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The solvability of quasilinear Brezis–Nirenberg-type problems with singular weights

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Abstract

In this paper, we consider the existence and non-existence of non-trivial solutions to quasilinear Brezis–Nirenberg-type problems with singular weights. First, we shall obtain a compact imbedding theorem which is an extension of the classical Rellich–Kondrachov compact imbedding theorem, and consider the corresponding eigenvalue problem. Secondly, we deduce a Pohozaev-type identity and obtain a non-existence result. Thirdly, thanks to the generalized concentration compactness principle, we will give some abstract conditions when the functional satisfies the $(PS)_c$ condition. Finally, basing on the explicit form of the extremal function, we will obtain some existence results.

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Keywords: Brezis–Nirenberg problem; Singular weights; Pohozaev-type identity; $(PS)_c$ condition

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1. Introduction

In this paper, we consider the existence and non-existence of non-trivial solutions to the following quasilinear Brezis–Nirenberg-type problems with singular weights:

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|Du|^{p-2}Du) = |x|^{-bq}|u|^{q-2}u + \lambda|x|^{-(a+1)p+c}|u|^{p-2}u & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, $1 < p < n$, $-\infty < a < \frac{n-p}{p}$, $a \leqslant b \leqslant a+1$, $q = p^*(a,b) = \frac{np}{n-dp}$, $d = 1+a-b \in [0,\ 1]$, $c > 0$.

The starting point of the variational approach to these problems is the following weighted Sobolev–Hardy inequality due to Caffarelli et al. [5], which is called the Caffarelli–Kohn–Nirenberg inequality. Let $1 < p < n$. For all $u \in C_0^\infty(\mathbb{R}^n)$, there is a constant $C_{a,b} > 0$ such that

$$\left(\int_{\mathbb{R}^n} |x|^{-bq}|u|^q \, dx\right)^{p/q} \leqslant C_{a,b} \int_{\mathbb{R}^n} |x|^{-ap}|Du|^p \, dx, \tag{1.2}$$

where

$$-\infty < a < \frac{n-p}{p}, \quad a \leqslant b \leqslant a+1, \quad q = p^*(a,b) = \frac{np}{n-dp}, \quad d = 1+a-b. \tag{1.3}$$

Let $\mathscr{D}_a^{1,p}(\Omega)$ be the completion of $C_0^\infty(\mathbb{R}^n)$, with respect to the norm $\|\cdot\|$ defined by

$$\|u\| = \left(\int_\Omega |x|^{-ap}|Du|^p \, dx\right)^{1/p}.$$

From the boundedness of Ω and the standard approximation arguments, it is easy to see that (1.2) holds for any $u \in \mathscr{D}_a^{1,p}(\Omega)$ in the sense:

$$\left(\int_\Omega |x|^{-\alpha}|u|^r \, dx\right)^{p/r} \leqslant C \int_\Omega |x|^{-ap}|Du|^p \, dx \tag{1.4}$$

for $1 \leqslant r \leqslant \frac{np}{n-p}$, $\frac{\alpha}{r} \leqslant (1+a) + n(\frac{1}{r} - \frac{1}{p})$, that is, the imbedding $\mathscr{D}_a^{1,p}(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is continuous, where $L^r(\Omega, |x|^{-\alpha})$ is the weighted L^r space with norm:

$$\|u\|_{r,\alpha} := \|u\|_{L^r(\Omega,\,|x|^{-\alpha})} = \left(\int_\Omega |x|^{-\alpha}|u|^r \, dx\right)^{1/r}.$$

On $\mathscr{D}_a^{1,p}(\Omega)$, we can define the energy functional

$$\begin{aligned} E_\lambda(u) = {} & \frac{1}{p}\int_\Omega |x|^{-ap}|Du|^p \, dx - \frac{1}{q}\int_\Omega |x|^{-bq}|u|^q \, dx \\ & - \frac{\lambda}{p}\int_\Omega |x|^{-(a+1)p+c}|u|^p \, dx. \end{aligned} \tag{1.5}$$

From (1.4), E_λ is well-defined in $\mathscr{D}_a^{1,p}(\Omega)$, and $E_\lambda \in C^1(\mathscr{D}_a^{1,p}(\Omega), \mathbb{R})$. Furthermore, the critical points of E_λ are weak solutions of problem (1.1).

We note that for $p=2,\ a=b=0$ and $c=2$, problem (1.1) becomes

$$\begin{cases} -\Delta u = |u|^{q-2}u + \lambda u & \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $q=2^*=2n/n-2$ is the critical Sobolev exponent. Problem (1.6) has been studied in a more general context in the famous paper by Brezis and Nirenberg [3]. Since the imbedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is not compact for $q=2n/n-2$, the corresponding energy functional does not satisfy the (PS) condition globally, which caused a serious difficulty when trying to find critical points by standard variational methods. By carefully analyzing the energy level of a cut-off function related to the extremal function of the Sobolev inequality in $\mathbb{R}^n$, Brezis and Nirenberg obtained that the energy functional does satisfy the $(\text{PS})_c$ for some energy level $c<\frac{1}{n}S^{n/2}$, where S is the best constant of the Sobolev inequality.

Brezis–Nirenberg type problems have been generalized to many situations (see [8–11,13, 16,18,23,24] and references therein). In [10,11,24], the results of [3] had been extended to the p-Laplace case; [18,23] extended the results of [3] to polyharmonic operators; Jannelli and Solomini [13] considered the case with singular potentials where $p=2, a=0, c=2, b\in[0,1]$; while [8] considered the weighted case where $p=2, a<n-2/2,\ b\in[a,a+1], c>0$, and [16] considered the case where $p=2,\ a=0$ and Ω is a ball.

All the above references are based on the fact that the extremal functions are symmetric and have explicit forms. In [7], based on a generalization of the moving plane method, Chou and Chu considered the symmetry of the extremal functions for $a\geqslant 0,\ p=2$; In [12], Horiuchi successfully treated the symmetry properties of the extremal functions for the more general case $p>1,\ a\geqslant 0$ by a clever reduction to the case $a=0$ (where Schwarz symmetrization gives the symmetry of the extremal functions); On the contrary, there are some symmetry breaking results (cf. [6,4]) for $a<0$. We define

$$S(a,b) = \inf_{u\in\mathscr{D}_a^{1,p}(\mathbb{R}^n)\setminus\{0\}} E_{a,b}(u), \tag{1.7}$$

to be the best embedding constants, where

$$E_{a,b}(u) = \frac{\int_{\mathbb{R}^n} |x|^{-ap}|Du|^p\,\mathrm{d}x}{\left(\int_{\mathbb{R}^n} |x|^{-bq}|u|^q\,\mathrm{d}x\right)^{p/q}} \tag{1.8}$$

and

$$S_R(a,b) = \inf_{u\in\mathscr{D}_{a,R}^{1,p}(\mathbb{R}^n)\setminus\{0\}} E_{a,b}(u),$$

where $\mathscr{D}_{a,R}^{1,p}(\mathbb{R}^n)=\{u\in\mathscr{D}_a^{1,p}(\mathbb{R}^n)\mid u \text{ is radial}\}$. It is well known that for $a<n-p/p$ and $b-a<1$, $S_R(a,b)$ is always achieved and the extremal functions are given by

$$U_{a,b}(r) = c_0\left(\frac{n-p-pa}{1+r^{\frac{dp(n-p-pa)}{(p-1)(n-dp)}}}\right)^{n-dp/dp}, \tag{1.9}$$

where

$$c_0 = \left(\frac{n}{(p-1)^{p-1}(n-dp)}\right)^{n-dp/dp^2}. \tag{1.10}$$

Under some condition on parameters a, b, n, p [6,4] obtain that $S(a,b) < S_R(a,b)$ for $a < 0$. In this case, it is very difficult to verify that the corresponding energy functional satisfies the $(\mathrm{PS})_c$ condition.

In Section 2, based on the Caffarelli–Kohn–Nirenberg inequality and the classical Rellich–Kondrachov compactness theorem, we will first deduce a compact imbedding theorem and then study the corresponding eigenvalue problem:

$$\begin{cases} -\mathrm{div}\,(|x|^{-ap}|Du|^{p-2}Du) = \lambda |x|^{-(a+1)p+c}|u|^{p-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

In Section 3, based on a Pohozaev-type identity, we obtained a non-existence result for problem (1.1) with $\lambda \leqslant 0$. In Section 4, based on a generalized concentration compactness principle, we shall give some abstract conditions when the functional satisfies the $(\mathrm{PS})_c$ condition. In Section 5, based on the explicit form of the extremal function, we will obtain some existence results to problem (1.1).

2. Eigenvalue problem in general domain

In this section, we first deduce a compact imbedding theorem which is an extension of the classical Rellich–Kondrachov compactness theorem.

