The Solvability Conditions for the Inverse Eigenvalue Problem of Hermitian and Generalized Skew-Hamiltonian Matrices and Its Approximation

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Abstract

In this paper, we first consider the inverse eigenvalue problem as follows: Find a matrix A with specified eigen-pairs, where A is a Hermitian and generalized skew-Hamiltonian matrix. The sufficient and necessary conditions are obtained, and a general representation of such a matrix is presented. We denote the set of such matrices by $\mathcal{L}_S$. Then the best approximation problem for the inverse eigenproblem is discussed. That is: Given an arbitrary $\tilde{A}$, find a matrix $A^* \in \mathcal{L}_S$ which is nearest to $\tilde{A}$ in the Frobenius norm. We show that the best approximation is unique and provide an expression for this nearest matrix.

Keywords. Inverse eigenvalue problem, Hermitian and generalized skew-Hamiltonian matrix, matrix norm, best approximation.

AMS subject classifications. 65F18, 65F15, 65F35.

1 Introduction

Let $J \in \mathbb{R}^{n\times n}$ be an orthogonal skew-symmetric matrix, i.e. $J \in \mathbb{R}^{n\times n}$ satisfies that $J^TJ = JJ^T = I_n$, $J^T = -J$. Then we have $J^2 = -I_n$ and $n = 2k, k \in N$. In the following, we give the definitions of generalized Hamiltonian and generalized skew-Hamiltonian matrices. Here, we denote the set of all n-by-m complex matrices by $\mathbb{C}^{n\times m}$.

Definition 1 *Given an orthogonal skew-symmetric matrix J.*

(1) A matrix $H \in \mathbb{C}^{n\times n}$ is called generalized Hamiltonian if $(HJ)^H = HJ$. The set of all n-by-n generalized Hamiltonian matrices is denoted by $\mathbb{GH}^{n\times n}$.

(2) A matrix $H \in \mathbb{C}^{n\times n}$ is called generalized skew-Hamiltonian if $(HJ)^H = -HJ$. The set of all n-by-n generalized skew-Hamiltonian matrices is denoted by $\mathbb{GSH}^{n\times n}$.

We observe that the sets $\mathbb{GH}^{n\times n}$ and $\mathbb{GSH}^{n\times n}$ depend on the choice of the matrix J. If $J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}$, then the sets $\mathbb{GH}^{n\times n}$ and $\mathbb{GSH}^{n\times n}$ are the well-known sets of Hamiltonian and skew-Hamiltonian matrices.

Definition 2 *Given an orthogonal skew-symmetric matrix J.*

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(1) *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be a Hermitian and generalized Hamiltonian matrix if $A^H = A$ and $(AJ)^H = AJ$. The set of all n-by-n Hermitian and generalized Hamiltonian matrices is denoted by $\mathbb{HH}^{n\times n}$.*

(2) *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be a Hermitian and generalized skew-Hamiltonian matrix if $A^H = A$ and $(AJ)^H = -AJ$. The set of all n-by-n Hermitian and generalized skew-Hamiltonian matrices is denoted by $\mathbb{HSH}^{n\times n}$.*

Hamiltonian and skew-Hamiltonian matrices play an important role in engineering, such as in linear-quadratic optimal control [13, 17], H_∞ optimization [24], and the related problem of solving algebraic Riccati equations [11].

In this paper, we will study two problems related to Hermitian and generalized skew-Hamiltonian matrices. The first problem is a kind of inverse eigenvalue problems. For decades, structured inverse eigenvalue problems have been of great value for many applications, see for instance the expository papers [7, 22]. There are also different types of inverse eigenproblem, for instances multiplicative type and additive type [22, Chapter 4]. In what follows, we consider the following type of inverse eigenproblem which appeared in the design of Hopfield neural networks [6, 12].

Problem I. Given $X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m] \in \mathbb{C}^{n\times m}$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^{m\times m}$, find a Hermitian and generalized skew-Hamiltonian matrix A in $\mathbb{HSH}^{n\times n}$ such that $AX = X\Lambda$.

We note from the above definition that the eigenvalues of a Hermitian and generalized skew-Hamiltonian matrix are real numbers. Hence we have $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^{m\times m}$.

The second problem we consider in this paper is the problem of best approximation:

Problem II. Let $\mathcal{L}_S$ be the solution set of Problem I. Given a matrix $\tilde{A} \in \mathbb{C}^{n\times n}$, find $A^* \in \mathcal{L}_S$ such that

$$\|\tilde{A} - A^*\| = \min_{A\in\mathcal{L}_S} \|\tilde{A} - A\|,$$

where $\|\cdot\|$ is the Frobenius norm.

