The Space of Contact Forms Adapted to an Open Book

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Abstract

We construct a complete set of two consecutive obstructions against homotopies of pointed S^n-families of adapted contact forms. Using these obstructions, we show that there is a manifold with an open book decomposition together with both infinitely many adapted contact forms that all induce the same Liouville form on one page but such that the underlying contact manifolds are not contactomorphic, and infinitely many non-homotopic adapted contact forms that all induce the same Liouville form on one page and such that the underlying contact manifolds are contactomorphic.

Following this, we use the neighbourhood theorem for the binding of an open book decomposition that we introduce in the construction of the obstructions to construct special generalised caps of contact manifolds. This leads us to a proof that, on closed manifold, the Reeb vector field of every contact form defining a contact structure supported by a tower of open book decompositions has a contractible orbit provided the binding in the lowest level of this tower embeds into a subcritical Stein manifold as a hypersurface of restricted contact type or is supported by an open book decomposition with trivial monodromy. Moreover, we show that the strong Weinstein conjecture holds for contact manifolds supported by an open book whose binding is planar.

Zusammenfassung

Wir konstruieren zwei Obstruktionen gegen Homotopien von S^n-Familien angepasster Kontaktformen, deren Verschwinden die Existenz einer solchen Homotopie garantiert. Mit ihrer Hilfe zeigen wir, dass eine Zerlegung einer Mannigfaltigkeit als offenes Buch gibt, zu der sowohl unendlich viele Kontaktformen angepasst sind, deren zu Grunde liegende Kontaktstrukturen nicht kontaktomorph sind, als auch unendlich viele Kontaktformen, deren zu Grunde liegende Kontaktstrukturen zwar kontaktomorph sind, die aber nicht homotop sind als angepasste Kontaktformen.

Danach benutzen wir den Umgebungssatz für die Bindung offener Bücher, welchen wir in der Konstruktion der Obstruktionen beweisen, um spezielle verallgemeinerte Kappen für Kontaktmannigfaltigkeiten zu konstruieren. Dies führt uns schließlich zu einem Beweis, dass auf geschlossenen Mannigfaltigkeiten das Reeb-Vektorfeld zu jeder Kontaktform, die eine Kontaktstruktur definiert, welche von einem Turm offener Bücher getragen ist, eine kontrahierbare geschlossene Bahn besitzt, sofern die Bindung in der untersten Ebene des Turms als Hyperfläche vom eingeschränkten Kontakttyp in eine subkritische Stein-Mannigfaltigkeit einbettet oder von einem offenen Buch getragen wird, dessen Monodromie trivial ist. Zudem zeigen wir, dass die starke Weinstein-Vermutung erfüllt ist für jede Kontaktmannigfaltigkeit, die von einem offenen Buch getragen wird dessen Bindung planar ist.

Contents

Introduction

In contact geometry, one examines odd-dimensional manifolds endowed with a maximally non-integrable hyperplane field, a *contact structure*. Such contact manifolds naturally appear as so-called hypersurfaces and boundaries of *contact type* of symplectic manifolds. These are hypersurfaces and boundaries which are transverse to a Liouville vector field. From another direction, contact manifolds also arise from *symplectic open book decompositions* of a manifold. The first result in this direction was the construction of a contact form by Thurston and Winkelnkemper [38] in dimension 3 based on an open book decomposition of the manifold. This reproved the result of Martinet [29] that every closed orientable 3 dimensional manifold carries a contact structure. The contact forms constructed by Thurston and Winkelnkemper had two special properties in relation to the open book decomposition, which now are known as the condition to be *adapted* to the open book decomposition.

In [22], Giroux presented a generalisation of the Thurston-Winkelnkemper construction to higher dimension. Now, the open book decompositions were not arbitrary anymore but the *pages* had to be Liouville manifolds and the *monodromy* had to be symplectic, conditions that are satisfied automatically in dimension 3.

Giroux did not only present this construction in [22] but also two correspondence results about the connection between contact structures and open books, one very strong one in dimension 3 and a weaker one in higher dimensions. Since then open books have become a major tool in contact topology, especially in dimension 3.

In addition to the existence of an open book decomposition supporting a given contact manifold, Giroux's higher-dimensional correspondence result provides a criterion under which two adapted contact forms are isotopic. Unfortunately, this criterion is not very precise. In the first part of this thesis, we take a closer look at the space of all contact forms adapted to a given open book decomposition with the goal to obtain a precise criterion that determines whether two adapted contact forms are homotopic as such forms. We find such a criterion in the form of a set of two consecutive obstruction with the property that two adapted contact forms are homotopic if and only if these two obstructions vanish. Then we use them to obtain two infinite families of examples of contact forms adapted to the same open book decomposition which are all non-homotopic but induce the same Liouville form on a fixed page of the open book decomposition. In one of the families, all underlying contact manifolds are contactomorphic, in the other family, they are pairwise non-contactomorphic. The latter examples, in particular, show that Giroux's result has to be read in the strictest way possible.

In addition to the obstructions for single adapted contact forms, we also construct obstructions against homotopies of pointed S^n-families of adapted contact forms. This provides an extension of Giroux's result to higher homotopy groups.

Apart from the mostly 3-dimensional applications, open books have an application to one of the most important open problems of contact geometry, the Weinstein conjecture. This conjecture has its origins in the theory of Hamiltonian dynamical systems. Because classical mechanics can be described as a Hamiltonian system on a symplectic manifold, early on the question arose whether closed orbits of the Hamilton vector field exist on special hypersurfaces in symplectic manifolds. First answers were given by Rabinowitz in [33] where he proved that such closed orbits exist on every star-shaped hypersurface in $\mathbb{R}^{2n}$. Since these are hypersurfaces of contact type, Weinstein [41] posed the question whether the statement is true for all such hypersurfaces. This question translates into the intrinsic question whether the *Reeb vector field* to every *contact form* has a closed orbit. This question is now known as the Weinstein conjecture.

By now, this conjecture has been proved in dimension 3 by Taubes [37]. Before this, the most important advance was the result of Abbas, Cieliebak, and Hofer [1] that every contact manifold *supported* by an open book decomposition with *planar* pages satisfies the Weinstein conjecture. Based on their results, they also introduced a stronger version of the conjecture: they conjectured the existence of a nullhomologous link of orbits of the Reeb vector field. This is now known as the strong Weinstein conjecture.

In the second part of this thesis, we follow Abbas *et al.* in spirit in the sense that we establish the strong Weinstein conjecture for manifold supported by suitable open books. We combine this approach with the method of Geiges and Zehmisch [19, 18] to obtain closed Reeb orbits through a study of pseudoholomorphic curves based on the existence of suitable *caps*.

In our approach, we do not use the topology of the pages of an open book to obtain orbits closed Reeb orbits as done by Abbas *et al.*, but properties of the *binding*. We prove that every contact manifold supported by an open book decomposition whose binding is planar satisfies the strong Weinstein conjecture. For contact manifolds whose binding embeds into a *subcritical Stein manifold* as a hypersurface of *restricted contact type* or whose binding is itself supported by an open book decomposition with trivial monodromy, we even show that there is a contractible orbit of the Reeb vector field. Furthermore, we show that these contractible orbits already exist if one of the two demands on the binding is satisfied for that of the lowest level of a *tower of open book decompositions* supporting the contact manifold.

This thesis is organised as follows. In the first chapter, we provide an introduction into the basic concepts of contact and symplectic geometry needed in the remainder of the thesis. Moreover, we present the basic properties of open books.

In Chapter 2, we introduce the space of contact forms adapted to an open book decomposition and present different ways to use abstract symplectic open books to construct contact manifolds.

We start in Section 2.1 with the introduction of adapted forms, *i.e.* forms satisfying the adaptedness condition but not necessarily the contact condition. Based on these, we provide the definition of adapted contact forms. Moreover, we extend an argument by Giroux [21] and prove that the space of contact adapted forms is a weak deformation retract of the space of adapted forms. Following this we examine the connection between the space of adapted contact forms and supported contact structures and prove that they are homotopy equivalent.

After this treatment of forms adapted to an open book decomposition, we present in Section 2.2 three ways to construct contact manifolds from symplectic open books. These include the generalised Thurston-Winkelnkemper construction by Giroux [22] and another construction by Giroux [30] using ideal Liouville domains. Following this, we show that the different spaces of Liouville forms involved in the constructions are all homotopy equivalent.

The heart of this thesis is Chapter 3, where we provide several neighbourhood theorems needed in the last two chapters. Most notably, we prove in Section 3.1 a neighbourhood theorem for the binding of an open book decomposition that improves upon our previous result [12, Proposition 3.1] in several ways. It provides a deformation of the entire space of contact forms adapted to an open book such that the time-1-map of this deformation is a homotopy equivalence to the space of adapted contact form with a special form in an adapted neighbourhood of the binding of our choice. Moreover, the parameters of the standard form can be chosen freely. As a first application, we use this neighbourhood theorem to provide the proof of a small part of Giroux's result in [22] that every contact manifold can be constructed by the generalised Thurston-Winkelnkemper construction applied to an abstract symplectic open book whose pages are Weinstein manifolds. Namely, we show that every contact manifold supported by an open book decomposition can be obtained via the generalised Thurston-Winkelnkemper construction from an abstract symplectic open book with the same pages.

Following this, we consider manifolds with boundary in Section 3.2. We prove that the space of diffeomorphisms with compact support in the interior is a weak deformation retract of the space of diffeomorphisms that fix the boundary pointwise. Then we prove a similar result about spaces of symplectic and Liouville forms. Namely, we show that there is a weak deformation retraction from the space of symplectic forms agreeing with the restriction of a fixed symplectic form on the tangent bundle of the boundary and inducing the same orientation to the space of symplectic forms that agree with the fixed form on a neighbourhood of the boundary. An analogous result for Liouville forms is proved, as well. Moreover, we prove that on symplectic manifolds with boundary the spaces of exact symplectic forms and of Liouville forms with the same boundary conditions are homotopy equivalent. Finally, we use these results to prove the existence of a long exact homotopy ladder diagram for the spaces of diffeomorphisms as above and their intersection with the symplectomorphisms of a fixed exact symplectic form. This ladder diagram, in particular, shows that the space of symplectomorphisms with compact support in the interior is weakly homotopy equivalent to the space of symplectomorphisms fixing the boundary pointwise.

Then, in Section 3.3, we present a slight generalisation of a well-known neighbourhood theorem for symplectic fibrations over S^1, which shows that symplectic manifolds can be glued along such a symplectic fibration, provided the holonomies of the fibrations are isotopic through symplectomorphisms.

Chapter 4 deals with our first main application of the neighbourhood theorems from the previous chapter. We construct a set of two consecutive obstructions against homotopies of pointed S^n-families of contact forms adapted to a fixed open book decomposition with the property that a homotopy exists if and only if these two obstructions vanish. We start with their definition on the spaces of adapted contact forms whose restriction to the binding is fixed. Applying the neighbourhood theorems from Section 2.1 essentially reduces the problem to a problem on a symplectic fibration. We use this to construct a long exact homotopy sequence, which can be used to define the obstructions. Next, we examine the connection to the contact forms induced on the binding. This leads to a further long exact homotopy ladder diagram. In Section 4.3, we combine this with the first long exact homotopy sequence to obtain a broken exact braid diagram. Based on this, we can extend the definition of our two obstructions to general adapted contact forms. This construction is followed by a discussion of the case that the monodromy is isotopic to the identity, which is the case in which our obstructions are most restrictive.

Following this, we connect our obstructions to the existence of certain families of symplectomorphisms and diffeomorphisms of the pages of the open book via the long exact sequence from Section 3.2. This allows us to find two infinite families of examples of non-homotopic adapted contact forms for the same open book that all induce the same Liouville form on a given page. In one of the families, the underlying contact manifolds are all contactomorphic, in the other one, they are pairwise non-contactomorphic.

Our second main application is contained in Chapter 5. There, we present a generalised version of our joint results with Geiges and Zehmisch in [12]. We prove that the strong Weinstein conjecture holds for contact manifolds supported by an open book decomposition whose binding is planar. Furthermore, we prove that there even is a contractible orbit of the Reeb vector field if the binding is at least 3-dimensional and embeds into a subcritical Stein manifold as a hypersurface of restricted contact type or is itself supported by an open book with trivial monodromy. We show that this contractible orbit even exists whenever the contact manifold is supported by something we call a tower of open book decompositions, provided that the binding in the lowest level of the tower has the properties demanded of the binding above.

We start the chapter with a presentation of basic results about pseudo-holomorphic curves we need in the remainder of the chapter. Then we introduce the concept of a generalised cap and provide a special construction that turns generalised caps of the binding of a supporting open book decomposition into generalised caps of the contact manifold. The major step of this construction is a generalisation to higher dimensions of [15, Theorem 1.1], which was the crucial step in Eliashberg's proof of the existence of symplectic caps for weakly fillable contact manifolds in dimension 3.

After this more general treatment, we construct special generalised caps for the contact manifolds for which we want to prove the existence of closed Reeb orbits. Finally, we use these caps in Section 5.4 to perform an argument using holomorphic curves as Geiges and Zehmisch did in [18] to obtain the desired Reeb orbits.

1. Preliminaries

The aim of this chapter is to provide the background in contact and symplectic topology needed in the remainder of this thesis. Readers familiar with theses topics may probably skip it, maybe except Theorem 1.1.8, which is a version of the *Gray stability theorem* (Theorem 1.1.4) that preserves contact submanifolds, or use it as a reference for notation.

1.1. Contact Forms and Contact Structures

The basic notion of contact topology is that of a *contact structure*, which is a special kind of hyperplane distribution on an odd-dimensional manifold. More precisely, it is defined as follows.

Definition 1.1.1. We say that a 1-form α on a $(2n+1)$-dimensional manifold M is a **contact form** if $\alpha \wedge (d\alpha)^n$ is a volume form. A **contact structure** ξ on M is a hyperplane distribution that is locally the kernel of a contact form. If ξ is the oriented kernel of a globally defined contact form, we say that it is a **cooriented contact structure**.

We call the pair (M, ξ) a **contact manifold** and the pair (M, α) a **strict contact manifold**. A diffeomorphism $\Psi\colon M_0 \to M_1$ between two contact manifolds (M_i, ξ_i), $i = 0, 1$, is called a **contactomorphism** if $\Psi_* \xi_0 = \xi_1$. If moreover $\xi_i = \ker \alpha_i$ for contact forms α_i and $\Psi^* \alpha_1 = \alpha_0$, we say that Ψ is a **strict contactomorphism** between the strict contact manifolds (M_0, α_0) and (M_1, α_1).

We denote the space of all contact structures on a manifold M by $\Xi(M)$ and the space of al contact forms on M by $\mathcal{A}(M)$.

In this thesis we will always assume that our contact structures are cooriented without specifically stating it.

Example 1.1.2. The basic example of a contact structure is the standard contact structure $\xi_{\mathrm{st}} = \ker(\alpha_{\mathrm{st}})$ on $\mathbb{R}^{2n+1}$, where $\alpha_{\mathrm{st}} = \sum x_i \, dy_i + dz$. Here, $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ are cartesian coordinates on $\mathbb{R}^{2n+1}$.

This example is fundamental because every point in a contact manifold has a neighbourhood that is contactomorphic to a neighbourhood of the origin in $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$; see [17, Theorem 2.5.1].

Theorem 1.1.3 (Darboux's theorem)**.** *Let α be a contact form on a $(2n+1)$-dimensional manifold and $p \in M$. Then there is a neighbourhood U of p with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ with origin p such that*

$$\alpha|_U = dz + \sum_{i=1}^{n} x_i \, dy_i.$$

This theorem shows that there can be no local invariants in contact topology. On closed contact manifolds, the situation is even better. On them, there is no difference between homotopy invariants and isotopy invariants.[1]

Theorem 1.1.4 (Gray stability)**.** *Let ξ_t, $t \in [0,1]$, be a smooth family of contact structures on a closed manifold M. Then there is an isotopy Ψ_t of M such that*

$$(\Psi_t)_* \, \xi_0 = \xi_t.$$

The modern proof of this theorem, which can be found for example in [17, Theorem 2.2.2], uses the *Moser trick*. By a slightly more cautious use of this trick we can achieve that the isotopy preserves common *contact submanifolds* of the contact manifolds (M, ξ_t).

Definition 1.1.5. A submanifold N of a contact manifold (M, ξ) is called a **contact submanifold** if $TN \cap \xi$ is a contact structure on N.

Before we prove our enhanced version of the Gray stability theorem, let us give a word of caution about contact forms. For these a statement analogous to that of the Gray stability theorem is false. This can be seen using another concept that plays an important role in contact topology, the *Reeb vector field*.

Definition 1.1.6. Let α be a contact form on a $(2n+1)$-dimensional manifold. Then the **Reeb vector field** R_α to α is the unique vector

[1] To be precise, in addition to Theorem 1.1.4 below, we also need to smoothen homotopies of contact structures. However, this is no problem as can be seen from the arguments in Appendix B.

field determined by the conditions

$$\iota_{R_\alpha} \alpha \equiv 1,$$
$$\iota_{R_\alpha} d\alpha \equiv 0.$$

Note that whenever a diffeomorphism Ψ satisfies $\Psi^* \alpha_1 = \alpha_0$ for two contact forms α_0 and α_1, it also satisfies $\Psi_* R_{\alpha_0} = R_{\alpha_1}$; see [17, Lemma 2.2.4]. In particular, the topology of the flow lines of R_{α_0} and R_{α_1} has to agree. However, in smooth families of contact forms the topology of these flow lines can change drastically as the following example illustrates.

Example 1.1.7 (See [17, Example 2.2.5]). On $S^3 \subset \mathbb{R}^4$ the family

$$\alpha_\epsilon = (x_1\, dy_1 - y_1\, dx_1) + \epsilon\, (x_2\, dy_2 - y_2\, dx_2)$$

with $\epsilon > 0$ is a family of contact forms with Reeb vector fields

$$R_\epsilon = (x_1\, \partial_{y_1} - y_1\, \partial_{x_1}) + \tfrac{1}{\epsilon}\, (x_2\, \partial_{y_2} - y_2\, \partial_{x_2})\,.$$

For $\epsilon = 1$, the flow of the Reeb vector field induces the Hopf fibration. Accordingly, all flow lines are closed. For $\epsilon \in \mathbb{R}^+ \setminus \mathbb{Q}$, however, there are only two closed flow lines.

After this small detour let us come back to our enhanced version of the Gray stability theorem.

Theorem 1.1.8. *Let ξ_t, $t \in [0,1]$, be a smooth family of contact structures on a $(2n+1)$-dimensional closed manifold M and B a closed contact submanifold for all ξ_t. Then there is an isotopy Ψ_t of M satisfying $(\Psi_t)_* \xi_0 = \xi_t$ that leaves B invariant. If moreover $\xi_t \cap TB = \xi_0 \cap TB$ for all $t \in [0,1]$, then Ψ_t may be chosen to fix B pointwise.*

Proof. As in the modern proof of the usual Gray stability theorem, we use the Moser trick: we assume that our isotopy Ψ_t exists and is generated by a time-dependent vector field X_t. Since M is closed, any such vector field can be integrated, yielding an isotopy. So we have to derive sufficient conditions for the existence of the vector field X_t.

Let α_t be any smooth family of contact forms such that $\xi_t = \ker \alpha_t$. Then we decompose the vector field X_t as

$$X_t = H_t R_t + Y_t$$

where R_t is the Reeb vector field of α_t, H_t a function and Y_t a section of ξ_t.

Now we use that the condition $(\Psi_t)_* \xi_0 = \xi_t$ is equivalent to demanding that there is a smooth family of positive functions λ_t such that $\Psi_t^* \alpha_t = \lambda_t \alpha_0$. Differentiating this with respect to the parameter t yields the following equation.

$$\begin{aligned}\dot{\lambda}_t \alpha_0 &= \tfrac{d}{dt} \Psi_t^* \alpha_t \\ &= \Psi_t^* \left(\dot{\alpha}_t + L_{X_t} \alpha_t\right) \\ &= \Psi_t^* \left(\dot{\alpha}_t + \iota_{X_t} d\alpha_t + d(\iota_{X_t} \alpha_t)\right) \\ &= \Psi_t^* (\dot{\alpha}_t + \iota_{Y_t} d\alpha_t + dH_t)\end{aligned}$$

Since every Ψ_t is a diffeomorphisms, this is equivalent to

$$\mu_t \alpha_t = \dot{\alpha}_t + dH_t + \iota_{Y_t} d\alpha_t \tag{1.1}$$

for some smooth family of real-valued functions $\mu_t = \frac{d}{dt} (\ln \lambda_t) \circ \Psi_t^{-1}$.

Equation 1.1 can be split into an equation concerning the direction of the Reeb vector field, which only fixes the function μ_t, and one on the contact structure, which can be solved for every choice of H_t thanks to the non-degeneracy of $d\alpha_t|_{\xi_t}$.

If we had chosen H_t to vanish, the arguments above would be exactly the modern proof of the Gray stability theorem as found in [17, Theorem 2.2.2]. However, we want to achieve a stronger result. So we have to find suitable functions H_t.

Since B is a contact submanifold of (M, ξ_t) for all t, the form $d\alpha_t|_{TB}$ is non-degenerate on $\ker \alpha_t|_{TB} = \xi_t \cap TB$. Hence, there is a section $\tilde{Y}_t$ of $\xi_t \cap TB$ and a function ν_t on B such that the restriction of

$$\dot{\alpha}_t + \iota_{\tilde{Y}_t} d\alpha_t - \nu_t \alpha_t =: \tilde{\alpha}_t$$

to TB vanishes. Consequently, there is a choice of H_t with $H_t|_B \equiv 0$ and $dH_t|_B = -\tilde{\alpha}_t$ such that Equation 1.1 can be solved with $\mu_t|_B = \nu_t$ and $X_t|_B = \tilde{Y}_t$. Because $\tilde{Y}_t$ is a section of TB, the isotopy generated by X_t leaves B invariant.

Now, assume that $\xi_t \cap TB = \xi_0 \cap TB$ for all $t \in [0, 1]$. Then $\alpha_t|_{TB} = \eta_t \alpha_0|_{TB}$ for some smooth family of positive functions η_t. This implies that, in the setup above, we get $\nu_t = \frac{d}{dt} \ln \eta_t$ and $X_t|_B = \tilde{Y}_t = 0$. Accordingly, the isotopy generated by X_t fixes B. □

1.2. Symplectic Forms and Almost Complex Structures

Contact topology has several close connections to symplectic topology of which we will use two in this thesis, namely the concept of *symplectic cobordisms* and that of *contact open books*. In this section we present the basics in symplectic topology needed in the remainder of this thesis. We start with the most basic concept, the concept of a *symplectic form*.

Definition 1.2.1. We say that a 2-form ω on a $2n$-dimensional manifold W is a **symplectic form** if it is closed and non-degenerate, *i.e.* if $d\omega = 0$ and ω^n is a volume form. If $\omega = d\beta$, we call β a **Liouville form** for ω.

We call the pair (W, ω) a **symplectic manifold**. A diffeomorphism $\Psi\colon W_0 \to W_1$ between two symplectic manifolds (W_i, ω_i), $i = 0, 1$, is called a **symplectomorphism** if $\Psi^*\omega_1 = \omega_0$. If $\omega_i = d\beta_i$, we say that Ψ is **exact** if moreover $\Psi^*\beta_1 = \beta_0 + dh$ for some function h on W_0.

Symplectic manifolds appear for example naturally in classical mechanics as phase spaces; see [17, Section 1.4] or [2] for a more extensive presentation.

