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The smallest g-supermartingale and reflected BSDE with single and double L^2 obstacles

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Abstract

In this paper we show how a solution of BSDE can be reflected by a very irregular L^2-obstacle. We prove that this problem is equivalent to find the smallest g-supermartingale of BSDE that dominates this obstacle. We then obtain the existence and uniqueness and continuous dependence theorem for this reflected BSDE. We also consider the problem of existence and uniqueness of reflected BSDE with double L^2 obstacles, by using a penalization method. A new monotonic limit theorem is developed to prove the convergence of the penalization sequence, and to prove the existence theorem. We also prove that this reflected BSDE with double obstacles is equivalent to a problem of the smallest g-supermartingale and the largest g-submartingale.

Résumé

Dans cet article, nous étudions comment une solution d'EDSR est réfléchie par un obstacle irrégulier qui est dans L^2. Nous montrons que ce problème est équivalent à trouver la plus petite g-surmartingale (ou g-sursolution) de l'EDSR qui majore cet obstacle. Nous obtenons des théorèmes d'existence, d'unicité et de dépendance continue pour ce problème. Nous considérons aussi l'unicité et l'existence de la solution pour l'EDSR avec deux obstacles par la méthode de pénalisation. Un nouveau théorème de limite monotone est développé pour montrer la convergence de la suite pénalisée, et pour obtenir le théorème d'existence. Nous montrons aussi que le problème de l'EDSR avec deux obstacles est équivalent à trouver la plus petite g-surmartingale et de la plus grande g-surmartingale.

Keywords: Reflected backward stochastic differential equation; Smallest g-supermartingale; Penalization method; Snell envelope

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1. Introduction

El Karoui, Kapoudjian, Pardoux, Peng and Quenez [10] studied the problem of BSDE (backward stochastic differential equation) with reflection, that is, a standard BSDE with an additional continuous, increasing process added in this equation to keep the solution above a certain given continuous boundary process. This increasing process must be chosen in an minimal way so that an integral condition, called Skorohod reflecting condition (cf. [24]), is satisfied. It was proved in this paper that the solution of the reflected BSDE is the smallest supersolution of this BSDE that dominates the given boundary process, called lower reflecting obstacle. An important observation of this paper is that the solution is the value function of an optimal stopping problem. Cvitanic and Karaztas (1996) [4] generalized the above results to the case of two reflecting obstacles: the solution of the BSDE has to remain between two prescribed continuous processes U and L, called lower and upper obstacle, respectively. Two continuous increasing processes was introduced in this reflected BSDE in order to force the solution to stay the region enveloped by the lower reflecting obstacle L and the upper reflecting obstacle U. Two Skorohod conditions are needed for the lower boundary L and the upper boundary U. They also established the connection of this problem and that of Dynkin games. We refer to [8,2,18,3,9,1,17,13] for interesting research works in this domain.

The advantage of introducing the above Skorohod condition is that it possesses a very interesting coercive structure that permits us to obtain many useful properties such as uniqueness, continuous dependence theories and other kind of regularities. It turns out to be a powerful tool to obtain the regularity properties of the corresponding solutions of PDE with obstacle such as free boundary PDE.

We recall that, when the lower boundary L is only an L^2-process, Peng [2] proved the existence of the smallest supersolution of BSDE with prescribed terminal condition that dominates this L and then applied this result to prove the a nonlinear decomposition of Doob–Meyer's type, i.e., a g-supermartingale is a g-supersolution. An interesting question is: in this situation, can we prove that this smallest supermartingale is the solution of the reflected BSDE with the lower obstacle L? In other words, can we find a new formulation of the Skorohod reflecting condition that characterizes this smallest solution? In the case where L has càdlàg (right continuous with lift limit) paths, a generalized Skorohod condition, similar to the original one, was given by Hamadene [12] and then, explicitly, by Lepeltier and Xu [16]. But their formulation cannot be applied to our L^2-case. In this paper we will give a generalized formulation of the Skorohod reflecting condition (see (7)) and then characterize the above smallest g-supermartingale as the unique solution of the related reflected BSDE.

We will also use this formulation to characterize the problem of BSDE with two reflecting L^2-obstacles L and U. For this purpose we first need to use a penalization method to prove the existence of the reflected solution. This is a constructive method in the sense that the solution of the reflected BSDE is proved to be the limit of a sequence of solutions of standard BSDEs called penalized BSDEs. Our penalization schemes might be useful since many numerical methods have been developed for these standard BSDEs (see our comments in Section 5 and Section 6.2). To prove the convergence, a new monotonic limit theorem, which generalizes a useful tool initially introduced in Peng [21], is developed. We also refer to [23,22,20,14,5] for some related studies on this subject.

The paper is organized as follows: In the next section, we state our main problems of the reflected BSDE, in Definition 2.2 for one L^2-obstacle and in Definition 2.3 for two L^2-obstacles. Both definitions will use the generalized notion of Skorohod reflecting conditions. We also present the notion of g-supersolutions. It will play a crucial role in this paper. The results of existence and uniqueness of these reflected BSDE and their equivalences to the corresponding g-supersolutions are given in Theorem 2.1 and Theorem 2.3. We also use these new formulation to prove the continuous dependence theorems. In Section 3 we will develop a monotonic limit theorem, i.e., Theorem 3.1 which is important in the proof of the existence part of Theorem 2.3 as well as in the proofs of the convergence our penalized BSDE schemes given in Section 5 and 6. In Section 4, we first present the results of existence of the smallest g-supersolution that dominates the process L by a penalization method of BSDE. Then we will prove the equivalence of the smallest g-supersolution that dominates L and the solution of the reflected BSDE with the obstacle L. This equivalence leads automatically the existence of the solution of the reflected BSDE with the lower obstacle L. To prove the existence of a solution of the reflected BSDE with two obstacles, we introduce a penalization scheme in Section 5 and give several important estimates. After these preparation, in Subsection 6.1, we will prove the existence of the reflected BSDE with two L^2-obstacles using the convergence of our penalized BSDEs. Subsection 6.2 is devoted to provide a direct penalization approach which is numerically more realistic.

The results of this paper can be regarded as a kind of nonlinear decomposition theorems (cf. [7,6]) of Doob–Meyer's type with a Brownian filtration (see Remark 2.2). It can be generalized to a more general filtration, using the existing results of BSDE with more general filtrations.

2. Statements and main results of reflected BSDE

2.1. Notations and preliminaries

On a given complete probability space $(\Omega, \mathcal{F}, P)$, let $(B_t, t \geqslant 0)$ be a standard d-dimensional Brownian motion defined on a finite interval $[0, T]$, and denote by $\mathbf{F} = \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ the augmentation of the natural filtration $\mathbf{F}^B = \{\mathcal{F}_t^B\}_{0\leqslant t\leqslant T}$ with $\mathcal{F}_t^B := \sigma\{B_s;\ 0 \leqslant s \leqslant t\}$, generated by B. The Euclidean norm of an element $x \in R^m$ will be denoted by $|x|$. We shall need the following notations. For each $p \geqslant 1$ and $t \in [0, T]$, let us introduce the following spaces:

- $L^p(\mathcal{F}_t; R^m) := \{\xi : \Omega \to R^m, \mathcal{F}_t\text{-measurable random variables } \xi \text{ with } E[|\xi|^p] < \infty\}$;
- $L^p_{\mathcal{F}}(0, t; R^m) := \{\varphi : \Omega \times [0, t] \to R^m;\ \mathbf{F}\text{-predictable processes with } E\int_0^t |\varphi_t|^p dt < \infty\}$;
- $D^p_{\mathcal{F}}(0, t; R^m) :=\{\varphi \in L^p_{\mathcal{F}}(0, t; R^m);\ \mathbf{F}$-progressively measurable càdlàg processes with $E[\sup_{0\leqslant t\leqslant \tau} |\varphi_t|^p] < \infty\}$.

In the real-value case, i.e., $m = 1$, they will be simply denoted by $L^p(\mathcal{F}_t)$, $L^p_{\mathcal{F}}(0, t)$ and $D^p_{\mathcal{F}}(0, t)$, respectively. We are mainly interested in the case $p = 2$.

We shall denote by $\mathcal{P}$ the σ-algebra of predictable sets in $[0, T] \times \Omega$.

2.2. Reflected BSDE with one L^2-obstacle

In this whole paper, $g : [0, T] \times \Omega \times R \times R^d \mapsto R$ is a given $\mathcal{P} \times \mathcal{B}(R) \times \mathcal{B}(R^d)$-measurable function. It satisfies the following standard condition (cf. Pardoux and Peng [19]:

$$E\int_0^T \left|g(t, \omega, 0, 0)\right|^2 \mathrm{d}t < \infty, \tag{1}$$

$$\begin{aligned}&\left|g(t, \omega, y_1, z_1) - g(t, \omega, y_2, z_2)\right| \leqslant k\big(|y_1 - y_2| + |z_1 - z_2|\big)\\ &\quad \forall (t, \omega) \in [0, T] \times \Omega,\ y_1, y_2 \text{ in } \mathbf{R},\ z_1, z_2 \text{ in } \mathbf{R}^d,\end{aligned} \tag{2}$$

for some given constant $k \in (0, \infty)$.

The following definition of g-supersolution is a notion parallel to that in PDE theory.

Definition 2.1 (*g-supersolution, cf. El Karoui, Peng and Quenez* [11] *and Peng* [21]). We say a triple

$$(Y, Z, V) \in D^2_{\mathcal{F}}(0, T) \times L^2_{\mathcal{F}}(0, T; \mathbf{R}^d) \times D^2_{\mathcal{F}}(0, T)$$

is a g-supersolution (resp. g-subsolution) if V is an increasing process in $D^2_{\mathcal{F}}(0, T)$

$$Y_t = Y_T + \int_t^T g(s, Y_s, Z_s)\,\mathrm{d}s + V_T - V_t - \int_t^T Z_s\,\mathrm{d}B_s, \quad t \in [0, T]. \tag{3}$$

We observe that if both (Y,Z,V) and (Y,Z',V') satisfy (3), then we have $Z=Z'$ and $V=V'$. For this reason we often simply call Y a g-supersolution.

Remark 2.1. We also observe that, given $\xi\in L^2(\mathcal{F}_T)$ and $V\in D^2_{\mathcal{F}}(0,T)$, there exists a unique solution $(Y,Z)\in D^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ of (3). This equivalent to solve

$$(\overline{Y},\overline{Z})=(Y+V,Z)\in D^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$$

of the following standard BSDE (cf. Pardoux and Peng [19])

$$\overline{Y}_t=\bar{\xi}+\int_t^T g(s,\overline{Y}_s-V_s,\overline{Z}_s)\,\mathrm{d}s-\int_t^T \overline{Z}_s\,\mathrm{d}B_s. \tag{4}$$

Remark 2.2. In Peng [21], we have obtained the following result: Y is a g-supersolution if and only if it is a g-supermartingale (a g-supermartingale is defined similarly as a classical supermartingale in which we use a notion of nonlinear expectations, called g-expectations, in the place of the classical linear expectations). It is a nonlinear version of decomposition theorems of Doob–Meyer's type. The increasing process A corresponds the one in the classical supermartingale (see, e.g., [6,7,15]). In this paper we consider a nonlinear version of decompositions of supermartingales and semimartingales.

We will first consider a reflected BSDE with a lower L^2-obstacles L. We assume that

$$L\in L^2_{\mathcal{F}}(0,T),\quad \xi\in L^2(\mathcal{F}_T)\quad\text{and}\quad E\Big[\operatorname*{ess\,sup}_{0\leqslant t\leqslant T}(L_t^+)^2\Big]<+\infty,\quad L_T\leqslant\xi,\ \text{a.s.} \tag{5}$$

Let us now introduce our generalized notion of RBSDE *with a single lower obstacle L*.

Definition 2.2. Let ξ be a given random variable in $L^2(\mathcal{F}_T)$ and $g:[0,T]\times\Omega\times R\times R^d\mapsto R$ be a given $\mathcal{P}\times\mathcal{B}(R)\times\mathcal{B}(R^d)$-measurable function satisfying (1) and (2). A triple $(Y,Z,A)\in D^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)\times D^2_{\mathcal{F}}(0,T)$ is called a solution of RBSDE with a lower obstacle $L\in L^2_{\mathcal{F}}(0,T)$ and terminal condition $\xi\in L^2(\mathcal{F}_T)$ if

(i) (Y,Z,A) is a g-supersolution with on $[0,T]$ with $Y_T=\xi$, i.e.

