}^0 = \overline{\Delta(H, \sigma)}/\mathscr{I} .$$

Proof. See Dixmier 1.9.13. The next result will be of interest later.

Proposition (3.10). *Any* $*$*-automorphism of* $\Delta(H, \sigma)$ *extends to a* $*$*-automorphism of* $\overline{\Delta(H, \sigma)}$.

$\overline{\Delta(H, \sigma)}$ we defined is actually the C^*-algebra of the Weyl group with the discrete topology; hence there is no connection between the topology of H and the topology of $\overline{\Delta(H, \sigma)}$; in this respect let us mention the following result:

Proposition (3.11). *Let* H' *a proper subgroup of* H*; then*

$$\overline{\Delta(H', \sigma)} \subsetneqq \overline{\Delta(H, \sigma)}$$

more precisely, we have

$$\|\delta_x - \boldsymbol{a}\| \geqq 1 , \qquad \forall \boldsymbol{a} \in \overline{\Delta(H', \sigma)}$$

if $x \in H$, $x \notin H'$.

For all $\boldsymbol{a} \in \Delta(H', \sigma)$, one has

$$\|\boldsymbol{a} - \delta_x\|^2 \geqq \omega_0((\boldsymbol{a} - \delta_x)^* (\boldsymbol{a} - \delta_x))$$

where ω_0 is the canonical central state of $\Delta(H, \sigma)$ (2.16); because $\omega_0(\boldsymbol{a}^* \boldsymbol{a}) \geqq 0$ and $\omega_0(\boldsymbol{a}^* \delta_x) = \omega_0(\delta_x \boldsymbol{a}) = 0$ (because $x \notin H'$). One then has $\|\boldsymbol{a} - \delta_x\|^2 \geqq 1$ and the proposition follows from the continuity of the map $\boldsymbol{a} \to \|\boldsymbol{a} - \delta_x\|$.

IV. Structure of the $*$-Ideals of $\overline{\Delta(H, \sigma)}$

According to Corollary (3.9), it is important to characterize the $*$-ideals of $\overline{\Delta(H, \sigma)}$ such that $\Delta(H, \sigma) \cap \mathscr{I} = \{0\}$ in order to find all the C^*-algebra norms on $\Delta(H, \sigma)$. For the moment we omit the conditions $\Delta(H, \sigma) \cap \mathscr{I} = \{0\}$ and characterize all $*$-ideals of $\overline{\Delta(H, \sigma)}$. In order to achieve this goal we shall introduce a mean over the algebra. This allows us to use similar techniques to those used in [6], Chapter III, § 5.2.

Definition (4.1). *Let $\mathscr{M}$ be the linear application of $\Delta(H, \sigma)$ onto $\Delta(H_0)$*[3] *defined by:*

$$\mathscr{M}\left(\sum_{i=1}^{N} \lambda_i \delta_{x_i}\right) = \sum_{j, x_j \in H_0} \lambda_j \delta_{x_j}.$$

One has the following theorem:

Theorem (4.2). i) *$\mathscr{M}$ is a linear continuous application with respect to both norm $\|\cdot\|$ and $\|\cdot\|_1$. Let us again denote by $\mathscr{M}$ the continuous extension of $\mathscr{M}$ to $\overline{\Delta(H, \sigma)}$ or $\overline{\Delta(H, \sigma)}^1$.*

ii) $\mathscr{M} \circ \tau_\chi = \tau_\chi \circ \mathscr{M} = \mathscr{M}, \quad \forall \chi \in \bar{K}$.

iii) $\mathscr{M} \circ \mathscr{M} = \mathscr{M}$.

iv) *$\mathscr{M}$ is faithfull and positive.*

v) $\mathscr{M}(\overline{\Delta(H, \sigma)}) = \overline{\Delta(H_0)}$.