Example 1.2.2. Let B be a smooth n-dimensional manifold. Denote by π the bundle projection of T^*B. Then there is a unique 1-form λ on the cotangent bundle T^*B satisfying $\lambda_u = u \circ \pi_*$ for all $u \in T^*B$. It is called the **canonical Liouville form** on T^*B. In local coordinates $(q_1, \ldots, q_n)$ on B and dual coordinates $(p_1, \ldots, p_n)$ in the fibres it can be written as

$$\lambda = \sum_{i=1}^{n} p_i \, dq_i .$$

From this local representation it is easy to see that λ indeed is a Liouville form.

The non-degeneracy of symplectic forms leads to the existence of two special kinds of vector fields, namely *Liouville vector fields* and *Hamilton vector fields*. Let us start with the first one.

Definition 1.2.3. Let $(W, d\beta)$ be a symplectic manifold. Then the **Liouville vector field** Y to β is the unique vector field satisfying

$$\iota_Y d\beta = \beta .$$

Remark 1.2.4. Often, Liouville vector fields Y are characterised by the condition $L_Y\omega = \omega$. This is equivalent to saying that Y is a Liouville vector field to some Liouville form for ω. To see this we apply the Cartan formula and obtain

$$L_Y\omega = \iota_Y d\omega + d(\iota_Y\omega) = d(\iota_Y\omega)\,.$$

Accordingly, $\beta = \iota_Y\omega$ is a Liouville form for ω.

Liouville vector fields are important in connection with contact topology because of the following observation.

Proposition 1.2.5. *Let (W,ω) be a symplectic manifold that contains a hypersurface M transverse to a Liouville vector field Y defined in a neighbourhood of M. Then $(M, \ker \iota_Y\omega|_{TM})$ is a contact manifold.*

Proof. Because ω^n is a volume form on W and Y is transverse to M the form

$$\iota_Y\omega^n = n\iota_Y\omega\wedge\omega^{n-1} = n\iota_Y\omega\wedge(d\iota_Y\omega)^{n-1}$$

must be a volume form on M. □

Remark 1.2.6. Because of the proposition above, a hypersurface M transverse to a Liouville vector field Y that is defined in a neighbourhood of M is called a **hypersurface of contact type**. If the Liouville vector field Y exists globally, then we say that M is of **restricted contact type**.

Indeed, every contact manifold without boundary can be realised as a hypersurface of contact type.

Example 1.2.7. Let $(M, \ker\alpha)$ be a contact manifold. Then the **symplectisation** of M is the symplectic manifold $(\mathbb{R}\times M, d(e^t\alpha))$. The coordinate vector field ∂_t is a Liouville vector field and the sections over M transverse to ∂_t are in one-to-one correspondence with the contact forms defining $\ker\alpha$.

Now we turn to the second important kind of vector fields on symplectic manifolds, the *Hamilton vector fields*.

Definition 1.2.8. Let (W,ω) be a symplectic manifold and $H\colon W\to\mathbb{R}$ a smooth function. Then the **Hamilton vector field** X_H to H is the unique vector field satisfying

$$\iota_{X_H}\omega = -dH.$$

Hamilton vector fields are widely studied in dynamics, *e.g.* on cotangent bundles they describe the dynamics of particles in classical mechanics corresponding to the energy function H; see again [17, Section 1.4] or [2].

One of the nice properties of these vector fields is that they preserve level sets of the corresponding Hamilton function H.

Proposition 1.2.9. *Let (W,ω) be a symplectic manifold and $H\colon W \to \mathbb{R}$ a smooth function. Then the local flow of the Hamilton vector field X_H preserves the level sets of H.*

Proof. The statement above is equivalent to the condition that the Lie derivative of H with respect to X_H vanishes. We have

$$L_{X_H} H = \iota_{X_H} dH = -\omega(X_H, X_H) = 0.$$

□

If one of the regular level sets of such a Hamilton is a contact type hypersurface, we have a direct connection to the Reeb vector field to the induced contact form.

Proposition 1.2.10. *Let (W,ω) be a symplectic manifold and $H\colon W \to \mathbb{R}$ a smooth function. Furthermore, let M be a regular level set of H that is of contact type. Then the restriction of the Hamilton vector field X_H to M is a non-vanishing multiple of the Reeb vector field to the induced contact form on M.*

Proof. Let Y be a Liouville vector field transverse to M and write α for the induced contact form $\iota_Y\omega|_{TM}$. Then we have

$$\iota_{X_H} d\alpha = \iota_{X_H}\omega|_{TM} = -\iota_{X_H} dH|_{TM} = 0$$

and

$$\iota_{X_H}\alpha = \omega(Y, X_H) = \iota_Y dH.$$

The latter does not vanish since M is a regular level set of H and Y transverse to M. □

By this proposition the flows of the Hamilton vector field and the Reeb vector field corresponding to the induced contact form only differ by a reparametrisation. This in combination with Example 1.2.7 inspired the formulation of the following conjecture by Weinstein [41].

Conjecture 1.2.11 (Weinstein conjecture). *On every closed contact manifold, the Reeb vector field to every contact form has a closed orbit.*

In dimension 3, this is by now a theorem due to Taubes [37]. In higher dimensions, however, it is still open. There is also a stronger version of this conjecture due to Abbas, Cieliebak, and Hofer [1].

Conjecture 1.2.12 (Strong Weinstein conjecture). *On every closed contact manifold, the orbits to the Reeb vector field to every contact form contain a nullhomologous link.*

Abbas *et al.* showed in [1] that that this conjecture holds for *planar* contact manifolds. In Chapter 5, we will describe sufficient conditions under which the conjecture holds. We will even show that, under some of these conditions, there are contractible orbits.

For the proofs in Chapter 5 we will need a further concept with connections to symplectic topology, that of *almost complex structures.*

Definition 1.2.13. An **almost complex structure** is a complex bundle structure J on the tangent bundle of an even-dimensional manifold W, *i.e.* a bundle endomorphism satisfying $J^2 = -\operatorname{id}_{TW}$. If ω is a symplectic form on W, we say that J is ω-**compatible** if $\omega(\cdot, J\cdot)$ defines a Riemannian metric.

Such an almost complex structure is a generalisation of a genuine complex structures in that it need not be integrable, *i.e.* that there need not be coordinates on W in which J is trivial. This weaker condition leads to the existence of almost complex structures on a much wider class of manifolds. In particular, for every symplectic form ω on a smooth manifold there is an ω-compatible almost complex structure.

More generally, this is also true for **symplectic bundles**, which are pairs (E, ω) of a smooth vector bundle E over a manifold B and a smooth section of the bundle $\Lambda^2 E$ that is symplectic in each fibre.

Theorem 1.2.14 (See [17, Proposition 2.4.5]). *For every symplectic bundle (E, ω), the space $\mathcal{J}(\omega)$ of ω-compatible almost complex structures on E is non-empty and contractible.*

In addition to the existence of ω-compatible complex structures, the theorem above shows that they are unique up to homotopy. Because of this, we can use ω-compatible almost complex structures to define Chern classes for symplectic vector bundles.

1.3. Symplectic Cobordisms

In differential topology, it turns out to be a fruitful idea to consider *cobordisms* between oriented manifolds. In contact topology, there is a version of this concept more adapted to contact structures, the *symplectic cobordism.* Before we explain this further, let us recall the definition of an *oriented cobordism.*

Definition 1.3.1. An **oriented cobordism** from a closed oriented manifold M_- to a closed oriented manifold M_+ is a compact oriented manifold W with boundary $\partial W \cong M_+ \sqcup \overline{M}_-$ where $\overline{M}_-$ denotes the manifold M_- with reversed orientation.

The condition on two manifolds to be **cobordant**, *i.e.* the existence of a cobordism between these two manifolds, is an equivalence relation. This is not true for the condition on two contact manifolds to be *symplectically cobordant.* However, it does only lose the symmetry property. To ensure transitivity, it is necessary to be able to glue *symplectic cobordisms* along their boundary. Therefore, a suitable boundary condition for *symplectic cobordisms* can be derived from the following neighbourhood theorem.

Lemma 1.3.2 (See [17, Lemma 5.2.4])**.** *Let M_i, $i = 0, 1$, be compact hypersurfaces of contact type in symplectic manifolds (W_i, ω_i) with corresponding Liouville vector fields Y_i. Furthermore, let there be a strict contactomorphism $\phi\colon\ (M_0, \iota_{Y_0}\omega_0) \to (M_1, \iota_{Y_1}\omega_1)$.*

Then ϕ can be extended to a symplectomorphism of neighbourhoods of M_0 and M_1 by sending the flow lines of Y_0 to those of Y_1.

In regard of the lemma above, one arrives at the following definition of a *symplectic cobordism.*

Definition 1.3.3. A **(strong) symplectic cobordism** from a strict contact manifold (M_-, α_-) to a strict contact manifold (M_+, α_+) is a pair (W, ω) where W is an oriented cobordism from M_- to M_+, both orientations induced by the contact forms, endowed with a symplectic form ω inducing the orientation of W and such that there is a Liouville vector field Y defined in a neighbourhood of ∂W satisfying $\iota_Y\omega|_{TM_\pm} = \alpha_\pm$.

We say that (W, ω) is a **(strong) symplectic cobordism** from a contact manifold (M_-, ξ_-) to a contact manifold (M_+, ξ_+) if there are contact forms $\alpha_\pm$ defining the contact structures $\xi_\pm$ such that (W, ω) is a symplectic cobordism from (M_-, α_-) to (M_+, α_+). In this case we say that (M_-, ξ_-) and (M_+, ξ_+) are **symplectically cobordant**.

(M_-, ξ_-) is called the **negative** or **concave** boundary of (W, ω) and (M_+, ξ_+) the **positive** or **convex** boundary.

Remark 1.3.4. By the orientation conditions, the Liouville vector field points inwards along the negative boundary of a symplectic cobordism and outwards along the positive boundary.

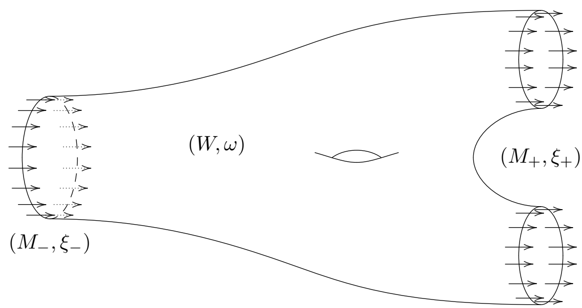

Figure 1.1.: Strong symplectic cobordism between contact 1-manifolds

The basic example of a symplectic cobordism is the following.

Example 1.3.5. Let (M, α) be a closed strict contact manifold, λ a smooth positive function on M, and C a constant such that $C\lambda > 1$. Then the subset $W = \{(t, x) \in \mathbb{R} \times M \mid 0 \leq t \leq \ln(C\lambda(x))\}$ of the symplectisation $(\mathbb{R} \times M, d(e^t\alpha))$ of (M, α) is a symplectic cobordism from (M, α) to $(M, C\lambda\alpha)$.

This example does not only show that the relation to be symplectically cobordant is reflexive but also provides the missing piece for the proof of the transitivity of this relation.

There are several stronger versions of symplectic cobordisms we will use in this thesis. We start with *exact cobordisms* and *Liouville cobordisms*.

Definition 1.3.6. A symplectic cobordism (W, ω) from a contact manifold (M_-, ξ_-) to a contact manifold (M_+, ξ_+) is called an **exact cobordism** if the symplectic form ω is exact. If, moreover, there is a primitive β of ω such that $\beta|_{TM_\pm}$ is a contact form defining $\xi_\pm$, we call (W, ω) a **Liouville cobordism**.

An even stronger form of symplectic cobordism derives from the concept of a *Weinstein manifold*, namely the *Weinstein cobordism.*

Definition 1.3.7. We say that a smooth function f is **exhausting** if it is proper and bounded from below. A vector field X is **gradient-like** for f if $L_X f > 0$ outside the critical points of f.

A **Weinstein manifold** is an exact symplectic manifold $(W, d\beta)$ with an exhausting Morse-function f that admits a gradient-like complete Liouville vector field Y.

A **Weinstein cobordism** is a Liouville cobordism (W, ω) such that the corresponding Liouville vector field Y is gradient-like for a proper Morse-function f on W for which the negative and the positive boundary of W are regular level sets.

Another special kind of symplectic cobordisms are *symplectic fillings.*

Definition 1.3.8. A **(strong) symplectic filling** of a contact manifold (M, ξ) is a symplectic cobordism from the empty set to (M, ξ). If a symplectic filling is exact, Liouville, or Weinstein, we call it an **exact**, **Liouville**, or **Weinstein filling**, respectively.

Sometimes we do not want to specify the contact manifold that is filled. Then we call a Liouville filling of its boundary a **Liouville domain** and a Weinstein filling of its boundary a **Weinstein domain**.

A somewhat different kind of filling is the *Stein filling* in the sense that it is derived from a *Stein manifold*, which itself is, *a priori*, not a symplectic manifold.

Definition 1.3.9. Let (W, J) be an almost complex manifold, *i.e.* J is an almost complex structure on the manifold W. Then we say that a real-valued function ρ on W is **plurisubharmonic** if $g_\rho = -d(d\rho \circ J)(\cdot, J\cdot)$ is a positive semi-definite symmetric linear form on W and **strictly plurisubharmonic** if it is a Riemannian metric on W.

We call the triple (W, J, ρ) a **Stein manifold** if (W, J) is a complex manifold, *i.e.* J is integrable, and ρ an exhausting strictly plurisubharmonic function on W.

If c is a regular value of such a function ρ, then we say that the subset $\rho^{-1}((-\infty, c])$ of (W, J, ρ) is a **Stein filling** of the the level set $\rho^{-1}(c)$ or, without emphasis on the contact manifold that is filled, a **Stein domain**.

A priori, it is not clear that the notion of a Stein filling is sensible, since we do not know yet that there is a natural contact structure on the regular level sets of a plurisubharmonic function. However, such a natural contact structure exists. One way to see this is to realise that Stein manifolds naturally carry the structure of Weinstein manifolds.

Proposition 1.3.10 (*Cf.* [39]). *Let (W, J, ρ) be a Stein manifold. Then (W, ω_ρ) with $\omega_\rho = -d(d\rho \circ J)$ is a Weinstein manifold and for every regular value c of ρ the symplectic manifold $\left(\rho^{-1}((-\infty, c]), \omega_\rho\right)$ is a Weinstein filling of its boundary.*

Proof. We know that $g_\rho = \omega_\rho(\cdot, J\cdot)$ is a Riemannian metric. So ω_ρ is non-degenerate. This has two consequences: first, ω_ρ is an exact symplectic form and, second, ρ only has isolated critical points. Consequently, it is a Morse function.

Now, let Y be the Liouville vector field to the primitive $-d\rho \circ J$ of ω_ρ. Then Y vanishes exactly in the critical points of ρ and we have

$$0 < g_\rho(Y, Y) = \omega_\rho(Y, JY) = -\left(d\rho \circ J\right)(JY) = \iota_Y d\rho = L_Y \rho.$$

So Y is gradient-like for ρ and, accordingly, (W, ω_ρ) a Weinstein manifold.

Moreover, if c is a regular value of ρ, then the level set $\rho^{-1}(c)$ must constitute the entire boundary of $\rho^{-1}((-\infty, c])$ because ρ is exhausting. Since the Liouville vector field Y is gradient-like for ρ and c is a regular value of ρ the vector field Y must point outwards along the boundary. So $\rho^{-1}((-\infty, c])$ is a Weinstein filling of its boundary. $\square$

Note that if (W, J, ρ) is a $2n$-dimensional Stein manifold, then ρ can only have critical points of index at most n, because g_ρ would fail to be positive definite at critical points of higher index. This puts restrictions on the topology of a Stein manifold and leads to the following distinction.

Definition 1.3.11. We say that a Stein manifold (W, J, ρ) of dimension $2n$ is **subcritical** if ρ has critical points of index at most $(n-1)$. Otherwise we call it **critical**.

Remark 1.3.12. Often a complex manifold (W, J) is called a Stein manifold if it admits an exhausting strictly plurisubharmonic function. However, from a symplectic viewpoint it is more natural to include the exhausting strictly plurisubharmonic function in the data.

As Cieliebak showed in [7], subcritical Stein manifolds are actually of a rather simple form.

Theorem 1.3.13 (*Cf.* [8, Theorem 14.16])**.** *Every subcritical Stein manifold is* deformation equivalent *to a* split *one.*

We have to clarify the two new concepts in the theorem above. We start with the second one.

Definition 1.3.14. A Stein manifold (W, J, ρ) is called **split** if (W, J) is of the form $(V \times \mathbb{C}, J \oplus i)$ and ρ of the form $\rho = \rho_V + |z|^2$ where (V, J) is a Stein manifold with an exhausting strictly plurisubharmonic function ρ_V.

This is a nice decomposition, which will play a role in Section 5.3.

Definition 1.3.15. We say that two Stein structures (J_i, ρ_i), $i = 0, 1$, on a manifold W are **Stein homotopic** if there is a homotopy J_t of complex structures on W from J_0 to J_1 together with a homotopy ρ_t of exhausting strictly plurisubharmonic functions with respect to J_t from ρ_0 to ρ_1 such that no critical points of ρ_t travel to infinity during the homotopy.

We call two Stein manifolds (W_i, J_i, ρ_i), $i = 0, 1$, **deformation equivalent** if there is a diffeomorphism $\Psi\colon W_0 \to W_1$ such that (J_0, ρ_0) and $(\Psi^* J_1, \Psi^* \rho_1)$ are Stein homotopic.

At first, this does not look too promising for our purposes; however, deformation equivalence implies symplectomorphism [14, Corollary 3.9].

Theorem 1.3.16. *If two Stein manifolds* (W_0, J_0, ρ_0) *and* (W_1, J_1, ρ_1) *are deformation equivalent, then the corresponding Weinstein manifolds* (W_0, ω_{ρ_0}) *and* (W_1, ω_{ρ_1}) *are symplectomorphic.*

1.4. Open Books

Throughout this thesis the concept of a *contact open book* plays an important role. In this section we recall some facts about the underlying topological concept of an *open book*, roughly following [17, Section 4.4.2].

Definition 1.4.1. An **abstract open book** is a pair (P, Ψ) where the **page** P is a compact manifold with boundary without closed components and the **monodromy** Ψ a diffeomorphism of P that agrees with the identity in a neighbourhood of the boundary.

We say that two monodromies are **isotopic** if they are isotopic as diffeomorphisms that agree with the identity on a neighbourhood of the boundary of the page.

Given such an abstract open book (P, Ψ), we can build the mapping torus $P(\Psi)$, which is given by the quotient space

$$P(\Psi) = P \times [0, 2\pi] /_{(x,2\pi)\sim(\Psi(x),0)}.$$

Since Ψ is trivial in a neighbourhood of the boundary, $P(\Psi)$ is a manifold with boundary $\partial P(\Psi) \cong \partial P \times S^1$. So we can glue $P(\Psi)$ along the boundary to $\partial P \times D^2$ in a natural way. We call the resulting closed manifold $M(P, \Psi)$.

Let us define $B = \partial P \times \{0\}$. Then the complement of B in $M(P, \Psi)$ can be endowed with the structure of a fibre bundle

$$P \cup (\partial P \times [0, 1)) \hookrightarrow M(P, \Psi) \setminus B \xrightarrow{\pi} S^1$$

by defining π to be given by the projection to S^1 on $P(\Psi)$ and by the angular coordinate on $D^2 \setminus \{0\}$ on $\big(\partial P \times D^2\big) \setminus B$. This yields an *open book decomposition* of $M(P, \Psi)$.

Definition 1.4.2. An **open book decomposition** of a closed manifold M is a pair (B, π) where the **binding** B is a submanifold of M of codimension 2 and $\pi\colon M \setminus B \to S^1$ a fibre bundle. Furthermore, there has to be a tubular neighbourhood $B \times D^2$ of B with polar coordinates on D^2 such that the angular coordinate agrees with π. The fibres $P_\varphi = \pi^{-1}(\varphi)$ of π are called the **pages** of the open book decomposition.

We will call a neighbourhood $B \times D^2$ of the binding as above together with the polar coordinates on D^2 an **adapted neighbourhood** of the binding.

In this thesis we will always assume that the pages of an open book decomposition are connected if not stated otherwise. In particular, they do not contain closed components.

Notation 1.4.3. We will denote by $d\varphi$ both the differential of the angular coordinate in an adapted neighbourhood of the binding and the form $\pi^* d\theta$ where $d\theta$ is the standard coordinate differential on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

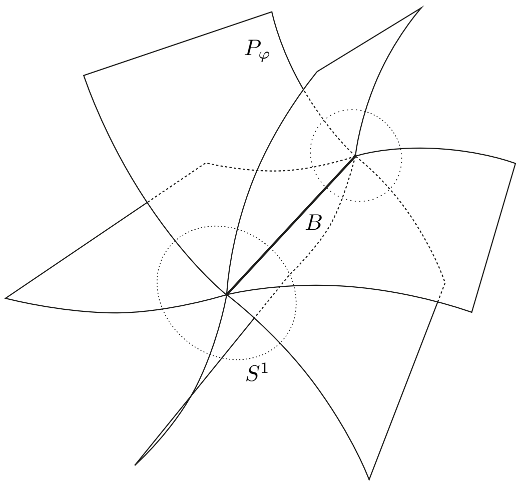

Figure 1.2.: Neighbourhood of the binding of an open book

We have seen how to obtain a closed manifold together with an open book decomposition of this manifold from an abstract open book. Now, we explain how to obtain an abstract open book from an open book decomposition (B, π) of a closed manifold M.

Take an adapted neighbourhood $U \cong B \times D^2$ of the binding and extend the coordinate vector field ∂_φ over M using an auxiliary Riemannian metric g. More precisely, choose g such that ∂_φ is orthogonal to the pages inside U and extend ∂_φ to all of M as a non-vanishing vector field orthogonal to the pages while holding it fixed on $B \times \bar{B}_{2/3}(0)$. Then the time-$2\pi$-flow of ∂_φ is a monodromy for an abstract open book for M with pages $P \cong P_\varphi \setminus \big(B \times B_{1/2}(0)\big)$.

If we start with an abstract open book (P, Ψ), construct the manifold $M(P, \Psi)$ with the corresponding open book decomposition, and then construct from this a new open book we do exactly obtain the old abstract open book (P, Ψ): we get an abstract open book with pages $P' = P \cup \partial P \times [0, 1/2]$ and a monodromy Ψ'. It is easy to see that this monodromy is isotopic to the diffeomorphism of P' obtained from Ψ by extending it over $\partial P \times [0, 1/2]$ as the identity.

On the other hand isotopic monodromies yield diffeomorphic manifolds.

Proposition 1.4.4 (See [17, Lemma 7.3.1])**.** *Let (P, Ψ_0) and (P, Ψ_1) be two abstract open books with isotopic monodromies. Then there is a diffeomorphism Φ from $M(P, \Psi_0)$ to $M(P, \Psi_1)$ that sends pages of the induced open book decomposition on $M(P, \Psi_0)$ to those on $M(P, \Psi_1)$.*

Proof. Let ψ_t be an isotopy from the identity to $\Psi_1^{-1} \circ \Psi_0$ and $\mu \colon [0, 2\pi] \to [0, 1]$ a smooth monotonously increasing function that evaluates vanishes close to 0 and is constant of value 1 close to 2π. Then we define the map

$$\begin{aligned} \Phi \colon P \times [0, 2\pi] &\to P \times [0, 2\pi] \\ (x, \varphi) &\mapsto \big(\psi_{\mu(\varphi)}(x)\,, \varphi\big)\,. \end{aligned}$$

This diffeomorphism has the property that $\Phi(x, \varphi) = (x, \varphi)$ for small φ and $\Phi\big(\Psi_0^{-1}(x)\,, \varphi\big) = \big(\Psi_1^{-1}(x)\,, \varphi\big)$ if φ is close to 2π. Consequently, it descends to a diffeomorphism $\Phi \colon P(\Psi_0) \to P(\Psi_1)$.