The best approximation problem occurs frequently in experimental design, see for instance [14, p.123]. Here the matrix $\tilde{A}$ may be a matrix obtained from experiments, but it may not satisfy the structural requirement (Hermitian and generalized skew-Hamiltonian) and/or spectral requirement (having eigenpairs X and Λ). The best estimate A^* is the matrix that satisfies both restrictions and is the best approximation of $\tilde{A}$ in the Frobenius norm, see for instance [2, 3, 10].

Problems I and II have been solved for different classes of structured matrices, see for instance [21, 23]. In this paper, we extend the results in [23] to the class of Hermitian and generalized skew-Hamiltonian matrices. We first give a solvability condition for Problem I and also the form of its general solution. Then in the case when Problem I is solvable, we show that Problem II has a unique solution and give a formula for the minimizer A^*.

In this paper, the notations are as follows. Let $\mathcal{U}(n)$ be the set of all n-by-n unitary matrices, and $\mathbb{H}^{n\times n}$ denote the set of all n-by-n Hermitian matrices. We denote the transpose, conjugate transpose and the Moore-Penrose generalized inverse of a matrix A by A^T, A^H and A^+ respectively, and the identity matrix of order n by I_n. We define the inner product in space $\mathbb{C}^{n\times m}$ by

$$(A, B) = \mathrm{tr}(A^H B), \quad \forall A, B \in \mathbb{C}^{n\times m}.$$

Then $\mathbb{C}^{n\times m}$ is a Hilbert inner product space. The norm of a matrix generated by the inner product space is the Frobenius norm.

This paper is outlined as follows. In §2 we first discuss the structure of the set $\mathbb{HSH}^{n\times n}$, and then present the solvability conditions and provide the general solution formula for Problem I. In §3 we first show the existence and uniqueness of the solution for Problem II, and then derive an expression of the solution when the solution set $\mathcal{L}_S$ is nonempty, and finally propose an algorithm to compute the solution to Problem II. In §4 we give some illustrative numerical examples.

2 Solvability Conditions of Problem I

We first discuss the structure of $\mathbb{HSH}^{n\times n}$. In what follows, we always assume that $n = 2k, k \in N$. By the definition of $\mathbb{HSH}^{n\times n}$, we have the following statement.

Lemma 1 *Let $A \in \mathbb{C}^{n\times n}$, then $A \in \mathbb{HSH}^{n\times n}$ if and only if $A^H = A$, $AJ - JA = 0$.*

Since J is orthogonal skew-symmetric, J is normal and skew-symmetric and then has only two multiple eigenvalues i and $-i$ with multiplicity k respectively, where i denotes the the imaginary unit, i.e. $i^2 = -1$. Thus we can easily show the following lemma.

Lemma 2 *Let $J \in \mathbb{R}^{n\times n}$ be orthogonal skew-symmetric, then there exists a matrix $U \in \mathcal{U}(n)$ such that*

$$J = U \begin{bmatrix} i \cdot I_k & 0 \\ 0 & -i \cdot I_k \end{bmatrix} U^H. \tag{1}$$

By the above two lemmas, we have the following result for the structure of $\mathbb{HSH}^{n\times n}$.

Theorem 1 *Let $A \in \mathbb{C}^{n\times n}$ and the spectral decomposition of J be given as (1). Then $A \in \mathbb{HSH}^{n\times n}$ if and only if*

$$A = U \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} U^H, \quad A_{11}, A_{22} \in \mathbb{H}^{k\times k}. \tag{2}$$

Proof: If $A \in \mathbb{HSH}^{n\times n}$, then by Lemma 1 and (1), we obtain

$$U^H AU \begin{bmatrix} i \cdot I_k & 0 \\ 0 & -i \cdot I_{n-k} \end{bmatrix} + \begin{bmatrix} i \cdot I_k & 0 \\ 0 & -i \cdot I_{n-k} \end{bmatrix} U^H AU = 0. \tag{3}$$

Since $A^H = A$, then $U^H AU \in \mathbb{H}^{n\times n}$. Let

$$A = U \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^H & A_{22} \end{bmatrix} U^H, \quad A_{11} \in \mathbb{H}^{k\times k}, A_{22} \in \mathbb{H}^{k\times k}.$$

Substituting it into (3) yields (2).

On the other hand, if A can be expressed as (2), then, obviously, $A^H = A$, $AJ - JA = 0$. By Lemma 1, $A \in \mathbb{HSH}^{n\times n}$. □

We now investigate the solvability of Problem I. We need the following lemma, see for instance [19].

Lemma 3 **[19, Lemma 1.4]** *Let $B, C \in \mathbb{C}^{n\times m}$ be given. Then $HB = C$ has a solution in $\mathbb{H}^{n\times n}$ if and only if*

$$C = CB^+B \quad \textit{and} \quad (BB^+CB^+)^H = BB^+CB^+.$$

In this case the general solution can be expressed by

$$Y = CB^+ + (B^+)^H C^H - (B^+)^H C^H BB^+ + (I - BB^+)Z(I - BB^+),$$

where $Z \in \mathbb{H}^{n\times n}$ is arbitrary.