Moreover $\overline{\Delta(H_0)}$ is the center $\mathscr{Z}$ of $\overline{\Delta(H, \sigma)}$.

vi) $\forall \boldsymbol{a} \in \overline{\Delta(H_0)}$, $\forall \boldsymbol{b} \in \overline{\Delta(H, \sigma)}$ *one has*

$$\mathscr{M}(\boldsymbol{a}\, \boldsymbol{b}) = \boldsymbol{a}\, \mathscr{M}(\boldsymbol{b}).$$

vii) *for every central state ω of $\overline{\Delta(H, \sigma)}$*

$$\omega \circ \mathscr{M} = \omega.$$

viii) *Let ω be a state over $\overline{\Delta(H_0)}$, there exists an unique central state $\bar{\omega}$ of $\Delta(H, \sigma)$ which extends ω:*

$$\bar{\omega} = \omega \circ \mathscr{M}.$$

[3] $\Delta(H_0)$ is exactly $\Delta(H_0, 0)$, see (2.5). It is an abelian $*$-algebra contained in $\Delta(H, \sigma)$.

Proof. Continuity with respect to the norm $\|\cdot\|_1$ is obvious. Moreover we note that from Proposition (3.10)

$$\|\tau_\chi(\boldsymbol{a})\| = \|\boldsymbol{a}\|, \quad \forall \chi \in \hat{H}, \quad \boldsymbol{a} \in \overline{\Delta(H,\sigma)}. \tag{4.3}$$

This and the continuity of $\chi \in \bar{K} \to \tau_\chi(\boldsymbol{a})$, $\boldsymbol{a} \in \Delta(H,\sigma)$ allow to define the following Bochner-convergent integral in $\overline{\Delta(H,\sigma)}$

$$\boldsymbol{m}(\boldsymbol{a}) = \int_{\bar{K}} d\chi\, \tau_\chi(\boldsymbol{a}), \quad \forall \boldsymbol{a} \in \Delta(H,\sigma) \tag{4.4}$$

where $d\chi$ is the normalized Haar measure on the compact group $\bar{K}$.

Moreover $\boldsymbol{m}$ coincides with $\mathcal{M}$ and

$$\|\boldsymbol{m}(\boldsymbol{a})\| \leqq \|\boldsymbol{a}\|, \quad \boldsymbol{a} \in \Delta(H,\sigma) \tag{4.5}$$

this proves i) and we have

$$\mathcal{M}(\boldsymbol{a}) = \int_{\bar{K}} d\chi\, \tau_\chi(\boldsymbol{a}), \quad \boldsymbol{a} \in \overline{\Delta(H,\sigma)}. \tag{4.6}$$

ii) and iii) and positivity of $\mathcal{M}$ are now obvious. The faithfullness of $\mathcal{M}$ can be proved as follows: Let $\boldsymbol{a} \neq 0$, $\boldsymbol{a} \in \overline{\Delta(H,\sigma)}$, there exists a state φ of $\overline{\Delta(H,\sigma)}$ such that:

$$\varphi(\boldsymbol{a}^*\boldsymbol{a}) = \varepsilon > 0. \tag{4.7}$$

Moreover since

$$\chi \to \varphi(\tau_\chi(\boldsymbol{a}^*\boldsymbol{a})) \tag{4.8}$$

is continuous, there exists an open neighborhood $\mathcal{V}$ in $\bar{K}$ of the identity such that

$$\varphi(\tau_\chi(\boldsymbol{a}^*\boldsymbol{a})) \geqq \frac{\varepsilon}{2}, \quad \forall \chi \in \mathcal{V} \tag{4.9}$$

hence

$$\begin{aligned} \varphi(\mathcal{M}(\boldsymbol{a}^*\boldsymbol{a})) &= \int_{\bar{K}} d\chi\, \varphi(\tau_\chi(\boldsymbol{a}^*\boldsymbol{a})) \\ &\geqq \int_{\mathcal{V}} d\chi\, \varphi(\tau_\chi(\boldsymbol{a}^*\boldsymbol{a})) \\ &\geqq \frac{\varepsilon}{2} \int_{\mathcal{V}} d\chi > 0. \end{aligned} \tag{4.10}$$

To prove v), we note that $\overline{\Delta(H_0)} \subsetneq \mathcal{Z}$. Moreover $\mathcal{M}(\Delta(H,\sigma)) = \Delta(H_0)$ hence by i)

$$\mathcal{M}(\overline{\Delta(H,\sigma)}) \subsetneq \overline{\Delta(H_0)}. \tag{4.11}$$