Since ψ_t agrees with the identity in a neighbourhood of the boundary of P we can further extend Φ over $\partial P \times D^2$ as the identity. This concludes the proof. □

1.5. Deformation Retractions

In this thesis, we construct many *deformation retractions.* Since the naming convention for the various types of these differ from author to author, we use this small section to clarify our own convention.

Definition 1.5.1. Let X be a topological space and $A \subset X$. We say that $D_t \colon X \to X$, $t \in [0, 1]$, is a **weak deformation retraction** from X into A if $D_0 = \mathrm{id}_X$, $D_1(X) \subset A$, and $D_t(A) \subset A$ for all $t \in [0, 1]$. If, moreover, $D_t|_A = \mathrm{id}_A$ for all $t \in [0, 1]$, then we say that D_t is a **strong deformation retraction** from X into A.

If there is a weak/strong deformation retraction from X into A, we call A a **weak/strong deformation retract** of X.

The basic property of a weak deformation retraction we use in this thesis is that the time-1-map is a homotopy equivalence between X and A.

Proposition 1.5.2. *Let D_t be a weak deformation retraction from a topological space X into a subspace A. Then the time-1-map D_1 is a homotopy equivalence between X and A with homotopy inverse the inclusion i_A of A into X.*

Proof. The deformation D_t is a homotopy from the identity on X to the map $D_1 = i_A \circ D_1$. Since $D_t(A) \subset A$ for all $t \in [0,1]$, the restriction of D_t to A is a homotopy from the identity on A to the map $D_1|_A = D_1 \circ i_A$. This proves that i_A is a homotopy inverse to D_1. □

2. Contact Open Books

Contact open books have come to major prominence in contact geometry since Giroux's result in [22]: every contact structure on a closed manifold is *supported* by an open book. In this chapter we explain the concept of a *contact open book*; we define what it means that a contact form is *adapted* to and a contact structure *supported* by an open book. Apart from this, we also generalise the concept of *adaptedness* to general 1-forms and lay the foundation for the chapters to come.

Following this we present three methods how to construct contact manifolds using open books and then show that, in a certain sense, they are equivalent.

2.1. Forms Adapted to an Open Book

2.1.1. Adapted Forms

Usually, in contact geometry the concept of *adaptedness* only exists for contact forms. However, as we will see throughout this thesis, in the study of *adapted* contact forms, it can be beneficial to drop the contact condition. Consequently, we make the following definition for general 1-forms.

Definition 2.1.1. Let (B, π) be an open book decomposition of a closed $(2n+1)$-dimensional manifold M. Then we say that a form $\alpha \in \Omega^1(M)$ is **adapted** to (B, π) if the following three conditions are satisfied.

- The restrictions of $d\alpha$ to the tangent bundles of the pages $P_\varphi = \pi^{-1}(\varphi)$ are symplectic forms.
- The restriction of α to TB is a contact form.
- The orientation of B induced by $\alpha|_{TB}$ agrees with the orientation as the boundary of the pages with the orientation induced by $d\alpha|_{TP_\varphi}$.

If there is a contact form α adapted to (B, π), we say that (B, π, α) is a **contact open book**.

We denote by $\Omega^1(\pi)$ the space of 1-forms adapted to (B, π) and by $\mathcal{A}(\pi)$ its subspace consisting of the adapted contact forms. Moreover, we denote by $\Omega^1(\pi, \alpha_B)$ and $\Omega^1(\pi, \xi_B)$ the subspaces of $\Omega^1(\pi)$ consisting of those forms whose restriction to TB agrees with the contact form α_B and those forms whose restriction to TB induces the contact structure ξ_B, respectively, and define $\mathcal{A}(\pi, \alpha_B)$ and $\mathcal{A}(\pi, \xi_B)$ to be the intersection of $\Omega^1(\pi, \alpha_B)$ and $\Omega^1(\pi, \xi_B)$ with $\mathcal{A}(\pi)$, respectively. As we will see in the next subsection, each of the pairs of spaces above defined by the same condition on the binding is homotopy equivalent.

Before we proceed to the next subsection and prove this, let us take a closer look at the space of adapted forms $\Omega^1(\pi)$. To do so, we first fix a manifold M and an open book decomposition (B, π) for the remainder of this subsection.

We know that for every $\alpha \in \Omega^1(\pi)$ the restriction of α to TP_φ is a Liouville form on P_φ for every $\varphi \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Moreover, we know that the orientation induced by $d\alpha|_{TP_\varphi}$ induces the same orientation on the binding $B = \partial \bar{P}_\varphi$. This is true, in particular, for the pairs of opposite pages P_φ and $P_{\varphi+\pi}$. The union of the closures of these two pages forms a single closed hypersurface of M. However, since the orientations induced on the binding agree, the orientations of P_φ and $P_{\varphi+\pi}$ do not match. This implies that the restriction of $(d\alpha)^n|_{T\bar{P}_\varphi}$ to the binding vanishes identically.

This fact motivates us to introduce the space $\mathcal{B}(\pi)$ of those 1-forms β on the closure of the page P_0 satisfying the following three conditions. The restriction of β to P_0 is a Liouville form, that to $TB = T\partial P_0$ a contact form, and the restriction of $(d\beta)^n$ to B vanishes identically. Furthermore, we introduce its subspace $\mathcal{B}(\pi, \alpha)$ where we fix the contact form α induced on ∂P_0. We call the space $\mathcal{B}(\pi)$, somewhat imprecisely, the space of **induced Liouville forms** on the page P_0.

Indeed, we can identify the pages of the open book decomposition (B, π) using the flow Ψ_t of some vector field X transverse to the pages, agreeing with ∂_φ on an adapted neighbourhood of the binding, and satisfying $d\varphi(X) = 1$. Then the restriction of an adapted form to the closure of any page is an element on $\mathcal{B}(\pi)$. However, because of smoothness issues at the binding, not every smooth path in $\mathcal{B}(\pi)$ from some β to $\Psi_{2\pi}^*\beta$ yields an adapted form, even if the contact forms induced on the binding agree.

Now, as a preparation for Section 3.1, we take a closer look at adapted forms in an adapted neighbourhood $U \cong B \times D^2$ of the binding B. Using the cartesian coordinates (x, y) on D^2 corresponding to the adapted polar coordinates (r, φ), we can decompose α as

$$\alpha = \beta + a\,dx + b\,dy.$$

Here, we adopt the convention to treat forms in such a decomposition on U as D^2-families of forms on B, *i.e.* a and b are families of functions on B, and β is a family of 1-forms on B, all parametrised over D^2. In particular, whenever we apply the exterior differential to one of the forms in such a decomposition, this is the exterior differential on B and not on $B \times D^2$.

Unfortunately, the more convenient decomposition

$$\alpha = \beta + u\,dr + v\,d\varphi$$

with respect to the polar coordinates is not well defined at the binding, unless the two functions u and v vanish there, *i.e.* unless we have $\alpha|_B = \alpha|_{TB}$ with respect to the splitting of $T\big(B \times D^2\big)$ induced by the product structure. Fortunately, we can always arrange this.

Before we engage the proof of the last statement, we have to inspect the condition a form α has to satisfy to make $d\alpha$ symplectic on the pages. Since $d\alpha$ is exact and hence closed, we only have to worry about $d\alpha$ being non-degenerate. In our coordinates the non-degeneracy of $d\alpha$ translates to

$$\begin{aligned} 0 < \tfrac{1}{n}\,(d\alpha)^n\,|_{TP_\varphi} &= (du - \beta_r) \wedge (d\beta)^{n-1} \wedge dr \\ &= (\cos\varphi\,(da - \beta_x) + \sin\varphi\,(db - \beta_y)) \wedge (d\beta)^{n-1} \wedge dr \end{aligned} \tag{2.1}$$

where subscripts denote the derivatives with respect to the corresponding parameters.

Equipped with this knowledge we are able to turn our attention again to arranging $\alpha|_B = \alpha|_{TB}$.

Proposition 2.1.2. *There is a strong deformation retraction D_t from $\Omega^1(\pi)$ into its subspace $\Omega^1_0(\pi)$ consisting of those forms additionally satisfying $\alpha|_B = \alpha|_{TB}$. Moreover, we may assume that the deformation is* *smooth in the deformation parameter and constant outside the adapted neighbourhood U of the binding.*

Proof. If $\Omega^1(\pi)$ is empty, so is its subspace $\Omega^1_0(\pi)$. So, let us assume that $\Omega^1(\pi)$ is non-empty.

Let α be a form adapted to the open book decomposition (B, π). To define the deformation retraction D we make the following ansatz.

$$D_t(\alpha) = \alpha_t \coloneqq a_t\, dx + b_t\, dy + \beta_t$$

with

$$\begin{aligned} a_t &= a - tg'(x)\, h(y)\, a|_B, \\ b_t &= b - tg'(y)\, h(x)\, b|_B, \\ \beta_t &= \beta - tg(x)\, h(y)\, da|_{TB} - tg(y)\, h(x)\, db|_{TB}, \end{aligned}$$

where $g, h\colon \mathbb{R} \to \mathbb{R}$ are smooth functions constructed as follows.

Choose a fixed cut-off function $h_0\colon \mathbb{R} \to [0,1]$ with $h_0|_{[-1/2,1/2]} \equiv 1$ and $h_0|_{\mathbb{R}\setminus[-1,1]} \equiv 0$ as in Figure 2.1 below.

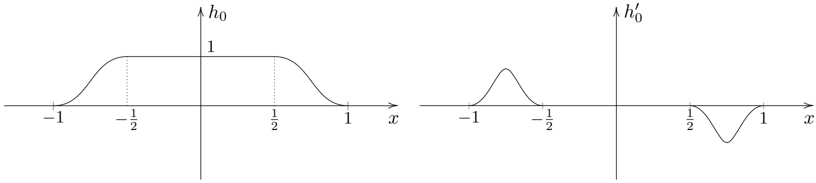

Figure 2.1.: The cut-off function and its derivative

Then set $h(x) \coloneqq \delta h_0(2x)$ and $g(x) \coloneqq \sin(x/\delta^2)\, h(x)$ for some $\delta \in \mathbb{R}^+$ that we still have to determine.

Note that by our choice of g and h we have $g'(0)\, h(0) = 1$ and $g(0)\, h(0) = 0$. This implies that $a_t|_B = (1-t)\, a|_B$, $b_t|_B = (1-t)\, b|_B$, and $\beta_t|_B \equiv \alpha|_{TB}$.

Inserting this into our ansatz, we see that

$$\begin{aligned} \big(da_t - (\beta_t)_x\big) &= (da - \beta_x) + tg(y)\, h'(x)\, db|_B, \\ \big(db_t - (\beta_t)_y\big) &= (db - \beta_y) + tg(x)\, h'(y)\, da|_B, \text{ and} \\ d\beta_t &= d\beta\,. \end{aligned}$$

Accordingly, Inequality 2.1, which has to hold in order for $d\alpha_t|_{TP_\varphi}$ to be symplectic, reads

$$0 < (d\alpha_t)^n\,|_{TP_\varphi} = (d\alpha)^n\,|_{TP_\varphi} + nt\big(\cos\varphi\; g(y)\,h'(x)\,db|_B + \sin\varphi\; g(x)\,h'(y)\,da|_B\big) \wedge (d\beta)^{n-1} \wedge dr.$$

We want to choose δ sufficiently small for this inequality to hold. To see that this is possible note that the support of the functions given by $(x,y) \mapsto g(y)\,h'(x)$ and $(x,y) \mapsto g(x)\,h'(y)$ is contained in the compact set

$$S := B \times \left([-1/2, 1/2]^2 \setminus (-1/4, 1/4)^2\right),$$

which is disjoint from $B = B \times \{0\}$, and that both functions are bounded from above by $2\delta^2\,\|h_0\|_\infty^2$.

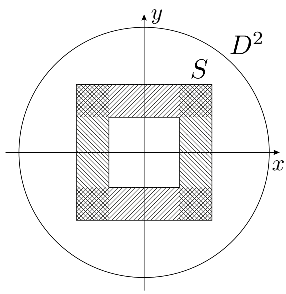

Figure 2.2.: The support of $g(x)\,h'(y)$ and $g(y)\,h'(x)$

Thus, the top-dimensional form $(d\alpha_t)^n\,|_{TP_\varphi} - (d\alpha)^n\,|_{TP_\varphi}$ vanishes in the limit $\delta \to 0$ and its support is contained in $S \cap P_\varphi$, where $(d\alpha)^n$ is bounded from below. Consequently, for a sufficiently small choice of δ, the restriction of $d\alpha_t$ to the pages is symplectic. We may for example set

$$\delta = \sqrt{\frac{\min_S \left|(d\alpha)^n\,|_{TP_\varphi}\right|}{2n\,\|h_0\|_\infty^2}} \Big(\max_S \left|da|_B \wedge (d\beta)^{n-1} \wedge dr\right| + \max_S \left|db|_B \wedge (d\beta)^{n-1} \wedge dr\right| + 1\Big)^{-1/2},$$

where we identified top-dimensional forms with functions using a fixed reference volume form. Here, $\min_S$ is the minimum in $S \cap P_\varphi$.

Since this choice of δ depends continuously on α, the map D_t is continuous for all $t \in [0,1]$. Furthermore, because the condition $\alpha|_B = \alpha|_{TB}$ is equivalent to $a|_B \equiv 0 \equiv b|_B$, the image of D_1 is contained in $\Omega^1_0(\pi)$, and whenever α already satisfies $\alpha|_B = \alpha|_{TB}$, the family α_t is constant. So D_t is a strong deformation retraction. That the deformation is constant outside U and the smoothness in the deformation parameter are clear from the definition of α_t. □

By the proposition above we know that we can always deform adapted forms α coherently such that $\alpha|_B = \alpha|_{TB}$. Then we can use the decomposition

$$\alpha = \beta + u\,dr + v\,d\varphi$$

in any adapted neighbourhood of the binding. Moreover, we know that u and v vanish identically on the binding. Together with our observation that

$$\tfrac{1}{n}\,(d\alpha)^n\,|_{TP_\varphi} = (du - \beta_r) \wedge (d\beta)^{n-1} \wedge dr$$

vanishes on the binding, as well, we see that the family of top-dimensional forms $-\beta_r \wedge (d\beta)^{n-1}$ on B does so, too. This will become important in Section 3.1.

2.1.2. From Adapted Forms to Adapted Contact Forms

In the last section we introduced the space $\Omega^1(\pi)$ of forms adapted to an open book decomposition (B,π) of a closed manifold. Though this space is more convenient than the corresponding space of contact forms $\mathcal{A}(\pi)$, we are usually interested in results about the latter one. Consequently, we have to find means to turn results about $\Omega^1(\pi)$ into results about $\mathcal{A}(\pi)$. These are provided by the following lemma, which is an easy improvement on a result of Giroux [21, Slide 39] that can also be found in [17, Proposition 4.4.9].

Theorem 2.1.3. *Let (B,π) be an open book decomposition of a $2n$-dimensional closed manifold M. Then there is a weak deformation retraction from $\Omega^1(\pi)$ into its subspace $\mathcal{A}(\pi)$. This deformation can be chosen to be smooth in the deformation parameter and to preserve the restriction of the forms to the binding and to the pages of the open book decomposition.*

Proof. First of all, if $\Omega^1(\pi)$ is empty, so is its subspace $\mathcal{A}(\pi)$. So let us assume that $\Omega^1(\pi)$ is non-empty.

The geometrical idea of the proof is to let the kernels of the adapted 1-forms approach the tangent spaces of the pages. Since the restriction of the forms to the pages are Liouville and the condition to be Liouville is open, the kernels become contact structures as soon as they are close enough to the pages. The adjustment of the kernels can be achieved by adding a further term to these forms; we describe it below.

Let $\alpha \in \Omega^1(\pi)$. For a number R and a function $f\colon M \to [0,1]$, which we construct later, we define

$$\alpha_R \coloneqq \alpha + Rf\, d\varphi.$$

For this adapted 1-form we have

$$\begin{aligned}\alpha_R \wedge (d\alpha_R)^n &= \alpha \wedge (d\alpha)^n + nRf\, d\varphi \wedge (d\alpha)^n + Rf'\, dr \wedge d\varphi \wedge \alpha \wedge (d\alpha)^{n-1}\\ &\eqqcolon \Omega_1 + R\Omega_2 + R\Omega_3\end{aligned}$$

where $'$ denotes the derivative with respect to the radial coordinate in an adapted neighbourhood $U \cong B \times D^2$ of the binding B.

We want this top-dimensional form to be positive for sufficiently large R. To achieve this, we have to find a choice of f such that α_R is well defined, and on all of M the sum of the two forms Ω_2 and Ω_3 is positive.

To construct a suitable function f, we start with a fixed smooth monotonously increasing function $f_0\colon \mathbb{R}_0^+ \to [0,1]$ that is given by $f_0(r) = r^2$ for $r \in [0, 1/3]$ and is constant of value 1 on $[2/3, \infty)$.

Since every form adapted to (B,π) restricts to a contact form on the binding B and the contact condition is an open condition on 1-forms on B, Corollary A.2 tells us that there is a function $\epsilon\colon \Omega^1(\pi) \to (0,1)$ such that for every $\alpha' \in \Omega^1(\pi)$ and $x \in B_{\epsilon(\alpha')}(0) \subset D^2$ the restriction of α' to $T\,(B \times \{x\})$ is a contact form.

We use this function ϵ to define our function f by $f((r,\varphi),b) = f_0(r/\epsilon(\alpha))$ for $((r,\varphi),b) \in B_{\epsilon(\alpha)}(0) \times B$ and extend it to the rest of M by 1.

For this choice of f, the form α_R is well defined for all $R \in \mathbb{R}$. Moreover, the form Ω_2 is positive everywhere except on the binding, where it vanishes, because α is adapted. Since the support of f' is contained inside $B_{\epsilon(\alpha)}(0) \times B$, the form Ω_3 is non-negative, too. Taking a closer look at this form for $r \leq \epsilon(\alpha)/3$, we observe that it is given by

$$\Omega_3 = \tfrac{n}{\epsilon(\alpha)^2} r\, dr \wedge d\varphi \wedge \alpha \wedge (d\alpha)^{n-1} \,,$$

which is strictly positive.

Because M is compact, the observations of the last paragraph imply that α_R is contact for sufficiently large $R \in \mathbb{R}$. More precisely, we may set

$$R(\alpha) \coloneqq 1 + \frac{\max_M |\Omega_1|}{\min\left\{\min_{M\setminus\left(B_{\epsilon(\alpha)/3}(0)\times B\right)} |\Omega_2| \,, \inf_{B_{\epsilon(\alpha)/3}(0)\times B} |\Omega_3|\right\}},$$

which continuously depends on the form α. Here, we identified top-dimensional forms with functions using some volume form as reference.

Now, notice that whenever $\alpha_R \wedge (d\alpha_R)^n > 0$, this will also be true for any larger choice of R. So setting $D_s(\alpha) \coloneqq \alpha_{sR(\alpha)}$ yields a weak deformation retraction with the desired properties. □

Remark 2.1.4. The statement of the lemma above also holds for the corresponding spaces of smooth paths α_t, $t \in [0,1]$: the sole thing that has to be changed is that in the definition of $R(\alpha_t)$, we also take the maximum over $t \in [0,1]$.

In dimension 3, Theorem 2.1.3 is nearly all we need to get a homotopy classification of the space of adapted contact forms. However, this relies on the low dimension of the pages.

Corollary 2.1.5. *If M is a 3-dimensional closed manifold with an open book decomposition (B,π), then the space $\mathcal{A}(\pi)$ of contact forms adapted to (B,π) is either empty or has 2^m contractible components where m is the number of components of the page P_0.*

Proof. Let us assume that $\mathcal{A}(\pi)$ and hence $\Omega^1(\pi)$ is non-empty and let $\alpha \in \Omega^1(\pi)$.

First of all, the number m of components of the pages P_φ does not vary with φ because they are all diffeomorphic via the flow of a vector field X satisfying $\iota_X d\varphi \equiv 1$.

In a 3-dimensional manifold, the binding B is 1-dimensional. As a result, the condition on α to induce a contact form on the binding B is the same as to induce a volume form on B. Analogously, the condition on $d\alpha$ to be symplectic on the pages is equivalent to inducing volume forms on them because they are 2-dimensional. So the components of $\Omega^1(\pi)$ are convex and hence contractible. Moreover, there are 2^m choices of the orientation of the pages so that $\Omega^1(\pi)$ has 2^m components. By Theorem 2.1.3 this implies that $\mathcal{A}(\pi)$ consists of 2^m contractible components, as well. □

As we will see in Section 2.2, the space of adapted contact forms is never empty in dimension 3.

In higher dimensions the space $\mathcal{A}(\pi)$ is more complicated. Because of this we turn our attention to the less restrictive space $\Omega^1(\pi)$ and then use Theorem 2.1.3 in the following form.

Corollary 2.1.6. *Let (B, π) be an open book decomposition of a closed manifold and D_t a deformation of $\Omega^1(\pi)$ into a subspace V that is invariant under the deformation from Theorem 2.1.3. Then there is a deformation of V into $V \cap \mathcal{A}(\pi)$. Moreover, if the time-1-map of the deformation on $\Omega^1(\pi)$ is a homotopy equivalence, so is the time-1-map of the deformation on $\mathcal{A}(\pi)$.*

Proof. Denote by $\tilde{D}_s$ the weak deformation retraction from Theorem 2.1.3. We construct the deformation of $\mathcal{A}(\pi)$ into $V \cap \mathcal{A}(\pi)$ as follows.

For given $\alpha \in \Omega^1(\pi)$, we first apply the deformation $\tilde{D}_s$ and then continue via the path $\big(\tilde{D}_1 \circ D_t\big)(\alpha)$.

Because the inclusion of $\mathcal{A}(\pi)$ into $\Omega^1(\pi)$ is a homotopy inverse of $\tilde{D}_1$ and the subspace V is invariant under $\tilde{D}_t$, the time-1-map of the deformation we have constructed is a homotopy equivalence if and only if D_1 is a homotopy equivalence. □

Remark 2.1.7. If D_t is smooth in the deformation parameter, this property can be preserved. To achieve this we have to replace $\tilde{D}_s$ by the weak deformation retraction from Remark 2.1.4 and reparametrise the two parts of the deformation such that they are constant in a neighbourhood of the connecting ends.

Two prominent examples of invariant subspaces of $\Omega^1(\pi)$ to which we will apply Corollary 2.1.6 are the spaces $\Omega^1(\pi, \alpha_B)$ and $\Omega^1(\pi, \xi_B)$ of those forms inducing the contact form α_B and the contact form ξ_B on B, respectively. Applying Corollary 2.1.6 to these spaces with the trivial deformation we get the following first result.

Corollary 2.1.8. *Let (B, π) be an open book decomposition of a closed manifold, α_B a contact form on B, and $\xi_B = \ker \alpha_B$. Then $\Omega^1(\pi, \alpha_B)$ is homotopy equivalent to $\mathcal{A}(\pi, \alpha_B)$, and $\Omega^1(\pi, \xi_B)$ to $\mathcal{A}(\pi, \xi_B)$.* □

2.1.3. Adapted Contact Forms and Supported Contact Structures

So far, we have considered adapted forms. In contact topology there is also the corresponding concept for contact structures, the *supported* contact structures.

Definition 2.1.9. We say that a contact structure ξ on a closed manifold M is **supported** by an open book decomposition (B, π) of M if it is the kernel of a contact form adapted to (B, π).

