# The Solutions and Periods of Some Considered Non-Linear Difference Equation Systems 

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#### Abstract

We find the periods of some non-linear discrete equation systems in this study. Then we obtain the solutions of these systems releted to initial values.


Keywords- The period of discrete system, high order non-linear discrete systems

## 1. INTRODUCTION

Difference equations system arise in many branches of mathematics as well as other sciences. It is a fascinating issue because it is used to solve some problems that appear to be very complicated in applied areas. It has a wide application area especially in biology, physics, social and engineering fields [1-9]. Some are following. Clark and Kulenovic examined the global stability properties and asymptotic behavior of recursive equation systems at [2]. Nasri M and at all, introduced a deterministic model for HIV infection in the presence of combination therapy with differential equations system [4]. In [1,3], Cınar C., Yalçınkaya I. and Iricanin B., Stevic S. have considered some system of nonlinear difference equations of higher order with periodic solutions. Kose H., Uslu K. and Taskara N., examined the solution dynamics of the nonlinear iteration system in [5]. In [5,7,8,9], some nonlinear discrete systems have been investigated and the dynamics of these systems have been investigated.

In this study, we consider following non-linear systems

$$
\begin{align*}
& p_{n+1}=\frac{1}{s_{n}-r_{n}-q_{n}}+\frac{1}{q_{n-1}-r_{n-1}}, \quad q_{n+1}=\frac{1}{s_{n}-r_{n}-q_{n}}+\frac{r_{n}\left(s_{n-1}-r_{n-1}-q_{n-1}\right)}{r_{n-1}} \\
& r_{n+1}=\frac{1}{s_{n}-r_{n}-q_{n}}, \quad s_{n+1}=\frac{2}{s_{n}-r_{n}-q_{n}}+\frac{r_{n}\left(s_{n-1}-r_{n-1}-q_{n-1}\right)}{r_{n-1}}+\frac{1}{p_{n}-r_{n}},(n \geq 0) \tag{1.1}
\end{align*}
$$

with initial values $p_{0}, q_{-1}, q_{0}, r_{-1}, r_{0}, s_{-1}, s_{0} \in \mathbb{R}-\{0\}, p_{0} \neq r_{0}, s_{i} \neq q_{i}+r_{i}, q_{i} \neq r_{i}, i \in\{-1,0\}$, and

$$
\begin{align*}
p_{n+1} & =\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{1}{q_{n-2}-r_{n-2}}, \quad q_{n+1}=\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{r_{n}\left(s_{n-2}-r_{n-2}-q_{n-2}\right)}{r_{n-1}} \\
r_{n+1} & =\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}, \quad s_{n+1}=\frac{2}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{r_{n}\left(s_{n-2}-r_{n-2}-q_{n-2}\right)}{r_{n-1}}+\frac{1}{p_{n-1}-r_{n-1}},(n \geq 0) \tag{1.2}
\end{align*}
$$

with initial values $p_{-1}, p_{0}, q_{-2}, q_{-1}, q_{0}, r_{-2}, r_{-1}, r_{0}, s_{-2}, s_{-1}, s_{0} \in \mathbb{R}-\{0\}, p_{i} \neq r_{i}, i \in\{-1,0\}, s_{i} \neq q_{i}+r_{i}, q_{i} \neq r_{i}, i \in$ $\{-2,-1,0\}$.