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s)\,\mathrm{d}s+A_T-A_t-\int_t^T Z_s\,\mathrm{d}B_s, \tag{6}$$

(ii) Y dominates L, i.e., $Y_t\geqslant L_t$, a.s. a.e.;

(iii) The following (generalized) Skorohod condition (cf. [24]) holds:

$$\int_0^T (Y_{s-}-L^*_{s-})\,\mathrm{d}A_s=0,\quad\text{a.s.},\quad \forall L^*\in D^2_{\mathcal{F}}(0,T)\text{ s.t. }L_t\leqslant L^*_t\leqslant Y_t,\ \text{a.s., a.e.} \tag{7}$$

The difference between the above definition and those of [10], with a continuous obstacle, and in [12,16], with a càdlàg obstacle, is in the Skorohod condition (iii). The following simple result linkes their notions and the ours.

Proposition 2.1. *If we assume further more that $L \in D^2_{\mathcal{F}}(0,T)$, then a triple $(Y,Z,A) \in D^2_{\mathcal{F}}(0,T) \times L^2_{\mathcal{F}}(0,T;\mathbf{R}^d) \times D^2_{\mathcal{F}}(0,T)$ is a solution of RBSDE with lower reflecting obstacle L and terminal condition $\xi \in L^2(\mathcal{F}_T)$ if and only if it satisfies the above conditions* (i), (ii) *and the following Skorohod condition*:

$$\int_0^T (Y_{s-} - L_{s-})\,\mathrm{d}A_s = 0, \quad a.s. \tag{8}$$

Proof. (7) $\Rightarrow$ (8) is obvious. To prove (8) $\Rightarrow$ (7), we only need to observe that, for each $L^*_t \in D^2_{\mathcal{F}}(0,T)$ such that $L_t \leqslant L^*_t \leqslant Y_t$, we have

$$0 \leqslant \int_0^T (Y_{s-} - L^*_{s-})\,\mathrm{d}A_s \leqslant \int_0^T (Y_{s-} - L_{s-})\,\mathrm{d}A_s = 0. \qquad \square$$

Remark 2.3. From the above definition, Y is a g-supersolution that dominates L. One may guess that this Y is, in fact, the smallest g-supersolution that dominates L. Indeed, we have

Theorem 2.1. *We assume that lower obstacle $L \in L^2_{\mathcal{F}}(0,T)$ and $\xi \in L^2(\mathcal{F}_T)$ satisfy* (5). *Then there exists a unique solution (Y,Z,A) of* RBSDE *with the lower obstacle L and the terminal condition $Y_T = \xi$. Moreover, Y is the smallest g-supersolution that dominates L with terminal condition $Y_T = \xi$.*

The proof of the existence will be given in Section 4. As we mentioned in the introduction, our formulation of the reflected BSDE permits us to derive easily the following continuous dependence theorem. This result also implies the proof of the uniqueness in Theorem 2.1.

Proposition 2.2. *We assume that lower obstacle $L \in L^2_{\mathcal{F}}(0,T)$ satisfies* (5). *Let $\varphi^i_s \in L^2_{\mathcal{F}}(0,T)$ and $\xi^i \in L^2(\mathcal{F}_T)$, $i = 1,2$, be given. Let (Y^i, Z^i, A^i) be the solution of RBSDEs with lower obstacle L, terminal condition ξ^i and the following coefficients*: $g^i(t,y,z) = g(t,y,z) + \varphi^i(t)$, *i.e., they are g^i-supersolutions of the following forms*:

$$Y^i_t = \xi^i + \int_t^T \left[g(s, Y^i_s, Z^i_s)\,\mathrm{d}s + \varphi^i_s\right]\mathrm{d}s + A^i_T - A^i_t - \int_t^T Z^i_s\,\mathrm{d}B_s, \tag{9}$$

and satisfy (7). *Then we have*

$$\begin{aligned} &E\Big[\sup_{0\leqslant t\leqslant T} |Y^1_t - Y^2_t|^2 + \sup_{0\leqslant t\leqslant T} |A^1_t - A^2_t|^2\Big] + E\left[\int_0^T |Z^1_t - Z^2_t|^2\right]\mathrm{d}t \\ &\quad \leqslant C E\left[|\xi^1 - \xi^2|^2 + \int_0^T |\varphi^1_s - \varphi^2_s|^2\,\mathrm{d}s\right], \end{aligned} \tag{10}$$

where the constant C depends only on T and the Lipschitz constant k of g, given in (2).

Proof. By setting $\widehat{Y} = Y^1 - Y^2$, $\widehat{Z} = Z^1 - Z^2$, $\hat{A} = A^1 - A^2$, $\hat{\xi} = \xi^1 - \xi^2$, $\hat{g} = g(s, Y^1, Z^1) - g(s, Y^2, Z^2)$ and $\hat{\varphi} = \varphi^1 - \varphi^2$, we have

$$\widehat{Y}_t = \hat{\xi} + \int_t^T [\hat{g}_s + \hat{\varphi}_s]\,\mathrm{d}s + \hat{A}_T - \hat{A}_t - \int_t^T \widehat{Z}_s\,\mathrm{d}B_s. \tag{11}$$

Their jumps satisfy $\Delta\widehat{Y} = -\Delta\hat{A}$. Apply Itô's rule to $|\widehat{Y}_t|^2$, we have

$$|\widehat{Y}_t|^2 + \int_t^T |\widehat{Z}_s|^2\,\mathrm{d}s + \sum_{t\leqslant s\leqslant T}(\Delta\hat{A}_s)^2 = \hat{\xi}^2 + 2\int_t^T \widehat{Y}_s(\hat{g}_s + \hat{\varphi}_s)\,\mathrm{d}s + 2\int_t^T \widehat{Y}_{s-}\,\mathrm{d}\hat{A}_s - 2\int_t^T \widehat{Y}_s\cdot\widehat{Z}_s\,\mathrm{d}B_s. \tag{12}$$

We set $L_t^* := Y_t^1 \wedge Y_t^2$. It is clear that $L^* \in D_{\mathcal{F}}^2(0,T)$ satisfy $L_t \leqslant L_t^* \leqslant Y_t^i$, a.e., a.s. $i = 1, 2$. Thanks to the generalized Skorohod condition (7), we have

$$\int_0^T (Y_{s-}^1 - L_{s-}^*)\,\mathrm{d}A_s^1 = \int_0^T (Y_{s-}^2 - L_{s-}^*)\,\mathrm{d}A_s^2 = 0.$$

The third term of the right hand of (12) is dominated by 0 since

$$\int_0^T \widehat{Y}_{s-}\,\mathrm{d}\hat{A}_s = \int_0^T (Y_{s-}^1 - L_{s-}^*)\,\mathrm{d}A_s^1 + \int_0^T (L_{s-}^* - Y_{s-}^2)\,\mathrm{d}A_s^1 + \int_0^T (Y_{s-}^2 - L_{s-}^*)\,\mathrm{d}A_s^2 + \int_0^T (L_{s-}^* - Y_{s-}^1)\,\mathrm{d}A_s^2.$$

It follows that

$$|\widehat{Y}_t|^2 + \int_t^T |\widehat{Z}_s|^2\,\mathrm{d}s + \sum_{t\leqslant s\leqslant T}(\Delta\hat{A}_s)^2 \leqslant \hat{\xi}^2 + \int_t^T \widehat{Y}_s(\hat{g}_s + \widehat{\varphi}_s)\,\mathrm{d}s - 2\int_t^T \widehat{Y}_s\cdot\widehat{Z}_s\,\mathrm{d}B_s. \tag{13}$$

By Lipschitz condition of g, we have $|\hat{g}_s| \leqslant k(|\widehat{Y}_s| + |\widehat{Z}_s|)$. Thus

$$\begin{aligned}
&|\widehat{Y}_t|^2 + \left(1 - \frac{1}{\alpha}\right)\int_t^T |\widehat{Z}_s|^2\,\mathrm{d}s + \sum_{t\leqslant s\leqslant T}(\Delta\hat{A}_s)^2 \\
&\quad\leqslant |\hat{\xi}|^2 + (2k + \alpha k^2 + \beta)\int_t^T |\widehat{Y}_s|^2\,\mathrm{d}s + \frac{1}{\beta}\int_t^T |\hat{\varphi}_s|^2\,\mathrm{d}s - 2\int_t^T \widehat{Y}_s\cdot\widehat{Z}_s\,\mathrm{d}B_s.
\end{aligned} \tag{14}$$

Set $\alpha = 2$, $\beta = 1$, it follows that

$$E\big[|\widehat{Y}_t|^2\big] \leqslant E[\hat{\xi}^2] + (2k + 2k^2 + 1)E\int_t^T |\widehat{Y}_s|^2\,\mathrm{d}s + E\int_t^T |\hat{\varphi}_s|^2\,\mathrm{d}s.$$

It then follows from Gronwell's inequality that

$$E\big[|\widehat{Y}_t|^2\big] \leqslant C\left(E[\hat{\xi}^2] + E\int_t^T |\hat{\varphi}_s|^2\,\mathrm{d}s\right).$$

We thus have

$$E\big[|\widehat{Y}_t|^2\big] + E\left[\int_0^T |\widehat{Z}_s|^2\,\mathrm{d}s\right] \leqslant C\left(E[\hat{\xi}^2] + E\int_0^T |\hat{\varphi}_s|^2\,\mathrm{d}s\right).$$

With this estimate and using Burkholder–Davis–Gundy inequality to (13), we deduce the estimate for $\mathbf{E}[\sup_t |\widehat{Y}_t|^2|]$ in (10). Then, using again Burkholder–Davis–Gundy inequality to (11), we deduce the estimate for $\mathbf{E}[\sup_t |\hat{A}_t|^2|]$. $\square$

The uniqueness part in Theorem 2.1 is proved by setting $\xi^1 = \xi^2 = \xi$, $\varphi^1 = \varphi^2 = 0$. We also have the following estimate:

Theorem 2.2. *We assume that $\xi \in L^2(\mathcal{F}_T)$ and the lower obstacle $L \in L^2_{\mathcal{F}}(0,T)$ satisfies* (5)*. Let (Y, Z, A) be the solution of* RBSDE *with the coefficient g, the terminal condition ξ and the lower obstacle L. Then we have*

$$E\Big[\sup_{0\leqslant t\leqslant T}|Y_t|^2 + \sup_{0\leqslant t\leqslant T}|A_t|^2\Big] + E\Bigg[\int_0^T |Z_t|^2\,\mathrm{d}t\Bigg] \leqslant CE\Bigg[|\xi|^2 + \int_0^T \big|g(s,0,0)\big|^2\,\mathrm{d}s + \sup_{0\leqslant t\leqslant T}(L_t^+)^2\Bigg].$$

Since the proof is similar to the previous one. We omit it.

2.3. Reflected BSDE with two L^2-obstacles

We now consider a BSDE reflected between a lower obstacles L and a upper obstacle U where L and U are L^2-processes. We still make the usual condition (1) and (2) for the coefficient g. The obstacles satisfy the following assumptions

(H) $L, U \in \mathbf{L}^2_{\mathcal{F}}(0,T)$ with

$$E\Big[\operatorname{ess}\sup_{0\leqslant t\leqslant T}(L_t^+)^2\Big] + E\Big[\operatorname{ess}\sup_{0\leqslant t\leqslant T}(U_t^-)^2\Big] < +\infty, \quad L_T \leqslant \xi \leqslant U_T, \text{ a.s.} \tag{15}$$

and there exists a process $X_t = X_0 + A_t^0 - K_t^0 + \int_0^t Z_s^0\,\mathrm{d}B_s$, $0 \leqslant t \leqslant T$ with $Z^0 \in L^2_{\mathcal{F}}(0,T)$, A^0, $K^0 \in D^2_{\mathcal{F}}(0,T)$, such that A^0 and K^0 are increasing with $A_0^0 = K_0^0 = 0$ and such that

$$L_t \leqslant X_t \leqslant U_t, \quad \text{a.e., a.s.} \tag{16}$$

The formulation of the RBSDE with two $\mathbf{L}^2$-obstacles is as follows.

