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THE SOBOLEV NORM OF CHARACTERISTIC FUNCTIONS WITH APPLICATIONS TO THE CALDERÓN INVERSE PROBLEM

DANIEL FARACO AND KEITH M. ROGERS

ABSTRACT. We consider Calderón's inverse problem on planar domains Ω with conductivities in fractional Sobolev spaces. When Ω is Lipschitz, the problem was shown to be stable in the L^2 -sense in [18]. We remove the Lipschitz condition on the boundary. To this end, we analyse the Sobolev regularity of the characteristic function of Ω . For Ω a quasi-ball, we compute $\|\chi_\Omega\|_{W^{s,p}(\mathbb{R}^d)}$ in terms of the δ -neighbourhoods of the boundary.

1. INTRODUCTION

The Calderón inverse problem consists of determining the conductivity of the interior of a body from voltage and current measurements on the surface. It arose originally in oil prospecting, however it now finds application in electrical impedance tomography (EIT) (see for example [17]).

Let Ω be a bounded, simply connected domain, and let γ be a measurable and bounded function on Ω representing the conductivity. The mathematical theory has been developed under the assumption of strong ellipticity;

$$K^{-1} \leq \gamma(x) \leq K, \quad \text{a.e. } x \in \Omega,$$

for some $K \geq 1$. Given an electric potential on the boundary $f \in H^{1/2}(\partial\Omega)$, there is a unique solution $u \in H^1(\Omega)$ to the Dirichlet problem;

$$(1) \quad \begin{cases} \nabla \cdot (\gamma \nabla u) = 0 \\ u|_{\partial\Omega} = f, \end{cases}$$

and, in the absence of sinks or sources, u describes the potential in the interior of Ω . If, in addition, one knows the current perpendicular to the boundary, then the Dirichlet-to-Neumann map Λ_γ can be defined by

$$(2) \quad \Lambda_\gamma(f) = (\gamma \partial_\nu u)|_{\partial\Omega},$$

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where ν denotes the exterior unit normal. In 2006, Astala and Päiväranta [10] established uniqueness in the plane for strongly elliptic conductivities with no smoothness conditions on the boundary. That is to say

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \Rightarrow \quad \gamma_1(x) = \gamma_2(x) \quad \text{a.e. } x \in \Omega \subset \mathbb{R}^2.$$

See [15, 32, 37] for results in higher dimensions and [16, 22, 24] for recent breakthroughs in the problem with partial data. Ultimately it is hoped that algorithms can be created to reconstruct the conductivity (see for example [14, 26, 27]). To this end, a number of stability results have been proven (see for example [3, 4, 11, 12]).

Unfortunately, there are counterexamples which show the problem is unstable for oscillating conductivities [4, 6, 29], necessitating some *a priori* control on the oscillation, in addition to ellipticity and measurability. An account of the state of the art is given in [5]. Recently, stability in the L^2 -sense was shown in [18] for conductivities with a small amount of Sobolev regularity.

In contrast with the work of Astala and Päiväranta, whose result held for general domains, the proof in [18] only works for domains which are Lipschitz. The missing ingredient in order to extend the result to general domains was control of the Sobolev regularity of characteristic functions. A number of results in this direction had already been proven by Sickel [34]. His results are summarized in the following lemma.

Lemma 1.1. [34] *Let $E \subset \mathbb{R}^d$ be a bounded set satisfying*

$$(3) \quad \int_0^1 |(\partial E)_\delta| \frac{d\delta}{\delta^{1+ps}} < \infty,$$

where $s > 0$. Then

$$\chi_E \in W^{s,p}(\mathbb{R}^d), \quad 1 \leq p < \infty.$$

On the other hand, if E is a John domain with $\dim_P(\partial E) > d - ps$. Then

$$\chi_E \notin W^{s,p}(\mathbb{R}^d), \quad 1 \leq p < \infty.$$

Here $(\partial E)_\delta = \{y \in \mathbb{R}^d : \text{dist}(y, \partial E) \leq \delta\}$ and \dim_P denotes packing dimension. We will also consider the Minkowski and Hausdorff dimensions, denoted respectively by \dim_M and \dim_H (see for example [19]). The definition of John domains can be found in [34].

Another result in the negative direction was proven by Triebel [38, Theorem 3 (iii)]. He proved that there exists a star-like domain E with ∂E an α -set, where $\alpha = d - ps$, such that

$$\chi_E \notin W^{s,p}(\mathbb{R}^d), \quad 1 \leq p < \infty.$$

For the definition of α -sets and star-like domains, see [38].

A set that has been used recurrently in the literature as a test for this question is the Koch snowflake, and the following corollary is easily deduced from Lemma 1.1.

