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THE SORGENFREY TOPOLOGY IS A JOIN OF ORDERABLE TOPOLOGIES

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The Sorgenfrey half open interval topology on the real line R is shown to be the join (in the lattice of all topologies on R) of two orderable topologies. This is in contrast with a result of Lutzer [3] who showed that this topology is not orderable.

To prove the result in the title, let $X = (]0, 1[\times\{0, 1\}) \cup \{(1, 0)\}$. We construct order topologies t_1 and t_2 on X such that (X, t) is homeomorphic to the Sorgenfrey line, where $t = t_1 \vee t_2$. Let t_1 be the lexicographic order topology and t_2 the usual euclidean topology, which is orderable. Since t-neighborhoods are formed by intersecting t_i-neighborhoods, we see that, locally, t is the Sorgenfrey topology. Since X contains the point $(1, 0)$ but not $(1, 1)$, the two intervals can be fitted together to form a single interval with the desired topology.

The Sorgenfrey topology is known to be a generalized order topology (GO topology) in the sense of Čech [1, page 245]. A join of orderable topologies need not be a GO topology; in fact an example in [4] shows that it need not even be a chain net topology. It is still an open question whether or not there exists a GO topology which is not a join of orderable topologies.

For connected spaces, it is known that the concepts of orderable and GO topologies are equivalent; the following proposition shows that the notion of a join of orderable topologies is also equivalent. Furthermore, this proposition extends the uniqueness criterion for possible orderings of connected orderable spaces [1, Corollary c, page 361]. (This uniqueness criterion can be used [1, page 844] to show that certain (GO) subspaces of the real line such as $Y =]0, 1[\cup \{2\}$ are not orderable. However, it is easy to construct two orderable topologies on Y whose join is the usual topology.)

Proposition. *If (X, t) is connected and t is the join of orderable topologies t_1 and t_2, then $t = t_1 = t_2$.*

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Proof. For $i = 1, 2$, let $\leqq_i$ denote an ordering of X which induces t_i. Since $t \supset t_i$, t_i is connected. It suffices to show $\leqq_1 = \leqq_2$ or $\leqq_1 = \leqq_2^{-1}$. If not, we may assume there are distinct points x, y, and z with $x \leqq_1 y \leqq_1 z$ and $x \leqq_2 z \leqq_2 y$. Since each t_i-separation is a t-separation, we can t-separate these three distinct points in more than one way. This is a contradiction [2, Lemma 3].

Note added in proof. The dual question has also been answered: the Sorgenfrey topology is not a finite intersection of orderable topologies, although it is the intersection of infinitely many orderable topologies. These follow from more general results to appear in: *Lattice operations on metric and order topologies,* Proceedings of 1973 topology conference, Blacksburg, Virginia.

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