Definition 2.3. A solution of BSDE reflected between a lower obstacle $L \in L^2_{\mathcal{F}}(0,T)$ and an upper obstacle $U \in L^2_{\mathcal{F}}(0,T)$ with parameters (ξ, g) is a quadruple $(Y, Z, A, K) \in D^2_{\mathcal{F}}(0,T) \times L^2_{\mathcal{F}}(0,T;\mathbf{R}^d) \times (D^2_{\mathcal{F}}(0,T))^2$ satisfying

(i) A, K are increasing: $\mathrm{d}A \geqslant 0$, $\mathrm{d}K \geqslant 0$;
(ii) (Y, Z) solves the following BSDE on $[0,T]$:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\,\mathrm{d}s + A_T - A_t - (K_T - K_t) - \int_t^T Z_s\,\mathrm{d}B_s; \tag{17}$$

(iii) $L_t \leqslant Y_t \leqslant U_t$, a.e. a.s.
(iv) (Generalized) Skorohod condition: for each $L^*, U^* \in D^2_{\mathcal{F}}(0,T)$ such that $L_t \leqslant L_t^* \leqslant Y_t \leqslant U_t^* \leqslant U_t$ a.e. a.s., we have

$$\int_0^T (Y_{s-} - L_{s-}^*)\,\mathrm{d}A_s = \int_0^T (Y_{s-} - U_{s-}^*)\,\mathrm{d}K_s = 0, \quad \text{a.s.} \tag{18}$$

For this reflected BSDE, we have the following main result of existence and uniqueness:

Theorem 2.3. *We make assumptions* (15) *and* (16) *of* (H). *Then there exists at least one solution* (Y, Z, A, K) *of* RBSDE *in the sense of Definition* 2.3. *The solution is unique in the following sense*: *if* (Y', Z', A', K') *is another solution, then* $Y'_t \equiv Y_t$, $Z'_t \equiv Z_t$, *and* $A_t - K_t \equiv A'_t - K'_t$, $\forall t \in [0, T]$, *a.s.*

Example 2.1. The following example shows that, while the uniqueness is true for (Y, Z), but not for (A, K).

$$L_t \equiv U_t \equiv 0, \quad g(t, y, z) \equiv 0, \quad \xi = 0.$$

In this case it is clear that $Y_t \equiv 0$ is the unique g-solution such that $L_t \leqslant Y_t \leqslant U_t$, a.e., a.s. Thus $(Y_t, Z_t, A_t, K_t) \equiv (0, 0, 0, 0)$. They satisfies (i)–(iv) of Definition 2.3. But $(Y_t, Z_t, A'_t, K'_t) \equiv (0, 0, t, t)$ also satisfies (i)–(iv).

Remark 2.4. It is easy to check that the assumption (5) for RBSDE with one obstacle, as well as (15) and (16) in (H) for RBSDE with two obstacles, are also necessary for the existence of the related RBSDE.

The uniqueness part of proof of Theorem 2.3 is a simple consequence of the following continuous dependence theorem, which once more, shows that our new Skorohod condition (18) is a very useful formulation.

Theorem 2.4. *We make assumptions* (15) *and* (16) *of* (H). *For* $i = 1, 2$, *let* $(Y^i, Z^i, A^i, K^i) \in D^2_{\mathcal{F}}(0, T) \times L^2_{\mathcal{F}}(0, T; R^d) \times D^2_{\mathcal{F}}(0, T) \times D^2_{\mathcal{F}}(0, T)$ *be the solutions of the* RBSDE

$$\begin{aligned} &\mathrm{d}Y^i_t = \big[g(t, Y^i_t, Z^i_t) + \varphi^i_t\big]\,\mathrm{d}t + \mathrm{d}A^i_t - \mathrm{d}K^i_t - Z^i_t\,\mathrm{d}B_t, \\ &Y^i_T = \xi^i, \quad i = 1, 2, \end{aligned} \tag{19}$$

with two obstacles $L, U \in L^2_{\mathcal{F}}(0, T)$, *i.e., in the sense of Definition* 2.3(i)–(iv). *Then we have*

$$\begin{aligned} &E\Big[\sup_{0\leqslant t\leqslant T}|Y^1_t - Y^2_t|^2\Big] + E\Big[\sup_{0\leqslant t\leqslant T}|A^1_t - A^2_t - (K^1_t - K^2_t)|^2\Big] + E\left[\int_0^T |Z^1_t - Z^2_t|^2\,\mathrm{d}t\right] \\ &\quad \leqslant CE\left[|\xi^1 - \xi^2|^2 + \int_0^T |\varphi^1_s - \varphi^2_s|^2\,\mathrm{d}s\right], \end{aligned} \tag{20}$$

the constant C *depends only on the Lipschitz constant of* g *and* T.

Proof. We set $\widehat{Y} = Y^1 - Y^2$, $\widehat{Z} = Z^1 - Z^2$, $\hat{A} = A^1 - A^2$, $\widehat{K} = K^1 - K^2$, $\hat{\xi} = \xi^1 - \xi^2$, and $\hat{g}_s = g(s, Y^1_s, Z^1_s) - g(s, Y^2_s, Z^2_s)$, $\hat{\varphi} = \varphi^1 - \varphi^2$, in following

$$\widehat{Y}_t = \hat{\xi} + \int_t^T [\hat{g}_s + \hat{\varphi}_s]\,\mathrm{d}s + \hat{A}_T - \hat{A}_t - (\widehat{K}_T - \widehat{K}_t) - \int_t^T \widehat{Z}_s\,\mathrm{d}B_s. \tag{21}$$

Obviously $\Delta\widehat{Y} = \Delta\widehat{K} - \Delta\hat{A}$. Apply Itô's formula to $|\widehat{Y}_t|^2$, then

$$\begin{aligned} &|\widehat{Y}_t|^2 + \int_t^T |\widehat{Z}_s|^2\,\mathrm{d}s + \sum_{t\leqslant s\leqslant T}(\Delta\widehat{K}_s - \Delta\hat{A}_s)^2 \\ &\quad = |\hat{\xi}|^2 + 2\int_t^T \widehat{Y}_s(\hat{g}_s + \hat{\varphi}_s)\,\mathrm{d}s + 2\int_t^T \widehat{Y}_{s-}\,\mathrm{d}\hat{A}_s - 2\int_t^T \widehat{Y}_{s-}\,\mathrm{d}\widehat{K}_s - 2\int_t^T \widehat{Y}_s\cdot\widehat{Z}_s\,\mathrm{d}B_s. \end{aligned} \tag{22}$$

We define $L_t^* = Y_t^1 \wedge Y_t^2$ and $U_t^* = Y_t^1 \vee Y_t^2$, it's clear that $L^*, U^* \in D_{\mathcal{F}}^2(0,T)$ and $L_t \leqslant L_t^* \leqslant Y_t^i \leqslant U_t^* \leqslant U_t$,. By the Generalized Skorohod condition (iv) of Definition 2.3, we have

$$\int_t^T (Y_{s-}^1 - L_{s-}^*)\,\mathrm{d}A_s^1 = \int_t^T (Y_{s-}^2 - L_{s-}^*)\,\mathrm{d}A_s^2 = 0,$$

$$\int_t^T (Y_{s-}^1 - U_{s-}^*)\,\mathrm{d}K_s^1 = \int_t^T (Y_{s-}^2 - U_{s-}^*)\,\mathrm{d}K_s^2 = 0.$$

Thus for the two last terms in (22), we have

$$\int_t^T \widehat{Y}_{s-}\,\mathrm{d}\hat{A}_s = \int_t^T (Y_{s-}^1 - L_{s-}^*)\,\mathrm{d}A_s^1 + \int_t^T (L_{s-}^* - Y_{s-}^2)\,\mathrm{d}A_s^1 + \int_t^T (Y_{s-}^2 - L_{s-}^*)\,\mathrm{d}A_s^2 \quad \int_t^T (L_{s-}^* - Y_{s-}^1)\,\mathrm{d}A_s^2 \leqslant 0$$

and, similarly, $\int_t^T \widehat{Y}_{s-}\,\mathrm{d}\widehat{K}_s \geqslant 0$. Applying these two inequalities to (22) yields

$$|\widehat{Y}_t|^2 + \int_t^T |\widehat{Z}_s|^2\,\mathrm{d}s + \sum_{t\leqslant s\leqslant T} (\Delta\widehat{K}_s - \Delta\hat{A}_s)^2 \leqslant \hat{\xi}^2 + 2\int_t^T \widehat{Y}_s(\hat{g}_s + \widehat{\varphi}_s)\,\mathrm{d}s - 2\int_t^T \widehat{Y}_s \cdot \widehat{Z}_s\,\mathrm{d}B_s. \tag{23}$$

We now arrive to a position similar to that of (13) in the proof of Theorem 2.2. We then can analogously obtain (20) by using Gronwall's inequality and Burkholder–Davis–Gundy inequality. □

3. A generalized monotonic limit theorem for Itô processes

In this section, we will develop a new convergence theorem for a monotonic sequence of Itô processes. It is a generalized version of a monotonic limit theorem obtained in Peng [21] (Theorem 2.1 of [21]). In Section 6, we will use this result to prove the existence part of Theorem 2.3 for reflected BSDE with two obstacles.

We consider the following sequence of Itô processes

$$y_t^i = y_0^i + \int_0^t g_s^i\,\mathrm{d}s - A_t^i + K_t^i + \int_0^t z_s^i\,\mathrm{d}B_s, \quad i = 1, 2, \ldots. \tag{24}$$

Here, for each i, the processes $g^i \in L_{\mathcal{F}}^2(0,T)$ and $A^i, K^i \in D_{\mathcal{F}}^2(0,T)$ are given. $(A^i, K^i)_{i=1}^{\infty}$ satisfy

(h1) A^i is continuous and increasing such that $A_0^i = 0$ and $\mathbf{E}[(A_T^i)^2] < \infty$;
(h2) K^i is increasing with $K_0^i = 0$;
(h3) $K_t^j - K_s^j \geqslant K_t^i - K_s^i$, $\forall 0 \leqslant s \leqslant t \leqslant T$, a.s., $\forall i \leqslant j$;
(h4) For each $t \in [0,T]$, $K_t^j \nearrow K_t$, with $\mathbf{E}[K_T^2] < \infty$.

For $(y^i, g^i, z^i)_{i=1}^{\infty}$, we assume

$$\begin{aligned} &\text{(i)}\ (g^i, z^i)_{i=i}^{\infty} \text{ weakly converges to } (g^0, z) \text{ in } L_{\mathcal{F}}^2(0,T;\mathbf{R}\times\mathbf{R}^d),\\ &\text{(ii)}\ (y_t^i)_{i=1}^{\infty} \text{ increasingly converges up to } (y_t) \text{ with } \mathbf{E}[\sup_{0\leqslant t\leqslant T} |y_t|^2] < \infty. \end{aligned} \tag{25}$$

It is clear that

$$\begin{aligned} &\text{(i)}\ \mathbf{E}[\sup_{0\leqslant t\leqslant T}|y_t^i|^2]\leqslant C,\\ &\text{(ii)}\ \mathbf{E}\int_0^T|y_t^i-y_t|^2\,\mathrm{d}s\to 0, \end{aligned} \tag{26}$$

where the constant C is independent of i.

Remark 3.1. It is easy to check that the limit y of $\{y^i\}_{i=1}^{\infty}$ is the following form of Itô processes

$$y_t=y_0+\int_0^t g_s^0\,\mathrm{d}s-A_t+K_t+\int_0^t z_s\,\mathrm{d}B_s, \tag{27}$$

where A_t is the weak limit in A_t^i in $L^2(\mathcal{F}_T)$. In general, we cannot prove the strong convergence of $\{z^i\}_{i=1}^{\infty}$ in $L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$. But as in Peng [21], we can prove that the convergence holds in some stronger sense: for each $p\in[1,2)$, $\{z^i\}$ converges strongly in $L^p_{\mathcal{F}}(0,T;\mathbf{R}^d)$.

Our monotonicity limit theorem is as following.

Theorem 3.1. *We assume that the sequence of Itô processes* (24) *satisfies* (h1)–(h4), (26) *and* (25). *Then the limit y of $\{y^i\}_{i=1}^{\infty}$ has a form* (27), *where A and K are increasing processes in $D^2_{\mathcal{F}}(0,T)$. Here, for each $t\in[0,T]$, A_t* (*resp.* K_t) *is the weak* (*resp. strong*) *limit of $\{A_t^i\}_{i=1}^{\infty}$* (*resp.* $\{K_t^i\}_{i=1}^{\infty}$) *in $L^2(\mathcal{F}_T)$. Furthermore, for any $p\in[0,2)$, $\{z^i\}_{i=1}^{\infty}$ strongly converges to z in $\mathbf{L}^p_{\mathcal{F}}(0,T,\mathbf{R}^d)$, i.e.,*

$$\lim_{i\to\infty}\mathbf{E}\int_0^T|z_s^i-z_s|^p\,\mathrm{d}s=0. \tag{28}$$

If furthermore $(A_t)_{t\in[0,T]}$ is continuous, then we have

$$\lim_{i\to\infty}\mathbf{E}\int_0^T|z_s^i-z_s|^2\,\mathrm{d}s=0. \tag{29}$$

Remark 3.2. A special situation of the above theorem is when $K_t^i\equiv 0$, $i=1,2,\ldots,$ and thus $K_t\equiv 0$. This result was obtained in [21]. This special case will be also applied in this paper.