Then we can establish the solvability of Problem I as follows.

Theorem 2 *Given $X \in \mathbb{C}^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^{m\times m}$. Let*

$$U^H X = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix}, \quad \tilde{X}_1, \tilde{X}_2 \in \mathbb{C}^{k\times m}. \tag{4}$$

Then there exists $A \in \mathbb{HSH}^{n\times n}$ such that $AX = X\Lambda$ if and only if

$$\tilde{X}_1\Lambda\tilde{X}_1^+\tilde{X}_1 = \tilde{X}_1\Lambda, \quad (\tilde{X}_1^+)^H\Lambda(\tilde{X}_1^+)^H = \tilde{X}_1\Lambda\tilde{X}_1^+, \tag{5}$$

and

$$\tilde{X}_2\Lambda\tilde{X}_2^+\tilde{X}_2 = \tilde{X}_2\Lambda, \quad (\tilde{X}_2^+)^H\Lambda(\tilde{X}_2^+)^H = \tilde{X}_2\Lambda\tilde{X}_2^+. \tag{6}$$

In this case the general solution is given by

$$A = A_0 + U \begin{bmatrix} (I_k - \tilde{X}_1\tilde{X}_1^+)Z_1(I_k - \tilde{X}_1\tilde{X}_1^+) & 0 \\ 0 & (I_k - \tilde{X}_2\tilde{X}_2^+)Z_2(I_k - \tilde{X}_2\tilde{X}_2^+) \end{bmatrix} U^H, \tag{7}$$

where $Z_1, Z_2 \in \mathbb{H}^{k\times k}$ are arbitrary and

$$A_0 = U \begin{bmatrix} \tilde{X}_1\Lambda\tilde{X}_1^+ & 0 \\ 0 & \tilde{X}_2\Lambda\tilde{X}_2^+ \end{bmatrix} U^H. \tag{8}$$

Proof: We assume that A is a solution to Problem I. By Theorem 1, there is a solution to Problem I if and only if there exist $A_{11}, A_{22} \in \mathbb{H}^{k\times k}$ such that

$$A = U \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} U^H, \quad AX = X\Lambda,$$

i.e.

$$U \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} U^H X = X\Lambda. \tag{9}$$

(9) is equivalent to

$$A_{11}\tilde{X}_1 = \tilde{X}_1\Lambda \quad \text{and} \quad A_{22}\tilde{X}_2 = \tilde{X}_2\Lambda. \tag{10}$$

By Lemma 3, (10) have solutions in $\mathbb{H}^{n\times n}$ if and only if

$$\tilde{X}_1\Lambda\tilde{X}_1^+\tilde{X}_1 = \tilde{X}_1\Lambda, \quad (\tilde{X}_1\tilde{X}_1^+\tilde{X}_1\Lambda\tilde{X}_1^+)^H = \tilde{X}_1\tilde{X}_1^+\tilde{X}_1\Lambda\tilde{X}_1^+, \tag{11}$$

and

$$\tilde{X}_2\Lambda\tilde{X}_2^+\tilde{X}_2 = \tilde{X}_2\Lambda, \quad (\tilde{X}_2\tilde{X}_2^+\tilde{X}_2\Lambda\tilde{X}_2^+)^H = \tilde{X}_2\tilde{X}_2^+\tilde{X}_2\Lambda\tilde{X}_2^+. \tag{12}$$

Since $\tilde{X}_1\tilde{X}_1^+\tilde{X}_1 = \tilde{X}_1$ and $\tilde{X}_2\tilde{X}_2^+\tilde{X}_2 = \tilde{X}_2$, (11) and (12) are equivalent to (5) and (6) respectively. Moreover in this case, the general solutions to (10) is given by

$$\begin{aligned} A_{11} &= \tilde{X}_1\Lambda\tilde{X}_1^+ + (\tilde{X}_1^+)^H\Lambda\tilde{X}_1^H - (\tilde{X}_1^+)^H\Lambda\tilde{X}_1^H\tilde{X}_1\tilde{X}_1^+ + (I_k - \tilde{X}_1\tilde{X}_1^+)Z_1(I_k - \tilde{X}_1\tilde{X}_1^+) \\ &= \tilde{X}_1\Lambda\tilde{X}_1^+ + (I_k - \tilde{X}_1\tilde{X}_1^+)Z_1(I_k - \tilde{X}_1\tilde{X}_1^+), \qquad (13) \\ A_{22} &= \tilde{X}_2\Lambda\tilde{X}_2^+ + (\tilde{X}_2^+)^H\Lambda\tilde{X}_2^H - (\tilde{X}_2^+)^H\Lambda\tilde{X}_2^H\tilde{X}_2\tilde{X}_2^+ + (I_k - \tilde{X}_2\tilde{X}_2^+)Z_2(I_k - \tilde{X}_2\tilde{X}_2^+) \\ &= \tilde{X}_2\Lambda\tilde{X}_2^+ + (I_k - \tilde{X}_2\tilde{X}_2^+)Z_2(I_k - \tilde{X}_2\tilde{X}_2^+), \qquad (14) \end{aligned}$$

where $Z_1, Z_2 \in \mathbb{H}^{k\times k}$ is arbitrary. Let

$$A_0 = U\begin{bmatrix} \tilde{X}_1\Lambda\tilde{X}_1^+ & 0 \\ 0 & \tilde{X}_2\Lambda\tilde{X}_2^+ \end{bmatrix}U^H.$$