Corollary 1.2. [34, Remark 3.10] *Let E be the interior of the Koch snowflake. Then*

$$\frac{\log 4}{\log 3} < 2 - ps \quad \Rightarrow \quad \chi_E \in W^{s,p}(\mathbb{R}^2), \quad 1 \leq p < \infty,$$

and

$$\frac{\log 4}{\log 3} \leq 2 - ps \quad \Leftarrow \quad \chi_E \in W^{s,p}(\mathbb{R}^2), \quad 1 \leq p < \infty.$$

Snowflakes are the canonical examples of quasiballs. In this work we show that for quasiballs (see for example [9]) the positive result of Sickel is optimal and one can almost compute $\|\chi_E\|_{W^{s,p}(\mathbb{R}^d)}$.

Theorem 1.3. *Let $1 \leq p < \infty$, $0 < s < 1$, and $E \subset \mathbb{R}^d$ be a K -quasiball. Then*

$$\|\chi_E\|_{W^{s,p}(\mathbb{R}^d)} \approx \left(|E| + \int_0^{\delta^*} |(\partial E)_\delta| \frac{d\delta}{\delta^{1+ps}} \right)^{1/p},$$

where $\delta^* = \inf\{\delta : E \subset (\partial E)_\delta\}$.

Here, $A \approx B$ denotes the existence of a constant $c = c(d, p, s, \delta^*, K)$ such that $c^{-1}B \leq A \leq cB$. The condition that E is a quasiball is unnecessary for the upper bound and this is due to Sickel [34], however we will provide a direct proof that will enable us to prove the lower bound.

In particular, $\dim_M(\partial E) < d - ps$ implies that the integral is finite and so

$$\chi_E \in W^{s,p}(\mathbb{R}^d), \quad 1 \leq p < \infty,$$

and $\dim_M(\partial E) > d - ps$ implies that the integral is unbounded and so

$$\chi_E \notin W^{s,p}(\mathbb{R}^d), \quad 1 \leq p < \infty.$$

This sidesteps the question of Remark 3.9 of [34], where it is asked for which boundaries the packing dimension and Minkowski dimension coincide.

To see that the lower bound of Theorem 1.3 is in some sense a refinement of the negative part of Lemma 1.1, we consider the lower Minkowski content of the boundary defined by

$$M_*^\alpha(\partial E) = \liminf_{\delta \rightarrow 0} \frac{|(\partial E)_\delta|}{\delta^{d-\alpha}}.$$

If E is a quasiball such that $M_*^{d-ps}(\partial E) > 0$, then we see from Theorem 1.3 that $\chi_E \notin W^{s,p}(\mathbb{R}^d)$. In particular for the snowflake, $M_*^{\frac{\log 4}{\log 3}}(\partial E) > 0$ (see for example [28]), so that $\chi_E \notin W^{s,p}(\mathbb{R}^2)$ when $2 - ps = \frac{\log 4}{\log 3}$, which yields the following strengthening of Corollary 1.2.

Corollary 1.4. *Let E be the interior of the Koch snowflake. Then*

$$\frac{\log 4}{\log 3} < 2 - ps \quad \Leftrightarrow \quad \chi_E \in W^{s,p}(\mathbb{R}^2), \quad 1 \leq p < \infty.$$

We remark that the previous discussion can be sharpened further by replacing the appearance of $\delta^{d-\alpha}$ in the definition of the Minkowski content with a gauge function $h : [0, 1] \rightarrow [0, 1]$ such that

$$\int_0^1 \frac{h(\delta)}{\delta^{1+ps}} d\delta = \infty,$$

and using lower and upper Minkowski dimensions.

According to [38, pp.466], one of the outstanding problems in the theory of function spaces is to characterize the sets whose characteristic functions are pointwise multipliers for Sobolev spaces (also Besov spaces; see [25]), and so we note the application to this area.

Corollary 1.5. *Let $E \subset \mathbb{R}^d$ be a quasiball. Then χ_E is a pointwise multiplier for $W^{s,p}(\mathbb{R}^d)$, with $1 \leq p < \infty$, $0 < s < 1$, if and only if (3) holds.*

This is a consequence Theorem 1.3 combined with the fact that $W^{s,p}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a multiplication algebra (see [31] or [23, Theorem A.12]).

These results are also of interest when determining how composition with quasiconformal maps affects Sobolev spaces (see [8, 18] for related results). In particular if we combine Lemma 1.1 with the recent estimates of Smirnov [9, 36] for the Hausdorff dimension of quasicircles, which by [7] imply the same bound for the Minkowski dimension, we obtain the following result.