The following two easy lemmas is applied to prove that A, K and thus y are càdlàg processes. We omit the proofs.

Lemma 3.1. *Let $\{x^i(\cdot)\}_{i=1}^{\infty}$ be a sequence of (deterministic) càdlàg processes defined on $[0,T]$ that increasingly converges to $x(\cdot)$ such that, for each $t\in[0,T]$, and $i=1,2,\ldots,$ $x^i(t)\leqslant x^{i+1}(t)$, with $x(t)=b(t)-a(t)$, where $b(\cdot)$ is an càdlàg process and $a(\cdot)$ is an increasing process with $a(0)=0$ and $a(T)<\infty$. Then $x(\cdot)$ and $a(\cdot)$ are also càdlàg processes.*

Lemma 3.2. *Let $\{a^i(t),\ 0\leqslant t\leqslant T\}_{i=1}^{\infty}$ be a sequence of* (*deterministic*) *càdlàg* (*resp. càglàd*) *and increasing processes defined on $[0,T]$ such that, for each $t\in[0,T]$, $a^i(t)\nearrow a(t)<\infty$ and such that $a^j(t)-a^i(t)\leqslant a^j(t')-a^i(t')$, for each $j\geqslant i$ and $0\leqslant t\leqslant t'\leqslant T$. Then the limit $a(\cdot)$ is also a càdlàg* (*resp. càglàd*) *process.*

Proof of Theorem 3.1. Since $\{g^i\}_{i=1}^\infty$ and $\{z^i\}_{i=1}^\infty$ weakly converge to g^0 and z in $L^2_{\mathcal{F}}(0,T)$ and $L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$, respectively, and $\{K_t^i\}_{i=1}^\infty$ converges up to K_t in $L^2(\mathcal{F}_t)$, thus, for each stopping time $\tau \leqslant T$, the following weak convergence holds in $L^2(\mathcal{F}_\tau)$.

$$\int_0^\tau z_s^i \,\mathrm{d}B_s \rightharpoonup \int_0^\tau z_s \,\mathrm{d}B_s, \quad \int_0^\tau g_s^i \,\mathrm{d}s \rightharpoonup \int_0^\tau g_s^0 \,\mathrm{d}s, \quad K_\tau^i \rightharpoonup K_\tau.$$

Since

$$A_\tau^i = -y_\tau^i + y_0^i + K_\tau^i + \int_0^\tau g_s^i \,\mathrm{d}s + \int_0^\tau z_s^i \,\mathrm{d}B_s$$

thus we also have the weak convergence in $L^2(\mathcal{F}_\tau)$:

$$A_\tau^i \rightharpoonup A_\tau := -y_\tau + y_0 + K_\tau + \int_0^\tau g_s^0 \,\mathrm{d}s + \int_0^\tau z_s \,\mathrm{d}B_s.$$

Obviously $\mathbf{E}[A_T^2] < \infty$. For any two stopping times $\sigma \leqslant \tau \leqslant T$, we have $A_\sigma \leqslant A_\tau$ since $A_\sigma^i \leqslant A_\tau^i$. From this it follows that A is an increasing process. Moreover, from Lemmas 3.1 and 3.2, K, A and y are càdlàg, thus y has a form of (27). Our key point is to show that $\{z^i\}_{i=1}^\infty$ converges to z in the strong sense of (28). In order to prove this we apply Itô's formula to $(y_t^i - y_t)^2$ on each given subinterval $(\sigma,\tau]$. Here $0 \leqslant \sigma \leqslant \tau \leqslant T$ are two stopping times. Observe that $\Delta y_t \equiv \Delta(K_t - A_t)$, $\Delta y_t^i = \Delta K_t^i$. We have

$$\begin{aligned}
&\mathbf{E}|y_\sigma^i - y_\sigma|^2 + \mathbf{E}\int_\sigma^\tau |z_s^i - z_s|^2 \,\mathrm{d}s \\
&\quad = \mathbf{E}|y_\tau^i - y_\tau|^2 - \mathbf{E}\sum_{t\in(\sigma,\tau]} \big(\Delta(A_t - K_t + K_t^i)\big)^2 - 2\mathbf{E}\int_\sigma^\tau (y_s^i - y_s)(g_s^i - g_s^0)\,\mathrm{d}s \\
&\qquad + 2\mathbf{E}\int_{(\sigma,\tau]} (y_s^i - y_s)\,\mathrm{d}A_s^i - 2\mathbf{E}\int_{(\sigma,\tau]} (y_{s-}^i - y_{s-})\,\mathrm{d}A_s - 2\mathbf{E}\int_{(\sigma,\tau]} (y_{s-}^i - y_{s-})\,\mathrm{d}(K_s^i - K_s) \\
&\quad = \mathbf{E}|y_\tau^i - y_\tau|^2 + \mathbf{E}\sum_{t\in(\sigma,\tau]} \big[(\Delta A_t)^2 - (\Delta K_t - \Delta K_t^i)^2\big] - 2\mathbf{E}\int_\sigma^\tau (y_s^i - y_s)(g_s^i - g_s^0)\,\mathrm{d}s \\
&\qquad + 2\mathbf{E}\int_{(\sigma,\tau]} (y_s^i - y_s)\,\mathrm{d}A_s^i - 2\mathbf{E}\int_{(\sigma,\tau]} (y_s^i - y_s)\,\mathrm{d}A_s - 2\mathbf{E}\int_{(\sigma,\tau]} (y_{s-}^i - y_{s-})\,\mathrm{d}(K_s^i - K_s).
\end{aligned}$$

Since $\int_{(\sigma,\tau]} (y_s^i - y_s)\,\mathrm{d}A_s^i \leqslant 0$ and $-2\mathbf{E}\int_{(\sigma,\tau]} (y_{s-}^i - y_{s-})\,\mathrm{d}(K_s^i - K_s) \leqslant 0$, we then have

$$\begin{aligned}
\mathbf{E}\int_\sigma^\tau |z_s^i - z_s|^2 \,\mathrm{d}s &\leqslant \mathbf{E}|y_\tau^i - y_\tau|^2 + \mathbf{E}\sum_{t\in(\sigma,\tau]} \Delta(A_t)^2 \\
&\quad + 2\mathbf{E}\int_\sigma^\tau |y_s^i - y_s||g_s^i - g_s^0|\,\mathrm{d}s + 2\mathbf{E}\int_{(\sigma,\tau]} |y_s^i - y_s|\,\mathrm{d}A_s.
\end{aligned} \tag{30}$$

Now we are in the same position as in that of proof of Theorem 2.1 of Peng [21] (see the first inequality in page 483, see also [22]). Thus we can follow that proof to prove (28) and (29). □

4. The proof of Theorem 2.1 through equivalence between the smallest g-supersolution and the related RBSDE

Theorem 2.1 is an easy consequence of Theorem 4.1 of this section in which the following equivalence is given: A triple (Y, Z, A) is the solution of RBSDE if and only if it is the related smallest g-supersolution. Using this result and the existence of the smallest g-supersolution given in Proposition 4.2, we then obtain the proof. We first claim

Proposition 4.1. *We assume that lower obstacle $L \in L^2_{\mathcal{F}}(0, T)$ satisfies* (5). *Let the function g satisfy* (1) *and* (2). *For a given process $Y \in D^2_{\mathcal{F}}(0, T)$ with $Y_T = \xi \in L^2(\mathcal{F}_T)$, the following claims are equivalent*:

(a) *Y is the smallest g-supersolution that dominates L*;
(b) *for each $L^* \in D^2_{\mathcal{F}}(0, T)$ such that $Y_t \geqslant L^*_t \geqslant L_t$, a.e., a.s., Y is the smallest g-supersolution that dominates L^*.*

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a): Let $\overline{Y} \in D^2_{\mathcal{F}}(0, T)$ be the smallest g-supersolution that dominates L with $\overline{Y}_T = \xi$. Then $Y_t \geqslant \overline{Y}_t \geqslant L_t$, a.e., a.s. Thus Y is the smallest g-supersolution that dominates $\overline{Y}$, i.e., $\overline{Y}_t \equiv Y_t$, $\forall t$, a.s. □

We now give the existence theorem of the smallest g-solution that dominates L. This theorem is proved in [10] for the situation where L has continuous paths. The case where $L \in L^2_{\mathcal{F}}(0, T)$ is a special situation of Theorem 4.2 in Peng [21]. This theorem claims the existence of the smallest g-supersolution (Y, Z) subject to the constraint

$$\Phi(t, Y_t, Z_t) = 0, \quad \text{a.e., a.s.,} \tag{31}$$

where the function $\Phi : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \to [0, \infty)$ satisfies the same assumptions (1) and (2) for g. In this paper we are only interested in the constraint $y \geqslant L_t$, or equivalently,

$$\Phi(t, y, z) := (y - L_t)^- = 0. \tag{32}$$

The main idea of the proof is to introduce the following so-called penalized BSDE, which will be frequently used in this paper,

$$\begin{aligned} Y^n_t &= \xi + \int_t^T g(s, Y^n_s, Z^n_s)\,\mathrm{d}s + A^n_T - A^n_t - \int_t^T Z^n_s\,\mathrm{d}B_s, \\ A^n_t &:= n \int_0^t \Phi(s, Y^n_s, Z^n_s)\,\mathrm{d}s. \end{aligned} \tag{33}$$

By comparison theorem of BSDE $Y^n_t \leqslant Y^{n+1}_t$, $t \in [0, T]$, a.s. As, $n \to \infty$, the limit is the smallest g-supersolution:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\,\mathrm{d}s + A_T - A_t - \int_t^T Z_s\,\mathrm{d}B_s, \tag{34}$$

$$\Phi(t, Y_t, Z_t) \equiv 0, \quad \text{a.e., a.s. } A \in D^2_{\mathcal{F}}(0, T), \quad \mathrm{d}A_t \geqslant 0. \tag{35}$$

More precisely, Theorem 4.2 of Peng [21] claims:

Proposition 4.2. *Let the function g satisfy* (1), (2). *We also assume that there exists a g-supersolution (Y^*, Z^*) constrained by $\Phi(t, Y_t^*, Z_t^*) \equiv 0$ with terminal condition $\xi \in L^2(\mathcal{F}_T)$. Then the smallest g-supersolution $Y \in D^2_{\mathcal{F}}(0,T)$ constrained by* (31) *with terminal condition ξ exists. It is the solution of BSDE* (34)*, where $A \in D^2_{\mathcal{F}}(0,T)$ is an increasing process. Moreover (Y, Z, A) is the limit of the sequence of penalized BSDEs* (33) *in the following sense, for each fixed $p \in [1,2)$,*

$$\begin{cases} E\int_0^T (|Y_t^n - Y_t|^2 + |Z_t^n - Z_t|^p)\,\mathrm{d}t \to 0, \\ E\int_0^T (Z_t^n - Z_t)\varphi_t\,\mathrm{d}t \to 0, \quad \forall \varphi \in L^2_{\mathcal{F}}(0,T;R^d), \\ E[(A_\tau^n - A_\tau)\zeta] \to 0, \quad \forall \zeta \in L^2(\mathcal{F}_T),\ \forall \tau (\textit{stopping time}). \end{cases} \tag{36}$$

Remark 4.1. The above convergence also imply the boundedness:

$$E\Big[\sup_{0\leqslant t\leqslant T} |Y_t^n|^2\Big] + E\left[\int_0^T |Z_t^n|^2\,\mathrm{d}t\right] + E\big[(A_T^n)^2\big] \leqslant C \tag{37}$$

where the constant C does not depends on n.

With this theorem we can obtain the existence of the smallest g-supersolution that dominates L:

Proposition 4.3. *Let the function g satisfy* (1), (2) *and let the lower obstacle L satisfies* (5). *Then the smallest g-supersolution $Y \in D^2_{\mathcal{F}}(0,T)$ that dominates L with terminal condition ξ exists. It is the solution of BSDE* (34) *with the constraint Φ defined in* (32)*, where $A \in D^2_{\mathcal{F}}(0,T)$ is the corresponding increasing process. Moreover, (Y, Z, A) is the limit of the sequence of penalized BSDEs* (33) *in the sense of* (36).