Theorem 2.1 (*Compact imbedding theorem*). *Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, $1 < p < n$, $-\infty < a < (n-p)/p$. The imbedding $\mathscr{D}_a^{1,p}(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is compact if $1 \leqslant r < np/(n-p)$, $\alpha < (1+a)r + n(1-\frac{r}{p})$.*

Proof. The continuity of the imbedding is a direct consequence of the Caffarelli–Kohn–Nirenberg inequality (1.2) or (1.4). To prove the compactness, let $\{u_m\}$ be a bounded sequence in $\mathscr{D}_a^{1,p}(\Omega)$. For any $\rho > 0$, let $B_\rho(0) \subset \Omega$ be a ball centered at the origin with radius ρ, it is easy to see that $\{u_m\} \subset W^{1,p}(\Omega \backslash B_\rho(0))$. Then the classical Rellich–Kondrachov compactness theorem guarantees the existence of a convergent subsequence of $\{u_m\}$ in $L^r(\Omega \backslash B_\rho(0))$. By taking a diagonal sequence, we can assume, without loss of generality, that $\{u_m\}$ converges in $L^r(\Omega \backslash B_\rho(0))$ for any $\rho > 0$.

On the other hand, for any $1 \leqslant r < np/n - p$, there exists a $b \in (a, a+1]$ such that $r < q = p^*(a,b) = np/n - dp$, $d = 1 + a - b \in [0,\ 1)$. From the Caffarelli–Kohn–Nirenberg inequality (1.2) or (1.4), $\{u_m\}$ is also bounded in $L^q(\Omega, |x|^{-bq})$. By the Hölder inequality, for any $\delta > 0$, it follows that

$$\int_{|x|<\delta} |x|^{-\alpha}|u_m - u_j|^r \,\mathrm{d}x \leqslant \left(\int_{|x|<\delta} |x|^{-(\alpha - br)q/(q-r)}\,\mathrm{d}x\right)^{1-(r/q)} \left(\int_\Omega |x|^{-bq}|u_m - u_j|^q \,\mathrm{d}x\right)^{r/q}$$

$$
\begin{aligned}
&\leqslant C\left(\int_0^{\delta} r^{n-1-(\alpha-br)q/(q-r)}\,\mathrm{d}r\right)^{1-(r/q)}\\
&= C\delta^{[n-(\alpha-br)q/(q-r)](1-r/q)}, \qquad (2.1)
\end{aligned}
$$

where $C>0$ is a constant independent of m. Since $\alpha<(1+a)r+n(1-(r/p))$, it follows that $n-(\alpha-br)q/(q-r)>0$. Therefore, for a given $\varepsilon>0$, we first fix $\delta>0$ such that

$$\int_{|x|<\delta}|x|^{-\alpha}|u_m-u_j|^r\,\mathrm{d}x\leqslant\frac{\varepsilon}{2}\quad\forall\, m,j\in\mathbb{N}.$$

Then we choose $N\in\mathbb{N}$ such that

$$\int_{\Omega\setminus B_\delta(0)}|x|^{-\alpha}|u_m-u_j|^r\,\mathrm{d}x\leqslant C_\alpha\int_{\Omega\setminus B_\delta(0)}|u_m-u_j|^r\,\mathrm{d}x\leqslant\frac{\varepsilon}{2}\quad\forall\, m,j\geqslant N,$$

where $C_\alpha=\delta^{-\alpha}$ if $\alpha\geqslant 0$ and $C_\alpha=(\operatorname{diam}(\Omega))^{-\alpha}$ if $\alpha<0$. Thus

$$\int_{\Omega}|x|^{-\alpha}|u_m-u_j|^r\,\mathrm{d}x\leqslant\varepsilon\quad\forall\, m,j\geqslant N,$$

that is, $\{u_m\}$ is a Cauchy sequence in $L^r(\Omega,|x|^{-\alpha})$. □

Remark 2.2. Chou and Chu [7] had obtained Theorem 2.1 for the case $p=2$.

In order to study the eigenvalue problem (1.11), let us introduce the following functionals in $\mathscr{D}_a^{1,p}(\Omega)$:

$$\Phi(u):=\int_{\Omega}|x|^{-ap}|Du|^p\,\mathrm{d}x\quad\text{and}\quad J(u):=\int_{\Omega}|x|^{-(a+1)p+c}|u|^p\,\mathrm{d}x.$$

For $c>0$, J is well-defined. Furthermore, $\Phi, J\in C^1(\mathscr{D}_a^{1,p}(\Omega),\mathbb{R})$, and a real value λ is an eigenvalue of problem (1.11) if and only if there exists $u\in\mathscr{D}_a^{1,p}(\Omega)\setminus\{0\}$ such that $\Phi'(u)=\lambda J'(u)$. At this point let us introduce set

$$\mathscr{M}:=\{u\in\mathscr{D}_a^{1,p}(\Omega)\ :\ J(u)=1\}.$$

Then $\mathscr{M}\neq\emptyset$ and $\mathscr{M}$ is a C^1 manifold in $\mathscr{D}_a^{1,p}(\Omega)$. It follows from the standard variational arguments that eigenvalues of (1.11) correspond to critical values of $\Phi|_{\mathscr{M}}$. From Theorem 2.1, Φ satisfies the (PS) condition on $\mathscr{M}$. Thus a sequence of critical values of $\Phi|_{\mathscr{M}}$ comes from the Ljusternik–Schnirelman critical point theory on C^1 manifolds. Let $\gamma(A)$ denote the Krasnoselski's genus on $\mathscr{D}_a^{1,p}(\Omega)$ and for any $k\in\mathbb{N}$, set

$$\Gamma_k:=\{A\subset\mathscr{M}\ :\ A\text{ is compact, symmetric and }\gamma(A)\geqslant k\}.$$

Then values

$$\lambda_k:=\inf_{A\in\Gamma_k}\max_{u\in A}\Phi(u) \qquad (2.2)$$

are critical values and thence are eigenvalues of problem (1.11). Moreover, $\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_k\leqslant\cdots\to+\infty$.

From the Caffarelli–Kohn–Nirenberg inequality (1.2) or (1.4), it is easy to see that

$$\lambda_1 = \inf\{\Phi(u) \ : \ u \in \mathscr{D}_a^{1,p}(\Omega),\, J(u) = 1\} > 0$$

and the corresponding eigenfunction $e_1 \geqslant 0$.

3. Pohozaev identity and non-existence result

In this section, we deduce a Pohozaev-type identity and obtain some non-existence results. First let us recall the following Pohozaev integral identity due to Pucci and Serrin [17]:

Lemma 3.1 (*Pohozaev-type identity*). *Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution of the Euler–Lagrange equation*

$$\begin{cases} \operatorname{div}\{\mathscr{F}_p(x,u,Du)\} = \mathscr{F}_u(x,u,Du) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $p = (p_1, \ldots, p_n) = Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ and $\mathscr{F}_u = \partial\mathscr{F}/\partial u$. Let A and h be, respectively, scalar and vector-value function of class $C^1(\Omega) \cap C(\bar{\Omega})$. Then it follows that

$$\begin{aligned} &\oint_{\partial\Omega} \left[\mathscr{F}(x,0,Du) - \frac{\partial u}{\partial x_i}\mathscr{F}_{p_i}(x,0,Du)\right](h \cdot \nu)\,\mathrm{d}s \\ &\quad = \int_\Omega \left\{\mathscr{F}(x,u,Du)\operatorname{div} h + h_i \mathscr{F}_{x_i}(x,u,Du) \right. \\ &\qquad - \left[\frac{\partial u}{\partial x_j}\frac{\partial h_j}{\partial x_i} + u\frac{\partial A}{\partial x_i}\right]\mathscr{F}_{p_i}(x,u,Du) \\ &\qquad \left. - A\left[\frac{\partial u}{\partial x_i}\mathscr{F}_{p_i}(x,u,Du) + u\mathscr{F}_u(x,u,Du)\right]\right\}\mathrm{d}x, \end{aligned} \tag{3.2}$$

where repeated indices i and j are understood to be summed from 1 *to* n.

Let us consider the following problem:

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|Du|^{p-2}Du) = g(x,u) \quad \text{in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{3.3}$$

where g satisfies $g(x,0)=0$. Suppose that $\mathscr{F}(x,u,Du) = \frac{1}{p}|x|^{-ap}|Du|^p - G(x,u)$, where $G(x,u) = \int_0^u g(x,t)\,\mathrm{d}t$ is the primitive of $g(x,u)$. If we choose $h(x)=x$, $A=(n/p)-(1+a)$, then (3.2) becomes

$$\begin{aligned} &\left(1 - \frac{1}{p}\right)\oint_{\partial\Omega}(x\cdot\nu)\left|\frac{\partial u}{\partial \nu}\right|^p \mathrm{d}s \\ &\quad = \int_\Omega \left[nG(x,u) + (x, G_x) + \left(1 + a - \frac{n}{p}\right)ug(x,u)\right]\mathrm{d}x. \end{aligned} \tag{3.4}$$

As to problem (1.1), suppose that $G(x,u) = (1/q)|x|^{-bq}|u|^q + (\lambda/p)|x|^{-p(1+a)+c}|u|^p$, then (3.2) or (3.4) becomes

$$\left(1-\frac{1}{p}\right)\oint_{\partial\Omega}(x\cdot\nu)\left|\frac{\partial u}{\partial\nu}\right|^p \mathrm{d}s = \frac{c\lambda}{p}\int_{\Omega}|x|^{-(a+1)p+c}|u|^p\,\mathrm{d}x. \tag{3.5}$$

Thus we obtain the following non-existence result:

Theorem 3.2 (*Non-existence theorem*). *There is no solution to problem* (1.1) *when* $\lambda\leqslant 0$ *and* Ω *is a* (*smooth*) *star-shaped domain with respect to the origin*.