Substituting (13) into (2) gives rise to (7). □

3 The Solution to Problem II

In this section, we solve Problem II over $\mathcal{L}_S$ when $\mathcal{L}_S$ is nonempty. We first recall the following statement.

Lemma 4 [8, Theorem 2] *Let $E, H \in \mathbb{C}^{n\times n}$. If $H \in \mathbb{H}^{n\times n}$, then*

$$\|E - \frac{E+E^H}{2}\| \leq \|E - H\|.$$

Then we have the following theorem for the solution to Problem II over $\mathcal{L}_S$.

Theorem 3 *Given $\tilde{A} \in \mathbb{C}^{n\times n}$, $X \in \mathbb{C}^{n\times m}$, and the notation of X, Λ and conditions are the same as in Theorem 2. Let*

$$U^H\tilde{A}U = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{A}_{11}, \tilde{A}_{22} \in \mathbb{C}^{k\times k}. \qquad (15)$$

If $\mathcal{L}_S$ is nonempty, then Problem II has a unique solution A^* and A^* can be represented as

$$A^* = A_0 + U\begin{bmatrix} P(\frac{\tilde{A}_{11}+\tilde{A}_{11}^H}{2})P & 0 \\ 0 & Q(\frac{\tilde{A}_{22}+\tilde{A}_{22}^H}{2})Q \end{bmatrix}U^H, \qquad (16)$$

where A_0 is given by (8) *and*

$$P = I_k - \tilde{X}_1\tilde{X}_1^+, \quad Q = I_k - \tilde{X}_2\tilde{X}_2^+. \qquad (17)$$

Proof: When $\mathcal{L}_S$ is nonempty, it is easy to verify from (7) that $\mathcal{L}_S$ is a closed convex set. Since $\mathbb{C}^{n\times n}$ is a uniformly convex Banach space under the Frobenius norm, there exists a unique solution for Problem II [5, p. 22]. Because the Frobenius norm is unitary invariant, Problem II is equivalent to

$$\min_{A\in\mathcal{L}_S} \|U^H\tilde{A}U - U^HAU\|^2. \qquad (18)$$

By Theorem 2, we have

$$\|U^H\tilde{A}U - U^HAU\|^2 = \left\| \begin{bmatrix} \tilde{A}_{11} - \tilde{X}_1\Lambda\tilde{X}_1^+ & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} - \tilde{X}_2\Lambda\tilde{X}_2^+ \end{bmatrix} - \begin{bmatrix} PZ_1P & 0 \\ 0 & QZ_2Q \end{bmatrix} \right\|^2$$

Thus (18) is equivalent to

$$\min_{Z_1\in\mathbb{H}^{k\times k}} \|\tilde{A}_{11}-\tilde{X}_1\Lambda\tilde{X}_1^{+}-PZ_1P\|^2+\min_{Z_2\in\mathbb{H}^{k\times k}} \|\tilde{A}_{22}-\tilde{X}_2\Lambda\tilde{X}_2^{+}-QZ_2Q\|^2.$$

By Lemma 4, the solution is given by Z_1^* and Z_2^* such that

$$\begin{aligned} PZ_1^*P &= \frac{\tilde{A}_{11}+\tilde{A}_{11}^H}{2}-\tilde{X}_1\Lambda\tilde{X}_1^{+}, \\ QZ_2^*Q &= \frac{\tilde{A}_{22}+\tilde{A}_{22}^H}{2}-\tilde{X}_2\Lambda\tilde{X}_2^{+}. \end{aligned}$$

Notice from (17) that P and Q are projection matrices, i.e. $P^2=P$ and $Q^2=Q$. Therefore $PZ_1^*P=P(\frac{\tilde{A}_{11}+\tilde{A}_{11}^H}{2}-\tilde{X}_1\Lambda\tilde{X}_1^{+})P$ and $QZ_2^*Q=Q(\frac{\tilde{A}_{22}+\tilde{A}_{22}^H}{2}-\tilde{X}_2\Lambda\tilde{X}_2^{+})Q$. Let $G_{11}=\frac{\tilde{A}_{11}+\tilde{A}_{11}^H}{2}$. Notice further that because $\tilde{X}_1^{+}\tilde{X}_1\tilde{X}_1^{+}=\tilde{X}_1^{+}$, we have

$$\begin{aligned} P(G_{11}-\tilde{X}_1\Lambda\tilde{X}_1^{+})P &= P(G_{11}-G_{11}\tilde{X}_1\tilde{X}_1^{+}-\tilde{X}_1\Lambda\tilde{X}_1^{+}+\tilde{X}_1\Lambda\tilde{X}_1^{+}\tilde{X}_1\tilde{X}_1^{+}) \\ &= P(G_{11}-G_{11}\tilde{X}_1\tilde{X}_1^{+})=PG_{11}P. \end{aligned}$$