If $z \in \mathscr{Z}$

$$\tau_\chi(z) = z, \qquad \forall \chi \in K \tag{4.12}$$

by the continuity of $\chi \to \tau_\chi(z)$; this is still true for $\chi \in \bar{K}$. Consequently by (4.6)

$$\mathscr{M}(z) = z \quad \text{for} \quad z \in \mathscr{Z} \tag{4.13}$$

so that

$$\mathscr{Z} = \mathscr{M}(\mathscr{Z}) \subset \mathscr{M}(\overline{\Delta(H, \sigma)}). \tag{4.14}$$

This completes the proof of v).

vi) and vii) are obvious and viii) follows immediately from vii). Now we are in position to prove the central result of this section, namely (see also [6] for similar techniques):

Theorem (4.15). *Let n be a closed $*$-ideal of $\overline{\Delta(H_0)}$; there exists a closed $*$-ideal $\mathscr{J}_n$ in $\overline{\Delta(H, \sigma)}$ such that*

i) $\mathscr{J}_n \cap \overline{\Delta(H_0)} = n$.

ii) *For every closed $*$-ideal $\mathscr{J}$ of $\overline{\Delta(H, \sigma)}$ such that*

$$\mathscr{J} \cap \overline{\Delta(H_0)} \subseteqq n$$

one has

$$\mathscr{J} \subseteqq \mathscr{J}_n.$$

Moreover there exists a closed $*$-ideal $\mathscr{J}'_n$ such that

iii) *For every closed $*$-ideal $\mathscr{J}$ such that*

$$\mathscr{J} \cap \overline{\Delta(H_0)} = n$$

one has

$$\mathscr{J} \supseteqq \mathscr{J}'_n.$$

Let $\mathscr{J}_n$ be the set of all $\boldsymbol{a} \in \overline{\Delta(H, \sigma)}$ such that

$$\mathscr{M}(\boldsymbol{a}\,\delta_x) \in n, \qquad \forall x \in H. \tag{4.16}$$

Since $\boldsymbol{a} \to \mathscr{M}(\boldsymbol{a}\,\delta_x)$ is linear and continuous, and by invariance of $\mathscr{M}$ with respect to the $*$-automorphisms τ_χ, $\chi \in K$, $\mathscr{J}_n$ is a closed $*$-ideal of $\overline{\Delta(H, \sigma)}$.

i) is obvious. In order to prove ii), let $\mathscr{J}$ be such that

$$\mathscr{J} \cap \overline{\Delta(H_0)} \subseteqq n$$

clearly since $\mathscr{J}$ is a closed $*$-ideal $\tau_\chi(\mathscr{J}) \subseteqq \mathscr{J}$ for $\chi \in \bar{K}$ hence

$$\mathscr{M}(\boldsymbol{a}) = \int_{\bar{K}} d\chi\, \tau_\chi(\boldsymbol{a}) \in \mathscr{J} \quad \text{for} \quad \boldsymbol{a} \in \mathscr{J}$$

and since $\boldsymbol{a}\,\delta_x \in \mathscr{J}$ for every $x \in H$:

$$\mathscr{M}(\boldsymbol{a}\,\delta_x) \in \mathscr{J}.$$

On the other hand

$$\mathscr{M}(\boldsymbol{a}\,\delta_x) \in \overline{\Delta(H_0)} \tag{4.17}$$

hence $\boldsymbol{a} \in \mathscr{I}_n$.

Let now $\mathscr{I}'_n$ be the closed $*$-ideal generated by n; the following $*$-ideal is dense in $\mathscr{I}'_n$:

$$i'_n = \left\{ \sum_{i=1}^{N} \boldsymbol{n}_i \delta_{x_i} \,|\, \boldsymbol{n}_i \in n,\ x_i \in H \right\}. \tag{4.18}$$

Let $\boldsymbol{a} \in i'_n$, $\mathscr{M}(\boldsymbol{a}\,\delta_x)$ $(x \in H)$ is in n [(4.2), vi)]. Consequently by continuity of $\boldsymbol{a} \to \mathscr{M}(\boldsymbol{a}\,\delta_x)$

$$\mathscr{I}'_n \subseteqq \mathscr{I}_n. \tag{4.19}$$

Moreover

$$n \subset \overline{\Delta(H_0)} \cap \mathscr{I}'_n \subset \overline{\Delta(H_0)} \cap \mathscr{I}_n = n. \tag{4.20}$$

iii) follows from the fact that $\mathscr{I}'_n$ is the intersection of the set of ideals whose intersection with $\overline{\Delta(H_0)}$ is n.