Proof. This is a simple corollary of Proposition 4.2 for $\Phi(t,y,z) = (y - L_t)^-$. We only need to check the existence of a g-supersolution Y^* with terminal condition $Y_T^* = \xi$ such that $(Y_t^* - L_t)^- \equiv 0$. By (5), we have

$$\zeta := \max\Big\{\operatorname{ess}\sup_{s\in[0,T)} L_s 1_{\{s<T\}}, \xi\Big\} \in L^2(\mathcal{F}_T).$$

Let (Y^*, Z^*) be the solution of the following BSDE

$$Y_t^* = \zeta + \int_t^T \big|g(s, Y_s^*, Z_s^*)\big|\,\mathrm{d}s - \int_t^T Z_s^*\,\mathrm{d}B_s.$$

It is easy to check that $Y_t^* \geqslant E[\zeta|\mathcal{F}_t] \geqslant L_t$. We then define an increasing process $A^* \in D^2_{\mathcal{F}}(0,T)$ by

$$A_t^* := \int_0^t \big(\big|g(s, Y_s^*, Z_s^*)\big| - g(s, Y_s^*, Z_s^*)\big)\,\mathrm{d}s + (\zeta - \xi)1_{\{t=T\}}.$$

The above Y^* is a g-supersolution that dominates L:

$$Y_t^* = \xi + A_T^* - A_t^* + \int_t^T g(s, Y_s^*, Z_s^*)\,\mathrm{d}s - \int_t^T Z_s^*\,\mathrm{d}B_s, \quad t \in [0,T]. \qquad \square$$

With the above existence theorem of the smallest g-supersolution, the existence and uniqueness of RBSDE with single obstacle L is merely a simple consequence of the following properties. As a main result, we will give the equivalence between the smallest g-supersolution dominated by L and RBSDE with lower obstacle L. First we consider a simple case.

Let $l \in D^2_{\mathcal{F}}(0,T)$ be a given process. For the case $g_0(t) \equiv 0$, a g_0-supersolution $Y \in D^2_{\mathcal{F}}(0,T)$ that dominates $l \in D^2_{\mathcal{F}}(0,T)$ with $Y_T = \xi \in L^2(\mathcal{F}_T)$ is simply defined by

$$Y_t = \xi + A_T - A_t - \int_t^T Z_s \, \mathrm{d}B_s, \quad Y_t \geqslant l_t, \quad \forall t \in [0,T], \quad \text{a.s.} \tag{38}$$

where $Z \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ and $A \in D^2_{\mathcal{F}}(0,T)$ is an increasing process with $A_0 = 0$. Thus Y is a merely a supermartingale that dominates l on $[0,T]$ with $Y_T = \xi$. We need the following result:

Lemma 4.1. *Let $Y \in D^2_{\mathcal{F}}(0,T)$ be the smallest g_0-supersolution that dominates l with $Y_T = \xi$. Then for each stopping time $\tau \leqslant T$, we have*

$$Y_{\tau-} = Y_\tau \vee l_{\tau-}. \tag{39}$$

Consequently

$$\sum_{0 \leqslant t \leqslant T} (Y_{t-} - l_{t-})(A_t - A_{t-}) = 0, \quad \textit{a.s.} \tag{40}$$

Proof. For any stopping times $\sigma, \tau \in \mathcal{T}_0$ such that $\sigma \leqslant \tau$, we denote by $\mathcal{T}_{\sigma,\tau}$ the set of stopping times $\rho \in \mathcal{T}_0$ such that $\sigma \leqslant \rho \leqslant \tau$. We define

$$\overline{Y_t} := \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_t} E\big[l_\sigma 1_{\{\sigma < T\}} + \xi 1_{\{\sigma = T\}} | \mathcal{F}_t\big].$$

It is known that $\overline{Y}$ is the smallest supermartingale that dominates l on $[0,T]$ with $Y_T = \xi$. Thus we have $Y \equiv \overline{Y}$. Moreover, for each stopping time $\tau \in \mathcal{T}_0$, $Y \in D^2_{\mathcal{F}}(0,T)$ is also the smallest g_0-supersolution on $[0,\tau]$ that dominates l with terminal condition Y_τ. We then can derive (39) by

$$Y_t = \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_{t,\tau}} E\big[l_\sigma 1_{\{\sigma < \tau\}} + Y_\tau 1_{\{\sigma = \tau\}} | \mathcal{F}_t\big].$$

But $Y_{\tau-} > l_{\tau-}$ implies $Y_{\tau-} = Y_\tau$, and thus $A_\tau = A_{\tau-}$. We then have (40). □

With the existence result of the smallest g-supermartingale given Proposition 4.3, the following equivalent conditions implies the proof of the existence part of Theorem 2.1.

Theorem 4.1. *Let the function g satisfy* (1), (2) *and let the lower obstacle L satisfies* (5). *Then the following conditions are equivalent*

(a) *The triple (Y,Z,A) is the unique solution of RBSDE with L^2-lower barrier L;*
(b) *Y is the smallest g-supersolution that dominates L with terminal condition $Y_T = \xi$;*
(c) *$\overline{Y}$ is the smallest $\bar{g}$-supersolution that dominates $\bar{L}$ with terminal condition $Y_T = \bar{\xi}$, where we set, for each $t \in [0,T]$,*

$$\bar{g}(t) := g(t, Y_t, Z_t), \qquad \overline{Y_t} := Y_t + \int_0^t \bar{g}(s)\,\mathrm{d}s,$$

$$\bar{L}_t := L_t + \int_0^t \bar{g}(s)\,\mathrm{d}s, \quad \bar{\xi} := \xi + \int_0^T \bar{g}(s)\,\mathrm{d}s;$$

(d) $\overline{Y} \in \mathbf{D}^2_{\mathcal{F}}(0,T)$ *is the smallest supermartingale that dominates* $\bar{L}$ *such that* $\overline{Y}_T = \bar{\xi}$;
(e) $\overline{Y} \in \mathbf{D}^2_{\mathcal{F}}(0,T)$ *is a supermartingale that dominates* $\bar{L}$ *with* $\overline{Y}_T = \bar{\xi}$, *of which* A *is the increasing process of the Doob–Meyer's decomposition, and the following reflecting condition holds*: *for each* $\bar{L}^* \in D^2_{\mathcal{F}}(0,T)$ *such that* $\overline{Y}_t \leqslant \bar{L}^*_t \leqslant \bar{L}_t$, *a.e., a.s.*,

$$\int_0^T (\overline{Y}_{t-} - \bar{L}^*_{t-})\,\mathrm{d}A_t = 0, \quad \textit{a.s.} \tag{41}$$

Proof of Theorem 4.1 and the existence part of Theorem 2.1. (c) $\Leftrightarrow$ (d) is easy to check.

We now prove (b) $\Leftrightarrow$ (c). We stress that in $\bar{g}(t)$ defined above, Y_t and Z_t are "fixed" or "frozen". We consider the solution $(\overline{Y}^n, \overline{Z}^n)$ of following penalized BSDE

$$\overline{Y}^n_t = \xi + \int_t^T \bar{g}(s)\,\mathrm{d}s + n\int_t^T (\overline{Y}^n_s - \bar{L}_s)^-\,\mathrm{d}s - \int_t^T \overline{Z}^n_s\,\mathrm{d}B_s.$$

Like g, the function $\bar{g}(t)$ satisfies also conditions (1) and (2). Thus, just as $\{(Y^n, Z^n)\}_{n=1}^\infty$ defined in (33), $\{(\overline{Y}^n, \overline{Z}^n)\}_{n=1}^\infty$ converges strongly to $(\overline{Y}, \overline{Z})$ in $L^2_{\mathcal{F}}(0,T) \times L^p_{\mathcal{F}}(0,T)$ for each $p \in [1,2)$. $\overline{Y} \in D^2_{\mathcal{F}}(0,T)$ is also the smallest $\bar{g}$-supersolution that dominates L with $\overline{Y}_T = \xi$:

$$\overline{Y}_t = \xi + \int_t^T \bar{g}(s)\,\mathrm{d}s + \bar{A}_T - \bar{A}_t - \int_t^T \overline{Z}_s\,\mathrm{d}B_s, \tag{42}$$

$$\overline{Y}_t \geqslant L_t, \qquad \mathrm{d}A_t \geqslant 0. \tag{43}$$

We now prove that $(\overline{Y}, \overline{Z}) = (Y, Z)$. Indeed, apply Itô's formula to $|Y^n_t - \overline{Y}^n_t|^2$, we have

$$\begin{aligned} E|Y^n_t - \overline{Y}^n_t|^2 + E\int_t^T |Z^n_t - \overline{Z}^n_t|^2\,\mathrm{d}t &= 2E\int_t^T (Y^n_s - \overline{Y}^n_s)\big(g(s, Y, Z^n_s) - \bar{g}(s)\big)\,\mathrm{d}s \\ &\quad + 2n\int_t^T (Y^n_s - \overline{Y}^n_s)\big[(Y^n_s - L_s)^- - (\overline{Y}^n_s - L_s)^-\big]\,\mathrm{d}s. \end{aligned}$$

For the last integrand, it is easy to check that $(Y^n_s - \overline{Y}^n_s)[(Y^n_s - L_s)^- - (\overline{Y}^n_s - L_s)^-] \leqslant 0$. We then have

$$\begin{aligned} &E|Y^n_t - \overline{Y}^n_t|^2 + E\int_t^T |Z^n_t - \overline{Z}^n_t|^2\,\mathrm{d}t \\ &\quad \leqslant 2E\int_t^T (Y^n_s - \overline{Y}^n_s)\big[g(s, Y^n_s, Z^n_s) - \bar{g}(s)\big]\,\mathrm{d}s \\ &\quad \leqslant 2E\int_t^T \big[|Y^n_s - Y_s| + |\overline{Y}^n_s - \overline{Y}_s|\big]\cdot\big|g(s, Y^n_s, Z^n_s) - \bar{g}(s)\big|\,\mathrm{d}s + 2E\int_t^T (Y_s - \overline{Y}_s)\big[g(s, Y^n_s, Z^n_s) - \bar{g}(s)\big]\,\mathrm{d}s. \end{aligned}$$

Since $|Y^n - Y| + |\overline{Y}^n - \overline{Y}| \to 0$ in $L^2_{\mathcal{F}}(0,T)$ and $|g(\cdot, Y^n_\cdot, Z^n_\cdot) - \bar{g}(\cdot)|$ is uniformly bounded in $L^2_{\mathcal{F}}(0,T)$, thus the first integral of the right side converges to zero as $n \to \infty$. For the second term, since $\{Y^n\}_{n=1}^\infty$ converges strongly to Y in $L^2_{\mathcal{F}}(0,T)$ and $\{Z^n\}_{n=1}^{\infty}$ converges strongly to Z in $L^p_{\mathcal{F}}(0,T)$, and g is Lipschitz in (Y,Z), thus $\{g(\cdot,Y^n,Z^n)\}_{n=1}^{\infty}$ converges strongly to $g(\cdot,Y.,Z.)=\bar{g}(\cdot)$ in $L^p_{\mathcal{F}}(0,T)$. But $\{g(\cdot,Y^n,Z^n)\}_{n=1}^{\infty}$ is also bounded in $L^2_{\mathcal{F}}(0,T)$. Thus it must converges weakly to $\bar{g}$ in $L^2_{\mathcal{F}}(0,T)$. Thus the second integral also converges to zero. It follows that $Y^n-\overline{Y}^n$ and $Z^n-\overline{Z}^n$ are both converges to zero. Thus $\overline{Y}\equiv Y$, $\overline{Z}\equiv Z$.

For (d) $\Leftrightarrow$ (e), we first prove (d) $\Rightarrow$ (e): Let $\overline{Y}$ be the smallest supersolution that dominates $\bar{L}$ with $\overline{Y}_T=\bar{\xi}$. Thus $(\overline{Y},\overline{Z},\bar{A})\in L^2(0,T;\mathbf{R}^d)\times D^2_{\mathcal{F}}(0,T)$ solves (i), (ii) in the Definition 2.2 of RBSDE. We only need to prove the Skorohod condition (iii), i.e., for each $\bar{L}^*\in D^2_{\mathcal{F}}(0,T)$ such that $\bar{L}_t\leqslant\bar{L}^*_t\leqslant\overline{Y}_t$, we have

$$\int_0^T(\overline{Y}_{t-}-\bar{L}^*_{t-})\,\mathrm{d}\bar{A}_t=0,\quad \text{a.s.}\tag{44}$$

We denote the discrete part of $\bar{A}$ by $\bar{A}^d$, and the continuous part by $\bar{A}^c$: $\bar{A}=\bar{A}^c+\bar{A}^d$. From (40), we have

$$\sum_{0\leqslant t\leqslant T}(\overline{Y}_{t-}-\bar{L}^*_{t-})(A_t-A_{t-})=\int_0^T(\overline{Y}_{s-}-\bar{L}_{s-})\,\mathrm{d}A^d_s=0.\tag{45}$$

The continuous part of $\overline{Y}$ is $\overline{Y}^c:=\overline{Y}+\bar{A}^d$. Then, with $g_0(t)\equiv 0$, Y^c is the smallest g_0-supersolution that dominates $L^c=L^*+A^d$ with terminal condition $Y^c_T=\xi+A^d_T$.