We denote by $\Xi(\pi)$ the space of all contact structures supported by (B, π) and by $\Xi(\pi, \xi_B)$ its subspace consisting of those contact structures ξ such that $\xi \cap TB = \xi_B$.

Often one is not really interested in the specific contact form but only the contact structure. Moreover, there are fundamental theorems that work for contact structures but fail for contact forms; a major example is Gray stability, including our version adapted to open books (Theorem 1.1.8). So it may be of interest to study supported contact structures instead of adapted contact forms. However, because of the very definition of a supported contact structure all our proofs work using adapted contact forms. Consequently, we need means to obtain families of adapted contact forms from families of supported contact structures. Ideally, these means would induce a homotopy equivalence between $\Xi(\pi)$ and $\mathcal{A}(\pi)$, and between $\Xi(\pi, \xi_B)$ and $\mathcal{A}(\pi, \xi_B)$. We show in this subsection that these means exist and that they have the desired property.

The strategy is to construct them from the corresponding tools that exist for ordinary contact structures and contact forms. The basic underlying observation is the following.

Lemma 2.1.10 (*Cf.* [17, Lemma 1.1.1])**.** *Let $\Omega^1_*(M)$ be the space of nowhere vanishing 1-forms on an oriented n-dimensional manifold M and $\mathrm{D}_1(M)$ the space of smooth oriented hyperplane distributions on M.*

Then the map $\ker\colon \Omega^1_*(M) \to \mathrm{D}_1(M)$ given by taking the kernel has a section s. Moreover, given a fixed $\alpha_0 \in \Omega^1_*(M)$ we can arrange that $s(\ker \alpha_0) = \alpha_0$.

Proof. If $\mathrm{D}_1(M)$ is empty, so is its preimage $\Omega^1_*(M)$ under ker. So, let us assume that $\mathrm{D}_1(M)$ is non-empty.

Choose a Riemannian metric g on M. Then the map on $\mathrm{D}_1(M)$ that associates to a hyperplane distribution ξ the positive unit orthogonal vector field η_ξ to ξ is a homeomorphism between $\mathrm{D}_1(M)$ and the space of smooth sections of the unit tangent bundle STM of M.

The unit tangent bundle STM is canonically isomorphic to the unit cotangent bundle ST^*M by dualising via the Riemannian metric g. Dualising η_ξ this way yields a smooth section of ST^*M whose kernel is precisely ξ. Since a section of ST^*M is a nowhere vanishing 1-form, this provides the desired section s_0.

Given a fixed $\alpha_0 \in \Omega^1_*(M)$ we know that the kernels of α_0 and $s_0(\ker \alpha_0)$ agree. So there is a positive function λ on M such that $s_0(\ker \alpha_0) = \lambda \alpha_0$. So our desired section is given by $s = \frac{1}{\lambda} s_0$. □

The existence of the section above immediately yields the following result about the space $\Xi(M)$ of contact structures on M and the space $\mathcal{A}(M)$ of contact forms on M.

Theorem 2.1.11. *There is a section s of the map* $\ker\colon \mathcal{A}(M) \to \Xi(M)$ *that is a homotopy equivalence. Moreover, given a contact form α_0 we may choose the section s such that $s(\ker \alpha_0) = \alpha_0$.*

Proof. If $\Xi(M)$ is empty, so is $\mathcal{A}(M)$. So, let us assume that $\Xi(M)$ is non-empty.

The section is simply the restriction of that from Lemma 2.1.10 to the subspace $\Xi(M)$ of the space of smooth oriented hyperplane distributions $\mathrm{D}_1(M)$. We only have to prove that it is a homotopy equivalence.

We already know that s is a section and hence $\ker \circ s = \mathrm{id}$. So it only remains to show that the identity on $\mathcal{A}(M)$ is homotopic to the map $s \circ \ker$.

Because two 1-forms with the same (oriented) kernel differ only by multiplication with a positive function and the space of positive functions is convex, the following map is a homotopy from the identity on $\mathcal{A}(M)$ to the map $s \circ \ker$.

$$\begin{aligned} H\colon \mathcal{A}(M) \times [0,1] &\to \mathcal{A}(M) \\ (\alpha, t) &\mapsto (1-t)\,\alpha + ts(\ker \alpha) \end{aligned}$$

□

The major problem in the adaptation of the corollary above to supported contact structures and adapted contact forms is that not every contact form defining a supported contact structure needs to adapted. Still we have the following lemma.

Lemma 2.1.12. *The fibres of the map* $\ker\colon \mathcal{A}(\pi) \to \Xi(\pi)$ *are convex.*

Proof. If $\Xi(\pi)$ is empty, the statement is void. So let us assume that it is not and let $\xi \in \Xi(\pi)$. Then, by definition, the fibre over ξ is non-empty, too.

Let $\alpha \in \ker^{-1}(\xi)$. Then every element $\alpha' \in \ker^{-1}(\xi)$ can be written as $\lambda\alpha$ for some positive function λ on M. The condition to restrict to a symplectic form on a page $P_\varphi = \pi^{-1}(\varphi)$ reads

$$0 < (\lambda\, d\alpha + d\lambda \wedge \alpha)^n \,|_{P_\varphi} = \lambda^{n-1} \left(\lambda\, d\alpha + n\, d\lambda \wedge \alpha\right) \wedge (d\alpha)^{n-1} \,|_{P_\varphi}.$$

Because λ is positive, this is equivalent to the condition

$$0 < (\lambda\, d\alpha + n\, d\lambda \wedge \alpha) \wedge (d\alpha)^{n-1} \,|_{P_\varphi},$$

which is convex in λ and hence in α'. Since the condition that B be a contact submanifold does only depend on the underlying contact structure ξ, this implies that the fibres of ker are convex. □

Thanks to the lemma above, we can adapt Theorem 2.1.11 to the space of supported contact structures.

Theorem 2.1.13. *There is a section s of the map* $\ker\colon \mathcal{A}(\pi) \to \Xi(\pi)$ *that is a homotopy equivalence. Moreover, given an adapted contact form α_0 we may choose the section s such that* $s(\ker\alpha_0) = \alpha_0$.

Proof. If $\Xi(\pi)$ is empty, so is $\mathcal{A}(\pi)$. So, let us assume that $\Xi(\pi)$ is non-empty.

Let $\alpha_0 \in \mathcal{A}(\pi)$ and s_0 be the restriction of the corresponding section from Theorem 2.1.11 to $\Xi(\pi)$. By Lemma 2.1.12 it is sufficient to modify s_0 such that its image is contained in $\mathcal{A}(\pi)$. Then the modified section is a homotopy equivalence by the same argument as in the proof of Theorem 2.1.11.

Now let $\xi \in \Xi(\pi)$. Then, by definition, there is a smooth positive function λ on M such that $\lambda s_0(\xi)$ is an adapted contact form. Moreover, we may choose $\lambda_{\ker\alpha_0}$ to be the constant function of value 1 because α_0 is adapted and $s_0(\ker\alpha_0) = \alpha_0$.

Both the conditions that the restriction to the binding B be contact and that the restrictions to the pages P_φ be Liouville forms are open. This shows that there is a small neighbourhood U_ξ of ξ in $\Xi(\pi)$ such that $\lambda_\xi s_0(\xi')$ is adapted for all $\xi' \in U_\xi$. Then the open sets $U_{\ker\alpha_0}$ and $U_\xi \setminus \{\ker\alpha\}$ for $\xi \in \Xi(\pi) \setminus \{\ker\alpha_0\}$ form an open cover $\mathcal{U}$ of $\Xi(\pi)$.

By the identification of $D_1(M)$ with the smooth sections of STM from the proof of Lemma 2.1.10, the space $\Xi(\pi)$ is homeomorphic to a subset of the smooth sections of TM. Thus, $\Xi(\pi)$ is metrizable and, hence, paracompact. Accordingly, there is a partition of unity $\{\mu_\xi\}_{\xi\in\Xi(\pi)}$ subordinate to $\mathcal{U}$. This allows us to define a map η from $\Xi(\pi)$ to the space of smooth positive functions on M by

$$\eta(\xi) = \sum_{\xi'\in\Xi(\pi)} \mu_{\xi'}(\xi)\,\lambda_{\xi'}.$$

By construction, $\eta(\xi)$ is a convex combination of finitely many smooth positive functions λ_{ξ_i}, $i = 1, \ldots, m$, such that $\lambda_{\xi_i} s_0(\xi)$ is adapted. So Theorem 2.1.3 tells us that $s(\xi) = \eta(\xi)\, s_0(\xi)$ is adapted. Moreover, all functions μ_ξ but $\mu_{\ker\alpha_0}$ vanish at $\ker\alpha_0$ so that $\eta(\ker\alpha_0) \equiv \lambda_{\ker\alpha_0} \equiv 1$.

Accordingly, $s(\xi) = \eta(\xi)\, s_0(\xi)$ defines a section with the desired properties. □

By restricting to $\Xi(\pi, \xi_B)$ we immediately get the following corollary.

Corollary 2.1.14. *Let ξ_B be a contact structure on B. Then there is a section s of the map* $\ker\colon \mathcal{A}(\pi, \xi_B) \to \Xi(\pi, \xi_B)$ *that is a homotopy equivalence. Moreover, given an $\alpha_0 \in \mathcal{A}(\pi, \xi_B)$ we may choose the section s such that $s(\ker\alpha_0) = \alpha_0$.*

2.2. Construction of Contact Open Books

In the preceding section we introduced the concept of a contact open book. Here, we present several constructions that can be used to build contact open books.

2.2.1. Generalised Thurston-Winkelnkemper Construction

In dimension 3, Thurston and Winkelnkemper described in [38] a method to construct a contact form on the manifold $M(P, \Psi)$ associated to an abstract open book (P, Ψ) with orientable page P. In fact, this construction yields for every volume form on the page a contact form adapted to the open book decomposition of $M(P, \Psi)$ associated to (P, Ψ); *cf.* [17, pages 151-154].

In higher dimensions this construction does not work anymore in its full generality because not every volume form on the page has to be induced by a symplectic form. However, Giroux showed in [22] and [21] that a similar construction is possible under more restrictive conditions. This construction is called the *generalised Thurston-Winkelnkemper construction.*

Before we describe this construction we need to introduce three concepts that appear in the construction. First, we need a symplectic version of an abstract open book.

Definition 2.2.1. We say that (P, Ψ, β) is a **symplectic open book** if P is an even-dimensional manifold with boundary, β a Liouville form on P such that the Liouville vector field to β points outwards along ∂P, and Ψ a symplectomorphism of $(P, d\beta)$ that agrees with the identity in a neighbourhood of ∂P. If moreover Ψ is an exact symplectomorphism, then we say that (P, Ψ, β) is an **exact symplectic open book**.

This provides for the symplectic structure on the mapping torus. Nevertheless, in the construction the usual mapping torus will be too restrictive. So, second, we introduce the *generalised mapping torus.*

Definition 2.2.2. Let P be a manifold with boundary, Ψ a diffeomorphism of P, and h a positive function on P that is constant in a neighbourhood of ∂P. Then the **generalised mapping torus** $P_h(\Psi)$ of Ψ with respect to h is defined as the quotient

$$P_h(\Psi) = \{(x, \varphi) \in P \times \mathbb{R} \mid 0 \leq \varphi \leq h(x)\} /_{(x,h(x)) \sim (\Psi(x),0)}.$$

Remark 2.2.3. Let C be the value of h in a neighbourhood of ∂P. Then the generalised mapping torus with respect to h is diffeomorphic to the ordinary one via the diffeomorphism

$$\begin{aligned}\Phi\colon P(\Psi) &\to P_h(\Psi)\\ (x,\varphi) &\mapsto (x,\mu(x,\varphi))\,,\end{aligned}$$

where $\mu\colon P\times[0,2\pi]\to\mathbb{R}_0^+$ is a smooth strictly monotonously increasing function such that $\mu(x,t)=Ct$ in a neighbourhood of $P\times\{0\}$ and $\partial P\times[0,2\pi]$, and $\mu(x,t)=(h(x)+C\,(t-2\pi))$ in a neighbourhood of $P\times\{2\pi\}$.

It remains to provide a contact structure on $B\times D^2$. This will essentially be the same as in the Thurston-Winkelnkemper construction in dimension 3. To be able to refer to such a contact structure easily we make the following third definition.

Definition 2.2.4. We say that a pair $\underline{h}=(h_1,h_2)$ of smooth functions $h_1\colon[0,1]\to\mathbb{R}^+$ and $h_2\colon[0,1]\to\mathbb{R}_0^+$ is a **Lutz pair** if these functions satisfy the following conditions.

- $h_1'(r)<0$ and $h_2'(r)\geq 0$ for $r>0$.
- $h_1(0)=1$ and h_2 vanishes like $r\mapsto r^2$ at $r=0$.
- $h_1^{(2n-1)}(0)=0$ for all $n\in\mathbb{N}$.

Remark 2.2.5. The condition $h_1(0)=1$ only serves as a normalisation.

Remark 2.2.6. We chose the name "Lutz pair" because such pairs of functions appear in the construction of the *Lutz twist*; *cf.* [17, Section 4.3].

Given a Lutz pair and a contact form α_B on B we can construct the following contact form on $B\times D^2$.

$$\alpha_{\underline{h},\alpha_B}=h_1(r)\,\alpha_B+h_2(r)\,d\varphi$$

Here, (r,φ) are polar coordinates on D^2. This is well defined and smooth because h_2 vanishes like $r\mapsto r^2$ and all odd derivatives of h_1 vanish at $r=0$.

The contact condition is given by

$$0\neq h_1^{n-1}\,(h_1h_2'-h_2h_1')\,\alpha_B\wedge(d\alpha_B)^{n-1}\wedge dr\wedge d\varphi,$$

which holds thanks to the properties of a Lutz pair. Moreover, the restriction of $d\alpha_{\underline{h},\alpha_B}$ to the sets $\{\varphi = \text{const}\}$ is symplectic because the corresponding condition (2.1) reads

$$0 < -h_1'\alpha_B \wedge (h_1\, d\alpha_B)^{n-1} \wedge dr = -h_1' h_1^{n-1} \alpha_B \wedge (d\alpha_B)^{n-1} \wedge dr.$$

Accordingly, the form $\alpha_{\underline{h},\alpha_B}$ has the features we need in the construction of an adapted contact form on the manifold $M(P,\Psi)$ associated to an abstract open book (P,Ψ).

Equipped with the three concepts above we are ready to describe the **generalised Thurston-Winkelnkemper construction**. We will mostly follow [17, Section 7.3].

Theorem 2.2.7 (Generalised Thurston-Winkelnkemper construction)**.** *Let (P,Ψ,β) be a symplectic open book. Then there is a contact form α on $M(P,\Psi)$ that is adapted to the open book decomposition associated to (P,Ψ). Moreover, we may assume that the restriction of $d\alpha$ to the tangent bundle of $P \subset P_\varphi$ is given by $d\beta$ with respect to the splitting of $T(M(P,\Psi))|_P$ inherited from $P(\Psi)$.*

Proof. By [17, Lemma 7.3.4] the symplectomorphism Ψ is isotopic to an exact symplectomorphism Ψ' through symplectomorphisms that agree with the identity in a neighbourhood of ∂P. Denote by Ψ_t the corresponding isotopy from Ψ to Ψ'.

Then the diffeomorphism from $M(P,\Psi)$ to $M(P,\Psi')$ we described in Proposition 1.4.4 is induced by the map

$$\begin{aligned} \Phi\colon P \times [0,2\pi] &\to P \times [0,2\pi] \\ (x,\varphi) &\mapsto \big(\psi_{\mu(\varphi)}(x)\,,\varphi\big) \end{aligned}$$

where $\mu\colon\ [0,2\pi] \to [0,2\pi]$ is a smooth monotonously increasing function that vanishes in a neighbourhood of 0 and is constant of value 2π in a neighbourhood of 2π, and $\psi_t = \Psi_t^{-1} \circ \Psi$.

Now, assume that we have a contact form α as in the assertion of the theorem but for Ψ' instead of Ψ. Then $\Phi^*\alpha$ is adapted to the open book decomposition of $M(P,\Psi)$ associated to (P,Ψ) and, moreover, the restriction of $\Phi^* d\alpha$ to $P \subset P_\varphi$ is given by

$$\psi_{\mu(\varphi)}^* d\beta = d\beta$$

because all ψ_t are symplectomorphisms of $(P, d\beta)$.

This shows that, without loss of generality, we may assume that Ψ is already an exact symplectomorphism.

Because Ψ is exact, we have

$$\left(\Psi^{-1}\right)^* \beta - \beta = dh$$

for some function h on P. Since P is compact, we may assume that h is positive.

To make the construction independent of the choice of h, we impose the condition that the minimum of h be 1, which fixes the function uniquely.

With this function h we can build the generalised mapping torus $P_h(\Psi)$ and on this the contact form $\alpha = \beta + d\varphi$. This is well defined since its canonical extension to $P \times \mathbb{R}$ is invariant under the pullback with the map $(x, \varphi) \mapsto \left(\Psi^{-1}(x), \varphi + h(x)\right)$.

Next, use the diffeomorphism Φ from Remark 2.2.3 to pull α back to the ordinary mapping torus $P(\Psi)$. Then $\Phi^*\alpha = \beta + d\mu$ for the function μ from the construction of Φ. In particular, $d(\Phi^*\alpha) = d\beta$.

Note that there is a neighbourhood U of ∂P on which the symplectomorphism Ψ agrees with the identity and for which we have $\Phi^*\alpha = \beta + C\, d\varphi$ on $U \times S^1$ where $C = (2\pi)^{-1} h|_{\partial P}$. Accordingly, we can glue $(P(\Psi), \Phi^*\alpha)$ along $\partial P \times S^1$ to $\left(\partial P \times D^2, \alpha_{(h_1, Ch_2), \alpha_B}\right)$ where $\underline{h}$ is any Lutz pair such that h_2 is constant close to $r = 1$ with value 1, and $\alpha_B = (h_1(1))^{-1} \beta|_{T\partial P}$.

This yields the desired contact form on $M(P, \Psi) = P(\Psi) \cup_{\partial P \times S^1} \left(\partial P \times D^2\right)$. □

Remark 2.2.8. Let us fix the Lutz pair $\underline{h}$ and assume that the symplectomorphism Ψ is already exact. Then the contact manifold constructed above only depends on the choice of the diffeomorphism from Remark 2.2.3.

In turn, this diffeomorphism solely depends on the function μ in its construction. The space of admissible functions μ is convex and hence contractible. Moreover, each pair of these functions agrees on a neighbourhood of $\partial P \times [0, 2\pi]$, of $P \times \{0\}$, and of $P \times \{2\pi\}$. Accordingly, a contact manifold resulting from the construction is determined up to an isotopy relative to $\partial P \times D^2$ and a neighbourhood of the page $P \times \{0\}$.

We will denote by $M(P, \Psi, \beta)$ any of these isotopic contact manifolds.

Above, we have seen a way to construct contact manifolds in higher dimensions. At first glance the contact manifolds we constructed look rather special. Nevertheless, according to Giroux [21], every contact manifold can be constructed this way.

Theorem 2.2.9 (See [21])**.** *Every contact manifold is contactomorphic to $M(P, \Psi, \beta)$ for some exact symplectic open book (P, Ψ, β) where $(P, d\beta)$ is a Weinstein domain.*

Unfortunately, so far no detailed proof of this has been published. In Subsection 3.1.4 we provide one small step of this theorem. Namely, we show that for every contact open book (B, π, α) there is a symplectic open book (P, Ψ, β) such that (B, π, α) and $M(P, \Psi, \beta)$ are contactomorphic via a contactomorphism preserving the binding.

2.2.2. Construction from Paths of Liouville Forms

There are three things about the generalised Thurston-Winkelnkemper construction that are not very satisfying. First, the monodromy has to be a symplectomorphism of the page $(P, d\beta)$, second, we have to modify the monodromy during the construction if it is not already exact, and, third, the construction performs a detour through the generalised mapping torus instead of constructing a contact form immediately on the mapping torus. In this subsection we present a simplified construction that addresses these points.

We start our construction with a Liouville domain (P, β_0). Given a suitable diffeomorphism Ψ of P that agrees with the identity in a neighbourhood of the binding we construct the contact form separately on the mapping torus $P(\Psi)$ and on $\partial P \times D^2$.

On the part $\partial P \times D^2$ we do not make any changes to the generalised Thurston-Winkelnkemper construction, *i.e.* we still choose some Lutz pair $\underline{h}$ such that h_2 is constant close to $r = 1$ with value 1 and endow $\partial P \times D^2$ with the contact form $\alpha_{(h_1, Ch_2), \alpha_B}$ with $\alpha_B = (h_1(1))^{-1} \beta_0|_{T\partial P}$ and some positive constant C we still have to determine.

Let us now denote by $\mathcal{B}_\infty(P, \beta_0)$ the space of all Liouville forms on P that agree with β_0 on ∂P including all derivatives. Suppose there is an adapted contact form on $P(\Psi)$ that restricts to β_0 on $P \times \{0\}$ and can be glued to the contact form on $\partial P \times D^2$. Then, by the nature of contact open books, there has to be a smooth path β_t in $\mathcal{B}_\infty(P, \beta_0)$ from β_0 to

$\Psi^*\beta_0$. If such a path exists, there is also always a corresponding path that, in addition, is **technical**, *i.e.* that is constant in a neighbourhood of its ends: we obtain it by reparametrising the original path. This is exactly the data we need for our construction.

Proposition 2.2.10. *Let (P, β_0) be a Liouville domain. Furthermore, let Ψ be a diffeomorphism of P that agrees with the identity in a neighbourhood of the boundary and β_t, $t \in [0, 2\pi]$, a technical smooth path in $\mathcal{B}_\infty(P, \beta_0)$ from β_0 to $\Psi^*\beta_0$.*

Then there is a contact form α on $M(P, \Psi)$ that is adapted to the open book decomposition associated to (P, Ψ). Moreover, we may assume that the restriction of α to the tangent bundle of $P \subset P_\varphi$ is given by β_φ with respect to the splitting of $T(M(P, \Psi))|_P$ inherited from $P(\Psi)$.

Proof. First, we endow the mapping torus $P(\Psi)$ with the adapted form α_P defined by $\alpha_P|_{P\times\{\varphi\}} = \beta_\varphi$, which is smooth because the path β_t is constant near its ends. Then, after choosing a Lutz pair $\underline{h}$ such that h_2 is constant close to $r = 1$ with value 1, we define the adapted form $\alpha_U = \frac{h_1(r)}{h_1(1)}(\beta|_{T\partial P})$ on U. The two forms α_P and α_U can be glued along $\partial P \times S^1$ using the Liouville vector fields on the pages. This yields an adapted form α_0 on $M(P, \Psi)$.

Since h_2 is constant close to $r = 1$ with value 1, we can extend it by 1 to a function on all of $M(P, \Psi)$. This function satisfies the conditions on the function f in the proof of Theorem 2.1.3. So this proof tells us that there is a positive number R such that $\alpha = \alpha_0 + Rh_2\, d\varphi$ is an adapted contact form. This concludes the construction. □

Remark 2.2.11. The number R in the proof of Theorem 2.1.3 is chosen in continuous dependence on the adapted form α_0. Accordingly, the construction above defines a continuous map from the space of technical paths from β_0 to $\Psi^*\beta_0$ in $\mathcal{B}_\infty(P, \beta_0)$ into the space $\mathcal{A}(\pi)$ of contact forms adapted to the open book decomposition associated to (P, Ψ).