Proof. The above deduction is formal. In fact, the solution to problem (1.1) may not be of class $C^2(\Omega)\cap C^1(\bar{\Omega})$. We need the approximation arguments in [11,8] (cf. Appendix). □

4. $(\mathrm{PS})_c$ condition

In this section, we first give a concentration compactness principle which is a weighted version of the Concentration Compactness Principle II due to Lions [14,15].

Theorem 4.1 (*Concentration compactness principle*). *Let* $1<p<n,\ -\infty<a<(n-p)/p,\ a\leqslant b\leqslant a+1,\ q=p^*(a,b)=np/(n-dp),\ d=1+a-b\in[0,\ 1]$, *and* $\mathscr{M}(\mathbb{R}^n)$ *be the space of bounded measures on* $\mathbb{R}^n$. *Suppose that* $\{u_m\}\subset\mathscr{D}_a^{1,p}(\mathbb{R}^n)$ *be a sequence such that*:

$$\begin{aligned}
&u_m \rightharpoonup u && \text{in } \mathscr{D}_a^{1,p}(\mathbb{R}^n),\\
&\mu_m := ||x|^a Du_m||^p\,\mathrm{d}x \rightharpoonup \mu && \text{in } \mathscr{M}(\mathbb{R}^n),\\
&\nu_m := ||x|^b u_m||^q\,\mathrm{d}x \rightharpoonup \nu && \text{in } \mathscr{M}(\mathbb{R}^n),\\
&u_m \to u && a.e.\ \text{on } \mathbb{R}^n.
\end{aligned}$$

Then there are the following statements:

(1) *There exists some at most countable set J, a family* $\{x^{(j)}\ :\ j\in J\}$ *of distinct points in* $\mathbb{R}^n$, *and a family* $\{\nu^{(j)}\ :\ j\in J\}$ *of positive numbers such that*

$$\nu = ||x|^{-b}u||^q\,\mathrm{d}x + \sum_{j\in J}\nu^{(j)}\delta_{x^{(j)}}, \tag{4.1}$$

where δ_x *is the Dirac-mass of mass* 1 *concentrated at* $x\in\mathbb{R}^n$.

(2) *The following inequality holds*

$$\mu \geqslant ||x|^{-a}Du||^p\,\mathrm{d}x + \sum_{j\in J}\mu^{(j)}\delta_{x^{(j)}} \tag{4.2}$$

for some family $\{\mu^{(j)}>0\ :\ j\in J\}$ *satisfying*

$$S(a,b)(\nu^{(j)})^{p/q}\leqslant\mu^{(j)}\quad \text{for all } j\in J. \tag{4.3}$$

In particular, $\sum_{j\in J}(\nu^{(j)})^{p/q}<\infty$.

Proof. The proof is similar to that of the concentration compactness principle II (see also [20]). □

Theorem 4.2 (*(PS)$_c$ condition in general domain*). *Let* $1 < p < n$, $-\infty < a < (n-p)/p$, $a \leqslant b < a+1$, $q = p^*(a,b) = np/(n-dp)$, $d = 1+a-b \in (0,\ 1]$, $c > 0$ *and* $0 < \lambda < \lambda_1$. *Then functional* E_λ *defined in* (1.5) *satisfies the* $(PS)_c$ *condition in* $\mathscr{D}_a^{1,p}(\Omega)$ *at the energy level* $M < \frac{d}{n} S(a,b)^{\frac{n}{dp}}$.

Proof. (1) The boundedness of (PS)$_c$ sequence.

Suppose that $\{u_m\} \subset \mathscr{D}_a^{1,p}(\Omega)$ is a (PS)$_c$ sequence of functional E_λ, that is,

$$E_\lambda(u_m) \to M \quad \text{and} \quad E'_\lambda(u_m) \to 0 \ \text{ in } (\mathscr{D}_a^{1,p}(\Omega))'.$$

Then as $m \to \infty$, it follows that

$$\begin{aligned} M + o(1) = E_\lambda(u_m) &= \frac{1}{p}\int_\Omega |x|^{-ap}|Du_m|^p \,\mathrm{d}x - \frac{1}{q}\int_\Omega |x|^{-bq}|u_m|^q \,\mathrm{d}x \\ &\quad - \frac{\lambda}{p}\int_\Omega |x|^{-(a+1)p+c}|u_m|^p \,\mathrm{d}x \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} o(1)\|\varphi\| &= (E_\lambda(u_m), \varphi) \\ &= \int_\Omega |x|^{-ap}|Du_m|^{p-2}Du_m \cdot D\varphi \,\mathrm{d}x - \int_\Omega |x|^{-bq}|u_m|^{q-2}u_m\varphi \,\mathrm{d}x \\ &\quad - \lambda\int_\Omega |x|^{-(a+1)p+c}|u_m|^{p-2}u_m\varphi \,\mathrm{d}x \end{aligned} \tag{4.5}$$

for any $\varphi \in \mathscr{D}_a^{1,p}(\Omega)$, where $o(1)$ denotes any quantity that tends to zero as $m \to \infty$. From (4.4) and (4.5), as $m \to \infty$, it follows that

$$\begin{aligned} qM + o(1) + o(1)\|u_m\| &= qE_\lambda(u_m) - (E_\lambda(u_m), v) \\ &= \left(\frac{q}{p} - 1\right)\int_\Omega |x|^{-ap}|Du_m|^p \,\mathrm{d}x \\ &\quad - \lambda\left(\frac{q}{p} - 1\right)\int_\Omega |x|^{-(a+1)p+c}|u_m|^{p-2}u_m v \,\mathrm{d}x \\ &= \left(\frac{q}{p} - 1\right)\left(1 - \frac{\lambda}{\lambda_1}\right)\|u_m\|^p, \end{aligned} \tag{4.6}$$

that is, $\{u_m\}$ is bounded in $\mathscr{D}_a^{1,p}(\Omega)$, since $q > p$, $\lambda < \lambda_1$. Thus up to a subsequence, we have the following convergence:

$$\begin{aligned} &u_m \rightharpoonup u \quad \text{in } \mathscr{D}_a^{1,p}(\Omega), \\ &u_m \rightharpoonup u \quad \text{in } L^q(\Omega, |x|^{-bq}), \\ &u_m \to u \quad \text{in } L^r(\Omega, |x|^{-\alpha}),\ \forall\, 1 \leqslant r < \tfrac{np}{n-p},\ \tfrac{\alpha}{r} < (1+a) + n(\tfrac{1}{r} - \tfrac{1}{p}) \\ &u_m \to u \quad \text{a.e. on } \Omega. \end{aligned}$$

From the concentration compactness principle—Theorem 4.1, there exist non-negative measures μ, ν and a countable family $\{x_j\} \subset \bar{\Omega}$ such that

$$|x|^{-b}|u_m|^q \,\mathrm{d}x \rightharpoonup \nu = ||x|^{-b}u||^q \,\mathrm{d}x + \sum_{j\in J} \nu^{(j)}\delta_{x^{(j)}},$$

$$||x|^{-a}Du_m||^p \,\mathrm{d}x \rightharpoonup \mu \geqslant ||x|^{-a}Du||^p \,\mathrm{d}x + S(a,b)\sum_{j\in J} (\nu^{(j)})^{p/q}\delta_{x^{(j)}}.$$

(2) Up to a subsequence, $u_m \to u$ in $L^q(\Omega, |x|^{-bq})$.

Since $\{u_m\}$ is bounded in $\mathscr{D}_a^{1,p}(\Omega)$, we may suppose, without loss of generality, that there exists $T \in (L^{p'}(\Omega, |x|^{-ap}))^n$ such that

$$|Du_m|^{p-2}Du_m \rightharpoonup T \quad \text{in } (L^{p'}(\Omega, |x|^{-ap}))^n.$$

On the other hand, $|u_m|^{q-2}u_m$ is also bounded in $L^{q'}(\Omega, |x|^{-bq})$ and

$$|u_m|^{q-2}u_m \rightharpoonup |u|^{q-2}u \quad \text{in } L^{q'}(\Omega, |x|^{-bq}).$$

Taking $m \to \infty$ in (4.5), we have

$$\int_\Omega |x|^{-ap}T\cdot D\varphi \,\mathrm{d}x = \int_\Omega |x|^{-bq}|u|^{q-2}u\varphi \,\mathrm{d}x + \lambda\int_\Omega |x|^{-(a+1)p+c}|u|^{p-2}u\varphi \,\mathrm{d}x \tag{4.7}$$

for any $\varphi \in \mathscr{D}_a^{1,p}(\Omega)$.