Corollary 1.6. *Let E be a K -quasidisc with $s < \frac{4K}{p(K+1)^2}$. Then*

$$\chi_E \in W^{s,p}(\mathbb{R}^2), \quad 1 \leq p < \infty.$$

We also mention that Brandolini, Hofmann, and Iosevich [13] proved that if ∂E is $C^{3/2}$, then

$$(4) \quad \int_{S^{d-1}} |\widehat{\chi}_E(R\omega)|^2 d\omega \leq C_E R^{-(d+1)}, \quad R \geq 1.$$

Their interest in this problem was motivated by a result of Landau regarding the distribution of lattice points, as well as the Falconer distance problem (see [30]). Using the fact that $W^{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$, and polar coordinates, Lemma 1.1 implies that

$$\int_1^\infty \int_{S^{d-1}} |\widehat{\chi}_E(R\omega)|^2 d\omega R^{d-1+2s} dR \leq C_E, \quad 2s < d - \overline{\dim}_M(\partial E),$$

where $\overline{\dim}_M$ denotes the upper Minkowski dimension. Thus, it seems plausible that (4) holds in further generality. We do not pursue this here.

Finally we note the application to our original motivation; stability for the Calderón inverse problem. We remove the Lipschitz condition from the result of [18].

Theorem 1.7. *Let Ω be a bounded planar domain and let $0 < s < \frac{2 - \overline{\dim}_M(\partial\Omega)}{2}$. Suppose also that γ_1, γ_2 are conductivities satisfying*

$$K^{-1} \leq \gamma_1, \gamma_2 \leq K \quad \text{and} \quad \|\gamma_1\|_{W^{s,2}(\mathbb{R}^2)}, \|\gamma_2\|_{W^{s,2}(\mathbb{R}^2)} \leq \Gamma.$$

Then there exist $c(K)$, $C(K, \Gamma, s, \text{diam}(\Omega), \overline{\dim}_M(\partial\Omega)) > 0$ such that

$$\|\gamma_1 - \gamma_2\|_{L^2(\Omega)} \leq \frac{C}{|\log(\rho^{-1})|^{cs^2}},$$

whenever $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)} \leq 1$.

Remark 1.8. As the conductivities are bounded on a bounded domain, L^2 -Sobolev regularity follows from L^p -Sobolev regularity, and vice versa.

The paper is structured as follows: In Section 2 we prove Theorem 1.3, in Section 3 we show how to deduce Theorem 1.7 as a consequence, and in Section 4 we return to Lemma 1.1, proving a refinement.

2. PROOF OF THEOREM 1.3

If ∂E had nonzero measure, then it is clear that

$$\left(|E| + \int_0^{\delta^*} |(\partial E)_\delta| \frac{d\delta}{\delta^{1+ps}} \right)^{1/p} = \infty.$$

On the other hand, in [34, Lemma 3.1] it was proven that in this case we also have $\|\chi_E\|_{W^{s,p}} = \infty$, and so we can assume that $|\partial E| = 0$.

Now, by Fubini's theorem,

$$\begin{aligned} \|\chi_E\|_{W^{s,p}(\mathbb{R}^d)}^p &= |E| + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\chi_E(x) - \chi_E(y)|^p}{|x - y|^{d+ps}} dx dy \\ &= |E| + \int_E \int_{\mathbb{R}^d \setminus E} \frac{dx dy}{|x - y|^{d+ps}} + \int_{\mathbb{R}^d \setminus E} \int_E \frac{dx dy}{|x - y|^{d+ps}} \\ &= |E| + 2 \int_E \int_{\mathbb{R}^d \setminus E} \frac{dx dy}{|x - y|^{d+ps}}. \end{aligned}$$

Thus, it will suffice to prove that

$$\int_E \int_{\mathbb{R}^d \setminus E} \frac{dx dy}{|x - y|^{d+ps}} \approx |E| + \int_0^{\delta^*} |(\partial E)_\delta| \frac{d\delta}{\delta^{1+ps}}.$$

First we prove the upper bound, without assuming that E is a quasiball, recovering the result of Sickel [34]. To this end, we fix $\lambda > 1$ and define the sets E_j by

$$E_j = \{ y \in E : \lambda^{-j+1} > \text{dist}(y, \partial E) \geq \lambda^{-j} \},$$

so that

$$\int_E \int_{\mathbb{R}^d \setminus E} \frac{dx dy}{|x - y|^{d+ps}} = \sum_{j \in \mathbb{Z}} \int_{E_j} \int_{\mathbb{R}^d \setminus E} \frac{dx dy}{|x - y|^{d+ps}}.$$