In general, there is no path from β_0 to $\Psi^*\beta_0$ and, if there is one, there is no canonical one. However, there is one notable exception: if Ψ is a symplectomorphism of $(P, d\beta_0)$, then

$$\beta_t = (1 - t)\,\beta_0 + t\Psi^*\beta_0 = \beta_0 + t\delta,$$

$t \in [0, 1]$, is always such a path. Here, $\delta = \Psi^*\beta_0 - \beta_0$ is a closed form that vanishes in a neighbourhood of ∂P because Ψ agrees with the identity there. After a reparametrisation, this path satisfies the conditions in Proposition 2.2.10 above. This shows that whenever the generalised Thurston-Winkelnkemper construction can be applied, this alternative construction can be applied, too.

Remark 2.2.12. In general, different paths β_t from β_0 to $\Psi^*\beta_0$ yield non-isotopic adapted contact forms on $M(P, \Psi)$. We will prove in Chapter 4 that this really happens.

2.2.3. Giroux Domains

So far, we have seen two constructions of adapted contact forms on $M(P, \Psi)$ in which the form is defined separately on the mapping torus $P(\Psi)$ and $\partial P \times D^2$. This separate construction is somewhat unsatisfying.

In this section we describe a further construction by Giroux, first published in [30, Section 5], that constructs a supported contact structure on $M(P, \Psi)$ by a *contact blow-down* along the boundary of $P(\Psi)$ instead of gluing in $\partial P \times D^2$. The major concept in this construction is that of an *ideal Liouville domain*.

Definition 2.2.13. Let P be a manifold with boundary. Then we denote by $C_r^\infty(P)$ the space of all smooth functions $f\colon P \to \mathbb{R}_0^+$ with regular level set $f^{-1}(0) = \partial P$ and by $\mathrm{i}\mathcal{B}(P)$ the space of all Liouville forms on the interior of P such that, for every $f \in C_r^\infty(P)$, the form $f\beta$ extends smoothly to ∂P and, there, induces a contact structure. Furthermore, we denote by $\mathrm{i}\mathcal{B}(P, \xi)$ the subspace of $\mathrm{i}\mathcal{B}(P)$ consisting of those β such that $\ker f\beta|_{T\partial P} = \xi$ for any $f \in C_r^\infty(P)$.

A triple (P, ω, ξ) is called an **ideal Liouville domain** if $\omega = d\beta$ for some $\beta \in \mathrm{i}\mathcal{B}(P, \xi)$.

Remark 2.2.14.

1) The quotient of two functions $f, g \in C_r^\infty(P)$ is a positive smooth function on the interior of P that can be extended smoothly to ∂P with value ${}^{\partial_t f}/{}_{\partial_t g}$ where t is a collar parameter of some collar neighbourhood of ∂P.

2) Because of our first remark, a Liouville form β on the interior of P is contained in $\mathrm{i}\mathcal{B}(P)$ if and only if there is some function $f \in C_r^\infty(P)$ such that $f\beta$ extends smoothly to ∂P and there induces a contact

structure. Moreover, the induced contact structure does not depend on f.

3) In consideration of our previous remark, we can infer that the intersection of $\mathrm{i}\mathcal{B}(P,\xi)$ with the space of primitives of a fixed symplectic form ω is convex.

4) We can obtain any contact form for ξ by choosing an appropriate function $f \in C_r^\infty(P)$.

Given an ideal Liouville domain, we can construct a contact structure ξ_β on $P \times S^1$ as the kernel of $\alpha_f = f\beta + f d\varphi$ for $f \in C_r^\infty(P)$ and $\beta \in \mathrm{i}\mathcal{B}(P)$. This does not depend on the specific function $f \in C_r^\infty(P)$ since any two of these functions do only differ by multiplication with a positive function. Moreover, it satisfies the contact condition in the interior because, there, f is positive and $\frac{1}{f}\alpha_f = \beta + d\varphi$ a contact form. On the boundary, we have $f\, d\varphi = 0$. So on ∂P, seen as a subset of a collar neighbourhood $[-\epsilon, 0] \times \partial P$ with collar parameter t, we have

$$\begin{aligned}\alpha_f \wedge (d\alpha_f)^n &= (f\beta) \wedge (d(f\beta) + \partial_t f\, dt \wedge d\varphi)^n \\ &= n\partial_t f\,(f\beta) \wedge (df\beta)^{n-1} \wedge dt \wedge d\varphi.\end{aligned}$$

This is positive because of the conditions on f and β. Accordingly, ξ_β is a contact form on $P \times S^1$. We say that $P \times S^1$ with this contact structure is the **Giroux domain** associated with the ideal Liouville domain (P, ω, ξ).

Remark 2.2.15. The ideal Liouville domain only determines the isotopy class of the contact structure ξ_β. To obtain a unique contact structure we have to know the auxiliary Liouville form $\beta \in \mathrm{i}\mathcal{B}(P)$.

According to [30, Remark 5.6], a Giroux domain is a special case of another construction by Giroux, namely the *suspension* of a symplectomorphism of (P, ω) with compact support in the interior of P. Unfortunately, no reference to the construction is given. So, in this thesis we use our own definition, which reads as follows.

Definition 2.2.16. Let (P, ω, ξ) be an ideal Liouville domain, $\beta \in \mathrm{i}\mathcal{B}(P)$ a corresponding Liouville form, $f \in C_r^\infty(P)$, and Ψ a symplectomorphism of (P, ω) with compact support in the interior of P. Furthermore, let $\mu\colon [0, 2\pi] \to [0, 1]$ be a smooth monotonously increasing function that vanishes near 0 and is constant of value 1 near 2π with $\mu' \leq 1$.

Define $\delta = \left(\Psi^{-1}\right)^* \beta - \beta$ and $R = 1 - n \min\{0, \min \delta \wedge \beta \wedge (d\beta)^{n-1}\}$ with respect to the reference volume form $(d\beta)^n = \omega^n$ on the interior of P.

Then we say that the **suspension** of Ψ is the mapping torus $P(\Psi)$ endowed with the contact structure $\xi_\beta = \ker \alpha_f$ where $\alpha_f = f\beta + f\mu(\varphi)\,\delta + Rf\,d\varphi$.

Remark 2.2.17. The constant R exists since Ψ has compact support in the interior of P and hence $\delta = \left(\Psi^{-1}\right)^* \beta - \beta$, as well.

To verify the contact condition for α_f, first note that δ vanishes in the neighbourhood of ∂P where Ψ agrees with the identity. Consequently, there, α_f agrees with the form on the corresponding Giroux domain and hence is contact.

On the complement of this neighbourhood we verify the contact condition for $\alpha = \frac{1}{f}\alpha_f = \beta + \mu(\varphi)\,\delta + R\,d\varphi$. It reads

$$\begin{aligned} 0 < \alpha \wedge (d\alpha)^n &= (\beta + \mu\delta + R\,d\varphi) \wedge (d\beta + \mu'\,d\varphi \wedge \delta)^n \\ &= R\,d\varphi \wedge (d\beta)^n + n\mu'\,d\varphi \wedge \delta \wedge \beta \wedge (d\beta)^{n-1}\,. \end{aligned}$$

This inequality holds because of the definition of the constant R. Thus, we constructed a contact structure.

Moreover, if Ψ is the identity, then the result of our construction is the Giroux domain associated to (P, ω, ξ).

Note that on $\partial P \times S^1$ the contact structure ξ_β is given by $\xi \oplus TS^1$ since f vanishes there and hence $\alpha_f = f\beta + f\,d\varphi = f\beta$. Thus, $\partial P \times S^1$ is a ξ_β-*round hypersurface* modelled on $(\partial P, \xi)$.

Definition 2.2.18. An oriented hypersurface H of a contact manifold (M, ξ) is called a ξ**-round** modelled on a closed contact manifold (B, ξ_B) if ξ is transverse to H and there is an identification of H with $S^1 \times B$ such that $\xi \cap TH = TS^1 \oplus \xi_B$. If H is contained in the boundary of M, then the orientation is assumed to be the opposite of the boundary orientation.

A contact manifold with a ξ-round component boundary can be *blown down* along this boundary component. Before we can describe this procedure we need the following lemma.

Lemma 2.2.19 (See [30, Lemma 5.1]). *Let $H \cong S^1 \times B$ be a ξ-round hypersurface modelled on (B, ξ_B) in the interior (or boundary) of a contact manifold (M, ξ). Then there is a neighbourhood $(-\epsilon, \epsilon) \times H$ (or $[0, \epsilon) \times H$ respectively) of H on which $\xi = \ker(\alpha_B + t\, d\varphi)$ where α_B is a contact form for ξ_B, t the coordinate on the interval, and φ the coordinate on S^1.*

Proof. Let α be a contact form for ξ. Then the restriction of α to $H \subset M$ is a positive multiple of the restriction of $\alpha_B + t\, d\varphi$ to H in $(-1, 1) \times H$ $([0, 1) \times H)$. By a straightforward adaption of the relevant part of the proof of [17, Theorem 2.5.23] there is a contactomorphism of a neighbourhood of H in the two manifolds. Thus, there is a possibly smaller neighbourhood of H in M that is contactomorphic to $(-\epsilon, \epsilon) \times H$ $([0, \epsilon) \times H)$ for some $0 < \epsilon \leq 1$. $\square$

Given a ξ-round boundary component H modelled on (B, ξ_B) in a contact manifold (M, ξ), we can perform the following construction.

Let $[0, \epsilon) \times S^1 \times M$ be the neighbourhood from Lemma 2.2.19 and $D_{\sqrt{\epsilon}}$ the disc of radius $\sqrt{\epsilon}$ around $0 \in \mathbb{R}^2$. Then the map

$$\begin{aligned} \Psi\colon \bigl(D_{\sqrt{\epsilon}} \setminus \{0\}\bigr) \times B &\to (0, \epsilon) \times S^1 \times B \\ \bigl(re^{i\varphi}, b\bigr) &\mapsto \bigl(r^2, \varphi, b\bigr) \end{aligned}$$

is a diffeomorphism that pulls $\alpha_B + t\, d\varphi$ back to $\alpha_B + r^2\, d\varphi$. Consequently, we can glue $\bigl(D_{\sqrt{\epsilon}} \times B, \alpha_B + r^2\, d\varphi\bigr)$ to $M \setminus H$. This procedure is called the **blow-down** of (M, ξ) along H. As mentioned in [30], it is equivalent to performing a *contact cut* (see [28]) of M with respect to the (local) S^1-action.

We have already seen that the boundary of the suspension of a symplectomorphism Ψ of an ideal Liouville domain $(P, d\beta, \xi)$ is a ξ_β-round hypersurface, after reversing the orientation. Thus, we can blow down the boundary to obtain a closed contact manifold such that the underlying manifold is diffeomorphic to $M(P, \Psi) = P(\Psi) \cup \bigl(\partial B \times D^2\bigr)$ via a page preserving diffeomorphism.

So far, we have seen how to construct a contact structure ξ_β on $M(P, \Psi)$ given an ideal Liouville domain $(P, d\beta, \xi)$ and a symplectomorphism Ψ of $(P, d\beta)$ with support in the interior of P. We did not see that this contact structure is supported by the open book decomposition associated to the abstract open book (P, Ψ). The problem is that in general for $f \in C^\infty_r(P)$

and $\beta \in \mathrm{i}\mathcal{B}(P)$ the form $f\beta$ is not a Liouville form on the interior of P. However, due to an argument by Giroux [30, Lemma 5.5] there always is a special choice of $f \in C_r^\infty(P)$ such that $f\beta$ is a Liouville form on the interior of P.

By a slight modification of the proof of [30, Lemma 5.5], we first show the following.

Lemma 2.2.20. *Let P be a manifold with boundary, $\beta \in \mathrm{i}\mathcal{B}(P)$, and $f \in C_r^\infty(P)$. Denote by Y the Liouville vector field to β. Then there is a smooth extension Y_f to ∂P of the vector field $\frac{1}{f}Y$.*

Moreover, there is a continuous map $(F,\alpha)\colon \mathrm{i}\mathcal{B}(P) \to C_r^\infty(P) \times \mathcal{A}(\partial P)$ such that the negative flow of $Y_{F(\beta)}$ induces a collar neighbourhood $(-1,0] \times \partial P$ with collar coordinate t such that $-t\beta = \alpha$.

Proof. Let us denote by γ the extension of the form $f\beta$ to all of P and define the top-dimensional form

$$\mu = f^{n+1}\,(d\beta)^n = f\,(d\gamma)^n - n\,df \wedge \gamma \wedge (d\gamma)^{n-1}\,.$$

Since f is positive on the interior of P and β a Liouville form, the form μ is non-degenerate there. On the boundary it is non-degenerate, too, because $f\,(d\gamma)^n$ vanishes there and the restriction of γ to $\ker df = T\partial P$ is a contact form on ∂P. This shows that there is a unique vector field Y_f on P such that

$$\iota_{Y_f}\mu = n\gamma \wedge (d\gamma)^{n-1}\,.$$

On the interior of P we have

$$\begin{aligned} n\gamma \wedge (d\gamma)^{n-1} &= nf\,\beta \wedge (df \wedge \beta + f\,d\beta)^{n-1} = nf^n\,\beta \wedge (d\beta)^{n-1} \\ &= f^n \iota_Y\,(d\beta)^n = \tfrac{1}{f}\iota_Y\mu. \end{aligned}$$

So the vector field Y_f is a smooth extension of $\frac{1}{f}Y$ to ∂P.

Note that on the boundary

$$\iota_{Y_f}\mu = -n\iota_{Y_f}df \wedge \gamma \wedge (d\gamma)^{n-1} = -n\left(\iota_{Y_f}df\right) \wedge \gamma \wedge (d\gamma)^{n-1}$$

and hence $\iota_{Y_f}df = -1$. This implies that Y_f points outwards along the boundary.

To construct the function $F\colon \mathrm{i}\mathcal{B}(P) \to C_r^\infty(P) \times A(\partial P)$ we first fix a reference function $f \in C_r^\infty(P)$. Then every other function $h \in C_r^\infty(P)$

can be written as $h = gf$ with a positive function g on P. Moreover, we can define α to be the restriction of $f\beta$ to the tangent bundle of ∂P.

Our goal is to find a function $h \in C_r^\infty(P)$ such that

$$L_{Y_h}(h\beta) = \iota_{Y_h} d(h\beta) + d(h\iota_{Y_h} d\beta) = \iota_{Y_h}(dh \wedge \beta + h\, d\beta) = (\iota_{Y_h} dh - 1)\,\beta$$

vanishes identically. Moreover, we would like h to have a decomposition as $h = gf$ with a function g of constant value 1 on ∂P.

Given such a function h we use the negative-time-flow Ψ_t of Y_h to identify a collar neighbourhood of ∂P with $(-1, 0] \times \partial P$. Because $L_{Y_h}(h\beta) \equiv 0$ we have

$$\Psi_t^*(h\beta) - h\beta = \int_0^t \tfrac{d}{ds}\Psi_s^*(h\beta)\, ds = \int_0^t \Psi_s^* L_{Y_h}(h\beta)\, ds = 0.$$

Consequently, the form $h\beta$ is pulled back to its restriction to ∂P. Since, moreover, $\iota_{Y_h} h\beta = 0$, this restriction agrees with the restriction to $T\partial P$. Because of the decomposition $h = gf$ for a function g of constant value 1 on ∂P this restriction agrees with that of $f\beta$, *i.e.* it is given by α.

Next, we identify the function h in the collar neighbourhood. Since h vanishes on ∂P and $\iota_{Y_h} dh = -1$ we have

$$\Psi_t^* h = \int_0^t (\Psi_s^* L_{Y_h} h)\, ds = \int_0^t (\Psi_s^* \iota_{Y_h} dh)\, ds = -t.$$

Accordingly,

$$\alpha = \Psi_t^*(h\beta) = -t\Psi_t^*\beta$$

as desired.

It remains to show that the function h exists and depends continuously on β.

Let us decompose h as gf with a positive function g on P. Then we can rewrite $\iota_{Y_h} dh$ as

$$\iota_{Y_h} dh = \tfrac{1}{g}\left(f\iota_{Y_f} dg + g\iota_{Y_f} df\right).$$

Thus the condition that $\iota_{Y_h} dh = -1$ is equivalent to

$$f\iota_{Y_f} dg = -\left(\iota_{Y_f} df + 1\right) g.$$

In the construction above we have seen that $\iota_{Y_f} df = -1$ on ∂P. Hence, the function $\frac{1}{f}\left(\iota_{Y_f} df + 1\right)$ is smooth on all of P. Consequently, we can divide by f on both sides to obtain the differential equation

$$\iota_{Y_f} dg = -\frac{\left(\iota_{Y_f} df + 1\right)}{f} g,$$

which is linear in g. So, there is a unique solution with initial values $g|_{\partial P} \equiv 1$. Moreover, the solution depends continuously on Y_f and hence on β. Accordingly, we set $F(\beta) = gf$. □

Given this lemma, we are able to construct a functions $f \in C_r^\infty(P)$ such that $f\beta$ is symplectic on the interior of P. However, we even have somewhat more control.

To be able to state this more precisely, let us denote by $\mathcal{B}_f(P)$ the space of all Liouville forms on P that provide P with the structure of a Liouville domain, *i.e.* that restrict to a contact form on ∂P. Furthermore, denote by $\mathcal{B}_f(P, \alpha)$ the subspace of $\mathcal{B}_f(P)$ consisting of those Liouville forms whose restriction to $T\partial P$ is given by the contact form α on ∂P. Then we have the following.

Proposition 2.2.21. *Let (B, π) be an open book decomposition of a closed manifold and P the closure of the page P_0. Then there are continuous functions $F_\pi, F_f \colon \mathrm{i}\mathcal{B}(P) \to C_r^\infty(P)$ such that for every $\beta \in \mathrm{i}\mathcal{B}(P)$ we have $F_\pi(\beta)\, \beta \in \mathcal{B}(\pi)$ and $F_f(\beta)\, \beta \in \mathcal{B}_f(P)$.*

Moreover, if α is a contact form on ∂P, then there are continuous functions $F_\pi^\alpha, F_f^\alpha \colon \mathrm{i}\mathcal{B}(P, \ker\alpha) \to C_r^\infty(P)$ such that $F_\pi^\alpha(\beta)\, \beta \in \mathcal{B}(\pi, \alpha)$ and $F_f^\alpha(\beta)\, \beta \in \mathcal{B}_f(P, \alpha)$ for every $\beta \in \mathrm{i}\mathcal{B}(P)$.

Proof. If $\mathrm{i}\mathcal{B}(P)$ is empty, the statement is trivially satisfied. So, let us assume that $\mathrm{i}\mathcal{B}(P)$ is non-empty.

Let $\beta \in \mathrm{i}\mathcal{B}(P)$. By Lemma 2.2.20 there is a collar neighbourhood $(-1, 0] \times \partial P$ of ∂P whose coordinates depend continuously on β and in which we have

$$-t\beta = \alpha,$$

where t is the collar coordinate and α a contact form on ∂P continuously depending on β.

We construct the functions $F_\pi(\beta)$ and $F_f(\beta)$ as functions of the collar coordinate and extend them to P by a constant. So let $\mu\colon (-1, 0] \to [0, 1]$

be a smooth monotonously increasing cut-off function that vanishes on $(-1, -2/3]$ and is constant of value 1 on $[-1/3, 1]$. Next, we choose fixed smooth functions $h_\pi, h_f \colon [-1, 0] \to \mathbb{R}^+$ such that $h_i(0) = 1$ and $h_i'(t) > 0$ for $t < 0$ and $i = \pi, f$. Moreover, we demand $h_\pi'(0) = 0$ and $h_f'(0) > 0$. To be able to deal with the case with fixed induced contact form, we also fix a positive function λ on ∂P.

Given the functions above, we define functions u_π and u_f on the collar neighbourhood by

$$u_i(t, x) = (1 - \mu(t))\, C_i - t\mu(t)\, h_i(t)\, \lambda(x)$$

where $C_i = \frac{1}{3} \min h_i \min \lambda$ and $i = \pi, f$. Since these two functions are constant for $t < -1/3$ with value C_i we can extend them with this value to all of P.

The resulting functions, which we still denote by u_f and u_π, are contained in $C_r^\infty(P)$. Consequently, the forms $\hat\beta_i = u_i\beta$ extend smoothly to all of P. We claim that these two forms are Liouville forms on the interior of P.

On the complement of the collar neighbourhood the forms $\hat\beta_i$ are constant positive multiples of β and hence Liouville forms. Inside the collar neighbourhood they are given by

$$\hat\beta_i = u_i\beta = \tfrac{u_i}{-t}\alpha = \left(-\tfrac{1-\mu}{t} C_i + \mu h_i \lambda\right) \alpha =: \hat h_i \alpha.$$

Because $(1 - \mu)$ vanishes close to $t = 0$ this is well-defined and $\hat h_i$ is positive. Moreover, due to our choice of C_i, the derivative $\partial_t \hat h_i$ is positive, too:

$$\partial_t \hat h_i = (1 - \mu)\, \frac{C_i}{t^2} + \mu h_i' \lambda + \mu' \left(h_i \lambda - \frac{C_i}{-t} \right) > 0.$$

Using the inequality above, we see that

$$\left(d\hat\beta_i\right)^n = n\left(\partial_t \hat h_i\right) dt \wedge \alpha \wedge \left(d\left(\hat h_i \alpha\right)\right)^{n-1} > 0$$

on the interior.

Note that $\partial_t \hat h_\pi(0, x) = h_\pi'(0)\, \lambda(x) = 0$ and $\partial_t \hat h_f(0, x) = h_f'(0)\, \lambda(x) > 0$. Consequently, $\hat\beta_\pi \in \mathcal{B}(\pi)$ and $\hat\beta_f \in \mathcal{B}_f(P)$ for every choice of λ. So we may define $F_\pi(\beta) = u_\pi$ and $F_f(\beta) = u_f$ for the choice $\lambda \equiv 1$.

Now, let α_0 be a fixed contact form on ∂P. Again, if $\mathrm{i}\mathcal{B}(P, \ker\alpha_0)$ is empty, the statement is trivially satisfied. So let us assume that the space $\mathrm{i}\mathcal{B}(P, \ker\alpha_0)$ is non-empty and let $\beta \in \mathrm{i}\mathcal{B}(P, \ker\alpha_0)$.

Then the contact form α above can be written as $\frac{1}{\lambda_0}\alpha_0$ for some positive function λ_0 on ∂P. Since this function depends continuously on β we may choose λ in the construction above to agree with λ_0. Then we have $\hat{\beta}_i|_{\partial P} = h_i(0)\,\lambda\alpha = \alpha_0$. Hence, we may set $F_i^{\alpha_0}(\beta) = u_i$. □

Remark 2.2.22. Formally, the space $\mathcal{B}(\pi)$ is only defined for manifolds P with boundary that appear as the closure of a page of an open book decomposition. However, this is no restriction because every compact manifold with boundary can be realised this way. In particular, the closure of the pages of the open book decomposition on the manifold $M(P, \mathrm{id})$ associated to the abstract open book (P, id) are diffeomorphic to P; *cf.* Section 1.4.

2.2.4. Homotopy Equivalence of the Spaces of Liouville Forms on the Pages

In Subsection 2.1.1 we have seen that, for a given open book decomposition (B, π) of a closed manifold M, the restriction of an adapted form to the tangent bundle of the closure P of the page P_0 is always contained in the space $\mathcal{B}(\pi)$. On the other hand we have used forms in $\mathcal{B}_f(P)$ and $\mathrm{i}\mathcal{B}(P)$ to construct manifolds together with an open book decomposition and an adapted contact form in Subsection 2.2.1 and Subsection 2.2.2, and Subsection 2.2.3, respectively. Here, we show that these three spaces are homotopy equivalent.