Let $\varphi = \psi u_m$ in (4.5), where $\psi \in C(\bar{\Omega})$, then it follows that

$$\int_\Omega |x|^{-ap}|Du_m|^{p-2}Du_m\cdot D\varphi \,\mathrm{d}x = \int_\Omega |x|^{-bq}|u_m|^{q-2}u_m\varphi \,\mathrm{d}x + \lambda\int_\Omega |x|^{-(a+1)p+c}|u_m|^{p-2}u_m\varphi + o(1). \tag{4.8}$$

Taking $m \to \infty$ in (4.8), we have

$$\int_\Omega \psi \,\mathrm{d}\mu + \int_\Omega |x|^{-ap}uT\cdot D\psi \,\mathrm{d}x = \int_\Omega \psi \,\mathrm{d}\nu + \lambda\int_\Omega |x|^{-(a+1)p+c}|u|^p\psi \,\mathrm{d}x. \tag{4.9}$$

Let $\varphi = \psi u$ in (4.7), then it follows that

$$\int_\Omega |x|^{-ap}uT\cdot D\psi \,\mathrm{d}x + \int_\Omega |x|^{-ap}\psi T\cdot Du \,\mathrm{d}x = \int_\Omega |x|^{-bq}|u|^q\psi \,\mathrm{d}x + \lambda\int_\Omega |x|^{-(a+1)p+c}|u|^p\psi \,\mathrm{d}x. \tag{4.10}$$

Thus (4.9)–(4.10) implies that

$$\int_\Omega \psi \,\mathrm{d}\mu = \sum_{j\in J} \nu_j\psi(x_j) + \int_\Omega |x|^{-ap}\psi T\cdot Du \,\mathrm{d}x, \tag{4.11}$$

which implies that

$$S(a,b)(v^{(j)})^{p/q} \leqslant \mu(x_j) = v_j.$$

Thence $v_j \geqslant S(a,b)^{n/dp}$ if $v_j \neq 0$.

On the other hand, from (4.4), (4.7) and (4.11), it follows that

$$\begin{aligned} M &= \frac{1}{p}\int_\Omega \mathrm{d}\mu - \frac{1}{q}\int_\Omega \mathrm{d}v - \frac{\lambda}{p}\int_\Omega |x|^{-(a+1)p+c}|u|^p\,\mathrm{d}x \\ &= \frac{1}{p}\sum_{j\in J} v_j + \frac{1}{p}\int_\Omega |x|^{-ap} T\cdot Du\,\mathrm{d}x - \frac{1}{q}\sum_{j\in J} v_j - \frac{1}{q}\int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x \\ &\quad - \frac{\lambda}{p}\int_\Omega |x|^{-(a+1)p+c}|u|^p\,\mathrm{d}x \\ &= \left(\frac{1}{p}-\frac{1}{q}\right)\sum_{j\in J} v_j + \left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x \\ &\geqslant \left(\frac{1}{p}-\frac{1}{q}\right)\sum_{j\in J} v_j = \frac{d}{n}\sum_{j\in J} v_j. \end{aligned} \tag{4.12}$$

Since it has been shown that $v_j \geqslant S(a,b)^{n/dp}$ if $v_j \neq 0$, the condition $M < (d/n)S(a,b)^{n/dp}$ implies that $v_j = 0$ for all $j \in J$. Hence we have

$$\int_\Omega |x|^{-bq}|u_m|^q\,\mathrm{d}x \to \int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x.$$

Thus the Brezis–Lieb Lemma [2] implies that $u_m \to u$ in $L^q(\Omega, |x|^{-bq})$.

(3) Existence of convergent subsequence.

To show that $u_m \to u$ in $\mathscr{D}_a^{1,p}(\Omega)$, from the Brezis–Lieb Lemma [2], it suffices to show that $Du_m \to Du$ a.e. in Ω and $\|u_m\| \to \|u\|$.

To show that $Du_m \to Du$ a.e. in Ω, first note that

$$|x|^{-ap}(|Du_m|^{p-2}Du_m - |Du|^{p-2}Du)\cdot(Du_m - Du) \geqslant 0, \tag{4.13}$$

the equality holds if and only if $Du_m = Du$.

Secondly, let $\varphi = u_m$ and $\varphi = u$ in (4.5) and then let $m \to \infty$, respectively, it follows that

$$\begin{aligned} \|u_m\|^p &= \int_\Omega |x|^{-ap}|Du_m|^p\,\mathrm{d}x \\ &= \int_\Omega |x|^{-bq}|u_m|^q\,\mathrm{d}x - \lambda\int_\Omega |x|^{-(a+1)p+c}|u_m|^p\,\mathrm{d}x + o(1)\|u_m\| \\ &\to \int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x - \lambda\int_\Omega |x|^{-(a+1)p+c}|u|^p\,\mathrm{d}x \end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
&\int_\Omega |x|^{-ap}|Du_m|^{p-2}Du_m\cdot Du\,\mathrm{d}x\\
&\quad=\int_\Omega |x|^{-bq}|u_m|^{q-2}u_m u\,\mathrm{d}x-\lambda\int_\Omega |x|^{-(a+1)p+c}|u_m|^{p-2}u_m u\,\mathrm{d}x+o(1)\|u\|\\
&\quad\to\int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x-\lambda\int_\Omega |x|^{-(a+1)p+c}|u|^p\,\mathrm{d}x.
\end{aligned}\tag{4.15}$$

From (4.14) and (4.15), it follows that

$$\begin{aligned}
&\int_\Omega |x|^{-ap}(|Du_m|^{p-2}Du_m-|Du|^{p-2}Du)\cdot(Du_m-Du)\,\mathrm{d}x\\
&\quad=\int_\Omega |x|^{-ap}|Du_m|^p\,\mathrm{d}x-\int_\Omega |x|^{-ap}|Du_m|^{p-2}Du_m\cdot Du\,\mathrm{d}x\\
&\qquad-\int_\Omega |x|^{-ap}|Du|^{p-2}Du\cdot(Du_m-Du)\,\mathrm{d}x\\
&\quad\to 0.
\end{aligned}\tag{4.16}$$

Eqs. (4.13) and (4.16) imply that $Du_m\to Du$ a.e. in Ω, hence $T=|Du|^{p-2}Du$, that is, $|Du_m|^{p-2}Du_m\rightharpoonup|Du|^{p-2}Du$ in $(L^{p'}(\Omega,|x|^{-ap}))^n$.

To show that $\|u_m\|\to\|u\|$, from (4.14) and (4.15), we have

$$\begin{aligned}
\|u\|^p\leftarrow&\int_\Omega |x|^{-ap}|Du_m|^{p-2}Du_m\cdot Du\,\mathrm{d}x\\
=&\int_\Omega |x|^{-bq}|u_m|^{q-2}u_m u\,\mathrm{d}x-\lambda\int_\Omega |x|^{-(a+1)p+c}|u_m|^{p-2}u_m u\,\mathrm{d}x\\
\to&\int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x-\lambda\int_\Omega |x|^{-(a+1)p+c}|u|^p\,\mathrm{d}x,
\end{aligned}$$

thus, $\|u_m\|^p\to\|u\|^p$. □

As indicated in the introduction, for $a<0$, $S(a,b)<S_R(a,b)$ and there is no explicit form of the minimizers of $S(a,b)$, so it is difficult to show that there exists a minimax value $M<(d/n)S(a,b)^{n/dp}$. But there does exist an explicit form of the extremal functions of $S_R(a,b)$, the method in [3] can be used to show that there exists a minimax value $M<(d/n)S_R(a,b)^{n/dp}$. Next theorem shows that in the space of radial functions, the functional E_λ defined in (1.5) satisfies the $(\mathrm{PS})_c$ condition in $\mathscr{D}^{1,p}_{a,R}(\Omega)$ at the energy level $M<(d/n)S_R(a,b)^{n/dp}$ in the case $p=2$.