The sets E_j with $\lambda^{-j} > \delta^* = \inf\{\delta : E \subset (\partial E)_\delta\}$ are of course empty. Now, with $y \in E_j$, by two changes of variables

$$\begin{aligned} \int_{\mathbb{R}^d \setminus E} \frac{dx}{|x-y|^{d+ps}} &= \int_{(\mathbb{R}^d \setminus E) - y} \frac{dz}{|z|^{d+ps}} \\ &\leq \int_{\mathbb{R}^d \setminus B(0, \lambda^{-j})} \frac{dz}{|z|^{d+ps}} \\ &= \lambda^{psj} \int_{\mathbb{R}^d \setminus B(0,1)} \frac{dz}{|z|^{d+ps}} \leq C \lambda^{psj}. \end{aligned}$$

Substituting in, we see that

$$\int_E \int_{\mathbb{R}^d \setminus E} \frac{dx dy}{|x-y|^{d+ps}} \leq C \sum_{j \in \mathbb{Z}} \lambda^{psj} |E_j|.$$

Now taking $\lambda = 2$ we have $|E_j| \leq |(\partial E)_{2^{-j+1}}|$, so that

$$\begin{aligned} \int_E \int_{\mathbb{R}^d \setminus E} \frac{dx dy}{|x-y|^{d+ps}} &\leq C \left(|E| + \sum_{j: 2^{-j+2} \leq \delta^*} 2^{psj} |(\partial E)_{2^{-j+1}}| \right) \\ &\leq C \left(|E| + \sum_{j: 2^{-j+2} \leq \delta^*} \int_{2^{-j+1}}^{2^{-j+2}} \delta^{-ps} |(\partial E)_\delta| \frac{d\delta}{\delta} \right) \\ (5) \quad &\leq C \left(|E| + \int_0^{\delta^*} |(\partial E)_\delta| \frac{d\delta}{\delta^{1+ps}} \right), \end{aligned}$$

and we are done.

For the lower bound, we recall that quasiconformal mappings are quasisymmetric (see for example [21]). We write $E = f(\mathbb{B}^d)$, where there exists a continuous strictly increasing bijection $\eta : [0, \infty] \rightarrow [0, \infty]$ such that for all triples $x, x', x'' \in \mathbb{R}^d$ it holds that

$$(6) \quad \frac{|f(x) - f(x')|}{|f(x) - f(x'')|} \leq \eta \left(\frac{|x - x'|}{|x - x''|} \right).$$

Taking $x' = x''$ we see that $\eta(1) \geq 1$, and $x = 0$ we see that

$$(7) \quad \delta^* \leq \sup_{x \in \partial \mathbb{B}^d} |f(x) - f(0)| \leq \eta(1) \inf_{x \in \partial \mathbb{B}^d} |f(x) - f(0)|.$$

Fix $\lambda = \eta(3)\eta(1)^3$. We claim that for every $y \in \partial E$ and every j satisfying $\lambda^{-j+1} \leq \delta^*$, there exists $y' \in E$ and $y'' \in E^c$ such that

$$(8) \quad \begin{aligned} B(y', \lambda^{-j}) &\subset B(y, \lambda^{-j+1}) \cap E_j, \\ B(y'', \lambda^{-j}) &\subset B(y, \lambda^{-j+1}) \cap (E^c)_j, \end{aligned}$$

where E_j is defined as before and $(E^c)_j$ is defined by

$$(E^c)_j = \{y \in E^c : \lambda^{-j+1} > \text{dist}(y, \partial E) \geq \lambda^{-j}\}.$$

To see this, we note that by definition $y = f(x)$ for some $x \in \partial\mathbb{B}^d$, and we let $y' = f(x')$ be such that $|x' - x| = \frac{2r}{3}$ and $B(x', \frac{r}{3}) \subset B(x, r) \cap \mathbb{B}^d$, and r is chosen so that

$$(9) \quad \frac{\lambda^{-j+1}}{\eta(3)\eta(1)} = \max_{z \in \partial B(x', \frac{r}{3})} |f(z) - f(x')|.$$

Now, for j satisfying $\lambda^{-j+1} \leq \delta^*$, we can suppose that $r \leq 3/2$. To see this we will show that when $r = 3/2$, the right hand side of (9) is larger than left hand side, so that r can be made smaller to find equality. When $r = 3/2$ we have that $x' = 0$ and that $B(x', \frac{r}{3}) = B(0, \frac{1}{2})$. Now by quasisymmetry we have that for $x \in \partial\mathbb{B}^d$ and $z \in \partial B(0, \frac{1}{2})$,

$$\begin{aligned} |f(x) - f(0)| &\leq \eta\left(\frac{|x-0|}{|z-0|}\right) |f(z) - f(0)| \\ &= \eta(2) |f(z) - f(0)|. \end{aligned}$$

Combined with (7), when $\lambda^{-j+1} \leq \delta^*$ this yields

$$\max_{z \in \partial B(0, \frac{1}{2})} |f(z) - f(0)| \geq \frac{\delta^*}{\eta(2)\eta(1)} \geq \frac{\lambda^{-j+1}}{\eta(3)\eta(1)}.$$

Thus we can suppose that $r \leq 3/2$.