Corollary (4.21). *Let $\mathscr{S}$ (resp. $\mathscr{S}_0$) be the set of maximal $*$-ideals of $\overline{\Delta(H,\sigma)}$ (resp. $\Delta(H_0)$) and $\{0\}$. The correspondence*

$$\mathscr{S} \ni \mathscr{I} \to \mathscr{I} \cap \overline{\Delta(H_0)}$$

is a one to one map of $\mathscr{S}$ onto $\mathscr{S}_0$.

More precisely we have

i) *If $\mathscr{I}$ is maximal in $\overline{\Delta(H,\sigma)}$ then $\mathscr{I} \cap \overline{\Delta(H_0)}$ is maximal in $\Delta(H_0)$.*

ii) *If n is maximal in $\overline{\Delta(H_0)}$ then there exists a unique maximal $*$-ideal $\mathscr{I}$ such that*

$$\mathscr{I} \cap \overline{\Delta(H_0)} = n.$$

iii) *For every closed $*$-ideal $\mathscr{I}$ of $\overline{\Delta(H,\sigma)}$, $\mathscr{I} = \{0\}$ iff*

$$\mathscr{I} \cap \overline{\Delta(H_0)} = \{0\}.$$

Let us show i): assume $\mathscr{I}$ is a maximal $*$-ideal of $\overline{\Delta(H,\sigma)}$ then $\mathscr{I}$ is closed; let n be a $*$-ideal of $\overline{\Delta(H_0)}$ such that $n \supsetneqq \mathscr{I} \cap \overline{\Delta(H_0)}$; consider $\mathscr{I}_n$ (Theorem 4.16), it contains $\mathscr{I}$ and it is strictly greater, hence $\mathscr{I}_n = \overline{\Delta(H,\sigma)}$ so that $n = \overline{\Delta(H_0)}$.

Let us show ii): Let n be a maximal $*$-ideal of $\overline{\Delta(H_0)}$. Let $\mathscr{I}$ be a $*$-ideal of $\overline{\Delta(H,\sigma)}$ such that $\mathscr{I} \supsetneqq \mathscr{I}_n$, then

$$\mathscr{I} \cap \overline{\Delta(H_0)} \supsetneqq n$$

hence

$$\overline{\Delta(H_0)} \subset \mathscr{I}$$

since $\overline{\Delta(H_0)}$ contains the identity $\mathscr{I} = \overline{\Delta(H,\sigma)}$. Consequently $\mathscr{I}_n$ is maximal.

Finally, let $\mathscr{J}$ be another $*$-ideal maximal in $\overline{\Delta(H,\sigma)}$ such that

$$\mathscr{J} \cap \overline{\Delta(H_0)} = n$$

then

$$\mathscr{J} \subseteqq \mathscr{J}_n$$

$\mathscr{J} = \mathscr{J}_n$ by maximality of $\mathscr{J}$.

To prove iii), it is sufficient to prove that $\mathscr{J}_{\{0\}} = \{0\}$; indeed if $\boldsymbol{x} \in \mathscr{J}_{\{0\}}$ then $\boldsymbol{x}^* \boldsymbol{x} \in \mathscr{J}_{\{0\}}$ and $\mathscr{M}(\boldsymbol{x}^* \boldsymbol{x}) = 0$, then $\boldsymbol{x}^* \boldsymbol{x} = 0$ and $\boldsymbol{x} = 0$ [Theorem (4.2), iv)].

In the following corollary we get the result which states precisely in what sense the C^*-algebra is the smallest amongst all C^*-algebra containing the Weyl operators.