Firstly, we give basic definitions. Let $\mathrm{I}_{1}, \mathrm{I}_{2}, \mathrm{I}_{3}$ and $\mathrm{I}_{4}$ be some intervals of real numbers and let $\mathrm{F}_{1}: \mathrm{I}_{2} \times \mathrm{I}_{3} \times \mathrm{I}_{4} \rightarrow \mathrm{I}_{1}$, $\mathrm{F}_{2}: \mathrm{I}_{2} \times \mathrm{I}_{3} \times \mathrm{I}_{4} \rightarrow \mathrm{I}_{2}, \mathrm{~F}_{3}: \mathrm{I}_{2} \times \mathrm{I}_{3} \times \mathrm{I}_{4} \rightarrow \mathrm{I}_{3}$ and $\mathrm{F}_{4}: \mathrm{I}_{1} \times \mathrm{I}_{2} \times \mathrm{I}_{3} \times \mathrm{I}_{4} \rightarrow \mathrm{I}_{4}$ be four continuously differentiable functions. It is obvious that the system of difference equations (1.3)

$$
\begin{array}{ll}
p_{n+1}=F_{1}\left(q_{n}, r_{n}, s_{n}\right), & q_{n+1}=F_{2}\left(q_{n}, r_{n}, s_{n}\right) \\
r_{n+1}=F_{3}\left(q_{n}, r_{n}, s_{n}\right), & s_{n+1}=F_{4}\left(p_{n}, q_{n}, r_{n}, s_{n}\right) \tag{1.3}
\end{array}
$$

has a unique solution $\left\{p_{n}, q_{n}, r_{n}, s_{n}\right\}$ for every initial condition $\left(p_{i}, q_{i}, r_{i}, s_{i}\right) \in \mathrm{I}_{1} \times \mathrm{I}_{2} \times \mathrm{I}_{3} \times \mathrm{I}_{4}$.
A solution $\left\{p_{n}, q_{n}, r_{n}, s_{n}\right\}$ of the system of difference equations (1.3) is periodic if there exist a positive integer $m$ such that

$$
p_{n+m}=p_{n}, \quad q_{n+m}=q_{n}, \quad r_{n+m}=r_{n}, \quad s_{n+m}=s_{n}
$$

the smallest such positive integer $m$ is called the prime period of the solution of difference equation system $(1.3)[2,4]$.

## 2. THE SOLUTIONS AND PERIODS OF SOME CONSIDERED NON-LINEAR DIFFERENCE EQUATION SYSTEMS

The following theorems show us the period of solutions of the systems (1.1) and (1.2).
Theorem 2.1. Suppose that $\left\{p_{n}, q_{n}, r_{n}, s_{n}\right\}$ are the solutions of the difference equation system (1.1) with initial values $p_{0}, q_{-1}, q_{0}, r_{-1}, r_{0}, s_{-1}, s_{0} \in \mathbb{R}-\{0\}, p_{0} \neq r_{0}, s_{i} \neq q_{i}+r_{i}, q_{i} \neq r_{i}, i \in\{-1,0\}$. Then all solutions of the system (1.1) are periodic with period 6 .