Now we apply the quasisymmetry condition to x , x' , and $z \in \partial B(x', \frac{r}{3})$, so that

$$\begin{aligned} |f(z) - f(x)| &\leq \eta\left(\frac{|z-x|}{|z-x'|}\right) |f(z) - f(x')| \\ &\leq \eta(3) |f(z) - f(x')| \leq \lambda^{-j+1}. \end{aligned}$$

In other words,

$$(10) \quad f\left(B(x', \frac{r}{3})\right) \subset B(f(x), \lambda^{-j+1}) \cap E.$$

On the other hand, if we let z_j denote the z which fulfills the maximum in (9),

$$(11) \quad \begin{aligned} |f(z_j) - f(x')| &\leq \eta\left(\frac{|z_j - x'|}{|z_j - x|}\right) |f(z_j) - f(x)| \\ &\leq \eta(1) |f(z_j) - f(x)|, \end{aligned}$$

and for $w \in \partial\mathbb{B}^d$,

$$(12) \quad \begin{aligned} |f(z) - f(x')| &\leq \eta\left(\frac{|z - x'|}{|z - w|}\right) |f(z) - f(w)| \\ &\leq \eta(1) |f(z) - f(w)|. \end{aligned}$$

Here we use that fact that $r \leq 3/2$. Combining (11) and (12), we get

$$|f(z) - f(w)| \geq \frac{\lambda^{-j+1}}{\eta(3)\eta(1)^3} = \lambda^{-j}.$$

Using (10), this yields

$$(13) \quad f\left(B\left(x', \frac{r}{3}\right)\right) \subset B(f(x), \lambda^{-j+1}) \cap E_j.$$

Now, $f\left(B\left(x', \frac{r}{3}\right)\right)$ is a quasiball and thus contains a ball comparable to its diameter. Indeed, by applying quasisymmetry to the points $z, z_j \in \partial B\left(x', \frac{r}{3}\right)$ and x' we obtain that

$$|f(z_j) - f(x')| \leq \eta(1)|f(z) - f(x')|.$$

We see that

$$|f(z) - f(x')| \geq \frac{\lambda^{-j+1}}{\eta(3)\eta(1)^2} = \eta(1)\lambda^{-j}.$$

That is to say,

$$B(f(x'), \eta(1)\lambda^{-j}) \subset f\left(B\left(x', \frac{r}{3}\right)\right).$$

Combining this with (13) this yields

$$B(f(x'), \eta(1)\lambda^{-j}) \subset B(f(x), \lambda^{-j+1}) \cap E_j$$

as desired. The argument for $(E_j)^c$ is slightly easier.

Now recall that

$$\|\chi_E\|_{W^{s,p}(\mathbb{R}^d)}^p \geq |E| + \sum_{j: \lambda^{-j+1} \leq \delta^*} \int_{E_j} \int_{\mathbb{R}^d \setminus E} \frac{dx dy}{|x - y|^{d+ps}},$$

where we define E_j with the constant $\lambda = \eta(3)\eta(1)^3$. By (8), for $y \in E_j$, there exists a ball $B(y'', \lambda^{-j}) \subset (E_j)^c$ such that $\text{dist}(y, y'') \leq 2\lambda^{-j+1}$. Thus,

$$\begin{aligned} \int_{\mathbb{R}^d \setminus E} \frac{dx}{|x - y|^{d+ps}} &\geq \int_{(E^c)_j} \frac{dx}{|x - y|^{d+ps}} \\ &\geq C \int_{B(y'', \lambda^{-j})} \frac{dx}{\lambda^{(-j+1)(d+ps)}} \geq C\lambda^{psj}, \end{aligned}$$

so that

$$(14) \quad \|\chi_E\|_{W^{s,p}(\mathbb{R}^d)}^p \geq |E| + C \sum_{j: \lambda^{-j+1} \leq \delta^*} \lambda^{psj} |E_j|.$$