We now follow Proposition 4.2 to construct a penalization sequence $(Y^n,Z^n,A^n)\in D^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;\mathbf{R}^d)\times D^2_{\mathcal{F}}(0,T)$ as follows

$$Y^n_t=Y^c_T+\int_t^T n(Y^n_s-L^c_s)^-\,\mathrm{d}s-\int_t^T Z^n_s\,\mathrm{d}B_s,$$

$$A^n_t=\int_0^t n(Y^n_s-L^c_s)^-\,\mathrm{d}s.$$

According to Proposition 4.2, the triple (Y^n,Z^n,A^n) converges to (Y^c,Z,A^c) in the sense of (36) and, for each stopping time $\tau\leqslant T$, as $n\to\infty$,

$$\begin{aligned}&Y^n_\tau\nearrow Y^c_\tau,\quad \forall t\in[0,T],\quad \text{a.s.},\\&A^n_\tau\to A^c_\tau,\quad \text{strongly in } L^2(\mathcal{F}_T).\end{aligned}\tag{46}$$

On the other hand, for each $m\leqslant n$, since

$$0=(Y^m_t-L^c_t)^+(Y^m_t-L^c_t)^-\geqslant(Y^m_t-L^c_t)^+(Y^n_t-L^c_t)^-,$$

we have

$$\int_0^T(Y^m_t-L^c_t)^+\,\mathrm{d}A^n_t=0.\tag{47}$$

For each $t\in[0,T]$, we define

$$\begin{aligned}&D^+_t:=\inf\bigl\{s\geqslant t\colon\ (Y^c_s-L^c_s)^+\wedge(Y^c_s-L^c_{s-})^+=0\bigr\}\wedge T,\\&D^m_t:=\inf\bigl\{s\geqslant t\colon\ (Y^m_s-L^c_s)^+\wedge(Y^m_s-L^c_{s-})^+=0\bigr\}\wedge T.\end{aligned}$$

Since $(Y_s^m - L_s^c)^+ \nearrow (Y_s - L_s^c)^+$ thus $D_t^m \leqslant D_t^{m+1} \leqslant D_t$. On the other hand, for a.s. $\omega \in \Omega$, if $D_t > t$, then for each $t < \bar{t} < D_t$, we have $(Y_s^c - L_s^c)^+ \geqslant \delta(\omega)$, $t \leqslant s \leqslant \bar{t}$, for a positive $\delta > 0$. Since

$$0 \leqslant (Y_s - L_s^c)^+ - (Y_s^m - L_s^c)^+ \leqslant Y_s^c - Y_s^m \searrow 0,$$

thus, for a sufficiently large $m(\omega)$, we have $(Y_s^m - L_s^c)^+ > 0$, $s \in [t, \bar{t}\,]$. Thus $D_t^m > \bar{t}$. It follows that, for each t, $\lim_{m\to\infty} D_t^m = D_t$, almost surely. On the other hand, by (47) we have

$$A^n_{D_t^m} - A^n_t = 0. \tag{48}$$

We let $n \to \infty$. By the convergence of A^n in the sense of (46), we derive $A^c_{D_t^m} - A^c_t = 0$. By letting $m \to \infty$, and with (48) we get

$$\{A^c_{D_t} - A^c_t\} = 0.$$

Thus

$$\int_0^T (Y_t^c - L_t^c)\,\mathrm{d}A_t^c = \int_0^T (Y_t^c - L_{t-}^c)\,\mathrm{d}A_t^c = 0, \quad \text{a.s.}$$

This with (45) it follows that (44) holds.

(e) $\Rightarrow$ (d): Since the solution (Y, Z, A) of RBSDE with the lower obstacle L is unique. Thus by (d) $\Rightarrow$ (e), Y must be the smallest g-supersolution that dominates L.

Through (e) $\Leftrightarrow$ (d) $\Leftrightarrow$ (c) $\Leftrightarrow$ (b) we can prove that the smallest g-supersolution Y with $Y_T = \xi$ that dominates L (given in (b)) must satisfied the generalized Skorohod reflecting condition (7). Thus we have (b) $\Rightarrow$ (a). This with the existence theorem, i.e., Proposition 4.3, of the smallest g-supersolution given in (b), it follows that the solution Y of RBSDE of type (a) exists. This proves the existence part of Theorem 2.1. Finally the uniqueness of RBSDE given in Proposition 2.2 gives (a) $\Rightarrow$ (b). The proof is complete. □

The following comparison theorem of RBSDEs is a by-product of the above results. It will be used in the proof of the existence of RBSDE with two reflecting barriers. This comparison theorem of RBSDE was introduced in [14], for the case where L is continuous.

Theorem 4.2 (Comparison). *We assume that lower obstacle $L \in L^2_{\mathcal{F}}(0,T)$ satisfies* (5). *Let g^1, g^2 be two coefficients of* BSDE *satisfying the standard condition* (1) *and* (2), *for $i = 1, 2$, let (Y^i, Z^i, A^i) be the solution of the* RBSDE *with the lower obstacle $L \in L^2_{\mathcal{F}}(0,T)$*:

$$Y_t^i = \xi^i + \int_t^T g^i(s, Y_s^i, Z_s^i)\,\mathrm{d}s + A_T^i - A_t^i - \int_t^T Z_s^i\,\mathrm{d}B_s. \tag{49}$$

Namely, the triple (Y^i, Z^i, A^i) satisfies (i)–(iii) *in Definition* 2.2. *Assume that*

$$g^1(t, y, z) \leqslant g^2(t, y, z), \quad \forall (y, z) \in \mathbf{R} \times \mathbf{R}^d,\ \textit{a.e., a.s.} \tag{50}$$

and $\xi^1 \leqslant \xi^2$, a.s. Then we have

$$Y_t^1 \leqslant Y_t^2, \quad \leqslant t \leqslant T,\ \textit{a.s.} \tag{51}$$

Moreover,

$$(A_t^1 - A_s^1) - (A_t^2 - A_s^2) \geqslant 0, \quad \forall 0 \leqslant s \leqslant t \leqslant T, \ a.s. \tag{52}$$

Proof. For each $i = 1, 2$, consider the following penalization BSDE for the RBSDE (49):

$$Y_t^{n,i} = \xi^i + \int_t^T g^i(s, Y_s^{n,i}, Z_s^{n,i})\,\mathrm{d}s + n\int_t^T (L_s - Y_s^{n,i})^+\,\mathrm{d}s - \int_t^T Z_s^{n,i}\,\mathrm{d}B_s.$$

By the comparison theorem of BSDEs we get $Y_t^{n,1} \leqslant Y_t^{n,2}, \forall n \in \mathbf{N}$. Thanks to Proposition 4.2, as $n \to \infty$, $Y^{n,i}$ converge to Y^i the solutions of RBSDE, for $i = 1, 2$. We immediately have (51).

Moreover the increasing processes $A_t^{n,i} := n\int_0^t (L_s - Y_s^{n,i})^+\,\mathrm{d}s$ satisfies

$$(A_t^{n,1} - A_s^{n,1}) - (A_t^{n,2} - A_s^{n,2}) \geqslant 0, \quad \text{for each } 0 \leqslant s \leqslant t \leqslant T.$$

Again by Proposition 4.2, $(A_t^{n,1})$ and $(A_t^{n,2})$ respectively convergence to A_t^1 and A_t^2 weakly in $L^2(\mathcal{F}_t)$. We then have (52). $\square$

5. Penalization method for RBSDE with two obstacles and some basic estimates

In the preceding section, the existence result of RBSDE is proved by a penalization approach. This is a constructive method since the penalized equation (33) is a standard BSDE to which many existing numerical results can be applied. We now proceed to prove the existence of RBSDE reflected by two obstacles by using this approach. The penalized BSDEs we need are:

$$Y_t^{m,n} = \xi + \int_t^T g(s, Y_s^{m,n}, Z_s^{m,n})\,\mathrm{d}s + m\int_t^T (L_s - Y_s^{m,n})^+\,\mathrm{d}s - n\int_t^T (Y_s^{m,n} - U_s)^+\,\mathrm{d}s - \int_t^T Z_s^{m,n}\,\mathrm{d}B_s$$

or

$$Y_t^{m,n} = \xi + \int_t^T g(s, Y_s^{m,n}, Z_s^{m,n})\,\mathrm{d}s + A_T^{m,n} - A_t^{m,n} - (K_T^{m,n} - K_t^{m,n}) - \int_t^T Z_s^{m,n}\,\mathrm{d}B_s \tag{53}$$

with

$$A_t^{m,n} = m\int_0^t (L_s - Y_s^{m,n})^+\,\mathrm{d}s, \qquad K_t^{m,n} = n\int_0^t (Y_s^{m,n} - U_s)^+\,\mathrm{d}s.$$

Here the basic idea is simple: we first fix an m and let $n \to \infty$, then let $m \to \infty$. The two increasing processes K and A, which are the limits of $K^{m,n}$ and $A^{m,n}$, will be proved to be the two increasing processes in RBSDE (17) we are looking for. In Section 6.2, we will prove that the quadruple $(Y^{m,m}, Z^{m,m}, A^{m,m}, K^{m,m})$ also converges to the solution (Y, Z, A, K) of RBSDE as $m \to \infty$.

We begin with establishing several basic estimates for $(Y^{m,n}, Z^{m,n}, A^{m,n}, K^{m,n})$. These estimates are useful not only to the proof the existence of RBSDE provided in the next section, also to the further development of numerical solutions.

Proposition 5.1. *We assume* (15) *and* (16) *of* (H). *Then there exists a constant C, independent from m and n, such that the following estimate hold for* (53):

$$E\Big[\sup_{0\leqslant t\leqslant T}(Y_t^{m,n})^2\Big]+E\left[\int_0^T|Z_s^{m,n}|^2\,\mathrm{d}s\right]+E\big[(A_T^{m,n})^2\big]+E\big[(K_T^{m,n})^2\big]\leqslant C. \tag{54}$$

To prove this result, we need the following lemma.

Lemma 5.1. *There exists a quadruple $(Y^*,Z^*,A^*,K^*)\in D^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;R^d)\times D^2_{\mathcal{F}}(0,T)\times D^2_{\mathcal{F}}(0,T)$, such that*

$$Y_t^*=\xi+\int_t^T g(s,Y_s^*,Z_s^*)\,\mathrm{d}s+A_T^*-A_t^*-(K_T^*-K_t^*)-\int_t^T Z_s^*\,\mathrm{d}B_s, \tag{55}$$

where A^*, K^* are both increasing, and $L_t\leqslant Y_t^*\leqslant U_t$, a.e., a.s.