In Proposition 2.2.21 we already constructed continuous maps F_π and F_f from $\mathrm{i}\mathcal{B}(P)$ to $C_r^\infty(P)$ such that for every $\beta \in \mathrm{i}\mathcal{B}(P)$ we have $F_\pi(\beta)\,\beta \in \mathcal{B}(\pi)$ and $F_f(\beta)\,\beta \in \mathcal{B}_f(P)$. Consequently, we can define continuous maps $\Phi_\pi \colon \mathrm{i}\mathcal{B}(P) \to \mathcal{B}(\pi)$ and $\Phi_f \colon \mathrm{i}\mathcal{B}(P) \to \mathcal{B}_f(P)$ by $\Phi_i(\beta) = F_i(\beta)\,\beta$ for $i = \pi, f$. Moreover, we have the corresponding maps $\Phi_\pi^\alpha \colon \mathrm{i}\mathcal{B}(P, \ker\alpha) \to \mathcal{B}(\pi, \alpha)$ and $\Phi_f^\alpha \colon \mathrm{i}\mathcal{B}(P, \ker\alpha) \to \mathcal{B}_f(P, \alpha)$ between the spaces with prescribed contact structure $\ker\alpha$ and contact form α on the boundary. We aim to show that all these maps are homotopy equivalences.

Theorem 2.2.23. *Let (B, π) be an open book decomposition of a closed manifold M and P the closure of the page P_0. Then the maps Φ_π and Φ_f are homotopy equivalences.*

Moreover, if α *is a contact form on* ∂P*, then the maps* Φ_π^α *and* Φ_f^α *are homotopy equivalences.*

We need to construct homotopy inverses to the maps above. More precisely, we require functions $g \in C_r^\infty(P)$ continuously depending on β such that $\frac{1}{g}\beta \in \mathrm{i}\mathcal{B}(P)$. These are provided by the following lemma.

Lemma 2.2.24. *There are continuous maps* $G_\pi \colon \mathcal{B}(\pi) \to C_r^\infty(P)$ *and* $G_f \colon \mathcal{B}_f(P) \to C_r^\infty(P)$ *such that* $G_\pi(\beta)^{-1}\,\beta \in \mathrm{i}\mathcal{B}(P)$ *for every* $\beta \in \mathcal{B}(\pi)$ *and* $G_f(\beta)^{-1}\,\beta \in \mathrm{i}\mathcal{B}(P)$ *for every* $\beta \in \mathcal{B}_f(P)$*. Moreover, if* α *is a contact form on* ∂P *and* β *an element of* $\mathcal{B}(\pi, \alpha)$ *or* $\mathcal{B}_f(P, \alpha)$*, then* $G_\pi(\beta)^{-1}\,\beta$ *or* $G_f(\beta)^{-1}\,\beta$ *is contained in* $\mathrm{i}\mathcal{B}(P, \ker\alpha)$*, respectively.*

Proof. If $\mathcal{B}(\pi)$ or $\mathcal{B}_f(P)$ is empty, the corresponding statement is trivially satisfied. So, let us assume that both $\mathcal{B}(\pi)$ and $\mathcal{B}_f(P)$ are non-empty.

Let $(-1, 0] \times \partial P$ be a fixed collar neighbourhood of ∂P.

Both $\mathcal{B}(\pi)$ and $\mathcal{B}_f(P)$ are subspaces of the space of smooth sections of a vector bundle over a compact manifold. Accordingly, they are metrizable and hence paracompact. Moreover, the restriction of every β in $\mathcal{B}(\pi)$ or $\mathcal{B}_f(P)$ to the tangent bundle of ∂P is a contact form. So, since the contact condition is open, Corollary A.2 shows that there is a continuous function $\epsilon\colon \mathcal{B}(\pi) \sqcup \mathcal{B}_f(P) \to (-1, 0)$ on the disjoint union of $\mathcal{B}(\pi)$ and $\mathcal{B}_f(P)$ such that $\beta|_{T(\{t\}\times\partial P)}$ is a contact form for every $\beta \in \mathcal{B}(\pi) \sqcup \mathcal{B}_f(P)$ and $t \geq \epsilon(\beta)$. We use this function to construct the desired functions on the collar neighbourhood.

Fix a smooth function $h\colon\ [-1, 0] \to \mathbb{R}_0^+$ such that $h \equiv 1$ on $[-1, -1/2]$, $h(0) = 0$, and $h'(t) < 0$ for $t \in (-1/2, 0]$. Then we define the function u_β on $[-\epsilon(\beta), 0]$ as

$$u_\beta(t, x) = h\left(\tfrac{t}{\epsilon(\beta)}\right).$$

Because this function is constant with value 1 in a neighbourhood of $t = \epsilon(\beta)$ we can extend it to all of P with this value.

By construction, the function u_β is an element of $C_r^\infty(P)$. We claim that on the interior of P the form $\hat{\beta} = 1/u_\beta\,\beta$ is a Liouville form. To see this we have to check whether $\hat{\beta}$ is non-degenerate. Inside the collar neighbourhood we compute

$$\begin{aligned}\left(d\hat{\beta}\right)^n &= u_\beta^{-(n-1)}\left(\tfrac{1}{u_\beta}d\beta - \tfrac{n}{u_\beta^2}du_\beta \wedge \beta\right) \wedge (d\beta)^{n-1}\\ &= u_\beta^{-n}\left(d\beta^n - \tfrac{nh'}{\epsilon(\beta)u_\beta}dt \wedge \beta \wedge (d\beta)^{n-1}\right).\end{aligned}$$

The first term is positive because β is a Liouville form on the interior of P, and the second one is non-negative since $\beta|_{T(\{t\}\times B)}$ is a contact form and $h'(t) < 0$ whenever $h'(t) \neq 0$. Accordingly, $\hat{\beta}$ is non-degenerate on the intersection of the collar with the interior of P.

On the complement of the collar neighbourhood, the function u_β has the constant value 1, *i.e.* the forms $\hat{\beta}$ and β agree. Accordingly, $\hat{\beta}$ is non-degenerate on the complement of the collar, as well. This shows that it is a Liouville form on the interior of P.

Since $u_\beta \in C_r^\infty(P)$ and $u_\beta\hat{\beta} = \beta$, we also know that $\hat{\beta} \in \mathrm{i}\mathcal{B}(P)$. More precisely, we have $\hat{\beta} \in \mathrm{i}\mathcal{B}(P, \ker\beta|_{T\partial P})$. Thus, we may set $G_\pi(\beta) = u_\beta$ for $\beta \in \mathcal{B}(\pi)$ and $G_f(\beta) = u_\beta$ for $\beta \in \mathcal{B}_f(P)$. □

With this lemma at hand, we are ready to prove Theorem 2.2.23.

Proof of Theorem 2.2.23. We know that any of the spaces $\mathrm{i}\mathcal{B}(P)$, $\mathcal{B}(\pi)$, and $\mathcal{B}_f(P)$ is empty if and only if the other two are empty, too, by Proposition 2.2.21 and Lemma 2.2.24. This proves the theorem for the case that any of them is empty. So, let us assume that none of them is. Then we may choose a contact form α on ∂P. Again, by Proposition 2.2.21 and Lemma 2.2.24 we know that any of the spaces $\mathrm{i}\mathcal{B}(P, \ker\alpha)$, $\mathcal{B}(\pi, \alpha)$, and $\mathcal{B}_f(P, \alpha)$ is empty if and only if the other two are so, as well. So, we may assume that these spaces are non-empty as well.

We claim that the maps $\Psi_\pi\colon \mathcal{B}(\pi) \to \mathrm{i}\mathcal{B}(P)$ and $\Psi_f\colon \mathcal{B}_f(P) \to \mathrm{i}\mathcal{B}(P)$ defined by $\Psi_i(\beta) = \frac{1}{G_i(\beta)}\beta$ are homotopy inverses of Φ_π and Φ_f, respectively, and that their restrictions Ψ_π^α and Ψ_f^α to $\mathcal{B}(\pi, \alpha)$ and $\mathcal{B}_f(P, \alpha)$ are homotopy inverses of Φ_π^α and Φ_f^α, respectively.

The quotient of two functions in $C_r^\infty(P)$ always extends to a unique smooth positive function on all of P. Thus, applying a composition of two of the maps above with the same indices to a form β only results in the multiplication of β with a positive function g on P. Moreover, we know that $g\beta$ still is a Liouville form on the interior of P. This is equivalent to

$$0 < (d\,(g\beta))^n = g^{n-1}\,(dg \wedge \beta - g d\beta) \wedge (d\beta)^{n-1} \tag{2.2}$$

or

$$0 < (dg \wedge \beta - gd\beta) \wedge (d\beta)^{n-1} \tag{2.3}$$

because g is a positive function.

Note that this condition is convex in g. Hence, it is also satisfied for $g_t = (1-t) + tg$, because β is a Liouville form on the interior, as well. We claim that $H(\beta, t) = g_t\beta$ defines a homotopy from the identity to our composition of choice. To see that this is true, we have to look separately at every choice of the concatenation.

We begin with $\beta \in \mathcal{B}_f(P)$. Here, we know that $g\beta$ and β are Liouville forms on all of P. Accordingly, the inequalities (2.2) and (2.3) show that this is also true for $g_t\beta$. Moreover, $g_t\beta|_{T\partial P}$ is a contact form for all $t \in [0,1]$ since g_t is positive. This shows that H defines a homotopy from the identity on $\mathcal{B}_f(P)$ to $\Phi_f \circ \Psi_f$. If moreover $\beta \in \mathcal{B}_f(P,\alpha)$, then we have $\Phi_f^\alpha \circ \Psi_f^\alpha(\beta) = g\beta$ with a positive function g whose restriction to ∂P is constant of value 1. Thus, this is also true for g_t and hence $g_t\beta|_{T\partial P} = \alpha$ for all $t \in [0,1]$. Consequently, H defines a homotopy from the identity on $\mathcal{B}_f(P,\alpha)$ to $\Phi_f^\alpha \circ \Psi_f^\alpha$.

Next, let $\beta \in \mathcal{B}(\pi)$. Then (2.2) and (2.3) with the inequality replaced by an equality show that $d\,(g_t\beta)^n$ vanishes on ∂P for all $t \in [0,1]$. Thus, H defines a homotopy from the identity on $\mathcal{B}(\pi)$ to $\Phi_\pi \circ \Psi_\pi$. By the same argument as in the last paragraph, H restricts to a homotopy from the identity on $\mathcal{B}(\pi,\alpha)$ to $\Phi_\pi^\alpha \circ \Psi_\pi^\alpha$.

Finally, let $\beta \in \mathrm{i}\mathcal{B}(P)$. Because g_t is a positive function on all of P the form $fg_t\beta$ with $f \in C_r^\infty(P)$ extends to a smooth function on all of P if and only if $f\beta$ does so. Moreover, the restriction of the corresponding extensions to the tangent bundle of ∂P only differ by a multiplication with the restriction of g_t. In particular $f\beta$ and $fg_t\beta$ induce the same contact structure on ∂P. This shows that H defines homotopies from the identity on $\mathrm{i}\mathcal{B}(P)$ to $\Psi_\pi \circ \Phi_\pi$ and $\Psi_f \circ \Phi_f$, and from the identity on $\mathrm{i}\mathcal{B}(P, \ker\alpha)$ to $\Psi_\pi^\alpha \circ \Phi_\pi^\alpha$ and $\Psi_f^\alpha \circ \Phi_f^\alpha$. $\square$

3. Neighbourhood Theorems

This chapter is the heart of this thesis, in which we provide several neighbourhood theorems. In Section 3.1 we derive a neighbourhood theorem for the binding of a contact open book that has several advantages over the one we proved in [12]: first, we may choose the neighbourhood and, second, the standardised appearance in the neighbourhood is provided uniformly by a deformation of the entire space of adapted contact forms. This is the corner stone for the further study of this space in Chapter 4. In addition, we use the neighbourhood to construct symplectic open books corresponding to given contact open books.

In Section 3.2 we provide neighbourhood theorems for diffeomorphisms and Liouville forms around the boundary of a manifold in the form of weak deformation retractions of the respective spaces. This is followed by a proof that there is a long exact homotopy sequence for exact symplectic forms and symplectomorphisms for manifolds with boundary, provided we impose suitable boundary conditions.

Finally, in Section 3.3 we provide a proof of a well-known neighbourhood theorem for symplectic fibrations over the circle contained in a symplectic manifold.

3.1. Open Books

Suppose a contact form is adapted to an open book decomposition of a closed manifold. Then, away from the binding, essentially all information about the contact form can be recovered from the Liouville forms on the pages. However, the knowledge that the form is adapted does not yield a lot of information in a neighbourhood of the binding. Since this is inconvenient in constructions involving open books, it is desirable to be able to bring adapted contact forms into a standardised form in a neighbourhood of the binding.

One such standardised form can be extracted from the generalised

Thurston-Winkelnkemper construction in Subsection 2.2.1. There, the neighbourhood $B \times D^2$ of the binding B, which is glued in, is endowed with the contact form

$$\alpha_{\underline{h},\alpha_B} = h_1(r)\,\alpha_B + h_2(r)\,d\varphi$$

where (r, φ) are polar coordinates on D^2, α_B is a contact form on B, and $\underline{h}$ a Lutz pair.

On this special form we base our definition of what it means that an adapted contact form is *standard*.

Definition 3.1.1. Let (B, π) be an open book decomposition of a closed manifold M and $U \cong B \times D^2$ an adapted neighbourhood of the binding. Furthermore, let $\underline{h}$ be a Lutz pair.

Then we say that a contact form α adapted to (B, π) is **standard** with respect to $\underline{h}$ for radius $r_0 > 0$ if

$$\alpha|_{B\times\bar{B}_{r_0}(0)} = h_1({}^r\!/_{r_0})\,\alpha|_{TB} + h_2({}^r\!/_{r_0})\,d\varphi.$$

If a form is said to be standard not stating h_1, h_2, or r_0, then the corresponding data is assumed to be arbitrary.

The space of all contact forms adapted to (B, π) that are standard with respect to a fixed Lutz pair $\underline{h}$ for radius $1/2$ we denote by $\mathcal{A}_{\underline{h}}(\pi)$. We denote by $\mathcal{A}_{\underline{h}}(\pi, \xi_B)$ and $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ its subspaces where we fix the contact structure ξ_B or the contact form α_B induced on the binding, respectively.

Remark 3.1.2. Whether a contact form is standard depends on the choice of the adapted neighbourhood U.

In [12] we have proved the folklore theorem that every adapted contact form is isotopic through adapted contact forms to *some* contact form that is standard for *some* Lutz pair for *some* radius in *some* adapted neighbourhood of the binding. Though this is sufficient for the applications in [12], it is unsatisfying in several regards: there is no control over the adapted neighbourhood or the Lutz pair and, moreover, the construction is not continuous in the adapted contact form. Consequently, this theorem is not suitable for a homotopy classification of the space of adapted contact forms.

The major goal of this section is to tackle these shortcomings of this neighbourhood theorem. Namely, we prove the following new neighbourhood theorem.

Theorem 3.1.3. *Let (B, π) be an open book decomposition of a closed manifold M and U an adapted neighbourhood of the binding. Furthermore, let $\underline{h}$ be a Lutz pair. Then there is a deformation D_t, $t \in [0, 1]$, of the space $\mathcal{A}(\pi)$ of contact forms adapted to (B, π) into its subspace $\mathcal{A}_{\underline{h}}(\pi)$ such that D_1 is a homotopy equivalence. Moreover, we may assume that the deformation is smooth in the deformation parameter t and that outside U the restrictions of $D_t(\alpha)$ and α to the tangent bundles of the pages agree for all $t \in [0, 1]$.*

Remark 3.1.4. Because of the smoothness in the deformation parameter, we obtain isotopies of the underlying supported contact structures by Gray stability.

The proof of Theorem 3.1.3 spans over the remainder of this section. In Subsection 3.1.1, we prove an analogous result for adapted contact forms. Then, in Subsection 3.1.3, we use Corollary 2.1.6 to turn this into a version of Theorem 3.1.3 in which the function h_2 is allowed to vary. Finally, in Lemma 3.1.20, we fix the function h_2.

3.1.1. Adapted Forms

In regard of Theorem 2.1.3 it is sensible to first prove a version of Theorem 3.1.3 for the larger space $\Omega^1(\pi)$ and then transform the corresponding deformation into the first part of that in Theorem 3.1.3.

Before we state the version of Theorem 3.1.3 for $\Omega^1(\pi)$ we have to define for general adapted forms what it means to be *standard*. In principle, we could copy the definition for contact forms. However, for notational reasons it is more convenient to define it as follows.

Definition 3.1.5. Let (B, π) be an open book decomposition of a closed manifold M and $U \cong B \times D^2$ an adapted neighbourhood of the binding. Furthermore, let $\underline{h}$ be a Lutz pair.

Then we say that a general 1-form α adapted to (B, π) is **standard** with respect to h_1 for radius $r_0 > 0$ if

$$\alpha|_{B \times \bar{B}_{r_0}(0)} = h_1({}^{r}/_{r_0})\, \alpha|_{TB}.$$

If a form is said to be standard not stating h_1 or r_0, then the corresponding data is assumed to be arbitrary.

The space of all general 1-forms adapted to (B, π) that are standard with respect to a fixed h_1 for radius $1/2$ we denote by $\Omega^1_{h_1}(\pi)$. We denote by $\Omega^1_{h_1}(\pi, \xi_B)$ and $\Omega^1_{h_1}(\pi, \alpha_B)$ its subspaces where we fix the contact structure ξ_B or the contact form α_B induced on the binding, respectively.

Remark 3.1.6. A general adapted 1-form that is standard does not satisfy the contact condition. Consequently, a contact form cannot be standard as a general adapted 1-form. So, no confusion should arise.

Given this new definition, we can state the version of Theorem 3.1.3 for $\Omega^1(\pi)$.

Theorem 3.1.7. *Let (B, π) be an open book decomposition of a closed manifold M and U an adapted neighbourhood of the binding. Furthermore, let $\underline{h}$ be a Lutz pair. Then there is a deformation D_t, $t \in [0,1]$, of the space $\Omega^1(\pi)$ into its subspace $\Omega^1_{h_1}(\pi)$ such that D_1 is a homotopy equivalence. Moreover, we may assume that the deformation is smooth in the deformation parameter t and that the deformation is constant outside U.*

For the remainder of this subsection, let us fix an open book decomposition (B, π) of a closed manifold M and an adapted neighbourhood $U \cong B \times D^2$ of the binding.

The proof of Theorem 3.1.7 will take several steps. In each of these steps we construct a weak deformation retraction from a subspace of $\Omega^1(\pi)$ into a smaller subspace, starting with the entire space $\Omega^1(\pi)$ and finishing in $\Omega^1_{h_1}(\pi)$. Finally, we piece these together to obtain the deformation D_t.

In Proposition 2.1.2 we have already seen that the space $\Omega^1_0(\pi)$ of those adapted forms α satisfying $\alpha|_B = \alpha|_{TB}$ with respect to the splitting of TU induced by the product structure is a strong deformation retract of $\Omega^1(\pi)$. So, we may start in this space instead of $\Omega^1(\pi)$.

Our first and hardest step is to construct a weak deformation retraction leading into the space $\Omega^1_L(\pi)$ of those adapted forms that are standard with respect to some h_1 for some radius $r_0 > 0$.

Proposition 3.1.8. *There is a weak deformation retraction D_t, $t \in [0,1]$, from $\Omega^1_0(\pi)$ into its subspace $\Omega^1_L(\pi)$ that is smooth in the deformation parameter and constant outside U. Moreover, there is a continuous* *function* $\rho\colon \Omega^1_0(\pi) \to (0, 1/2]$ *such that* $D_1(\alpha)$ *is standard for radius* $\rho(\alpha)$ *for all* $\alpha \in \Omega^1_0(\pi)$.

Proof. If $\Omega^1_0(\pi)$ is empty, so is its subspace $\Omega^1_L(\pi)$. So, we may assume that $\Omega^1_0(\pi)$ is non-empty.

Let $\alpha \in \Omega^1_0(\pi)$. As we have seen in Subsection 2.1.1, inside the adapted neighbourhood $U \cong B \times D^2$ of the binding we can write α as

$$\alpha = u\,dr + v\,d\varphi + \beta$$

where u and v are D^2-families of functions on B that vanish identically on $B \times \{0\}$, and β a D^2-family of 1-forms on B whose restriction to $B \times \{0\}$ is given by $\alpha_B = \alpha|_{TB}$.

In this notation the condition on α to be adapted translates to

$$0 < (du - \beta_r) \wedge (d\beta)^{n-1} \tag{3.1}$$

as a top-dimensional form on $B \times \{x\}$ for every $x \in D^2$. Here, β_r is the derivative of β with respect to the radial coordinate and the inequality is meant with respect to the reference volume form $\alpha_B \wedge (d\alpha_B)^{n-1}$. Throughout this proof we will use this reference volume form to identify top-dimensional forms on B with functions on B.

We construct the deformation in three steps. In the first step we arrange the correct function v in the decomposition, in the second step we add the form $h_1(r)\,\alpha_B$ on a small neighbourhood of the binding where $h_1(r) = \big(1 - Kr^2\big)$ for some (possibly very large) constant K that depends continuously on α, and in the third and final step we subtract all terms but $h_1(r)\,\alpha_B$ on an even smaller neighbourhood.

For the first step let us fix a smooth monotonously increasing cut-off function $\lambda\colon \mathbb{R}^+_0 \to [0,1]$ that is constant of value 1 on $[0, 1/2]$ and vanishes on $[3/4, \infty)$. Then we define

$$\alpha^1_t = \alpha - t\lambda(r)\,v\,d\varphi = u\,dr + (1 - t\lambda)\,v\,d\varphi + \beta$$

inside U and extend this by α outside U. Then we have

$$\alpha^1_1 = u\,dr + \beta$$

in the smaller closed neighbourhood $U_1 = B \times \bar{B}_{1/2}(0)$.

The remaining steps require a much more technical argument.

Before we are able to start with these steps, we have to take a closer look at the family of forms β. In order to do so, we decompose it as

$$\beta = h_0\,\alpha_B + \beta_\Delta$$

where h_0 is a family of functions on B and β_Δ a family of 1-forms on B satisfying $\beta \wedge (d\alpha_B)^{n-1} \equiv 0$. Immediately from the definition we see that $h_0|_B \equiv 1$, $\beta_\Delta|_B \equiv 0$, and $(\beta_\Delta)_r \wedge (d\alpha_B)^{n-1} = \left(\beta_\Delta \wedge (d\beta_B)^{n-1}\right)_r \equiv 0$.

As we have seen at the end of Subsection 2.1.1 the form $\beta_r \wedge (d\beta)^n$ vanishes on the binding. Writing this form in our decomposition we get

$$\beta_r \wedge (d\beta)^n = \left((h_0)_r\,\alpha_B + (\beta_\Delta)_r\right) \wedge \left(h_0\,d\alpha_B + dh_0 \wedge \alpha_B + d\beta_\Delta\right)^{n-1}.$$

By our observations above this implies that both the function $(h_0)_r$ and the form $\beta_r \wedge (d\alpha_B)^{n-1}$ vanish on the binding as well. This will be important in the two steps to come.

As already mentioned, in the following steps we restrict the changes to α_1^1 to a small neighbourhood. To this end we introduce two further cut-off functions λ_2 and λ_3 defined by $\lambda_2(r) = \lambda({}^{r}\!/\!{}_{\epsilon})$ and $\lambda_2(r) = \lambda({}^{2r}\!/\!{}_{\epsilon})$. Here, ${}^1\!/\!{}_2 \geq \epsilon > 0$ is a small constant continuously depending on α that we still have to determine. Note that λ_2 evaluates to 1 wherever λ_3 is non-zero.

Now we are ready to engage the second step of our construction. We define the second deformation by

$$\alpha_t^2 = \alpha_1^1 + t\lambda_2(r)\,h_1(r)\,\alpha_B = \alpha_1^1 + t\lambda_2(r)\left(1 - Kr^2\right)\alpha_B.$$

Outside $U_2 = B \times B_\epsilon(0)$ this family is constant. So, we may restrict our attention to U_2.