Theorem 4.3 (*$(PS)_c$ condition in ball*). *Let $\Omega=B_1(0)$ be the unit ball in $\mathbb{R}^n$, $p=2<n$, $-\infty<a<(n-2)/2$, $a\leqslant b\leqslant a+1$, $q=2^*(a,b)=2n/(n-2d)$, $d=1+a-b\in[0,1]$, $c>0$ and $0<\lambda<\lambda_1$. Then functional E_λ defined in* (1.5) *satisfies the $(PS)_c$ condition in $\mathscr{D}^{1,2}_{a,R}(\Omega)$ at the energy level $M<(d/n)S_R(a,b)^{n/2d}$.*

Proof. (1) As in the proof of Theorem 4.2, any $(\mathrm{PS})_c$ sequence is bounded in $\mathscr{D}^{1,2}_{a,R}(\Omega)$, and up to a subsequence, we have

$$\begin{aligned}
&u_m \rightharpoonup u && \text{in } \mathscr{D}^{1,2}_{a,R}(\Omega),\\
&u_m \rightharpoonup u && \text{in } L^q(\Omega, |x|^{-bq}),\\
&u_m \to u && \text{in } L^r(\Omega, |x|^{-\alpha}),\ \forall\, 1 \leqslant r < 2n/(n-2),\ \tfrac{\alpha}{r} < (1+a) + n(\tfrac{1}{r} - \tfrac{1}{2})\\
&u_m \to u && \text{a.e. on } \Omega.
\end{aligned}$$

Thence u satisfies the following equation in weak sense:

$$\begin{cases} -\mathrm{div}\,(|x|^{-2a} Du) = |x|^{-bq}|u|^{q-2}u + \lambda |x|^{-2(a+1)+c} u & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.17}$$

Thus it follows that

$$\begin{aligned}
E_\lambda(u) &= \frac{1}{2}\int_\Omega |x|^{-2a}|Du|^2\,\mathrm{d}x - \frac{1}{q}\int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega |x|^{-2(a+1)+c}u^2\,\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{q}\right)\left(\int_\Omega |x|^{-2a}|Du|^2\,\mathrm{d}x - \lambda\int_\Omega |x|^{-2(a+1)+c}u^2\,\mathrm{d}x\right) \geqslant 0.
\end{aligned} \tag{4.18}$$

(2) Let $v_m := u_m - u$, the Brezis–Lieb Lemma [2] leads to

$$\int_\Omega |x|^{-bq}|u_m|^q\,\mathrm{d}x = \int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x + \int_\Omega |x|^{-bq}|v_m|^q\,\mathrm{d}x + o(1).$$

From $E_\lambda(u_m) \to M$ and $(E'_\lambda(u_m), u_m) \to 0$, we have

$$\begin{aligned}
E_\lambda(u_m) &= E_\lambda(u) + \frac{1}{2}\int_\Omega |x|^{-2a}|Dv_m|^2\,\mathrm{d}x \\
&\quad - \frac{1}{q}\int_\Omega |x|^{-bq}|v_m|^q\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega |x|^{-2(a+1)+c}v_m^2\,\mathrm{d}x \\
&\to M
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
&\int_\Omega |x|^{-2a}|Dv_m|^2\,\mathrm{d}x - \int_\Omega |x|^{-bq}|v_m|^q\,\mathrm{d}x - \lambda\int_\Omega |x|^{-2(a+1)+c}v_m^2\,\mathrm{d}x \\
&\quad \to \int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x + \lambda\int_\Omega |x|^{-2(a+1)+c}u^2\,\mathrm{d}x - \int_\Omega |x|^{-2a}|Du|^2\,\mathrm{d}x \\
&\quad = -(E'_\lambda(u), u) = 0.
\end{aligned} \tag{4.20}$$

Up to a subsequence, we may assume that

$$\int_\Omega |x|^{-2a}|Dv_m|^2\,\mathrm{d}x - \lambda\int_\Omega |x|^{-2(a+1)+c}v_m^2\,\mathrm{d}x \to b, \qquad \int_\Omega |x|^{-bq}|v_m|^q\,\mathrm{d}x \to b$$

for some $b \geqslant 0$. From Theorem 2.1, $v_m \to 0$ in $L^2(\Omega, |x|^{-2(a+1)+c})$, then

$$\int_\Omega |x|^{-2a}|Dv_m|^2\,\mathrm{d}x \to b.$$

On the other hand, we have

$$\int_\Omega |x|^{-2a}|Dv_m|^2\,\mathrm{d}x \geqslant S_R(a,b)\left(\int_\Omega |x|^{-bq}|v_m|^q\,\mathrm{d}x\right)^{2/q}.$$

Thus it follows that $b \geqslant S_R(a,b)b^{2/q}$, either $b \geqslant S_R(a,b)^{n/2d}$ or $b=0$. If $b=0$, the proof is complete. Assume that $b \geqslant S_R(a,b)^{n/2d}$, from (4.18) and (4.19), it follows that

$$\frac{d}{n}\,S_R(a,b)^{n/2d} \leqslant \left(\frac{1}{2}-\frac{1}{q}\right)b \leqslant M < \frac{d}{n}S_R(a,b)^{n/2d}$$

a contradiction. □

5. Existence results

In this section, by verifying that there exists a minimax value M such that $M<(d/n)$ $S(a,b)^{n/dp}$ or $M<(d/n)S_R(a,b)^{n/dp}$, we obtain some existence results to (1.1). We need some asymptotic estimates on the truncation function of the extremal function of $S_R(a,b)$. Let

$$U_\varepsilon(x) = \frac{1}{(\varepsilon + |x|^{dp(n-p-pa)/(p-1)(n-dp)})^{n-dp/dp}},$$

$$k(\varepsilon) = c_0(\varepsilon(n-p-ap))^{n-dp/dp}$$

and c_0 is defined by (1.9). Then $y_\varepsilon(x) := k(\varepsilon)U_\varepsilon(x)$ is the extremal function of $S_R(a,b)$. Furthermore, we have

$$\|Dy_\varepsilon\|^p_{L^p(\mathbb{R}^n,|x|^{-ap})} = S_R(a,b)^{q/q-p} = k(\varepsilon)^p\|DU_\varepsilon\|^p_{L^p(\mathbb{R}^n,|x|^{-ap})} \tag{5.1}$$

and

$$\|y_\varepsilon\|^q_{L^q(\mathbb{R}^n,|x|^{-bq})} = S_R(a,b)^{q/(q-p)} = k(\varepsilon)^q\|U_\varepsilon\|^q_{L^q(\mathbb{R}^n,|x|^{-bq})}. \tag{5.2}$$

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^1 boundary and $0 \in \Omega$, $R>0$ such that $B_{2R} \subset \Omega$. Denote $u_\varepsilon(x) = \psi(x)U_\varepsilon(x)$ where $\psi(x) \equiv 1$ for $|x|<R$ and $\psi(x) \equiv 0$ for $|x| \geqslant 2R$. As $\varepsilon \to 0$, the behavior of u_ε has to be the same as that of U_ε.

Lemma 5.1. *Assume* $1<p<n$, $-\infty<a<(n-p)/p$, $a \leqslant b \leqslant a+1$, $q=p^*(a,b)=np/(n-dp)$, $d=1+a-b \in [0,\ 1]$, $c>0$. *Let*

$$v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{L^q(\Omega,|x|^{-bq})}}.$$

Then $\|v_\varepsilon\|^q_{L^q(\Omega,|x|^{-bq})}=1$. *Furthermore, we have*

1. $\|Dv_\varepsilon\|^p_{L^p(\Omega,|x|^{-ap})} = S_R(a,b) + O(\varepsilon^{(n-dp)/d})$;
2. $\|Dv_\varepsilon\|^\alpha_{L^\alpha(\Omega,|x|^{-ap})} = O(\varepsilon^{\alpha(n-dp)/dp})$ *for* $\alpha = 1, 2, p-2, p-1$;

3. $$\|v_\varepsilon\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})} = \begin{cases} O(\varepsilon^{(n-dp)/d}) \text{ if } c > (n-p-ap)/(p-1), \\ O(\varepsilon^{(n-dp)/d}|\log\varepsilon|) \text{ if } c = (n-p-ap)/(p-1), \\ O(\varepsilon^{(p-1)(n-dp)(n+c-(a+1)p)/dp(n-p-ap)}) \\ \quad \text{if } c < (n-p-ap)/(p-1). \end{cases}$$

The proof of Lemma 5.1 is given in the Appendix.

In the case where $a \geqslant 0$, $1 < p < n$, the results in [12] and [7] show that the minimizers of $S(a,b)$ are symmetric and given by (1.9). Combining Theorem 4.2 and Lemma 5.1, there is the following existence result:

Theorem 5.2 (*Existence Theorem in general domain*). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^1 boundary and* $0 \in \Omega$, $1 < p < n$, $0 \leqslant a < (n-p)/p$, $a \leqslant b \leqslant a+1$, $q = p^*(a,b) = np/(n-dp)$, $d = 1+a-b \in (0,\ 1]$, $c \leqslant (n-p-ap)/(p-1)$, *and* $0 < \lambda < \lambda_1$. *Then there exists a non-trivial solution $u \in \mathscr{D}^{1,p}_a(\Omega)$ to problem* (1.1).