Proof. For the process X satisfying (16), we set $X_t^*=X_t+(\xi-X_T)1_{\{t=T\}}$. We have $L_t\leqslant X_t^*\leqslant U_t$ and

$$\begin{aligned}
X_t^*&=\xi-\int_t^T Z_s^0\,\mathrm{d}B_s+(\xi-X_T)1_{\{t=T\}}+(A_T^0-A_t^0)-(K_T^0-K_t^0)\\
&=\xi+\int_t^T g(s,X_s^*,Z_s^0)\,\mathrm{d}s+(A_T^0-A_t^0)+(\xi-X_t)1_{\{t=T\}}+\int_t^T\big[g(s,X_s^*,Z_s^0)\big]^-\,\mathrm{d}s\\
&\quad-(K_T^0-K_t^0)-\int_t^T\big[g(s,X_s^*,Z_s^0)\big]^+\,\mathrm{d}s-\int_t^T Z_s^0\,\mathrm{d}B_s.
\end{aligned}$$

We denote $Z^*=Z^0$ and

$$\begin{aligned}
A_t^*&:=A_t^0+(\xi-X_t)^+1_{\{t=T\}}+\int_0^t\big[g(s,X_s^*,Z_s^0)\big]^-\,\mathrm{d}s,\\
K_t^*&:=K_t^0+(\xi-X_t)^-1_{\{t=T\}}+\int_0^t\big[g(s,X_s^*,Z_s^0)\big]^+\,\mathrm{d}s.
\end{aligned}$$

Then (Y^*,Z^*,A^*,K^*) satisfies (55) and $L\leqslant Y^*\leqslant U$. □

Proof of Proposition 5.1. Let (Y^*,Z^*,A^*,K^*) be given as in Lemma 5.1 and let (Y^+,Z^+) and (Y^-,Z^-) be respectively the solutions of following two BSDEs:

$$Y_t^+=\xi+\int_t^T g(s,Y_s^+,Z_s^+)\,\mathrm{d}s+(A_T^*-A_t^*)-\int_t^T Z_s^+\,\mathrm{d}B_s,$$

and

$$Y_t^-=\xi+\int_t^T g(s,Y_s^-,Z_s^-)\,\mathrm{d}s-(K_T^*-K_t^*)-\int_t^T Z_s^-\,\mathrm{d}B_s.$$

From the comparison theorem of a standard BSDE, we have $Y_t^- \leqslant Y_t^* \leqslant Y_t^+$, thus $Y_t^+ \geqslant L_t$, $Y_t^- \leqslant U_t$. For $m, n \in \mathbf{N}$, (Y^+, Z^+) and (Y^-, Z^-) satisfy respectively,

$$Y_t^+ = \xi + \int_t^T g(s, Y_s^+, Z_s^+)\,\mathrm{d}s + (A_T^* - A_t^*) + m\int_t^T (L_s - Y_s^+)^+\,\mathrm{d}s - \int_t^T Z_s^+\,\mathrm{d}B_s,$$

$$Y_t^- = \xi + \int_t^T g(s, Y_s^-, Z_s^-)\,\mathrm{d}s - (K_T^* - K_t^*) - n\int_t^T (Y_s^- - U_s)^+\,\mathrm{d}s - \int_t^T Z_s^-\,\mathrm{d}B_s.$$

Always by comparison theorem,

$$Y_t^- \leqslant Y_t^{m,n} \leqslant Y_t^+, \quad \forall t \in [0, T], \text{ a.s.}$$

It follows that

$$E\Big[\sup_{0\leqslant t\leqslant T} (Y_t^{m,n})^2\Big] \leqslant \max\Big\{E\Big[\sup_{0\leqslant t\leqslant T} (Y_t^+)^2\Big],\ E\Big[\sup_{0\leqslant t\leqslant T} (Y_t^-)^2\Big]\Big\} \leqslant C. \tag{56}$$

In order to obtain the uniform estimate for $A_T^{m,n}$, we consider the following BSDE:

$$\widetilde{Y}_t^m = \xi + \int_t^T g(s, \widetilde{Y}_s^m, \widetilde{Z}_s^m)\,\mathrm{d}s - (K_T^* - K_t^*) + m\int_t^T (L_s - \widetilde{Y}_s^m)^+\,\mathrm{d}s - \int_t^T \widetilde{Z}_s^m\,\mathrm{d}B_s. \tag{57}$$

We compare it with the BSDE (55). Observe that the solution Y^* of (55) satisfies $L_t \leqslant Y_t^* \leqslant U_t$, thus we can add the zero sum $m\int_t^T (L_s - Y_s)^-\,\mathrm{d}s$ to the right side of (55). Since $\mathrm{d}A_t \geqslant 0$, it then follows again from the comparison theorem of BSDE that $Y_t^* \geqslant \widetilde{Y}_t^m$ and thus $U_t \geqslant \widetilde{Y}_t^m$, $t \in [0, T]$. Consequently, the term $-m\int_t^T (\widetilde{Y}_s^m - U_s)^+\,\mathrm{d}s$ is zero and thus can be add to the right side of BSDE (57). We then compare this BSDE, with the mentioned additional terms, with BSDE (53). With $\mathrm{d}K_t^* \geqslant 0$, we can, once again, apply the comparison theorem to derive $\widetilde{Y}_t^m \leqslant Y_t^{m,n}$. But this implies that

$$0 \leqslant A_t^{m,n} := m\int_0^t (L_s - Y_s^{m,n})^+\,\mathrm{d}s \leqslant m\int_0^t (L_s - \widetilde{Y}_s^m)^+\,\mathrm{d}s = \widetilde{A}_t^m.$$

Consequently $E[(A_T^{m,n})^2] \leqslant E[(\widetilde{A}_T^m)^2]$, for each $m, n = 1, 2, \ldots$.

Thus it suffices to estimate $E[(\widetilde{A}_T^m)^2]$. By (57), the pair $(\widehat{Y}^m, \widetilde{Z}^m)$, with $\widehat{Y}^m := \widetilde{Y}^m - K^*$, satisfies the BSDE

$$\widehat{Y}_t^m = \xi - K_T + \int_t^T g_{K^*}(s, \widehat{Y}_s^m, \widetilde{Z}_s^m)\,\mathrm{d}s + m\int_t^T (L_s - K_s^* - \widehat{Y}_s^m)^+\,\mathrm{d}s - \int_t^T \widetilde{Z}_s^m\,\mathrm{d}B_s, \tag{58}$$

where we denote $g_{K^*}(s, y, z) = g(s, y + K_s^*, z)$. This g_{K^*} satisfies also the usual conditions (1) and (2) of BSDE. But this sequence of BSDEs (58), for $m = 1, 2, \ldots$, is just the penalized BSDE for g_{K^*}-supersolution that dominates $L - K^*$ with terminal condition $\widehat{Y}_T^m = \xi - K_T^*$. We then can apply the boundedness estimate (37) in Proposition 4.2 to derive

$$E\big[(\widetilde{A}_T^m)^2\big] \leqslant C, \quad \text{and thus} \quad E\big[(A_T^{m,n})^2\big] \leqslant C.$$

Here the constant C does not depend on m, n.

The proof of the estimate $E[(K_T^{m,n})^2] \leqslant C$ is similar. We then use the standard technique of BSDE to apply Itô's formula to $|Y_t^{m,n}|^2$:

$$E\big[|Y_t^{m,n}|^2\big] + E\left[\int_t^T |Z_s^{m,n}|^2\,\mathrm{d}s\right] \leqslant C\left(1 + \int_t^T |Y_s^{m,n}|^2\,\mathrm{d}s\right) + \alpha\int_t^T |Z_s^{m,n}|^2\,\mathrm{d}s + E\Big[\operatorname*{ess\,sup}_{0\leqslant t\leqslant T}(L_t^+)^2\Big] + E\Big[\operatorname*{ess\,sup}_{0\leqslant t\leqslant T}(U_t^-)^2\Big] + E\big[(A_T^{m,n})^2\big] + E\big[(K_T^{m,n})^2\big].$$

Let $\alpha = \frac{1}{3}$, we finally get the estimate for $E[\int_0^T |Z_s^{m,n}|^2\,\mathrm{d}s]$. □

We now pass limit in the penalization BSDE (53). By the comparison theorem of BSDEs, we know that $(Y^{m,n})$ is increasing in m for each fixed n, and decreasing in n for each fixed m. In (53) we fix m and set $g^m(s,y,z) = g(s,y,z) + m(L_s - y)^+$. Like g itself, the function g^m also satisfies the standard conditions (1) and (2), with Lipschitz constant $k+m$ in the place of k. Thanks to Proposition 4.2, we have the following convergence:

Lemma 5.2. *When $n \to \infty$, the triple $(Y^{m,n}, Z^{m,n}, K^{m,n})$ converges to $(Y^m, Z^m, K^m) \in D^2_{\mathcal{F}}(0,T) \times L^2_{\mathcal{F}}(0,T;R^d) \times D^2_{\mathcal{F}}(0,T)$ in the following sense*:

$$\begin{cases} E\int_0^T (|Y_t^{m,n} - Y_t^m|^2 + |Z_t^{m,n} - Z_t^m|^p)\,\mathrm{d}t \to 0, \quad p\in[1,2), \\ E[|Y_t^{m,n} - Y_t^n|^2] \to 0, \quad \forall t\in[0,T], \\ E\int_0^T (Z_t^{m,n} - Z_t^n)\varphi_t\,\mathrm{d}t \to 0, \quad \forall\varphi\in L^2_{\mathcal{F}}(0,T), \\ E[(K_t^{m,n} - K_t^n)\zeta] \to 0, \quad \forall\zeta\in L^2(\mathcal{F}_T),\ \forall t\in[0,T]. \end{cases} \tag{59}$$

The limit (Y^m, Z^m, K^m) is the solution of the following RBSDE with one upper obstacle U,

$$Y_t^m = \xi + \int_t^T g(s, Y_s^m, Z_s^m)\,\mathrm{d}s + m\int_t^T (L_s - Y_s^m)^+\,\mathrm{d}s - (K_T^m - K_t^m) - \int_t^T Z_s^m\,\mathrm{d}B_s. \tag{60}$$

We also have, for each $i \leqslant j$, $0 \leqslant t \leqslant t' \leqslant T$,

$$K_{t'}^j - K_t^j \geqslant K_{t'}^i - K_t^i \geqslant 0. \tag{61}$$

Moreover, with $A_t^m = m\int_0^t (L_s - Y_s^m)^+\,\mathrm{d}s$, we have the following estimate: *there exists a constant C, independent of m, such that*

$$\sup_{0\leqslant t\leqslant T} E(Y_t^m)^2 + E\int_0^T \|Z_t^m\|^2\,\mathrm{d}t + E(A_T^m)^2 + E(K_T^m)^2 \leqslant C, \tag{62}$$

where $A_t^m := m\int_0^t (L_s - Y_s^m)^+\,\mathrm{d}s$.

Proof. The convergence of (59) and Eq. (60) result directly from Proposition 4.2 and Proposition 4.3 in which the coefficient $g(t,y,z)$ is replaced by $g(t,y,z) + m(L_t - y)^+$ and the lower obstacle L by the upper obstacle U.

Observe that (60) can be regarded as the following RBSDE with upper obstacle U:

$$Y_t^m = \xi + \int_t^T g^m(s, Y_s^m, Z_s^m)\,\mathrm{d}s - (K_T^m - K_t^m) - \int_t^T Z_s^m\,\mathrm{d}B_s,$$

where $g^m(s,y,z) := g^m(s,y,z) + m(L_s - y)^+$. Since $g^i(t,y,z) \leqslant g^j(t,y,z)$, for $i \leqslant j$, thus (61) is a direct result of comparison Theorem 4.2.

Since by $(Y^{m,n}, Z^{m,n}, K^{m,n})$ are uniformly bounded by (54), their strong and weak limits in L^2 are also uniformly bounded. □

6. Proof of Theorem 2.3: the existence of RBSDE with two obstacles

6.1. Proof of Theorem 2.3 and some results of convergence

We now proceed the

Proof of Theorem 2.3 – the part of existence and some results of convergence. We write Eq. (60) in the forward form:

$$Y_t^m = Y_0^m - \int_0^t g(s, Y_s^m, Z_s^m)\,\mathrm{d}s + K_t^m - A_t^m + \int_0^t Z_s^m\,\mathrm{d}B_s. \tag{63}$$

Using the Burkholder–Davis–Gundy inequality and (62), we have

$$E\Big(\sup_{0\leqslant t\leqslant T}(Y_t^m)^2\Big) \leqslant C.$$

From the comparison Theorem 4.2, Y^m is increasing in m. This with $Y^m \leqslant U$, it follows that, there exists a process Y, such that $Y^m \nearrow Y \leqslant U$ and thus

$$E\Big(\sup_{0\leqslant t\leqslant T}(Y_t)^2\Big) \leqslant C. \tag{64}$$

We also have the following L^2-convergence:

$$E\left(\int_0^T |Y_t^m - Y_t|^2\,\mathrm{d}t\right) \to 0. \tag{65}$$

By Lemma 5.2 the sequence $(Y^m)_{m=1}^\infty$ satisfy all conditions of the monotonic limit Theorem 3.1. It follows that its limit Y is in $D^2_{\mathcal{F}}(0,T)$ and has the following form:

$$Y_t = \xi + \int_t^T g_s^0\,\mathrm{d}s + A_T - A_t - (K_T - K_t) - \int_t^T Z_s\,\mathrm{d}B_s,$$

where $(g^0, Z) \in L^2_{\mathcal{F}}(0,T; R\times R^d)$ is the weak limit of $\{(g(\cdot, Y^m, Z^m), Z^m)\}_{m=1}^\infty$ in $L^2_{\mathcal{F}}(0,T; R\times R^d)$. For each $t\in[0,T]$, A_t is a weak limit of $\{A_t^m\}_{m=1}^\infty$ in $L^2(\mathcal{F}_t)$, K_t is the strong limit of $\{K_t^m\}_{m=0}^\infty$ in $L^2(\mathcal{F}_t)$. A and K are increasing processes in $D^2_{\mathcal{F}}(0,T)$. Furthermore, for any $p\in[0,2)$, we have

$$\lim_{m\to\infty} E\int_0^T |Z_s^m - Z_s|^p\,\mathrm{d}s = 0. \tag{66}$$

It follows that $g(\cdot, Y_\cdot^m, Z_\cdot^m) \to g(\cdot, Y_\cdot, Z_\cdot)$ in L^p, and thus

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\,\mathrm{d}s + A_T - A_t - (K_T - K_t) - \int_t^T Z_s\,\mathrm{d}B_s, \tag{67}$$

i.e. condition (ii) of Definition 2.3 is satisfied.