Our aim is to determine the constants ϵ and K such that the family stays in $\Omega_0^1(\pi)$ under the additional condition that they depend continuously on α.

A first observation is that h_1 can only be the first function in a Lutz pair if $h_1 > 0$, and hence that $K\left({}^{3\epsilon}\!/\!{}_4\right)^2$ should be smaller than 1. In regard of this we define

$$K = K(\epsilon) = \tfrac{1}{2\epsilon^2}.$$

Doing so, we can simultaneously arrange large K and small ϵ while keeping h_1 positive.

Let us now take a closer look at the forms α_t^2 inside U_2. There they are given by

$$\alpha_t^2 = (h_0 + t\lambda_2 h_1)\,\alpha_B + \beta_\Delta + u\,dr.$$

Inserting this into the condition (3.1) that α_t^2 be adapted, we obtain the inequality

$$\begin{aligned} 0 < \Big(((du - \beta_r) - t\lambda_2 h_1'\,\alpha_B) - t\lambda_2' h_1 \alpha_B\Big) \\ \wedge \big(\tilde{h}_{t\lambda_2}\,d\alpha_B + (dh_0 \wedge \alpha_B + d\beta_\Delta)\big)^{n-1}. \end{aligned} \tag{3.2}$$

Here, we used the abbreviation $\tilde{h}_{t\lambda_2} = h_0 + t\lambda_2 h_1$.

We show that this inequality holds for a sufficiently small choice of ϵ. First, we only consider the terms containing the derivative of λ_2. Since $t\lambda_2' h_1$ is a non-positive function, we have to show that the rest of these terms is a family of positive volume forms on B.

To see this, consider the points on which $|h_0 - 1| < 1/2$. There, the function $\tilde{h}_{t\lambda_2}$ only takes values in $[1/2, 5/2]$. So, if we have

$$0 < \alpha_B \wedge (s\,d\alpha_B + (dh_0 \wedge \alpha_B + d\beta_\Delta))^{n-1}$$

for all $s \in [1/2, 5/2]$, then the terms we consider define a positive volume form whenever $|h_0 - 1| < 1/2$.

The two conditions above are open conditions on the restrictions of h_0 and $\sigma_s = \alpha_B \wedge (s\,d\alpha_B + (dh_0 \wedge \alpha_B + d\beta_\Delta))^{n-1}$ to the sets $B \times \{x\}$ with $x \in D^2$ which are satisfied for $x = 0$. Consequently, Corollary A.2 provides a continuous function $\epsilon_1 \colon \Omega_0^1(\pi) \times [1/2, 5/2] \to (0, 1/2)$ such that the two conditions are satisfied for h_0 and σ_s inside the set $B \times B_{\delta(\alpha,s)}(0)$. Because $[1/2, 5/2]$ is compact, for fixed α, this function has a minimum $\epsilon(\alpha)$. Consequently, choosing $\epsilon \leq \epsilon_1(\alpha)$ guarantees that the terms involving the derivative of λ_2 are non-negative.

Next, let us take a closer look at the remaining terms, *i.e.* those not containing λ_2'. To understand them better, we expand them as a polynomial in $t\lambda_2$. This expansion reads as follows.

$$
\begin{aligned}
&(du-\beta_r)\wedge (d\beta)^{n-1}\\
&+\sum_{k=1}^{n-1}(t\lambda_2)^k\Bigg(\binom{n-1}{k}(du-\beta_r)\wedge(d\beta)^{n-k-1}\wedge(h_1\,d\alpha_B)^k\\
&\quad+\binom{n-1}{k-1}Kr\,\alpha_B\wedge(d\beta)^{n-k}\wedge(h_1\,d\alpha_B)^{k-1}\Bigg)\\
&+\big(t\lambda_2\big)^n Kr\,\alpha_B\wedge(h_1\,d\alpha_B)^{n-1}
\end{aligned}
\tag{3.3}
$$

The zeroth order coefficient is positive because this is the condition $d\alpha$ has to satisfy to be symplectic on the pages. Since furthermore $t\lambda_2 \geq 0$, it suffices to show that there is a sufficiently small choice of ϵ such that all coefficients are non-negative.

For the highest order coefficient this is evident, since h_1 is positive. For the remaining coefficients, the situation is more complicated. Each of these coefficients consists of two terms; about the first term we do not know a lot and the second term is the product of the coordinate function r with a form that is positive on a neighbourhood of the binding. The key observation is that the second term contains a factor of K and the first one does not. Consequently, we show that the coefficients are positive for a sufficiently large choice of K, which is equivalent to a sufficiently small choice of ϵ.

There is one problem with our observation about the occurrence of K in the two terms of the coefficients: the function h_1 does depend on K. However, this problem is not severe: by our definition of K as a function of ϵ we know that $h_1(r) = \big(1-Kr^2\big) \in [1/2, 1]$ for $r \leq \epsilon$ because $K = 1/2\epsilon^2$. So, since only powers of h_1 occur in the two terms of the coefficients, both times as a factor, and the power in the first term is larger, we can replace h_1 by its largest value 1.

Now, let us choose a positive continuous function δ on $\Omega^1_0(\pi)$ such that the second term in each of the coefficients is the product of r with a volume form on $B_\delta(0)\times B$, which exists by Corollary A.2. With this function at hand, we would like to set

$$
K_2(\alpha) = 2 + \max_{k\in\{1,\dots,n-1\}}\ \sup_{B_\delta(0)\setminus\{0\}} \frac{\left|\frac{1}{r}\binom{n-1}{k}(du-\beta_r)\wedge(d\beta)^{n-k-1}\wedge(d\alpha_B)^k\right|}{\left|\binom{n-1}{k-1}\alpha_B\wedge(d\beta)^{n-k}\wedge(d\alpha_B)^{k-1}\right|}
$$

and define $\epsilon_2(\alpha) \coloneqq \min\{1/\sqrt{2K_2}, \delta, \epsilon_1\}$. Then, for every choice of ϵ smaller than or equal to ϵ_2, the family α_t^2 will remain in $\Omega_0^1(\pi)$.

However, *a priori*, the supremum of the numerator does not have to be finite. To see that it is, first notice that, up to the factor of $1/r$, the numerator is well defined on the binding and vanishes there because both du and $\beta_r \wedge (d\alpha_B)^{n-1}$ vanish on the binding and $d\beta|_B = d\alpha_B$. Now, remember that the function u and the families of forms β and α_B are smooth on D^2. So they are still smooth when restricted to rays from 0 and the restriction and its derivatives continuously depend on the ray, *i.e.* the corresponding angle $\varphi \in S^1$. As a result, on each ray, the numerator has a smooth continuation to the origin that continuously depends on the ray. Thus, the supremum above is finite and our choices of K_2 and ϵ_2 are valid.

The third step is similar to the second one. We perform a further deformation of α via the family α_t^3, $t \in [0,1]$, given by

$$\alpha_t^3 = (1 + t\lambda_3(r))^8 \left(\alpha_t^2 - t\lambda_3(r)\,(\beta + u\,dr)\right).$$

This family is constant outside $U_3 = B \times B_{\epsilon/2}(0)$ and inside this set given by

$$\begin{aligned}\alpha_t^3 = (1 + t\lambda_3(r))^8 \big(&(h_1(r) + (1 - t\lambda_3(r))\,h_0)\,\alpha_B \\ &+ (1 - t\lambda_3(r))\,(u\,dr - \beta_\Delta)\big).\end{aligned}$$

This time the condition (3.1) that α_t^3 be adapted reads

$$\begin{aligned}0 < (1 + t\lambda_3)^7 \Big(&(1 + t\lambda_3)\,(-h_1'\alpha_B + (1 - t\lambda_3)\,(du - \beta_r)) \\ &- t\lambda_3'\big(-(1 + t\lambda_3)\,(h_0\,\alpha_B + \beta_\Delta) + 8\hat{h}_{t\lambda_3}\,\alpha_B + 8(1 - t\lambda_3)\,\beta_\Delta\big)\Big) \\ &\wedge \left(\hat{h}_{t\lambda_3}\,d\alpha_B + (1 - t\lambda_3)\,(dh_0 \wedge \alpha_B + d\beta_\Delta)\right)^{n-1}\end{aligned} \tag{3.4}$$

where we have used the abbreviation $\hat{h}_{t\lambda_3} = h_1 + \big(1 - t\lambda_3\big)h_0$. Because $\big(1 + t\lambda_3\big)^7$ is positive we may drop this factor.

As in the second step, we first consider the terms containing a factor of λ_3'. Since $t\lambda_3' \leq 0$, we have to show that the rest of these terms is non-negative. Remember that for our previous choice of ϵ we already know that $|h_0 - 1| < 1/2$ and $h_1(r) \in [1/2, 1]$. Accordingly, the function $\hat{h}_{t\lambda_3}$ only takes values in $[1/2, 5/2]$. In particular, $8\hat{h}_{t\lambda_3} - (1 + t\lambda_3)\, h_0$ is bounded from below by 1.

To ensure that inequality (3.4) is satisfied for the terms containing a factor of λ_3', we proceed by replacing $\hat{h}_{t\lambda_3}$ and $(1 - t\lambda_3)$ by parameters $s \in [1/2, 5/2]$ and $\tilde{s} \in [0, 1]$. Then the inequality reads

$$\begin{aligned} 0 < \big((8s - (2 - \tilde{s})\, h_0)\, \alpha_B + (9\tilde{s} - 2)\, \beta_\Delta\big) \\ \wedge \big(s\, d\alpha_B + \tilde{s}\, (dh_0 \wedge \alpha_B + d\beta_\Delta)\big)^{n-1}, \end{aligned} \tag{3.5}$$

which is independent of t and λ_3. Moreover, the inequality is satisfied on the binding for all parameters because the forms dh_0 and β_Δ vanish there. Thus, Corollary A.2 tells us that there is a continuous map $\epsilon_3 \colon \Omega_0^1(\pi) \times [1/2, 5/2] \times [0, 1] \to (0, \epsilon_2)$ such that the inequality holds on $B \times B_{\epsilon_3(\alpha, s, \tilde{s})}(0)$ for fixed s and $\tilde{s}$. Taking the minimum over the compact parameter space yields a function ϵ_3 on $\Omega_0^1(\pi)$ such that the inequality holds on $B \times B_{\epsilon_3(\alpha)}(0)$ for all values of s and $\tilde{s}$. In particular, the inequality holds for $s = \hat{h}_{t\lambda_3}(r)$ and $\tilde{s} = (1 - t\lambda_3(r))$.

Let us now turn our attention to the remaining terms. After dividing them by the common positive factor $\big(1 + t\lambda_3\big)$, an expansion as a polynomial in $\big(1 - t\lambda_3\big)$ reads as follows.

$$\begin{aligned} & Kr\, \alpha_B \wedge (h_1\, d\alpha_B)^{n-1} \\ & + \sum_{k=1}^{n-1} \big(1 - t\lambda_3\big)^k \Bigg(\binom{n-1}{k} Kr\, \alpha_B \wedge (d\beta)^k \wedge (h_1\, d\alpha_B)^{n-k-1} \\ & \quad + \binom{n-1}{k-1} (du - \beta_r) \wedge (d\beta)^{k-1} \wedge (h_1\, d\alpha_B)^{n-k} \Bigg) \\ & + \big(1 - t\lambda_3\big)^n\, (du - \beta_r) \wedge (d\beta)^{n-1} \end{aligned}$$

Here, the same arguments as in the second step apply. The zeroth order term is positive because α_B is a contact form on B, and the highest order term because this is the condition $d\alpha$ has to satisfy to be symplectic on the pages. By the same arguments as in the second step we may define

$$K_3(\alpha) = 2 + \max_{k \in \{1, \dots, n-1\}} \sup_{B_\delta(0) \setminus \{0\}} \frac{\left| \frac{2}{r} \binom{n-1}{k-1} (du - \beta_r) \wedge (d\beta)^{k-1} \wedge (d\alpha_B)^{k-1} \right|}{\left| \binom{n-1}{k} \alpha_B \wedge (d\beta)^k \wedge (d\alpha_B)^{n-k-1} \right|}$$

where $\delta = \delta(\alpha)$ is the same function as in the second step, which guarantees that the denominator is positive. Here, we have approximated h_1 by $1/2$ because, this time, the term containing the higher power of h_1 is the one containing K. Defining $\epsilon = \min\left\{1/\sqrt{2K_3}, \epsilon_3\right\}$ then ensures that both families α_t^2 and α_t^3 stay inside $\Omega_0^1(\pi)$ for all $t \in [0,1]$.

For $r \leq \epsilon/4$ the function λ_3 is constant of value 1. As a result, for these radii the form α_1^3 is given by

$$\alpha_1^3 = 2^8\left(1 - Kr^2\right)\alpha_B = \left(1 - \frac{1}{32}\left(\frac{r}{\epsilon/4}\right)^2\right)\left(2^8\alpha_B\right).$$

Hence, it is standard for radius $\epsilon/4$ with respect to the function $\bar{h}_1\colon [0,1] \to \mathbb{R}_0^+$ given by $\bar{h}_1(r) = \left(1 - \frac{1}{32}r^2\right)$.

We define the deformation $D_t(\alpha)$ by piecing together the three deformations above and set $\rho(\alpha) = \epsilon(\alpha)/4$. More precisely, to guarantee the smoothness in t, we choose some smooth monotonously increasing cut-off function $\mu\colon [0, 1/3] \to [0,1]$ that vanishes in a neighbourhood of 0 and is constant of value 1 in a neighbourhood of $1/3$. Then we define

$$D_t(\alpha) = \begin{cases} \alpha^1_{\mu(t)} & , \text{ for } t \in [0, 1/3] \\ \alpha^2_{\mu(t-1/3)} & , \text{ for } t \in [1/3, 2/3] \\ \alpha^3_{\mu(t-2/3)} & , \text{ for } t \in [2/3, 1]. \end{cases}$$

It remains to show that the deformation D_t leaves the space $\Omega_L^1(\pi)$ invariant. So, let $\alpha \in \Omega^1_{(B,\pi),L}(M)$ and $r_0 \in (0, 1/2]$ be some radius for which α is standard for some function h_1 as in the definition of a Lutz pair. Then α is also standard for every radius $0 < r_1 \leq r_0$, albeit with respect to another function $\check{h}_1$. More precisely, this function is given by $\check{h}_1(r) = h\big((r_0/r_1)\,r\big)$. In particular, it is standard for $\tilde{r} = \min\left\{\rho(\alpha), r_0\right\}$.

Because α is standard for radius $\tilde{r}$, we know that for all $r \leq \tilde{r}$ the function v from the decomposition vanishes. Consequently, the first deformation does not change α at all inside $B \times B_{\tilde{r}}(0)$. Changes inside this set only occur in the two remaining deformations.

As both λ_2 and λ_3 are constant for $r \leq \tilde{r} \leq \epsilon/4$, for these radii the families are given by

$$\begin{aligned} \alpha_t^2 &= \left(\check{h}_1\left(r/\tilde{r}\right) + t\,h_1(r)\right)\alpha_B \\ &= \left(\tfrac{1}{1+t}\,\check{h}_1\left(r/\tilde{r}\right) + \tfrac{t}{1+t}\,h_1(r)\right)\left((1+t)\,\alpha_B\right) \end{aligned}$$

and

$$\begin{aligned}\alpha_t^3 &= (1+t)^8 \left(h_1(r) + (1-t)\,\check{h}_1\left({}^r\!/_{\tilde{r}}\right)\right)\alpha_B \\ &= \left(\tfrac{1}{2-t}\,h_1(r) + \tfrac{1-t}{2-t}\,\check{h}_1\left({}^r\!/_{\tilde{r}}\right)\right)\left((1+t)^8\,(2-t)\,\alpha_B\right).\end{aligned}$$

Because the conditions on the functions in a Lutz pair are convex, these two families stay standard for radius $\tilde{r}$ for all times $t \in [0,1]$. This shows that D_t is a weak deformation retraction from $\Omega^1_0(\pi)$ into $\Omega^1_L(\pi)$. $\square$

In the remainder of the proof, we will need to know for which radius a form is standard. Unfortunately, this radius is not unique for forms in $\Omega^1(\pi)$. There is not even a way to determine one such radius in continuous dependence on the form itself. So, we will have to keep track of such a radius separately.

Here, it pays off that we constructed a function ρ on $\Omega^1_0(\pi)$ such that the forms $D_1(\alpha)$ are standard for radius $\rho(\alpha) \leq 1/2$. This enables us to define the continuous function (D_1, ρ) from $\Omega^1_0(\pi)$ into the subset $\tilde{\Omega}^1_L(\pi)$ of $\Omega^1_L(\pi) \times (0, 1/2]$ consisting of those pairs (α, r_0) such that α is standard for radius r_0.

Taking a closer look at the proof of Proposition 3.1.8, we see that this map is a homotopy equivalence.

Corollary 3.1.9. *There is a weak deformation retraction D'_t from the space $\Omega^1_0(\pi) \times (0, 1/2]$ into its subspace $\tilde{\Omega}^1_L(\pi)$ such that $D'_1(\alpha, r_0) = (D_1(\alpha), \rho(\alpha))$. Moreover, we may assume that the deformation is smooth in the deformation parameter and constant outside U.*

Proof. If $\Omega^1_0(\pi) \times (0, 1/2]$ is empty, so is its subspace $\tilde{\Omega}^1_L(\pi)$. So, let us assume that $\Omega^1_0(\pi) \times (0, 1/2]$ is non-empty.

Let $(\alpha, r_0) \in \Omega^1_0(\pi) \times (0, 1/2]$ and $\tilde{r}_0 = \min\{r_0, \rho(\alpha)\}$.

As we have seen already in the proof of Proposition 3.1.8, whenever α is standard for r_0, this is also true for all $r \leq r_0$. Moreover, in this case we also know that $D_t(\alpha)$ is standard for $\tilde{r}_0$ for all $t \in [0,1]$. Consequently, we can define the deformation retraction in the following three parts, which we patch together as in the proof of Proposition 3.1.8.

First, change r_0 through the family $r^1_t = (1-t)\,r_0 + t\tilde{r}_0$, holding α fixed. Second, change α through the family $D_t(\alpha)$, holding r_0 fixed. Finally, change r_0 through the family $r^2_t = (1-t)\,\tilde{r}_0 + t\rho(\alpha)$, holding α fixed.

After the second step we are already in $\tilde{\Omega}^1_L(\pi)$ and we know that $D_1(\alpha)$ is standard for $\rho(\alpha)$. This enables us to raise r_0 again from $\tilde{r}_0$ to $\rho(\alpha)$ without leaving $\tilde{\Omega}^1_L(\pi)$. $\square$

Because the inclusion of $\Omega^1_0(\pi)$ into $\Omega^1_0(\pi) \times (0, 1/2]$ given by $\alpha \mapsto (\alpha, 1/2)$ is a homotopy equivalence, the corollary above proves that the map (D_1, ρ) is one, as well.

Now that we can keep track of the radii for which our forms are standard, we can fix the function h_1 with respect to which the forms are standard.

Lemma 3.1.10. *Let $\underline{h}$ be a Lutz pair. Then there is a weak deformation retraction from $\tilde{\Omega}^1_L(\pi)$ into its subspace $\tilde{\Omega}^1_{h_1}(\pi)$ consisting of those pairs (α, r_0) such that α is standard with respect to h_1 for radius r_0. Moreover, we may assume that the deformation is smooth in the deformation parameter and constant outside U.*

Proof. If $\tilde{\Omega}^1_L(\pi)$ is empty, so is its subspace $\tilde{\Omega}^1_{h_1}(\pi)$. So, let us assume that $\tilde{\Omega}^1_L(\pi)$ is non-empty.

Let $(\alpha, r_0) \in \tilde{\Omega}^1_L(\pi)$. Our goal is to construct a deformation of α that depends continuously on the pair (α, r_0) while holding r_0 fixed.

We already know that inside $B \times \bar{B}_{r_0}(0) \subset U$ the form α is given by

$$\alpha = \tilde{h}_1(r/r_0)\, \alpha_B$$

for some function $\tilde{h}_1$ such that $(\tilde{h}_1, h_2)$ is a Lutz pair and $\alpha_B = \alpha|_{TB}$. Because the space of Lutz pairs is convex, this leads us to the idea to use a convex interpolation between $\tilde{h}_1$ and h_1 to arrange that α be standard with respect to h_1 for radius r_0.

The problem is that we have to cut off this convex interpolation outside of $B \times \bar{B}_{r_0}(0)$. Consequently, we need suitable extensions of h_1 and $\tilde{h}_1$. We start with h_1.

Let $h_{1,\infty} \colon \mathbb{R}^+_0 \to \mathbb{R}^+$ be a smooth extension of h_1 such that $h'_{1,\infty}(r) < 0$ for $r > 0$. Then we define h_{1,r_0} by $h_{1,r_0}(r) = h_{1,\infty}(r/r_0)$. By construction, an adapted form $\tilde{\alpha}$ is standard with respect to h_1 for radius r_0 if and only if it is given by

$$\tilde{\alpha} = h_{1,r_0}(r)\, \tilde{\alpha}|_{TB}$$

for $r \leq r_0$.

As an extension for $\tilde{h}_1$ we define the function $\bar{h}_1 := \iota_{R_{\alpha_B}} \alpha$, which is defined on U. Inside $B \times \bar{B}_{r_0}(0)$ it depends only on r and is given by

$$\bar{h}_1(r) = \tilde{h}_1({}^r\!/_{r_0}) \,.$$

Next, we need a cut-off function. We want to cut off the interpolation for radii slightly larger than r_0; the exact amount of how much larger they might be should depend continuously on the form α. So, we first define a smooth reference function $\lambda_0 \colon \mathbb{R} \to [0,1]$ that is monotonously decreasing, is constant of value 1 on $(-\infty, 0]$, and vanishes on $[1, \infty)$. Then we define the actual cut-off function by $\lambda(r) = \lambda_0\left(\frac{1}{\delta}\,(r - r_0)\right)$ where δ is a constant that we still have to determine.

Given the preparations above, we set

$$\alpha_t = \alpha - \bar{h}_1\,\alpha_B + (1 + t\lambda(r))^m \left((1 - t\lambda(r))\,\bar{h}_1 + t\lambda(r)\,h_{1,r_0}(r)\right)\alpha_B$$

for some constant m we still have to determine.

For $r \leq r_0$ the function λ is constant of value 1. Thus, we have

$$\begin{aligned} \alpha_t &= (1+t)^m \left((1-t)\,\bar{h}_1 + t h_{1,r_0}(r)\right)\alpha_B \\ &= \Big((1-t)\,\tilde{h}_1({}^r\!/_{r_0}) + t h_1({}^r\!/_{r_0})\Big)\,(1+t)^m\,\alpha_B \end{aligned}$$

inside $B \times \bar{B}_{r_0}(0)$. This shows that α_t is standard for radius r_0 for all $t \in [0,1]$. Moreover, if α is standard with respect to h_1 for radius r_0, *i.e.* $\tilde{h}_1 = h_1$, then the interpolation is constant for $r \leq r_0$.

By the observation above we know that α_t defines a weak deformation retraction, provided we can find suitable constants δ and m such that $d\alpha_t$ is non-degenerate on the pages.