Proof. It is trivial that functional

$$E_\lambda(u) = \frac{1}{p}\int_\Omega |x|^{-ap}|Du|^p\,\mathrm{d}x - \frac{1}{q}\int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega |x|^{-(a+1)p+c}|u|^p\,\mathrm{d}x$$

satisfies the geometric condition of the mountain pass lemma without (PS) condition due to Ambrosetti and Rabinowitz [1]. From Theorem 4.2, it suffices to show that there exists a minimax value $M < (d/n)S(a,b)^{n/dp}$. In fact, we will show that $\max_{t\geqslant 0} E_\lambda(tv_\varepsilon) < (d/n) S(a,b)^{n/dp}$ for ε small enough. Let

$$\begin{aligned} g(t) &= E_\lambda(tv_\varepsilon) \\ &= \frac{t^p}{p}\int_\Omega |x|^{-ap}|Dv_\varepsilon|^p\,\mathrm{d}x - \frac{t^q}{q}\int_\Omega |x|^{-bq}|v_\varepsilon|^q\,\mathrm{d}x \\ &\quad - \frac{\lambda t^p}{p}\int_\Omega |x|^{-(a+1)p+c}|v_\varepsilon|^p\,\mathrm{d}x \\ &= \frac{t^p}{p}\int_\Omega |x|^{-ap}|Dv_\varepsilon|^p\,\mathrm{d}x - \frac{t^q}{q} - \frac{\lambda t^p}{p}\int_\Omega |x|^{-(a+1)p+c}|v_\varepsilon|^p\,\mathrm{d}x. \end{aligned}$$

Since $0 < \lambda < \lambda_1$, it follows that $g(t) > 0$ when t is close to 0, and $\lim_{t\to\infty} g(t) = -\infty$ if $d = 1+a-b \in (0,\ 1]$, $q = p^*(a,b) = np/(n-dp) > p$. Thus $g(t)$ attains its maximum at some $t_\varepsilon > 0$. From

$$g'(t) = t^{p-1}\left(\int_\Omega |x|^{-ap}|Dv_\varepsilon|^p\,\mathrm{d}x - t^{q-p} - \lambda\int_\Omega |x|^{-(a+1)p+c}|v_\varepsilon|^p\,\mathrm{d}x\right) = 0,$$

it follows that

$$t_\varepsilon = \left(\int_\Omega |x|^{-ap}|Dv_\varepsilon|^p\,\mathrm{d}x - \lambda\int_\Omega |x|^{-(a+1)p+c}|v_\varepsilon|^p\,\mathrm{d}x\right)^{1/(q-2)}$$

and

$$g(t_\varepsilon)=\left(\frac{1}{p}-\frac{1}{q}\right)\left(\int_\Omega |x|^{-ap}|Dv_\varepsilon|^p\,\mathrm{d}x-\lambda\int_\Omega |x|^{-(a+1)p+c}|v_\varepsilon|^p\,\mathrm{d}x\right)^{q/(q-2)}$$

$$=\begin{cases}\dfrac{d}{n}S(a,b)^{n/dp}+O(\varepsilon^{(n-dp)/d}) & \\ \quad -O(\varepsilon^{\frac{(p-1)(n-dp)(n-(a+1)p+c)}{dp(n-p-ap)}}) & \text{if } c<\dfrac{n-p-ap}{p-1}\\ \dfrac{d}{n}S(a,b)^{n/dp}+O(\varepsilon^{(n-dp)/d}) & \\ \quad -O(\varepsilon^{(n-dp)/d}|\log\varepsilon|) & \text{if } c=\dfrac{n-p-ap}{p-1}.\end{cases}$$

Note that for $c<(n-p-ap)/(p-1)$, we have $(n-dp)/d>(p-1)(n-dp)(n-(a+1)p+c)/dp(n-p-ap)$. Thus for ε small enough, it follows that $g(t_\varepsilon)<(d/n)S(a,b)^{n/dp}$. □

In the case where $p=2$, combining Theorem 4.3 and Lemma 5.1, there is the following existence result:

Theorem 5.3 (*Existence of radial solution in ball*). *Let* $\Omega=B_1(0)$ *is the unit ball in* $\mathbb{R}^n$, $-\infty<a<(n-2)/2$, $a\leqslant b\leqslant a+1$, $q=2^*(a,b)=2n/(n-2d)$, $d=1+a-b\in(0,\ 1]$, $c\leqslant n-2-2a$, *and* $0<\lambda<\lambda_1$. *Then there exists a nontrivial radial solution* $u\in\mathscr{D}^{1,2}_{a,R}(\Omega)$ *to problem* (1.1).

Proof. It is trivial that functional

$$E_\lambda(u)=\frac{1}{2}\int_\Omega |x|^{-2a}|Du|^2\,\mathrm{d}x-\frac{1}{q}\int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x-\frac{\lambda}{2}\int_\Omega |x|^{-2(a+1)+c}|u|^2\,\mathrm{d}x$$

satisfies the geometric condition of the mountain pass lemma without (PS) condition due to Ambrosetti and Rabinowitz [1]. From Theorem 4.3, it suffices to show that there exist a minimax value $c<(d/n)S_R(a,b)^{n/2d}$. In fact, the same process in Theorem 5.2 shows that $\max_{t\geqslant 0}E_\lambda(tv_\varepsilon)<(d/n)S_R(a,b)^{n/2d}$ for ε small enough for $c\leqslant n-2-2a$. □

From the result in [7], that is, $S(a,b)=S_R(a,b)$ for $p=2$, $a\geqslant 0$, Theorem 4.2 and the proofs of Lemma 5.1 and Theorem 5.2 imply that

Corollary 5.4. *Let* $\Omega\subset\mathbb{R}^n$ *be an open bounded domain with* C^1 *boundary and* $0\in\Omega$, $0\leqslant a<(n-2)/2$, $a\leqslant b\leqslant a+1$, $q=2^*(a,b)=2n/(n-2d)$, $d=1+a-b\in(0,\ 1]$, $c\leqslant n-2-2a$, *and* $0<\lambda<\lambda_1$. *Then there exists a nontrivial solution* $u\in\mathscr{D}^{1,2}_a(\Omega)$ *to problem* (1.1).

Remark 5.5. The results for the case where $a\geqslant 0$, $p=2$ had been obtained in [8] and [16] for $a=0$, $p=2$. But the results for the cases where $a<0$ or $p\neq 2$ had not been covered there.

Appendix

Proof of Theorem 3.2. Let $\{g_\varepsilon\}$ be a sequence of $C^2(\bar{\Omega}\backslash\{0\})$ functions converging to $g(\cdot, u)$ as ε goes to 0^+ and u_ε the solution of

$$\begin{cases} -\operatorname{div}(|x|^{-ap}(\varepsilon+|Du_\varepsilon|^2)^{(p-2)/2}Du_\varepsilon) = g_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.1}$$

Then from the standard regularity results in [21], u_ε is of class $C^3(\bar{\Omega}\backslash\{0\})$ and converges to u in $C^{1,\alpha}(\bar{\Omega}\backslash\{0\})$, for some $\alpha \in (0,1)$. For problem (A.1), we apply the Pohozaev integral identity–Lemma 3.1 in $\Omega_\delta = \Omega\backslash\overline{B_\delta(0)}$, $0<\delta<\operatorname{dist}(0,\partial\Omega)$, noting that u_ε may not vanish on the boundary $\partial B_\delta(0) = \{x \in \mathbb{R}^n \;:\; |x|=\delta\}$, or deduce directly by multiplying (A.1) by $(Au_\varepsilon - h\cdot Du_\varepsilon)$ with $A=(n/p)-(1+a)$, $h=x$, we have

$$\begin{aligned} &-\int_{\Omega_\delta} \operatorname{div}(|x|^{-ap}(\varepsilon+|Du_\varepsilon|^2)^{(p-2)/2}Du_\varepsilon)(Au_\varepsilon - x\cdot Du_\varepsilon)\,\mathrm{d}x \\ &\quad = \int_{\Omega_\delta} g_\varepsilon(Au_\varepsilon - x\cdot Du_\varepsilon)\,\mathrm{d}x. \end{aligned} \tag{A.2}$$

Integrating by parts over Ω_δ, we get

$$\begin{aligned} LHS = & -\int_{\partial\Omega_\delta} |x|^{-ap}(\varepsilon+|Du_\varepsilon|^2)^{(p-2)/2}(Au_\varepsilon - x\cdot Du_\varepsilon)(Du_\varepsilon\cdot \nu)\,\mathrm{d}\sigma \\ & + \int_{\Omega_\delta} |x|^{-ap}(\varepsilon+|Du_\varepsilon|^2)^{(p-2)/2}Du_\varepsilon\cdot D(Au_\varepsilon - x\cdot Du_\varepsilon)\,\mathrm{d}x \\ = & -A\int_{|x|=\delta} |x|^{-ap}(\varepsilon+|Du_\varepsilon|^2)^{(p-2)/2}u_\varepsilon(Du_\varepsilon\cdot\nu)\,\mathrm{d}\sigma \\ & + \int_{\partial\Omega} |x|^{-ap}(\varepsilon+|Du_\varepsilon|^2)^{(p-2)/2}|Du_\varepsilon|^2(x\cdot\nu)\,\mathrm{d}\sigma \\ & + \int_{|x|=\delta} |x|^{-ap}(\varepsilon+|Du_\varepsilon|^2)^{(p-2)/2}|Du_\varepsilon|^2(x\cdot\nu)\,\mathrm{d}\sigma \\ & + A\int_{\Omega_\delta} |x|^{-ap}(\varepsilon+|Du_\varepsilon|^2)^{(p-2)/2}|Du_\varepsilon|^2\,\mathrm{d}x \\ & - \int_{\Omega_\delta} |x|^{-ap}(\varepsilon+|Du_\varepsilon|^2)^{(p-2)/2}Du_\varepsilon\cdot D(x\cdot Du_\varepsilon)\,\mathrm{d}x. \end{aligned} \tag{A.3}$$