Corollary (4.22). *If* $|||\cdot|||$ *is a* C^**-algebra norm which coincides with the usual norm on* $\Delta(H_0)$, *then they coincide everywhere.*

Proof. By (3.8)

$$|||\boldsymbol{a}||| \leqq \|\boldsymbol{a}\|, \qquad \forall \boldsymbol{a} \in \Delta(H,\sigma)$$

hence $|||\cdot|||$ extends to $\overline{\Delta(H,\sigma)}$ as a regular pseudo-norm and

$$|||\boldsymbol{a}||| = \|\boldsymbol{a}\|, \qquad \forall \boldsymbol{a} \in \overline{\Delta(H_0)}\,.$$

Let $\mathscr{J} = \{\boldsymbol{a} \in \overline{\Delta(H,\sigma)} \mid |||\boldsymbol{a}||| = 0\}$; it is a closed $*$-ideal. Assume that $\boldsymbol{a} \in \mathscr{J} \cap \overline{\Delta(H_0)}$ and $\boldsymbol{a} \neq 0$, then

$$0 = |||\boldsymbol{a}||| = \|\boldsymbol{a}\| \neq 0\,.$$

We get a contradiction. Hence the corollary.

Corollary (4.23). *If* σ *is regular* ($H_0 = \{0\}$), *then there exists a unique* C^**-algebra norm on* $\overline{\Delta(H,\sigma)}$: *it is the minimal regular norm.*

Corollary (4.24). $\overline{\Delta(H,\sigma)}$ *is simple iff* $H_0 = \{0\}$.

This follows from [(4.21), iii)] and the Gelfand-Mazur Theorem [5], p. 175.

Corollary (4.25). *Let* $\overline{\Delta(H,\sigma)}^2$ *be the closure of* $\Delta(H,\sigma)$ *with respect to the norm (see* [2]*)*

$$\|\boldsymbol{a}\|_2 = \sqrt{\omega_0(\boldsymbol{a}^* \boldsymbol{a})}$$

where ω_0 *is the canonical central state* (2.17), *then:*

$$\overline{\Delta(H,\sigma)}^1 \subseteqq \overline{\Delta(H,\sigma)} \subseteqq \overline{\Delta(H,\sigma)}^2\,.$$

The first inclusion has been already given in § 3; the second one can be proved as follows: $(\boldsymbol{a} \mid \boldsymbol{b}) = \omega_0(\boldsymbol{a}\,\boldsymbol{b}^*)$ is a scalar product on $\overline{\Delta(H,\sigma)}$ and $\Delta(H,\sigma)$ owing to the faithfulness of ω_0. Moreover, $\|\boldsymbol{a}\| \geqq \|\boldsymbol{a}\|_2$ shows that $\Delta(H,\sigma)$ is dense with respect to the norm $\|\cdot\|_2$.

V. $\mathcal{M}$-abelianness of $\bar{K}$

We previously mentionned that the set of $\bar{K}$ invariant states of $\overline{\Delta(H,\sigma)}$ is just the set of central states. We take advantage of this fact to prove the following result:

Lemma (5.1). *$\bar{K}$ is a group of $*$-automorphisms of $\overline{\Delta(H,\sigma)}$ which is $\mathcal{M}$-abelian. See [7], p. 430.*

This follows immediately from the fact that if μ is the mean over $\bar{K}$ for any state ϱ of $\overline{\Delta(H,\sigma)}$ one has:

$$\mu(\varrho(\tau_\chi(\boldsymbol{a}))) = \varrho(\mathcal{M}(\boldsymbol{a})), \qquad \boldsymbol{a} \in \overline{\Delta(H,\sigma)} \tag{5.2}$$

and Theorem (4.2), v).

This allows us to make use of the now classical results about asymptotically abelian systems, see e.g. [8]. In particular we have the following results.