That is, $PZ_1^*P=P(\frac{\tilde{A}_{11}+\tilde{A}_{11}^H}{2})P$. Similarly, $QZ_2^*Q=Q(\frac{\tilde{A}_{22}+\tilde{A}_{22}^H}{2})Q$. Hence the unique solution for Problem II is given by (16). $\square$

Based on Theorem 3, we propose the following algorithm for solving Problem II over $\mathcal{L}_S$.

Algorithm I

(1) Compute $\tilde{X}_1$ and $\tilde{X}_2$ by (4).

(2) Compute $\tilde{X}_1^{+}$ and $\tilde{X}_2^{+}$.

(3) If $\tilde{X}_1\Lambda\tilde{X}_1^{+}\tilde{X}_1=\tilde{X}_1\Lambda$, $(\tilde{X}_1^{+})^H\Lambda(\tilde{X}_1)^H=\tilde{X}_1\Lambda\tilde{X}_1^{+}$, $\tilde{X}_2\Lambda\tilde{X}_2^{+}\tilde{X}_2=\tilde{X}_2\Lambda$, $(\tilde{X}_2^{+})^H\Lambda(\tilde{X}_2)^H$ $=\tilde{X}_2\Lambda\tilde{X}_2^{+}$, then the solution set $\mathcal{L}_S$ to Problem I is nonempty and we continue. Otherwise we stop.

(4) Compute $\tilde{A}_{11}$ and $\tilde{A}_{22}$ by (15).

(5) Compute $G_{11}=\frac{\tilde{A}_{11}+\tilde{A}_{11}^H}{2}$ and $G_{22}=\frac{\tilde{A}_{22}+\tilde{A}_{22}^H}{2}$.

(6) Compute

$$\begin{aligned} M_{11} &= \tilde{X}_1\Lambda\tilde{X}_1^{+}+G_{11}-G_{11}\tilde{X}_1\tilde{X}_1^{+}-\tilde{X}_1\tilde{X}_1^{+}G_{11}-\tilde{X}_1\tilde{X}_1^{+}G_{11}\tilde{X}_1\tilde{X}_1^{+}, \\ M_{22} &= \tilde{X}_2\Lambda\tilde{X}_2^{+}+G_{22}-G_{22}\tilde{X}_2\tilde{X}_2^{+}-\tilde{X}_2\tilde{X}_2^{+}G_{22}+\tilde{X}_2\tilde{X}_2^{+}G_{22}\tilde{X}_2\tilde{X}_2^{+}. \end{aligned}$$

(7) Compute $A^*=U\begin{bmatrix} M_{11} & 0 \\ 0 & M_{22}\end{bmatrix}U^H$.

Now, we consider the computational complexity of our algorithm. We observe from Lemma 2 that, for different choice of J, the structure of $U\in\mathcal{U}(n)$ may be varied. Thus the total computational complexity may be changed.

We first consider the case when given a fixed J with $U \in \mathcal{U}(n)$ dense. For Step (1), since U is dense, it requires $O(n^2m)$ operations to compute $\tilde{X}_1$ and $\tilde{X}_2$. For Step (2), using singular value decomposition to compute $\tilde{X}_1^+$ and $\tilde{X}_2^+$ requires $O(n^2m+m^3)$ operations. Step (3) obviously requires $O(n^2m)$ operations. For Step(4), because of the density of U, the operations required is $O(n^3)$. Step(5) requires O(n) operations only. For Step(6), if we compute $G_{ii}\tilde{X}_i\tilde{X}_i^+$ as $[(G_{ii}\tilde{X}_i)\tilde{X}_i^+]$, $\tilde{X}_i\tilde{X}_i^+G_{ii}$ as $[\tilde{X}_i(\tilde{X}_i^+G_{ii})]$, and $\tilde{X}_i\tilde{X}_i^+G_{ii}\tilde{X}_i\tilde{X}_i^+$ as $\{\tilde{X}_i[(\tilde{X}_i^+(G_{ii}\tilde{X}_i))\tilde{X}_i^+]\}$, then the cost will only be of $O(n^2m)$ operations. Finally, because of the density of U again, Step (7) requires $O(n^3)$ operations. Thus the total cost of the algorithm is $O(n^3+n^2m+m^3)$.