Since for each $m \in \mathbf{N}$, $Y^m \leqslant U$, thus $Y \leqslant U$. Notice that $E[(A_T^m)^2] \leqslant C$. As $m \to 0$, we have

$$0 \leqslant E\left[\left(\int_0^T (L_s - Y_s)^+ \,\mathrm{d}s\right)^2\right] = E\left[\left(\int_0^T (L_s - Y_s^m)^+ \,\mathrm{d}s\right)^2\right] \leqslant \frac{C}{m^2} \to 0$$

and thus $Y \geqslant L$. So (iii) of Definition 2.3 holds.

It remains to prove the two Skorohod reflecting conditions (iv) in Definition 2.3. For the upper obstacle, U and a process $U^* \in D^2_{\mathcal{F}}(0,T)$ such that $Y^m \leqslant U^* \leqslant U$, we have $\int_0^T (U^*_{t-} - Y^m_{t-})\,\mathrm{d}K_t^m = 0$. This with $(U^*_{t-} - Y^m_{t-}) \geqslant (U^*_{t-} - Y_{t-}) \geqslant 0$ yields $\int_0^T (U^*_{t-} - Y_{t-})\,\mathrm{d}K_t^m = 0$. We recall that $d(K_t - K_t^m) \geqslant 0$ and $K_T^m \nearrow K_T$ in $L^2(\mathcal{F}_T)$. It follows from

$$0 \leqslant \int_0^T (U^*_{t-} - Y_{t-})\,\mathrm{d}(K_t - K_t^m) \leqslant (K_T - K_T^m) \max_{t\in[0,T]} (U^*_{t-} - Y_{t-})$$

that the reflecting condition in Definition 2.3 for the upper boundary holds:

$$\int_0^T (U^*_{t-} - Y_{t-})\,\mathrm{d}K_t = 0.$$

We now proceed to prove the reflecting condition for the lower obstacle L. We consider the following BSDE:

$$\widetilde{Y}_t^m = \xi + \int_t^T g(s, \widetilde{Y}_s^m, \widetilde{Z}_s^m)\,\mathrm{d}s + m\int_t^T (L_s - \widetilde{Y}_s^m)^+\,\mathrm{d}s - (K_T - K_t) - \int_t^T \widetilde{Z}_s^m\,\mathrm{d}B_s. \tag{68}$$

We denote $\overline{Y}^m := \widetilde{Y}^m - K$ and rewrite the above BSDE

$$\overline{Y}_t^m = \xi - K_T + \int_t^T g_K(s, \overline{Y}_s^m, \widetilde{Z}_s^m)\,\mathrm{d}s + m\int_t^T (L_s - K_s - \overline{Y}_s^m)^+\,\mathrm{d}s - \int_t^T \widetilde{Z}_s^m\,\mathrm{d}B_s,$$

where we set $g_K(t,y,z) := g(t, y + K_t, z)$. We observe that this is just the penalization equation of the form (33), (32) with g_K in the place of g and $L - K$ in the place of L. From Theorem 4.2, as $m \to \infty$, we have the limit:

$$\overline{Y}_t = \xi - K_T + \int_t^T g_K(s, \overline{Y}_s, \widetilde{Z}_s)\,\mathrm{d}s + \tilde{A}_T - \tilde{A}_t - \int_t^T \widetilde{Z}_s\,\mathrm{d}B_s. \tag{69}$$

Here $\overline{Y}$ is the $L^2_{\mathcal{F}}(0,T)$-strong limit of $\overline{Y}^m$, $\widetilde{Z}$ is the $L^2_{\mathcal{F}}(0,T;R^d)$-weak limit and $L^p_{\mathcal{F}}(0,T;R^d)$-strong limit of $\widetilde{Z}^m$ and, for each t, $\tilde{A}_t$ is the $L^2(\mathcal{F}_t)$-weak limit of

$$\tilde{A}_t^m := m\int_0^t (L_s - \widetilde{Y}_s^m)^+\,\mathrm{d}s = m\int_t^T (L_s - K_s - \overline{Y}_s^m)^+\,\mathrm{d}s.$$

Theorem 4.2 also tells us that the limit $\overline{Y} \in D^2_{\mathcal{F}}(0,T)$ is the smallest g_K-supersolution with $\overline{Y}_T = \xi - K_T$ that dominates $L - K$. But on the other hand, using comparison theorem of BSDE to (68) and (60), we have $Y_t^m \geqslant \widetilde{Y}_t^m$. Thus, for each $s \leqslant t$,

$$\tilde{A}_t^m - \tilde{A}_s^m = m\int_s^t (L_r - \widetilde{Y}_r^m)^+\,\mathrm{d}r \geqslant m\int_s^t (L_r - Y_r^m)^+\,\mathrm{d}r = A_t^m - A_s^m.$$

Thus their weak limits satisfies $\tilde{A}_t - \tilde{A}_s \geqslant A_t - A_s$. Observe that, by (67), $Y - K$ is also a g_K-supersolution:

$$Y_t - K_t = \xi - K_T + \int_t^T g_K(s, Y_s - K_s, Z_s)\,\mathrm{d}s + A_T - A_t - \int_t^T Z_s\,\mathrm{d}B_s.$$

Compare this with (69), we have $Y \leqslant \widetilde{Y}$. Thus $Y - K$ must be $\widetilde{Y} - K$, the smallest g_K-supersolution with terminal condition $\xi - K_T$ that dominates $L - K$. It follows from Theorem 4.1, (a) $\Leftrightarrow$ (b) that $Y - K$ satisfies the Skorohod reflecting condition (iii) of Definition 2.2 with the obstacle $L^* - K$. But this implies that, for each $L^* \in D^2_{\mathcal{F}}(0, T)$ such that $Y \geqslant L^* \geqslant L$, we have

$$\int_0^T (Y_{t-} - L^*_{t-})\,\mathrm{d}A_t = \int_0^T \big(Y_{t-} - K_{t-} - (L^*_{t-} - K_{t-})\big)\,\mathrm{d}A_t = 0,$$

namely, the reflecting condition for the lower bound L. Thus all conditions in Definition 2.3 are satisfied. The proof is complete. □

We now give an equivalent relation between a double obstacles reflected BSDE and smallest g-solution. We observe that the solution (Y, Z, A, K) of the reflected BSDE with double obstacles can be rewrite to

$$Y_t - K_t = \xi - K_T + \int_t^T g_K(s, Y_s, Z_s)\,\mathrm{d}s + A_T - A_t - \int_t^T Z_s\,\mathrm{d}B_s \tag{70}$$

where $g_K(t, y, z) := g(t, y + K_t, z)$. We have

Theorem 6.1. *The following two claims are equivalent* (a). *The quadruple (Y, Z, A, K) is the solution of the reflected BSDE with double obstacles L and U*; (b) *$Y - K$ is the smallest g_K-supersolution that dominates $L - K$ with terminal condition $\xi - K_T$, and $Y + A$ is the largest g_{-A}-subsolution that dominated by $U + A$ with terminal condition $\xi + A_T$.*

Proof. If (Y, Z, A, K) is the solution of the reflected BSDE with double obstacles L and U. Then it is clear that the triple $(Y - K, Z, A)$ is the solution of the reflected BSDE (70) with the lower obstacle $Y - K \geqslant L - K$. But by Theorem 4.1 (a) $\Leftrightarrow$ (b), this is equivalent to say that $Y - K$ is the smallest g_K-supersolution that dominates $L - K$ with terminal condition $\xi - K_T$. The same argument is applied for the upper obstacle. □

6.2. A direct penalization scheme for RBSDE with two obstacles

A shortcoming of the sequence of the penalized BSDEs (53) is that we have to pass limit two times. Numerically, it is not easy to be realized. But we can apply our established results to prove that, when we force $m = n$ (53) and let $m \to \infty$, the penalization BSDE still converges to the RBSDE with two obstacles. In this setting, (53) becomes

$$\begin{aligned} Y_t^{m,m} = \xi &+ \int_t^T g(s, Y_s^{m,m}, Z_s^{m,m})\,\mathrm{d}s + m\int_t^T (L_s - Y_s^{m,m})^+\,\mathrm{d}s - m\int_t^T (U_s - Y_s^{m,m})^-\,\mathrm{d}s \\ &- \int_t^T Z_s^{m,m}\,\mathrm{d}B_s. \end{aligned} \tag{71}$$

We then claim

Theorem 6.2. *Let* (Y, Z, A, K) *be the solution of the double obstacle RBSDE* (17) *formulated in Definition* 2.3. *Then, as* $m \to \infty$*, we have the following convergence*: $Y_t^{m,m} \to Y_t$, $\forall t \in [0, T]$*, a.s., and*

$$\lim_{m\to\infty} E\left(\int_0^T |Y_t^{m,m} - Y_t|^2 \,\mathrm{d}t\right) \to 0, \tag{72}$$

$$\lim_{m\to\infty} E\int_0^T |Z_s^{m,m} - Z_s|^p \,\mathrm{d}s = 0, \quad p \in [1, 2). \tag{73}$$

Sketch of proof. To prove the convergence of $(Y^{m,m}, Z^{m,m}, A^{m,m}, K^{m,m})$, we rewrite the solution $(\bar{y}^m, \bar{z}^m, \bar{k}^m)$ of RBSDE (60) with one upper obstacle U to

$$\bar{y}_t^m = \xi + \int_t^T g(s, \bar{y}_s^m, \bar{z}_s^m)\,\mathrm{d}s + m\int_t^T (L_s - \bar{y}_s^m)^+ \,\mathrm{d}s - \bar{k}_T^m - \bar{k}_t^m - \int_t^T \bar{z}_s^m \,\mathrm{d}B_s,$$

and, symmetrically, the solution $(\underline{y}^m, \underline{z}^m, \underline{a}^m)$ of the RBSDE with one lower obstacle L:

$$\underline{y}_t^m = \xi + \int_t^T g(s, \underline{y}_s^m, \underline{z}_s^m)\,\mathrm{d}s + (\underline{a}_T^m - \underline{a}_t^m) - m\int_t^T (U_s - \underline{y}_s^m)^- \,\mathrm{d}s - \int_t^T \underline{z}_s^m \,\mathrm{d}B_s.$$

Since $\underline{y}_t^m \geqslant L_t$ and $\bar{y}_t^m \leqslant U_t$, we can add $m\int_t^T (U_s - \bar{y}_s^m)^- \,\mathrm{d}s$ to the first BSDE and $m\int_t^T (L_s - \underline{y}_s^m)^+ \,\mathrm{d}s$ to the second one. By comparison theorem of RBSDE, we have $\bar{y}^m \leqslant Y^{m,m} \leqslant \underline{y}^m$. But this with $\underline{y}^m \searrow Y$ and $\bar{y}^m \nearrow Y$ it follows that, almost surely, $Y_t^{m,m} \to Y_t$, $t \in [0, T]$. From (65) for $\bar{y}^m$ and the corresponding result for $\underline{y}^m$, we obtain (72).

Applying a technique similar to the proof of Theorem 3.1, i.e., for each pair of stopping times $0 \leqslant \sigma \leqslant \tau \leqslant T$, we apply Itô's formula to $E|Y_t^{m,m} - Y_t|^2$ on $[\sigma, \tau]$. We can obtain (73). □

Remark 6.1. We can also prove that, for each stopping time $\tau \leqslant T$, we have $(A_\tau^{m,m}, K_\tau^{m,m}) \to (A_\tau, K_\tau)$, weakly in $L^2(\mathcal{F}_T)$.

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