Now, we consider a maximal collection of disjoint balls $\{B(y_k, \lambda^{-j+2})\}$ with centres $y_k \in \partial E$, and denote the cardinality by Υ_j . Then we have that

$$|(\partial E)_{\lambda^{-j+2}}| \leq C\Upsilon_j \lambda^{-dj}.$$

On the other hand, when $\lambda^{-j+1} \leq \delta^*$, by (8), there exist corresponding balls $\{B(y'_k, \lambda^{-j})\}$ of radius λ^{-j} contained in E_j which must be disjoint due to the fact that $\{B(y_k, \lambda^{-j+2})\}$ are disjoint. Thus,

$$|E_j| \geq \Upsilon_j \lambda^{-dj} \geq C|(\partial E)_{\lambda^{-j+2}}|.$$

Substituting into (14) we see that

$$\begin{aligned}
\|\chi_E\|_{W^{s,p}(\mathbb{R}^d)}^p &\geq |E| + C \sum_{j:\lambda^{-j+1} \leq \delta^*} \lambda^{psj} |(\partial E)_{\lambda^{-j+2}}| \\
&\geq |E| + C \sum_{j:\lambda^{-j+1} \leq \delta^*} \int_{\lambda^{-j+1}}^{\lambda^{-j+2}} \delta^{-ps} |(\partial E)_\delta| \frac{d\delta}{\delta} \\
&\geq C \left(|E| + \int_0^{\delta^*} |(\partial E)_\delta| \frac{d\delta}{\delta^{1+ps}} \right)
\end{aligned}$$

and we are done. \square

Remark 2.1. It is clear from the previous proof that the δ -neighbourhoods of ∂E can be replaced by the δ -neighbourhoods of ∂E relative to E defined by

$$(\partial E)^\delta = \{y \in E : \text{dist}(y, \partial E) \leq \delta\}.$$

However, we also see that when E is a K -quasiball, we have that

$$|(\partial E)^\delta| \approx |(\partial E)_\delta| \quad \text{when } \delta \leq c(K)\delta^*.$$

3. PROOF OF THEOREM 1.7

We begin by recalling the weak formulation of the problem. For non-smooth domains, $H^{1/2}(\partial\Omega)$ is defined to be $H^1(\Omega)/H_0^1(\Omega)$, where $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. The dual is denoted by $H^{-1/2}(\partial\Omega)$. For $\varphi \in H^1(\Omega)$ we denote $\varphi + H_0^1(\Omega)$ by $\varphi|_{\partial\Omega}$.

Now $\Lambda_\gamma(\psi|_{\partial\Omega})$ is defined to be the unique element of $H^{-1/2}(\partial\Omega)$ which satisfies

$$\langle \Lambda_\gamma(\psi|_{\partial\Omega}), \varphi|_{\partial\Omega} \rangle = \int_\Omega \gamma \nabla u \cdot \nabla \varphi$$

whenever $\varphi \in H^1(\Omega)$, where $u \in H^1(\Omega)$ is the unique solution to

$$\begin{cases} \int_\Omega \gamma \nabla u \cdot \nabla \phi = 0 \\ u - \psi \in H_0^1(\Omega) \end{cases}$$

whenever $\phi \in H_0^1(\Omega)$. When the boundary and solution are sufficiently smooth, by the trace and divergence theorems, these definitions correspond with those of the introduction.

The following version of Theorem 1.7 on balls was proven in [18].

Proposition 3.1. [18] *Let $C > 1$ and $0 < s < 1/2$. Suppose that $\tilde{\gamma}_1, \tilde{\gamma}_2$ are conductivities satisfying*

$$K^{-1} \leq \tilde{\gamma}_1, \tilde{\gamma}_2 \leq K \quad \text{and} \quad \|\tilde{\gamma}_1\|_{W^{s,2}(\mathbb{R}^2)}, \|\tilde{\gamma}_2\|_{W^{s,2}(\mathbb{R}^2)} \leq \Gamma.$$

Then there are constants $a(K), A(C, K, \Gamma) > 0$ such that

$$\|\tilde{\gamma}_1 - \tilde{\gamma}_2\|_{L^2(\mathbb{D})} \leq \frac{A}{|\log(\rho^{-1})|^{as^2}},$$

whenever $\rho = \|\Lambda_{\tilde{\gamma}_1} - \Lambda_{\tilde{\gamma}_2}\|_{H^{1/2}(\partial\mathbb{D}) \rightarrow H^{-1/2}(\partial\mathbb{D})} \leq CK^2$.