In particular, if we choose that $J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}$ with $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & I_k \\ i\cdot I_k & -i\cdot I_k \end{bmatrix} \in \mathcal{U}(n)$. Then, because of the sparsity of U, Steps (1), (4) and (7) will require $O(nm)$, $O(n^2)$ and $O(n^2)$ respectively. Therefore the total complexity of the algorithm is $O(n^2m+m^3)$.

Finally, we remark that in practice, $m \ll n$. In addition, it is easy to verify that our algorithm is stable.

4 Numerical Experiments

In this section, we will give some numerical examples to illustrate our results. All the tests are performed by MATLAB which has a machine precision around 10^{-16}. In the following, we let $n = 2k, k \in N$ and $J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}$. Then it is clear that the spectral decomposition of J is given by

$$J = U \begin{bmatrix} i\cdot I_k & 0 \\ 0 & -i\cdot I_k \end{bmatrix} U^H,$$

where $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & I_k \\ i\cdot I_k & -i\cdot I_k \end{bmatrix}$, $U^HU = UU^H = I_n$.

Example 1. We choose a random matrix A in $\mathbb{HSH}^{n\times n}$:

$$A = \begin{bmatrix} 1.9157 & \text{-0.5359 + 5.5308i} & \text{0 + 0.0596i} & \text{4.2447 + 0.1557i} \\ \text{-0.5359 - 5.5308i} & \text{-0.5504} & \text{-4.2447 + 0.1557i} & \text{0 + 0.8957i} \\ \text{0 - 0.0596i} & \text{-4.2447 - 0.1557i} & 1.9157 & \text{-0.5359 + 5.5308i} \\ \text{4.2447 - 0.1557i} & \text{0 - 0.8957i} & \text{-0.5359 - 5.5308i} & \text{-0.5504} \end{bmatrix}.$$

Then the eigenvalues of A are $-9.7331, -0.4090, 2.7296, 10.1431$. We let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ denote the eigenvectors of A associated with $-9.7331, -0.4090, 2.7296, 10.1431$ respectively. Now we take $X = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4]$, i.e.

$$X = \begin{bmatrix} \text{-0.4554 - 0.0322i} & \text{0.3324 + 0.0983i} & \text{0.5910 + 0.1747i} & \text{0.5386 + 0.0381i} \\ \text{0.0000 - 0.5399i} & \text{-0.0000 + 0.6163i} & \text{-0.0000 - 0.3466i} & \text{0.0000 - 0.4566i} \\ \text{0.0322 - 0.4554i} & \text{0.0983 - 0.3324i} & \text{0.1747 - 0.5910i} & \text{-0.0381 + 0.5386i} \\ 0.5399 & 0.6163 & -0.3466 & 0.4566 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} -9.7331 & 0 & 0 & 0 \\ 0 & -0.4090 & 0 & 0 \\ 0 & 0 & 2.7296 & 0 \\ 0 & 0 & 0 & 10.1431 \end{bmatrix}.$$

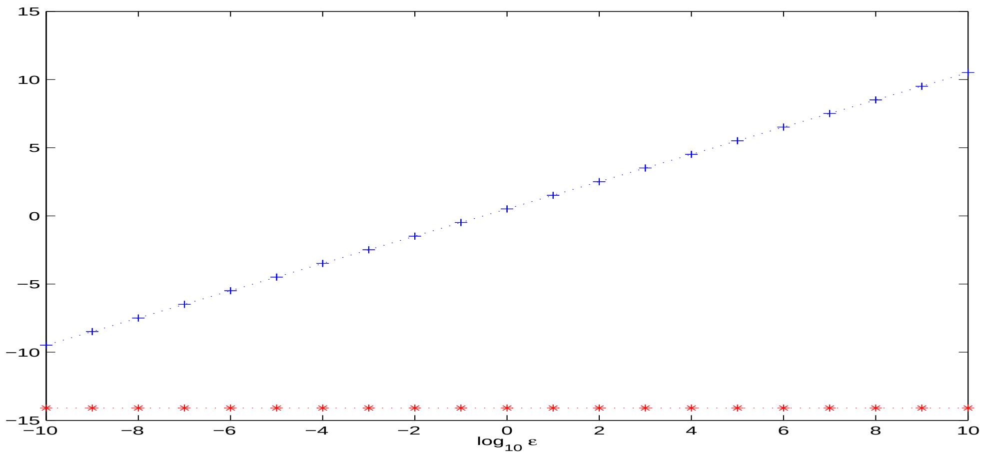

Figure 1: $\log_{10} \|\tilde{A}(\epsilon) - A^*(\epsilon)\|$ ("+") and $\log_{10} \|A - A^*(\epsilon)\|$ ("$*$") versus $\log_{10} \epsilon$ for Example 1.