Proof: From the system (1.1), it is obtained the following equalities

$$
\begin{aligned}
& p_{n+1}=\frac{1}{s_{n}-r_{n}-q_{n}}+\frac{1}{q_{n-1}-r_{n-1}}, \quad q_{n+1}=\frac{1}{s_{n}-r_{n}-q_{n}}+\frac{r_{n}\left(s_{n-1}-r_{n-1}-q_{n-1}\right)}{r_{n-1}} \\
& r_{n+1}=\frac{1}{s_{n}-r_{n}-q_{n}}, \quad s_{n+1}=\frac{2}{s_{n}-r_{n}-q_{n}}+\frac{r_{n}\left(s_{n-1}-r_{n-1}-q_{n-1}\right)}{r_{n-1}}+\frac{1}{p_{n}-r_{n}}, \\
& p_{n+2}=p_{n}-r_{n}+\frac{1}{q_{n}-r_{n}}, \quad q_{n+2}=p_{n}-r_{n}+\frac{1}{r_{n}} \\
& r_{n+2}=p_{n}-r_{n}, \quad s_{n+2}=2\left(p_{n}-r_{n}\right)+\frac{1}{r_{n}}+q_{n-1}-r_{n-1}, \\
& p_{n+3}=\frac{1}{q_{n-1}-r_{n-1}}+\frac{r_{n-1}}{r_{n}\left(s_{n-1}-r_{n-1}-q_{n-1}\right)}, \quad q_{n+3}=\frac{1}{q_{n-1}-r_{n-1}}+s_{n}-r_{n}-q_{n} \\
& r_{n+3}=\frac{1}{q_{n-1}-r_{n-1}}, \quad s_{n+3}=\frac{2}{q_{n-1}-r_{n-1}}+s_{n}-r_{n}-q_{n}+q_{n}-r_{n}, \\
& p_{n+4}=\frac{1}{q_{n}-r_{n}}+r_{n}, \quad q_{n+4}=\frac{1}{q_{n}-r_{n}}+\frac{1}{p_{n}-r_{n}} \\
& r_{n+4}=\frac{1}{q_{n}-r_{n}}, \quad s_{n+4}=\frac{2}{q_{n}-r_{n}}+\frac{1}{p_{n}-r_{n}}+\frac{r_{n}\left(s_{n-1}-r_{n-1}-q_{n-1}\right)}{r_{n-1}}, \\
& p_{n+5}=\frac{r_{n-1}}{r_{n}\left(s_{n-1}-r_{n-1}-q_{n-1}\right)}+\frac{1}{\left(s_{n}-r_{n}-q_{n}\right)}, \quad q_{n+5}=\frac{r_{n-1}}{r_{n}\left(s_{n-1}-r_{n-1}-q_{n-1}\right)}+q_{n-1}-r_{n-1}
\end{aligned}
$$

$$
\begin{gathered}
r_{n+5}=\frac{r_{n-1}}{r_{n}\left(s_{n-1}-r_{n-1}-q_{n-1}\right)}, \quad s_{n+5}=\frac{2 r_{n-1}}{r_{n}\left(s_{n-1}-r_{n-1}-q_{n-1}\right)}+q_{n-1}-r_{n-1}+\frac{1}{r_{n}}, \\
p_{n+6}=p_{n}, \quad q_{n+6}=q_{n} \\
r_{n+6}=r_{n}, \quad s_{n+6}=s_{n} .
\end{gathered}
$$

Thus all solutions of the system (1.1) are periodic with 6 period.

Theorem 2.2. Suppose that $\left\{p_{n}, q_{n}, r_{n}, s_{n}\right\}$ are the solutions of the difference equation system (1.2) with initial values $p_{-1}, p_{0}, q_{-2}, q_{-1}, q_{0}, r_{-2}, r_{-1}, r_{0}, s_{-2}, s_{-1}, s_{0} \in \mathbb{R}-\{0\}, p_{i} \neq r_{i}, i \in\{-1,0\}, s_{i} \neq q_{i}+r_{i}, q_{i} \neq r_{i}, i \in\{-2,-1,0\}$. Then all solutions of the system (1.2) are periodic with period 9 .
Proof: From the system (1.2), it is obtained the following equalities