Since $Du_\varepsilon \cdot D(x \cdot Du_\varepsilon) = |Du_\varepsilon|^2 + \frac{1}{2}(x \cdot D(|Du_\varepsilon|^2))$, from (A.1), it follows that

$$\begin{aligned}&\int_{\Omega_\delta} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{(p-2)/2}|Du_\varepsilon|^2 \,\mathrm{d}x \\ &\quad = \int_{\Omega_\delta} g_\varepsilon u_\varepsilon \,\mathrm{d}x + \int_{|x|=\delta} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{(p-2)/2} u_\varepsilon (Du_\varepsilon \cdot v) \,\mathrm{d}\sigma \end{aligned} \tag{A.4}$$

and

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_\delta} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{(p-2)/2}(x \cdot D(|Du_\varepsilon|^2)) \,\mathrm{d}x \\ &\quad = \frac{1}{p}\int_{\Omega_\delta} |x|^{-ap} x \cdot D((\varepsilon + |Du_\varepsilon|^2)^{p/2}) \,\mathrm{d}x \\ &\quad = \frac{1}{p}\int_{\partial\Omega} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{p/2}(x \cdot v) \,\mathrm{d}\sigma \\ &\qquad + \frac{1}{p}\int_{|x|=\delta} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{p/2}(x \cdot v) \,\mathrm{d}\sigma \\ &\qquad - \frac{1}{p}(n - ap)\int_{\Omega_\delta} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{p/2} \,\mathrm{d}x, \end{aligned} \tag{A.5}$$

where v is the unit outer normal vector. Substituting (A.4) and (A.5) into (A.3) implies that

$$\begin{aligned} LHS = &\int_{\partial\Omega} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{(p-2)/2}|Du_\varepsilon|^2 (x \cdot v) \,\mathrm{d}\sigma \\ &+ \int_{|x|=\delta} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{(p-2)/2}|Du_\varepsilon|^2 (x \cdot v) \,\mathrm{d}\sigma \\ &- \frac{1}{p}\int_{\partial\Omega} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{p/2}(x \cdot v) \,\mathrm{d}\sigma \\ &- \frac{1}{p}\int_{|x|=\delta} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{p/2}(x \cdot v) \,\mathrm{d}\sigma \\ &+ (A - 1)\int_{\Omega_\delta} g_\varepsilon u_\varepsilon \,\mathrm{d}x \\ &+ \frac{1}{p}(n - ap)\int_{\Omega_\delta} |x|^{-ap}(\varepsilon + |Du_\varepsilon|^2)^{p/2} \,\mathrm{d}x. \end{aligned} \tag{A.6}$$

On the other hand, we have

$$RHS = A\int_{\Omega_\delta} g_\varepsilon u_\varepsilon \,\mathrm{d}x - \int_{\Omega_\delta} g_\varepsilon x \cdot Du_\varepsilon \,\mathrm{d}x. \tag{A.7}$$

Letting $\varepsilon \to 0^+$, we get

$$LHS = \left(1-\frac{1}{p}\right)\int_{\partial\Omega}|x|^{-ap}|Du|^p(x\cdot\nu)\,\mathrm{d}\sigma + \left(1-\frac{1}{p}\right)\int_{|x|=\delta}|x|^{-ap}|Du|^p(x\cdot\nu)\,\mathrm{d}\sigma + (A-1)\int_{\Omega_\delta} gu\,\mathrm{d}x + \frac{1}{p}(n-ap)\int_{\Omega_\delta}|x|^{-ap}|Du|^p\,\mathrm{d}x \tag{A.8}$$

and

$$RHS = A\int_{\Omega_\delta} gu\,\mathrm{d}x - \int_{\Omega_\delta} gx\cdot Du\,\mathrm{d}x = A\int_{\Omega_\delta} gu\,\mathrm{d}x - \int_{\partial\Omega_\delta} G(x,u)(x\cdot\nu)\,\mathrm{d}\sigma + \int_{\Omega_\delta}(x\cdot G_x)\,\mathrm{d}x + n\int_{\Omega_\delta} G(x,u)\,\mathrm{d}x. \tag{A.9}$$

From (A.8) and (A.9), noting that $G(x,u)=(1/q)|x|^{-bq}|u|^q+(\lambda/p)|x|^{-p(1+a)+c}|u|^p$, it follows that

$$\left(1-\frac{1}{p}\right)\int_{\partial\Omega}|x|^{-ap}|Du|^p(x\cdot\nu)\,\mathrm{d}\sigma + \left(1-\frac{1}{p}\right)\int_{|x|=\delta}|x|^{-ap}|Du|^p(x\cdot\nu)\,\mathrm{d}\sigma + \frac{1}{p}(n-ap)\int_{\Omega_\delta}|x|^{-ap}|Du|^p\,\mathrm{d}x = \int_{\Omega_\delta} gu\,\mathrm{d}x - \frac{1}{q}\int_{|x|=\delta}|x|^{-bq}|u|^q(x\cdot\nu)\,\mathrm{d}\sigma - \frac{\lambda}{p}\int_{|x|=\delta}|x|^{-p(1+a)+c}|u|^p(x\cdot\nu)\,\mathrm{d}\sigma + \left(\frac{n}{q}-b\right)\int_{\partial\Omega}|x|^{-bq}|u|^q\,\mathrm{d}x + \lambda\frac{n-p(1+a)+c}{p} \times \int_{\partial\Omega}|x|^{-p(1+a)+c}|u|^p\,\mathrm{d}x. \tag{A.10}$$

Next, we need to get rid of the boundary integrals along $|x|=\delta$ in (A.10). In fact, let u be a solution of (1.1), from the Caffarelli–Kohn–Nirenberg inequality (1.2) or (1.4), and Theorem 2.1, we know that

$$\int_\Omega |x|^{-ap}|Du|^p\,\mathrm{d}x,\quad \int_\Omega |x|^{-bq}|u|^q\,\mathrm{d}x \quad\text{and}\quad \int_\Omega |x|^{-p(1+a)+c}|u|^p\,\mathrm{d}x$$

are finite. Therefore, by the mean-value theorem there exists a sequence $\{\delta_m\}$, $\delta_m \to 0^+$ such that integrals

$$\int_{|x|=\delta} |x|^{-ap}|Du|^p(x\cdot v)\,\mathrm{d}\sigma, \quad \int_{|x|=\delta} |x|^{-bq}|u|^q(x\cdot v)\,\mathrm{d}\sigma,$$
$$\int_{|x|=\delta} |x|^{-p(1+a)+c}|u|^p(x\cdot v)\,\mathrm{d}\sigma \to 0$$

as $m\to\infty$. Thus, letting $m\to\infty$ and noting (A.2), we obtain (3.5) from (A.10). □

Proof of Lemma 5.1. (1) It is easy to see that

$$Du_\varepsilon(x) = \begin{cases} DU_\varepsilon(x) & \text{if } |x|<R, \\ U_\varepsilon(x)D\psi(x)+\psi(x)DU_\varepsilon(x) & \text{if } R\leqslant|x|<2R \\ 0 & \text{if } |x|\geqslant 2R \end{cases}$$
$$= \begin{cases} -\dfrac{n-p-ap}{p-1}\dfrac{x}{(\varepsilon+|x|^{dp(n-p-pa)/(p-1)(n-dp)})^{n/dp}|x|^{2-(dp(n-p-ap)/(p-1)(n-dp))}} & \text{if } |x|<R, \\ U_\varepsilon(x)D\psi(x)+\psi(x)DU_\varepsilon(x) & \text{if } R\leqslant|x|<2R \\ 0 & \text{if } |x|\geqslant 2R, \end{cases}$$

$$\int_\Omega \frac{|Du_\varepsilon|^p}{|x|^{ap}}\,\mathrm{d}x = O(1)+\int_{|x|<R}\frac{|DU_\varepsilon|^p}{|x|^{ap}}\,\mathrm{d}x$$
$$= O(1)+\int_{\mathbb{R}^n}\frac{|DU_\varepsilon|^p}{|x|^{ap}}\,\mathrm{d}x$$
$$= O(1)+S_R(a,b)^{\frac{q}{q-p}}k(\varepsilon)^{-p}$$

and

$$\int_\Omega \frac{|u_\varepsilon|^q}{|x|^{bq}}\,\mathrm{d}x = O(1)+S_R(a,b)^{q/(q-p)}k(\varepsilon)^{-q}.$$