Corollary (5.3). *Let $\mathscr{C}$ be the set of central states and $\omega \in \mathscr{C}$; let $\mathscr{H}_\omega$, Π_ω, Ω_ω, U^ω be the representation space, the representation, the cyclic vector induced by ω and the unitary representation of $\bar{K}$ implementing the τ_χ's. Let E_ω be the projection onto the set of vectors in $\mathscr{H}_\omega$ invariant by all the $U^\omega(\chi)$'s, $\mathscr{Z}_\omega$ the center of $\Pi_\omega(\Delta(H,\sigma))''$; then the following are equivalent:*

i) *ω is extremal in $\mathscr{C}$.*

ii) *$\Pi_\omega(\overline{\Delta(H,\sigma)})'' \cup U^\omega(\bar{K})$ is irreducible.*

iii) *$\mathscr{Z}_\omega$ is the scalars (ω is a factor state).*

iv) *E_ω is one-dimensional.*

v) $\omega(\boldsymbol{a}\cdot\boldsymbol{b}) = \omega(\boldsymbol{a})\,\omega(\boldsymbol{b}), \quad \forall \boldsymbol{a} \in \overline{\Delta(H_0)},\ \boldsymbol{b} \in \overline{\Delta(H,\sigma)}$.

vi) *The restriction of ω to $\overline{\Delta(H_0)}$ is a pure state.*

vii) *The set $\{\boldsymbol{x} \mid \boldsymbol{x} \in \overline{\Delta(H,\sigma)},\ \omega(\boldsymbol{x}^*\boldsymbol{x}) = 0\}$ is a maximal ideal.*

Moreover if ω_i, $i=1,2$ satisfy one of the previous equivalent conditions, then the following are equivalent:

a) *ω_1 is quasi-equivalent to ω_2.*

b) *ω_1 is unitarily equivalent to ω_2.*

c) $\omega_1 = \omega_2$.

d) $\omega_1 \restriction \overline{\Delta(H_0)} = \omega_2 \restriction \overline{\Delta(H_0)}$.

e) *$\omega_1 \restriction \overline{\Delta(H_0)}$ is equivalent to $\omega_2 \restriction \overline{\Delta(H_0)}$.*

Proofs of i) to v) are immediate if one notes that

$$U^\omega(\chi_x) = \Pi_\omega(\delta_x)\,J_\omega\,\Pi_\omega(\delta_x)\,J_\omega$$

where J_ω is the canonical involution given by the Tomita theory [9], so that $\mathscr{Z}_\omega$ is contained in $U^\omega(\bar{K})'$

vi) is equivalent to v) by (4.2).

vii) is quasi-equivalent to vi) by Theorem (4.15). For the last part, see e.g. [8].

The decomposition into extremal $\bar{K}$ invariant states or equivalently in extremal trace states is explicitly given by the following

Proposition (5.4). *There exists a canonical one to one map between the set of central states and the Radon measures over the spectrum $\mathscr{X}$ of the abelian C*-algebra $\overline{\Delta(H_0)}$ such that*

$$\omega(\boldsymbol{a}) = \int_{\mathscr{X}} \varrho(\boldsymbol{a})\, dm_\omega(\varrho) \qquad \forall \boldsymbol{a} \in \overline{\Delta(H,\sigma)}\,.$$

Moreover, let us define the following central state $\bar{\omega}^0$ of $\overline{\Delta(H,\sigma)}$

$$\bar{\omega}^0\left(\sum_{i=1}^{N} \lambda_i \delta_{x_i}\right) = \sum_{i,x_i \in H_0} \lambda_i \tag{5.5}$$

[see Proposition (4.2), viii)]; its restriction to $\overline{\Delta(H_0)}$ is pure according to Corollary (5.3), v). Now let ϱ be an arbitrary extremal central state; then one has

$$\varrho = \bar{\omega}^0 \circ \tau_\chi \tag{5.6}$$

where χ is the character of H_0 defined by

$$\varrho(\delta_x) = \chi(x)\,. \tag{5.7}$$

Hence we may rewrite Proposition (5.4):

Proposition (5.8).[4] *Let ω be an arbitrary central state; then there exists a probability measure on $\hat{H}_0$ such that*

$$\omega(\boldsymbol{a}) = \omega(\mathscr{M}(\boldsymbol{a})) = \int_{\hat{H}_0} \bar{\omega}^0 \circ \tau_\chi(\mathscr{M}(\boldsymbol{a}))\, dm_\omega(\chi)\,.$$

This decomposition is into disjoint factor states.

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[4] It has been pointed out to us by M. Winnink that this decomposition (i.e. the central decomposition) can be obtained without explicit reference to the $\mathscr{M}$-abelian character of the automorphism of $\overline{\Delta(H,\sigma)}$ induced by $\bar{K}$.

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