Given such X and Λ, it is easy to see that there exists a solution for Problem I, i.e. A. Thus $\mathcal{L}_S$ is nonempty. If we perturb A to obtain a matrix $\tilde{A}(\epsilon) = A + \epsilon \cdot C \notin \mathbb{HSH}^{n\times n}$, where

$$C = \begin{bmatrix} 0.2476 + 0.7668i & 0.3006 + 0.8790i & 0.8569 + 0.4963i & 0.2968 + 0.3608i \\ 0.4358 + 0.5740i & 0.2659 + 0.9058i & 0.2429 + 0.3921i & 0.3903 + 0.3135i \\ 0.9776 + 0.7098i & 0.1334 + 0.0886i & 0.1949 + 0.5583i & 0.1873 + 0.7436i \\ 0.8600 + 0.8126i & 0.7425 + 0.3055i & 0.3908 + 0.6318i & 0.8957 + 0.2838i \end{bmatrix},$$

then the conditions in Theorem 2 and Theorem 3 are satisfied. Using Algorithm I in §3, we get the solution $A^*(\epsilon)$ of Theorem 3 corresponding to $\tilde{A}(\epsilon)$. In Figure 1, we plot the following two quantities for ϵ from 10^{-10} to 10^{10}: $\log_{10} \|\tilde{A}(\epsilon) - A^*(\epsilon)\|$ (marked by "+") and $\log_{10} \|A - A^*(\epsilon)\|$ (marked by "$*$"). We observe from Figure 1 that $A^*(\epsilon)$ approaches gradually $\tilde{A}(\epsilon)$ as ϵ goes to zero. While for any ϵ between 10^{-10} and 10^{10}, $A^*(\epsilon) = A$ almost up to the machine precision.

Example 2. We solve Problems I and II with multiple eigenvalues. The following is one of various eigenpairs we have tested:

$$X = \begin{bmatrix} 0.0553 + 0.2344i & 0.1528 + 0.6470i & 0.2859 + 0.3281i & 0.3662 + 0.4202i \\ -0.6648 & 0.2408 & -0.5573 & 0.4352 \\ -0.2344 + 0.0553i & -0.6470 + 0.1528i & 0.3281 - 0.2859 & 0.4202 - 0.3662i \\ 0 - 0.6648i & 0 + 0.2408i & 0 + 0.5573i & 0 - 0.4352i \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Given such X and Λ, it is easy to verify that there exists a solution for Problem I, i.e.

$$A = \begin{bmatrix} 1.1946 & 0.2329 + 0.4945i & 0 + 0.4266i & 0.1288 + 0.0858i \\ 0.2329 - 0.4945i & 0.3054 & -0.1288 + 0.0858i & 0 + 1.0734i \\ 0 - 0.4266i & -0.1288 - 0.0858i & 1.1946 & 0.2329 + 0.4945i \\ 0.1288 - 0.0858i & 0 - 1.0734i & 0.2329 - 0.4945i & 0.3054 \end{bmatrix}.$$

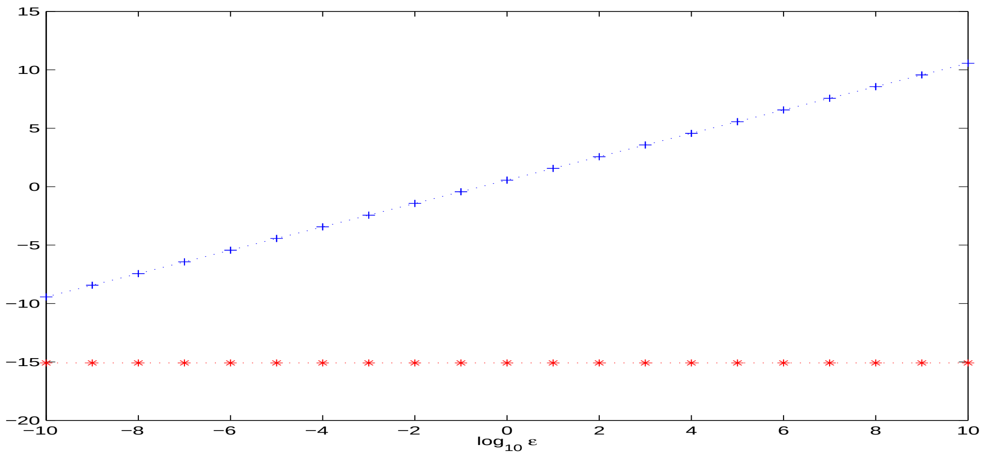

Figure 2: $\log_{10} \|\tilde{A}(\epsilon) - A^*(\epsilon)\|$ ("+") and $\log_{10} \|A - A^*(\epsilon)\|$ ("$*$") versus $\log_{10} \epsilon$ for Example 2.