Lemma 3.2. [18] *Let $\Omega \subset \mathbb{C}$ be a bounded domain, and suppose that γ_1, γ_2 are conductivities on Ω satisfying*

$$K^{-1} \leq \gamma_1, \gamma_2 \leq K.$$

Let $\tilde{\gamma}_1 = \gamma_1\chi_\Omega + \chi_{\mathbb{D}\setminus\Omega}$ and $\tilde{\gamma}_2 = \gamma_2\chi_\Omega + \chi_{\mathbb{D}\setminus\Omega}$. Then

$$\|\Lambda_{\tilde{\gamma}_1} - \Lambda_{\tilde{\gamma}_2}\|_{H^{1/2}(\partial\mathbb{D}) \rightarrow H^{-1/2}(\partial\mathbb{D})} \leq CK^2 \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)}.$$

Combining the results to obtain Theorem 1.7 is now easy. By scaling, we can suppose that $\Omega \subset \mathbb{D}$, and by hypothesis

$$\|\gamma_1\|_{W^{s,2}(\mathbb{R}^2)}, \|\gamma_2\|_{W^{s,2}(\mathbb{R}^2)} \leq \Gamma, \quad s < \frac{2 - \overline{\dim}_M(\partial\Omega)}{2}.$$

By Lemma 1.1,

$$\chi_\Omega, \chi_{\mathbb{D}\setminus\Omega} \in W^{s,2}(\mathbb{R}^2), \quad s < \frac{2 - \overline{\dim}_M(\partial\Omega)}{2},$$

and so, as $W^{s,2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a multiplication algebra (see [23, Theorem A.12] or [31]),

$$\chi_\Omega\gamma_1, \chi_\Omega\gamma_2 \in W^{s,2}(\mathbb{R}^2), \quad s < \frac{2 - \overline{\dim}_M(\partial\Omega)}{2}.$$

Thus, by the triangle inequality, we have that

$$\|\tilde{\gamma}_1\|_{W^{s,2}(\mathbb{R}^2)}, \|\tilde{\gamma}_2\|_{W^{s,2}(\mathbb{R}^2)} \leq C(\Gamma, s, \overline{\dim}_M(\partial\Omega)), \quad s < \frac{2 - \overline{\dim}_M(\partial\Omega)}{2},$$

and we are in position to apply Proposition 3.1. Writing $\kappa(\rho) = \frac{A}{|\log(\rho^{-1})|^{as^2}}$, we observe that

$$\begin{aligned} \|\gamma_1 - \gamma_2\|_{L^2(\Omega)} &= \|\tilde{\gamma}_1 - \tilde{\gamma}_2\|_{L^2(\mathbb{D})} \\ &\leq \kappa\left(\|\Lambda_{\tilde{\gamma}_1} - \Lambda_{\tilde{\gamma}_2}\|_{H^{1/2}(\partial\mathbb{D}) \rightarrow H^{-1/2}(\partial\mathbb{D})}\right) \\ &\leq \kappa\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)}\right), \end{aligned}$$

where the final inequality is by Lemma 3.2, and we are done. \square

4. REFINEMENTS OF LEMMA 1.1

The negative part of Theorem 1.3 for quasiballs relied on the property (8), that quasiballs are ‘biporous’ (see for example [7, 39] for definitions and the relations between porosity, quasisymmetric maps and the Cauchy integral). A set in some sense opposite to one enjoying the property (8) is a *hairy ball*; a set whose boundary consists of the unit circle together with pieces of Koch curves which protrude inward and outwards.

On the other hand, for such balls, a stronger version of Lemma 1.1 holds. We can remove the hairs, as they have measure zero, and so in fact the characteristic function of a hairy ball belongs to $W^{s,p}(\mathbb{R}^d)$ for $s < 1/p$.

To capture this phenomena in general we consider the following definition.

Definition 4.1. We say that $x \in \partial^* E$ if

$$0 < |E \cap B(x, r)| < |B(x, r)| \quad \text{for all } r > 0.$$

It is clear that points in the interior of E , denoted by E° , do not belong to $\partial^* E$. Similarly this is true of $\overline{E}^c = \mathbb{R}^d \setminus \overline{E}$. Thus,

$$\partial^* E \subset \mathbb{R}^d \setminus (E^\circ \cup \overline{E}^c) = \partial E,$$

which yields

$$\dim(\partial^* E) \leq \dim(\partial E).$$

In particular, for a hairy ball the inequality is strict, and so the following lemma is in some sense stronger than the positive part of Lemma 1.1.