$$
\begin{aligned}
& p_{n+1}=\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{1}{q_{n-2}-r_{n-2}}, \quad q_{n+1}=\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{r_{n}\left(s_{n-2}-r_{n-2}-q_{n-2}\right)}{r_{n-1}}, \\
& r_{n+1}=\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}, \quad s_{n+1}=\frac{2}{s_{n-1}-r_{n-1}-q_{n-1}}+\frac{r_{n}\left(s_{n-2}-r_{n-2}-q_{n-2}\right)}{r_{n-1}}+\frac{1}{p_{n-1}-r_{n-1}}, \\
& p_{n+2}=\frac{1}{s_{n}-r_{n}-q_{n}}+\frac{1}{q_{n-1}-r_{n-1}}, \quad q_{n+2}=\frac{1}{s_{n}-r_{n}-q_{n}}+\frac{1}{r_{n}}, \\
& r_{n+2}=\frac{1}{s_{n}-r_{n}-q_{n}}, \quad s_{n+2}=\frac{2}{s_{n}-r_{n}-q_{n}}+\frac{1}{r_{n}}+\frac{1}{p_{n}-r_{n}}, \\
& p_{n+3}=p_{n-1}-r_{n-1}+\frac{1}{q_{n}-r_{n}}, \quad q_{n+3}=p_{n-1}-r_{n-1}+s_{n-1}-r_{n-1}-q_{n-1}, \\
& r_{n+3}=p_{n-1}-r_{n-1}, \quad s_{n+3}=2\left(p_{n-1}-r_{n-1}\right)+s_{n-1}-r_{n-1}-q_{n-1}+q_{n-2}-r_{n-2}, \\
& p_{n+4}=p_{n}-r_{n}+\frac{r_{n-1}}{r_{n}\left(s_{n-2}-r_{n-2}-q_{n-2}\right)}, \quad q_{n+4}=p_{n}-r_{n}+s_{n}-r_{n}-q_{n}, \\
& r_{n+4}=p_{n}-r_{n}, \quad s_{n+4}=2\left(p_{n}-r_{n}\right)+s_{n}-r_{n}-q_{n}+q_{n-1}-r_{n-1}, \\
& p_{n+5}=\frac{1}{q_{n-2}-r_{n-2}}+r_{n}, \quad q_{n+5}=\frac{1}{q_{n-2}-r_{n-2}}+\frac{1}{p_{n-1}-r_{n-1}}, \\
& r_{n+5}=\frac{1}{q_{n-2}-r_{n-2}}, \quad s_{n+5}=\frac{2}{q_{n-2}-r_{n-2}}+\frac{1}{p_{n-1}-r_{n-1}}+q_{n}-r_{n}, \\
& p_{n+6}=\frac{1}{q_{n-1}-r_{n-1}}+\frac{1}{s_{n-1}-r_{n-1}-q_{n-1}}, \quad q_{n+6}=\frac{1}{q_{n-1}-r_{n-1}}+\frac{1}{p_{n}-r_{n}}, \\
& r_{n+6}=\frac{1}{q_{n-1}-r_{n-1}}, \quad s_{n+6}=\frac{2}{q_{n-1}-r_{n-1}}+\frac{1}{p_{n}-r_{n}}+\frac{r_{n}\left(s_{n-2}-r_{n-2}-q_{n-2}\right)}{r_{n-1}},
\end{aligned}
$$

$$
\begin{gathered}
p_{n+7}=\frac{1}{q_{n}-r_{n}}+\frac{1}{s_{n}-r_{n}-q_{n}}, \quad q_{n+7}=\frac{1}{q_{n}-r_{n}}+q_{n-2}-r_{n-2}, \\
r_{n+7}=\frac{1}{q_{n}-r_{n}}, \quad s_{n+7}=\frac{2}{q_{n}-r_{n}}+q_{n-2}-r_{n-2}+\frac{1}{r_{n}}, \\
p_{n+8}=\frac{r_{n-1}}{r_{n}\left(s_{n-2}-r_{n-2}-q_{n-2}\right)}+p_{n-1}-r_{n-1}, \quad q_{n+8}=\frac{r_{n-1}}{r_{n}\left(s_{n-2}-r_{n-2}-q_{n-2}\right)}+q_{n-1}-r_{n-1}, \\
r_{n+8}=\frac{r_{n-1}}{r_{n}\left(s_{n-2}-r_{n-2}-q_{n-2}\right)}, \quad s_{n+8}=\frac{2 r_{n-1}}{r_{n}\left(s_{n-2}-r_{n-2}-q_{n-2}\right)}+q_{n-1}-r_{n-1}+s_{n-1}-r_{n-1}-q_{n-1}, \\
p_{n+9}=p_{n}, \quad q_{n+9}=q_{n} \\
r_{n+9}=r_{n}, \quad s_{n+9}=s_{n} .
\end{gathered}
$$

Thus all solutions of the system (1.1) are periodic with 9 period.

This study is related to the master's thesis of Vural Deniz.

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