Thus, it follows that

$$\|Dv_\varepsilon\|^p_{L^p(\Omega,|x|^{-ap})} = \frac{\|Du_\varepsilon\|^p_{L^p(\Omega,|x|^{-ap})}}{\|u_\varepsilon\|^p_{L^q(\Omega,|x|^{-bq})}}$$
$$= \frac{O(1)+S_R(a,b)^{q/(q-p)}k(\varepsilon)^{-p}}{O(1)+S_R(a,b)^{p/(q-p)}k(\varepsilon)^{-p}}$$
$$= S_R(a,b)+O(k(\varepsilon)^p) = S_R(a,b)+O(\varepsilon^{(n-dp)/d}).$$

(2) A direct computation shows that

$$\begin{aligned}
&\int_\Omega \frac{|Du_\varepsilon|^\alpha}{|x|^{ap}}\,\mathrm{d}x \\
&\quad = O(1) + \int_{|x|<R} \frac{|DU_\varepsilon|^\alpha}{|x|^{ap}}\,\mathrm{d}x \\
&\quad = O(1) + \int_{|x|<R} \left(\frac{n-p-ap}{p-1}\right)^\alpha \\
&\qquad \times \frac{|x|^{\alpha-ap}}{(\varepsilon+|x|^{dp(n-p-pa)/(p-1)(n-dp)})^{\alpha n/dp}|x|^{\alpha(2-(dp(n-p-ap)/(p-1)(n-dp)))}}\,\mathrm{d}x \\
&\quad = O(1) + \omega_n \int_0^R \left(\frac{n-p-ap}{p-1}\right)^\alpha \\
&\qquad \times \frac{r^{\alpha-ap+n-1-\alpha(2-(dp(n-p-ap)/(p-1)(n-dp)))}}{(\varepsilon+r^{dp(n-p-pa)/(p-1)(n-dp)})^{\alpha n/dp}}\,\mathrm{d}r \\
&\quad \leqslant O(1) + \omega_n \left(\frac{n-p-ap}{p-1}\right)^\alpha \\
&\qquad \times \int_0^R r^{\alpha-ap+n-1-\alpha(2-(dp(n-p-ap)/(p-1)(n-dp)))-(\alpha(n-p-ap)/(p-1)(n-dp))}\,\mathrm{d}r
\end{aligned}$$

and the order of r in the integrand is

$$\begin{aligned}
&\alpha - ap + n - 1 - \alpha\left(2 - \frac{dp(n-p-ap)}{(p-1)(n-dp)}\right) - \frac{\alpha(n-p-ap)}{(p-1)(n-dp)} \\
&\quad = \frac{np - n + \alpha - \alpha n - ap^2 + ap + \alpha ap}{p-1} - 1 > -1
\end{aligned}$$

for $\alpha = 1, 2, p-2, p-1$. Thus

$$\int_\Omega \frac{|Du_\varepsilon|^\alpha}{|x|^{ap}}\,\mathrm{d}x = O(1)$$

and

$$\begin{aligned}
\|Dv_\varepsilon\|^\alpha_{L^\alpha(\Omega,|x|^{-ap})} &= \frac{\|Du_\varepsilon\|^\alpha_{L^\alpha(\Omega,|x|^{-ap})}}{\|u_\varepsilon\|^\alpha_{L^q(\Omega,|x|^{-bq})}} \\
&= \frac{O(1)}{O(1) + S_R(a,b)^{\alpha/(q-p)} k(\varepsilon)^{-\alpha}} \\
&= O(k(\varepsilon)^\alpha) = O(\varepsilon^{\alpha(n-dp)/dp}).
\end{aligned}$$

(3) If $c = (n - p - ap)/(p - 1)$, then we have

$$\begin{aligned}
&\int_\Omega |x|^{-(a+1)p+c}|u_\varepsilon|^p \,\mathrm{d}x \\
&\quad = O(1) + \int_{|x|<R} \frac{1}{(\varepsilon + |x|^{dp(n-p-pa)/(p-1)(n-dp)})^{(n-dp)/d}|x|^{(a+1)p-c}} \,\mathrm{d}x \\
&\quad = O(1) + \omega_n \int_0^R \frac{r^{n-1-(a+1)p+c}}{(\varepsilon + r^{dp(n-p-pa)/(p-1)(n-dp)})^{(n-dp)/d}} \,\mathrm{d}r \\
&\quad = O(1) + \omega_n \int_0^{R\varepsilon^{-(p-1)(n-dp)/dp(n-p-pa)}} \frac{r^{n-1-(a+1)p+c}}{(1 + r^{dp(n-p-pa)/(p-1)(n-dp)})^{(n-dp)/d}} \,\mathrm{d}r \\
&\quad \leqslant O(1) + \omega_n \int_0^{R\varepsilon^{-(p-1)(n-dp)/dp(n-p-pa)}} \frac{1}{r} \,\mathrm{d}r \\
&\quad = O(1) + O(|\log \varepsilon|).
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\|v_\varepsilon\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})} &= \frac{\|u_\varepsilon\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})}}{\|u_\varepsilon\|^p_{L^q(\Omega,|x|^{-bq})}} \\
&= \frac{O(1) + O(|\log \varepsilon|)}{O(1) + S_R(a,b)^{p/(q-p)}k(\varepsilon)^{-p}} \\
&= O(k(\varepsilon)^p |\log \varepsilon|) = O(\varepsilon^{(n-dp)/d} |\log \varepsilon|).
\end{aligned}$$

If $c > (n - p - ap)/(p - 1)$, then we have

$$\begin{aligned}
&\int_\Omega |x|^{-(a+1)p+c}|u_\varepsilon|^p \,\mathrm{d}x \\
&\quad = O(1) + \int_{|x|<R} \frac{1}{(\varepsilon + |x|^{dp(n-p-pa)/(p-1)(n-dp)})^{(n-dp)/d}|x|^{(a+1)p-c}} \,\mathrm{d}x \\
&\quad = O(1) + \omega_n \int_0^R \frac{r^{n-1-(a+1)p+c}}{(\varepsilon + r^{dp(n-p-pa)/(p-1)(n-dp)})^{(n-dp)/d}} \,\mathrm{d}r \\
&\quad \leqslant O(1) + \omega_n \int_0^R r^{n-1-(a+1)p+c-(p(n-p-ap))/p-1)} \,\mathrm{d}r \\
&\quad = O(1),
\end{aligned}$$

the last equality is due to that $n - 1 - (a + 1)p + c - p(n - p - ap)/(p - 1) > -1$ if $c > (n - p - ap)/(p - 1)$. Thus it follows that

$$\begin{aligned}
\|v_\varepsilon\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})} &= \frac{\|u_\varepsilon\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})}}{\|u_\varepsilon\|^p_{L^q(\Omega,|x|^{-bq})}} \\
&= \frac{O(1)}{O(1) + S_R(a,b)^{p/(q-p)}k(\varepsilon)^{-p}} \\
&= O(k(\varepsilon)^p) = O(\varepsilon^{(n-dp)/d}).
\end{aligned}$$

If $c<(n-p-ap)/(p-1)$, then $-(n-dp)/d+(n-(a+1)p+c)(p-1)(n-dp)/dp(n-p-ap)<0$ and $n-1-(a+1)p+c-p(n-p-ap)/(p-1)<-1$, we have

$$\begin{aligned}
&\int_\Omega |x|^{-(a+1)p+c}|u_\varepsilon|^p\,\mathrm{d}x\\
&= O(1)+\int_{|x|<R}\frac{1}{(\varepsilon+|x|^{dp(n-p-pa)/(p-1)(n-dp)})^{(n-dp)/d}|x|^{(a+1)p-c}}\,\mathrm{d}x\\
&= O(1)+\omega_n\varepsilon^{-(n-dp)/d+(n-(a+1)p+c)((p-1)(n-dp)/dp(n-p-ap))}\\
&\quad\times\int_1^\infty\frac{r^{n-1-(a+1)p+c}}{(1+r^{dp(n-p-pa)/(p-1)(n-dp)})^{(n-dp)/d}}\,\mathrm{d}r\\
&= O(\varepsilon^{-(n-dp)/d+(n-(a+1)p+c)((p-1)(n-dp)/dp(n-p-ap))})
\end{aligned}$$

and

$$\begin{aligned}
\|v_\varepsilon\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})} &= \frac{\|u_\varepsilon\|^p_{L^p(\Omega,|x|^{-(a+1)p+c})}}{\|u_\varepsilon\|^p_{L^q(\Omega,|x|^{-bq})}}\\
&= \frac{O(\varepsilon^{-(n-dp)/d+(n-(a+1)p+c)(p-1)(n-dp)/dp(n-p-ap)})}{O(1)+S_R(a,b)^{p/(q-p)}k(\varepsilon)^{-p}}\\
&= O(\varepsilon^{(p-1)(n-dp)(n-(a+1)p+c)/dp(n-p-ap)}). \qquad \square
\end{aligned}$$

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