Lemma 4.2. *Let $E \subset \mathbb{R}^d$ be bounded with $\overline{\dim}_M(\partial^* E) < d - ps$. Then*

$$\chi_E \in W^{s,p}(\mathbb{R}^d), \quad 1 \leq p < \infty.$$

Proof. We appeal to Proposition 3.1 of [20]. It states that for all sets E there exists a second set \tilde{E} , differing from E on a set of zero Lebesgue measure, such that

$$(15) \quad \partial \tilde{E} \subset \partial^* E.$$

Writing

$$\chi_E = \chi_{\tilde{E}} + \chi_{E \setminus \tilde{E}} - \chi_{\tilde{E} \setminus E},$$

we have

$$\|\chi_{E \setminus \tilde{E}}\|_{W^{s,p}(\mathbb{R}^d)} = \|\chi_{\tilde{E} \setminus E}\|_{W^{s,p}(\mathbb{R}^d)} = 0.$$

Thus, the result follows by proving

$$(16) \quad \chi_{\tilde{E}} \in W^{s,p}(\mathbb{R}^d), \quad 1 \leq p < \infty,$$

for p and s which satisfy $\overline{\dim}_M(\partial^* E) < d - ps$. However this condition implies that $\overline{\dim}_M(\partial \tilde{E}) < d - ps$ by (15), and so (16) follows from Lemma 1.1. \square

Remark 4.3. A refined version of Theorem 1.7, with $s < 1 - \overline{\dim}_M(\partial E)/2$ replaced by $s < 1 - \overline{\dim}_M(\partial^* E)/2$ follows by using Lemma 4.2 instead of Lemma 1.1.

It is a natural to ask what are the weakest conditions under which results of this type hold. To this end, one can remove points from the boundary and/or change the definition of fractal dimension. The following definition of a boundary not only excludes the hairs, but also the peaks of cusps.

Definition 4.4. We say that $x \in \partial^{**} E$ if

$$0 < \liminf_{r \rightarrow 0} \frac{|E \cap B(x, r)|}{|B(x, r)|} \leq \limsup_{r \rightarrow 0} \frac{|E \cap B(x, r)|}{|B(x, r)|} < 1.$$

Again we note that,

$$\partial^{**}E \subset \partial E^*,$$

so that

$$\dim(\partial^{**}E) \leq \dim(\partial^*E).$$

Lemma 4.5. *Let $E \subset \mathbb{R}^d$ be bounded and suppose that $\dim_H(\partial^{**}E) > d - ps$. Then*

$$\chi_E \notin W^{s,p}(\mathbb{R}^d), \quad 1 \leq p < \infty.$$

Proof. We prove the contrapositive. That is to say, if

$$\chi_E \in W^{s,p}(\mathbb{R}^d), \quad 1 \leq p < \infty,$$

then $\dim_H(\partial^{**}E) \leq d - ps$, where \dim_H denotes the Hausdorff dimension.

To see this, we note that the boundary $\partial^{**}E$ consists of non-Lebesgue points. That is to say if $x \in \partial^{**}E$, then

$$(17) \quad \liminf_{r \rightarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\chi_E(y) - \chi_E(x)| dy > 0.$$

On the other hand, it is a classical result in potential theory (see for example Theorem 6.2.1 of [1]) that the set of non-Lebesgue points Σ_f of a Sobolev function $f \in W^{s,p}(\mathbb{R}^d)$ defined by

$$\Sigma_f = \left\{ x : \liminf_{r \rightarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)| dy > 0 \right\},$$

has (s,p) -capacity zero. Thus, if $\chi_E \in W^{s,p}(\mathbb{R}^d)$, then the (s,p) -capacity of $\partial^{**}E$ would be less than or equal to the (s,p) -capacity of Σ_{χ_E} which would be zero. Finally, it is well known (see for example Theorem 5.1.13 of [1]) that this implies that $\dim_H(\partial^{**}E) \leq d - ps$. \square

We remark that a consequence of (8) is that $\partial E = \partial^*E = \partial^{**}E$ for quasiballs. Thus Lemma 4.5 is weaker than the negative part of Theorem 1.3 for quasiballs, however it has the advantage of being true for general sets. This suggests that the Hausdorff dimension of the boundary may characterize the smoothness of a characteristic function. Sickel [35] proved however that neither Hausdorff nor Minkowski dimension, combined with any of our definitions of a boundary, can serve for this purpose. He constructed a bounded set E (in a certain sense a limit of the classical Nikodym domain) satisfying

$$\chi_E \in W^{s,p}(\mathbb{R}^2), \quad 1 \leq p < \infty \quad \Leftrightarrow \quad \alpha < 2 - ps,$$

where $\dim_H(\partial E) < \alpha < \underline{\dim}_M(\partial^{**}E)$. Thus, the Hausdorff dimension is too small to characterise the smoothness of characteristic functions in general, and the Minkowski dimension too large. In particular we see that there exist bounded sets E with $\underline{\dim}_M(\partial^{**}E) > d - ps$ such that

$$\chi_E \in W^{s,p}(\mathbb{R}^d), \quad 1 \leq p < \infty.$$

Thus, the condition that E is a quasiball in the negative parts of our results cannot be removed completely.

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