Thus $\mathcal{L}_S$ is nonempty. We now perturb A to obtain a matrix $\tilde{A}(\epsilon) = A + \epsilon \cdot F \notin \mathbb{HSH}^{n\times n}$, where

$$F = \begin{bmatrix} 0.8408 + 0.4910\mathrm{i} & 0.7168 + 0.5550\mathrm{i} & 0.9106 + 0.6066\mathrm{i} & 0.8739 + 0.6959\mathrm{i} \\ 0.6463 + 0.9427\mathrm{i} & 0.8112 + 0.5147\mathrm{i} & 0.2761 + 0.3202\mathrm{i} & 0.7105 + 0.7889\mathrm{i} \\ 0.0559 + 0.5107\mathrm{i} & 0.1534 + 0.7272\mathrm{i} & 0.9571 + 0.4688\mathrm{i} & 0.9746 + 0.9407\mathrm{i} \\ 0.2057 + 0.3490\mathrm{i} & 0.0864 + 0.1896\mathrm{i} & 0.7400 + 0.7850\mathrm{i} & 0.1543 + 0.6763\mathrm{i} \end{bmatrix}.$$

Then the conditions in Theorem 2 and Theorem 3 are satisfied. Using Algorithm I in §3, we get the solution $A^*(\epsilon)$ of Theorem 3 corresponding to $\tilde{A}(\epsilon)$. In Figure 2, we plot the following two quantities for ϵ from 10^{-10} to 10^{10}: $\log_{10} \|\tilde{A}(\epsilon) - A^*(\epsilon)\|$ (marked by "+") and $\log_{10} \|A - A^*(\epsilon)\|$ (marked by "$*$"). We can see from Figure 2 that $A^*(\epsilon)$ approximates to $\tilde{A}(\epsilon)$ as ϵ goes to zero. However, for any ϵ between 10^{-10} and 10^{10}, $A^*(\epsilon) = A$ almost up to the machine precision.

Example 3. Let $T(1:n)$ denote a n-by-n Hermitian Toeplitz matrix whose first row is $(1, 2+2\cdot i, \ldots, n+n\cdot i)$, and $T(1:1/n)$ be a n-by-n Hermitian Toeplitz matrix whose first row is $(1, 1/2+1/2\cdot i, \ldots, 1/n+1/n\cdot i)$. For example

$$T(1:4) = \begin{bmatrix} 1 & 2+2i & 3+3i & 4+4i \\ 2-2i & 1 & 2+2i & 3+3i \\ 3-3i & 2-2i & 1 & 2+2i \\ 4-4i & 3-3i & 2-2i & 1 \end{bmatrix},$$

and

$$T(1:1/4) = \begin{bmatrix} 1 & 1/2+1/2i & 1/3+1/3i & 1/4+1/4i \\ 1/2-1/2i & 1 & 1/2+1/2i & 1/3+1/3i \\ 1/3-1/3i & 1/2-1/2i & 1 & 1/2+1/2i \\ 1/4-1/4i & 1/3-1/3i & 1/2-1/2i & 1 \end{bmatrix}.$$

By Theorem 1, if

$$A = U \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} U^H, \quad A_{11}, A_{22} \in \mathbb{H}^{n\times n},$$

then $A \in \mathbb{HSH}^{n\times n}$. We assume that $\lambda_j, \mathbf{x}_j$ are eigenpairs of A. Now we take $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Let $\tilde{A} = A + \Delta A$, $\Delta A = 10^{-3} \cdot C$, where C is a complex

k	A_{11}	A_{22}	$\|\Delta A\| = \|\tilde{A} - A^*\|$	Time (s)
25	T(1:25)	T(1:1/25)	0.2930	0.1300
50	T(1:50)	T(1:1/50)	0.8226	0.6500
100	T(1:100)	T(1:1/100)	2.3181	4.3300
150	T(1:150)	T(1:1/150)	4.2532	13.6500
200	T(1:200)	T(1:1/200)	6.5442	31.2800

Table 1: Numerical results for Example 3.

matrix of order n whose first column is $(1, 2, \ldots, n)^T$ and whose first row is $(1, 2 \cdot i, \ldots, n \cdot i)$ and the other entries are zeros. Then the Frobenius norm of ΔA becomes larger as n increases. We can theoretically show that A^* approaches to A as the rank of X is greater. In particular, when the rank of X is n, it is clear that $A^* = A$. We take $A_{11} = T(1 : k)$ and $A_{22} = T(1 : 1/k)$. We test Algorithm I in §3 using MATLAB 6.1.

In Table 1, we list our numerical results, where 'Time' is the CPU timings.

The above three examples and many other examples we have tested by MATLAB confirm our theoretical results in this paper. We also note from the numerical experiments that as $\tilde{A}$ approximates a solution of Problem I, $\tilde{A}$ becomes closer to the unique solution A^* of Problem II. This also agrees with our prediction.

Acknowledgment: We thank the referees for their helpful and valuable comments.

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