To find such constants, we proceed in analogy with the proof of Proposition 3.1.8 and define a parametric version of α_t where we replace $t\lambda$ by a constant $s \in [0,1]$, *i.e.* we set

$$\tilde{\alpha}_s = \alpha - \bar{h}_1\,\alpha_B + (1+s)^m \left((1-s)\,\bar{h}_1 + s h_{1,r_0}(r)\right)\alpha_B .$$

Then we can write the differential of α_t as

$$\begin{aligned} d\alpha_t = d\tilde{\alpha}_{t\lambda} + t\lambda'\,(1+t\lambda)^{m-1} \Big(&(1+t\lambda)\left(h_{1,r_0} - \bar{h}_1\right) \\ &+ m\big((1-t\lambda)\,\bar{h}_1 + t\lambda h_{1,r_0}\big)\Big)\,dr \wedge \alpha_B . \end{aligned}$$

Accordingly, the condition on $d\alpha_t$ to be non-degenerate on the pages reads

$$\begin{aligned} 0 < (d\alpha_t)^n = (d\tilde{\alpha}_{t\lambda})^n - nt\lambda' (1+t\lambda)^{m-1} \Big((1+t\lambda)\left(h_{1,r_0} - \bar{h}_1\right) \\ + m\big((1-t\lambda)\,\bar{h}_1 + t\lambda h_{1,r_0}\big)\Big)\, \alpha_B \wedge (d\tilde{\alpha}_{t\lambda})^{n-1} \wedge dr \end{aligned} \tag{3.6}$$

with respect to the reference volume form $\alpha_B \wedge (d\alpha_B)^{n-1} \wedge dr$.

Because, for all $s \in [0,1]$, the forms $\tilde{\alpha}_s$ and α_s agree for $r \leq r_0$ and the forms α_s are standard for radius r_0, we know that $(d\tilde{\alpha}_s)^n$ and $\alpha_B \wedge (d\tilde{\alpha}_s)^{n-1} \wedge dr$ are volume forms for $0 < r \leq r_0$. Thus, the proof of Corollary A.2 tells us that there is a continuous function $\delta_1 \colon \tilde{\Omega}^1_L(\pi) \times [0,s] \to (r_0, 1)$ such that this is also true for all radii $0 < r \leq \delta_1(\alpha, r_0, s)$. Taking the minimum over s we obtain a function $\delta_1 \colon \tilde{\Omega}^1_L(\pi) \to (r_0, 1)$ with the same property.

The discussion above provides a suitable choice of δ such that $(d\tilde{\alpha}_{t\lambda})^n$ is positive. It remains to show that we can also arrange that the second term in (3.6) is non-negative. By construction λ' is non-positive and by our previous choice of δ we know that $\alpha_B \wedge (d\tilde{\alpha}_{t\lambda})^{n-1} \wedge dr$ is a volume form on the support of λ'. So, it remains to show that

$$(1+t\lambda)\left(h_{1,r_0} - \bar{h}_1\right) + m\big((1-t\lambda)\,\bar{h}_1 + t\lambda h_{1,r_0}\big) \tag{3.7}$$

is non-negative on said support.

To see this, we have to take a closer look at the functions h_{1,r_0} and $\bar{h}_1$. Inside $B \times \bar{B}_{r_0}(0)$ both only depend on the radial coordinate, with respect to which they have a non-positive derivative. Furthermore, the values of the restrictions to this set of both functions are contained in $(0,1]$. Consequently, we know for $r \leq r_0$ that $\bar{h}_1(r) > \frac{1}{2}\bar{h}_1(r_0)$, $h_{1,r_0}(r) > \frac{1}{2}h_{1,r_0}(r_0) = \frac{1}{2}h_1(1)$, and $\left|\bar{h}_1(r) - h_{1,r_0}(r)\right| < 1$. These conditions are open. So, again by the proof of Corollary A.2, we get a function $\delta \colon \tilde{\Omega}^1_L(\pi) \to (r_0, \delta_1)$ such that these three inequalities also hold for all $r \leq r_0 + \delta$.

With this choice of δ, the first term in (3.7) is bounded from below by -2. Moreover, we can approximate the second term by

$$m\big((1-t\lambda)\,\bar{h}_1 + t\lambda h_{1,r_0}\big) \geq m \min\left\{\bar{h}_1, h_{1,r_0}\right\} \geq \tfrac{m}{2} \min\left\{\bar{h}_1(r_0)\,, h_1(1)\right\}.$$

Thus, setting $m = 6/\min\{\tilde{h}_1(r_0), h_1(1)\}$ ensures that (3.7) is positive on the support of λ, *i.e.* for $r \leq r_0 + \delta$, and hence that α_t is adapted for all $t \in [0,1]$.

Since all choices in the construction depend continuously on the pair (α, r_0) the path (α_t, r_0) defines the desired weak deformation retraction. $\square$

So far, we had to separately keep track of the radius for which a form is standard, because there was no continuous way to obtain this information from the form. Now that we have entered the space $\hat{\Omega}^1_{h_1}(\pi)$ of those adapted 1-forms that are standard with respect to a fixed h_1 for some radius smaller or equal to $1/2$, the radius for which the forms are standard with respect to h_1 is unique and depends continuously on the forms.

Lemma 3.1.11. *Let $\underline{h}$ be a Lutz pair. Then the function $r_0 \colon \hat{\Omega}^1_{h_1}(\pi) \to (0, 1/2]$ assigning to a form the radius for which it is standard with respect to h_1 is well-defined and continuous.*

Proof. If $\hat{\Omega}^1_{h_1}(\pi)$ is empty, the assertion holds. So, let us assume that $\hat{\Omega}^1_{h_1}(\pi)$ is non-empty.

Let $\alpha_0 \in \hat{\Omega}^1_{h_1}(\pi)$ and $0 < r_0 \leq 1/2$ be a radius for which α_0 is standard with respect to h_1. Furthermore, fix a bundle metric on $\Omega^1(M)$ respecting the product structure on $U \cong D^2 \times B$.

Seeking a contradiction, assume there was an $r_1 \neq r_0 \in (0, 1/2]$ such that for every $\delta > 0$ there is an $\alpha_1 \in \hat{\Omega}^1_{h_1}(\pi)$ such that α_1 is standard with respect to h_1 for radius r_1 and $\|\alpha_1 - \alpha_0\|_{C^0(\Omega^1(M))} < \delta \|\alpha_0|_{TB}\|_{C^0(\Omega^1(B))}$. Using that $h_1 \leq 1$, we have

$$\begin{aligned}
\delta \|\alpha_0|_{TB}\|_{C^0(\Omega^1(B))} &> \|\alpha_1 - \alpha_0\|_{C^0(\Omega^1(M))} \\
&\geq \|h_1(r/r_1)\, \alpha_1|_{TB} - h_1(r/r_0)\, \alpha_0|_{TB}\|_{C^0(\Omega^1(B_{\bar{r}}(0)\times B))} \\
&\geq \tfrac{1}{2} \|(h_1(r/r_1) - h_1(r/r_0))\,(\alpha_1|_{TB} + \alpha_0|_{TB})\| \\
&\quad - \tfrac{1}{2} \|(h_1(r/r_1) + h_1(r/r_0))\,(\alpha_1|_{TB} - \alpha_0|_{TB})\| \\
&\geq \left(\tfrac{2-\delta}{2} \|h_1(r/r_1) - h_1(r/r_0)\|_{C^0([0,\bar{r}])} - \delta\right) \\
&\quad \cdot \|\alpha_0|_{TB}\|_{C^0(\Omega^1(B))}
\end{aligned}$$

where $\bar{r} = \min\{r_0, r_1\}$.

Because we assumed that this is true for all $\delta > 0$, we see that $h_1(r/r_0) = h_1(r/r_1)$ for all $r \in (0, \bar{r}]$. However, this is impossible since $h_1' < 0$ on $(0, 1)$.

This shows two things: the radius for which a form is standard with respect to h_1 is unique, and the function assigning this radius to the form is continuous on $\hat{\Omega}^1_{h_1}(\pi)$. □

Immediately, we get the following consequence.

Lemma 3.1.12. *The projection from $\tilde{\Omega}^1_{h_1}(\pi)$ to $\hat{\Omega}^1_{h_1}(\pi)$ is a homeomorphism.*

Now that we know that our forms are standard with respect to a fixed function h_1 and we are able to continuously assign to it the radius for which it is standard with respect to h_1, we would like to change this radius to a fixed value.

Lemma 3.1.13. *Let $\underline{h}$ be a Lutz pair. Then there is a strong deformation retraction from $\hat{\Omega}^1_{h_1}(\pi)$ to the space $\Omega^1_{h_1}(\pi)$ of adapted 1-forms standard with respect to h_1 for radius $1/2$. Moreover, we may assume that the deformation is smooth in the deformation parameter and constant outside U.*

Proof. If $\hat{\Omega}^1_{h_1}(\pi)$ is empty, so is its subspace $\Omega^1_{h_1}(\pi)$. So, let us assume that $\hat{\Omega}^1_{h_1}(\pi)$ is non-empty.

We know by Lemma 3.1.11 that the radius r_0 for which a 1-form $\alpha \in \hat{\Omega}^1_{h_1}(\pi)$ is standard with respect to h_1 depends continuously on α. So, for a continuous family of isotopies $\Psi_{r_0,t}$ of M, a deformation of the form

$$D_t(\alpha) = \left(\Psi^{-1}_{r_0(\alpha),t}\right)^* \alpha$$

is continuous.

For our purposes, we choose $\Psi_{r_0,t}$ to be constant outside the adapted neighbourhood $U \cong D^2 \times B$ of the binding and inside this set to be defined by

$$\Psi_{r_0,t}(((r,\varphi),b)) = (((1-t)\,r + t f_{r_0}(r)\,,\varphi)\,,b)$$

where $f_{r_0}\colon [0,1] \to [0,1]$, $r_0 \in (0, 1/2]$, is a continuous family of smooth functions with the following properties.

1) $f_{1/2}(r) = r$.

2) $f_{r_0}|_{[0,r_0]}(r) = \frac{r}{2r_0}$ and $f_{r_0}|_{[3/4,1]}(r) = r$.

3) $f'_{r_0} > 0$.

Such a family of functions can be defined by patching together three affine linear functions. An explicit example is given by

$$f_{r_0}(r) = \lambda_1(r)\,\tfrac{r}{2r_0} + (1-\lambda_1(r))\,(1-\lambda_2(r))\left(\tfrac{1}{3}\tfrac{1+4r_0}{3-4r_0}r + \tfrac{1}{12}\tfrac{21-38r_0-8r_0^2}{3-4r_0}\right) + \lambda_2(r)\,r$$

where $\lambda_1(r) = \lambda_0({}^{20}/_{r_0}\,(r-r_0))$ and $\lambda_2(r) = \lambda_0(96\,({}^3/_4 - r))$ for a fixed smooth monotonously decreasing cut-off function $\lambda_0\colon \mathbb{R} \to [0,1]$ that is constant of value 1 on $\mathbb{R}^-$ and vanishes on $[1,\infty)$.

For fixed $r_0 \in (0, {}^1/_2]$, the function f_{r_0} essentially looks like the one in Figure 3.1 below.

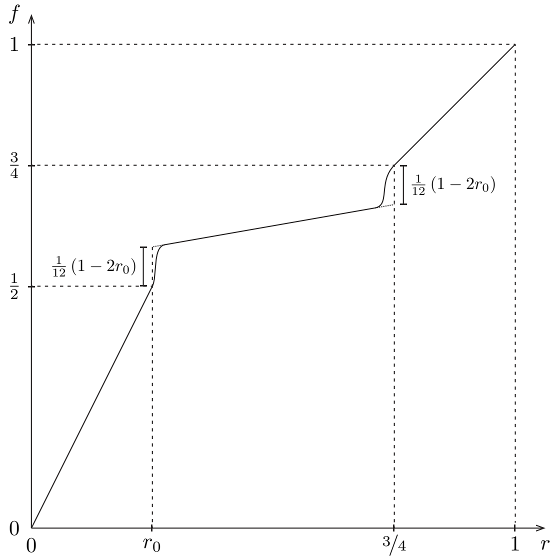

Figure 3.1.: The function f_{r_0}

Since the isotopies $\Psi_{r_0,t}$ are constant at the binding and restrict to isotopies of the pages, pulling back an adapted 1-form yields an adapted 1-form. So, all we have to verify is that $D_t(\alpha)$ is standard for some radius $r_1(t)$ with respect to h_1 and that $r_1(1) = {}^1/_2$.

Because the functions f_{r_0} are linear on $[0, r_0]$, we see that for $r \leq {}^1\!/_2\left(\left(1-t\right)2r_0+t\right) =: r_1(t)$ we have

$$\Psi^{-1}_{r_0,t}\left(\left(\left(r,\varphi\right),b\right)\right) = \left(\left(\tfrac{2r_0}{(1-t)2r_0+t}r,\varphi\right),b\right).$$

Using that $\alpha \in \hat{\Omega}^1_{h_1}(\pi)$, we deduce that for $r \leq r_1(t)$ we have

$$D_t(\alpha) = h_1\left(\tfrac{2r}{(1-t)2r_0+t}\right)\alpha|_B = h_1\left(\tfrac{r}{r_1(t)}\right)\alpha|_B.$$

This is exactly what we wanted to prove. □

Connecting the deformations from Proposition 2.1.2 and the three lemmata Proposition 3.1.8, Lemma 3.1.10, and Lemma 3.1.13 the same way we connected the steps in the proof of Proposition 3.1.8 yields a deformation with the properties asserted in Theorem 3.1.7.

Remark 3.1.14. In the construction above, the restriction to TB of the adapted forms only changes by multiplication with a positive constant. This has two consequences: first, all the deformations above restrict to the corresponding subspaces in which the induced contact structure on the binding is fixed. Second, if we modify the deformations by multiplying the forms with the inverse of this constant at each time of the deformations, we get corresponding deformations of the subspaces in which the induced contact form on the binding is fixed. However, in this case the deformations are not constant outside U anymore.

3.1.2. Induced Liouville Forms

In the preceding subsection, we constructed several weak deformation retractions of spaces of adapted 1-forms. Taking a closer look at the construction of these deformation retractions we see that we never explicitly use that we are working with adapted forms rather than with families of induced forms, *i.e.* rather than with families in $\mathcal{B}(\pi)$. Consequently, the weak deformation retractions of the corresponding subspaces of $\mathcal{B}(\pi)$ exist, as well. Therefore, we would like to infer that a version of Theorem 3.1.7 is true for the space $\mathcal{B}(\pi)$.

To be able to formulate a suitable version of Theorem 3.1.7, we first have to define what being *standard* means for an induced Liouville form.

Definition 3.1.15. Let (B, π) be an open book decomposition of a closed manifold and $C \cong B \times [0,1)$ a collar neighbourhood of the binding B in the closure of the page P_0. Furthermore, let $\underline{h}$ be a Lutz pair. Then we say that an induced Liouville form $\beta \in \mathcal{B}(\pi)$ is **standard** with respect to h_1 for distance s_0 if

$$\beta = h_1(s)\,\beta|_{TB}$$

on $B \times [0, s_0) \subset C$.

If a form is said to be standard not stating h_1 or s_0, then the corresponding data is assumed to be arbitrary.

The space of all induced Liouville forms on $\bar{P}_0$ that are standard with respect to a fixed h_1 for distance $1/2$ we denote by $\mathcal{B}_{h_1}(\pi)$. We denote by $\mathcal{B}_{h_1}(\pi, \xi_B)$ and $\mathcal{B}_{h_1}(\pi, \alpha_B)$ its subspaces where we fix the contact structure ξ_B or the contact form α_B induced on the binding, respectively.

Given the notation above, a version of Theorem 3.1.7 for induced Liouville forms can be formulated as follows.

Theorem 3.1.16. *Let (B, π) be an open book decomposition of a closed manifold M and C a collar of the boundary of the closure of the page P_0. Furthermore, let $\underline{h}$ be a Lutz pair. Then there is a deformation D_t, $t \in [0,1]$, of the space $\mathcal{B}(\pi)$ into its subspace $\mathcal{B}_{h_1}(\pi)$ such that D_1 is a homotopy equivalence. Moreover, we may assume that the deformation is smooth in the deformation parameter t and that the deformation is constant outside C.*

By the discussion at the beginning of this subsection, we know that the proof of Theorem 3.1.7 carries over to that of the theorem above nearly verbatim. It only remains to prove an analogue of Proposition 2.1.2, since the proof of this proposition uses a construction adapted to cartesian coordinates, rather than polar coordinates, on an adapted neighbourhood $B \times D^2$ of the binding. Furthermore, not every induced Liouville forms can appear as the restriction of an adapted form to the pages because of smoothness issues. Fortunately, we are still able to adapt the proof to the situation in $\mathcal{B}(\pi)$; this even simplifies the proof.

Lemma 3.1.17. *Let (B, π) be an open book decomposition of a closed manifold M and $C \cong B \times [0,1)$ a collar of the boundary of the closure of the page P_0. Then there is a strong deformation retraction from $\mathcal{B}(\pi)$ into its subspace $\mathcal{B}_0(\pi)$ consisting of those induced Liouville forms β that* *satisfy* $\beta|_B = \beta|_{TB}$ *with respect to the splitting of* TC *induced by the product structure. Moreover, we may assume that the deformation is smooth in the deformation parameter and constant outside* C*.*

Proof. If $\mathcal{B}(\pi)$ is empty, so is its subspace $\mathcal{B}_0(\pi)$. So, let us assume that $\mathcal{B}(\pi)$ is non-empty.

Let $\beta \in \mathcal{B}(\pi)$. Then we can write β inside C as

$$\beta = \alpha + f\,ds$$

where α is a $[0,1)$-family of 1-forms on B and f a $[0,1)$-family of functions on B.

In analogy with the proof of Proposition 2.1.2, we make the ansatz

$$\beta_t = \alpha_t + f_t\,ds$$

with $f_t = f - t\lambda'(s)\,f|_B$ and $\alpha_t = \alpha - t\lambda(s)\,|_{TB}$. Here, $\lambda(s) = \sin(s)\,\lambda_0(s)$ where $\lambda_0\colon\ [0,1] \to [0,1]$ is a smooth function that vanishes for $s \geq {}^1\!/\!_2$ and takes the value 1 in a neighbourhood of $s = 0$.

By construction, we have $\lambda'(0) = 1$ and $\lambda(0) = 0$. This implies that $\beta_t|_B = \beta|_{TB} + (1-t)\,f|_B\,ds$. Consequently, it remains only to show that β_t is a Liouville form on P_0 for all $t \in [0,1]$.

To see this, note that the exterior differential $d\alpha_t$ an B agrees with $d\alpha$ for all t and that

$$df_t - \partial_s(\alpha_t) = (df - t\lambda'\,df|_{TB}) - \partial_s(\alpha - t\lambda\,df|_{TB}) = df - \partial_s\alpha.$$

This shows that

$$\begin{aligned}(d\beta_t)^n &= n\,(df_t - \partial_s(\alpha_t)) \wedge (d\alpha_t)^{n-1} \wedge ds \\ &= n\,(df - \partial_s\alpha) \wedge (d\alpha)^{n-1} \wedge ds \\ &= (d\beta)^n\,.\end{aligned}$$

Accordingly, β_t is a Liouville form because β is a Liouville form.

Since β_t is constant whenever $f|_B \equiv 0$, *i.e.* whenever $\beta|_B = \beta|_{TB}$, and the construction above is continuous in the form β, it defines a strong deformation retraction from $\mathcal{B}(\pi)$ into $\mathcal{B}_0(\pi)$. That it is smooth and constant outside C is immediate from the construction. □

Because the deformation retractions from Subsection 3.1.1 still work after we replace the adapted forms by families in $\mathcal{B}_0(\pi)$, the lemma above completes the proof of Theorem 3.1.16.

Remark 3.1.18. Analogous to Remark 3.1.14, all the deformations in the proof of Theorem 3.1.16 change the contact form induced on the binding only by multiplication with a positive constant. Thus, the deformations restrict to the corresponding subspaces in which the induced contact structure on the binding is fixed. Moreover, if we modify the deformations by multiplying the forms with the inverse of this constant at each time of the deformations, then we get corresponding deformations of the subspaces in which the induced contact form on the binding is fixed. However, in this case the deformations are not constant outside C anymore.

3.1.3. Adapted Contact Forms

In Subsection 3.1.1, for a given open book decomposition (B, π) of a closed manifold M, adapted neighbourhood $U \cong B \times D^2$ of the binding, and Lutz pair $\underline{h}$, we constructed a deformation D_t of $\Omega^1(\pi)$ into its subspace $\Omega^1_{h_1}(\pi)$ such that D_1 is a homotopy equivalence. We would like to use Corollary 2.1.6 to turn this into a deformation of the space of adapted contact forms $\mathcal{A}(\pi)$. Unfortunately, this is not possible immediately because $\Omega^1_{h_1}(\pi)$ is not invariant under the deformation from Theorem 2.1.3. Consequently, we have to find a suitable invariant subspace of $\Omega^1(\pi)$ homotopy equivalent to $\Omega^1_{h_1}(\pi)$.

Such a subspace is given by the space $\bar{\Omega}^1_{h_1}(\pi)$ consisting of those adapted forms α that have a decomposition

$$\alpha = \alpha_0 + f\, d\varphi$$

with the following properties:

1) $\alpha_0 \in \Omega^1_{h_1}(\pi)$
2) $f \in C^\infty(M)$
3) Either there is a Lutz pair $\left(\bar{h}_1, \bar{h}_2\right)$ such that f is an extension of the function $\bar{h}_2(2r)$ on $B \times B_{1/2}(0) \subset U$, or f vanishes identically.

Now, we prove that this subspace of $\Omega^1(\pi)$, indeed, is homotopy equivalent to the space $\Omega^1_{h_1}(\pi)$.

Lemma 3.1.19. $\Omega^1_{h_1}(\pi)$ *is a strong deformation retract of* $\bar{\Omega}^1_{h_1}(\pi)$.

Proof. By definition, if $\Omega^1_{h_1}(\pi)$ is empty, so is $\bar{\Omega}^1_{h_1}(\pi)$, and vice versa. So, let us assume that $\bar{\Omega}^1_{h_1}(\pi)$ is non-empty.

Let $\mu\colon [0,1] \to [0,1]$ be a smooth function that is constant of value 1 on $[0, 1/2]$ and vanishes on $[3/4, 1]$.

Using this function, for $\alpha \in \bar{\Omega}^1_{h_1}(\pi)$ we set

$$\alpha_t = \alpha - t\lambda(r)\, \iota_{\partial_\varphi}\alpha\, d\varphi$$

inside U and extend this by α on the complement of U.

Since $\alpha = \alpha_0 + f\, d\varphi$ for some $\alpha_0 \in \Omega^1_{h_1}(\pi)$ and some smooth function f, for $r \leq 1/2$ we have

$$\alpha_t = \alpha_0 + (1-t)\, f\, d\varphi.$$

Hence, α_1 is contained in $\Omega^1_{h_1}(\pi)$. Moreover, if $f \equiv 0$, then the family α_t is constant.

This shows that α_t defines a strong deformation retraction of $\bar{\Omega}^1_{h_1}(\pi)$ into $\Omega^1_{h_1}(\pi)$. □

The lemma above, in particular, implies that the inclusion of $\Omega^1_{h_1}(\pi)$ into $\bar{\Omega}^1_{h_1}(\pi)$ is a homotopy equivalence. Thus, Corollary 2.1.6 shows that the deformation from Theorem 3.1.7 can be turned into a deformation D_t of $\mathcal{A}(\pi)$ into its subspace $\mathcal{A}_{h_1}(\pi) = \bar{\Omega}^1_{h_1}(\pi) \cap \mathcal{A}(\pi)$ consisting of those adapted contact forms that are standard with respect to h_1 for radius $1/2$ such that D_1 is a homotopy equivalence. Moreover, this deformation is smooth in the deformation parameter and the restrictions of $D_t(\alpha)$ and α to the tangent bundle of the pages agree outside U for all $t \in [0,1]$ and $\alpha \in \mathcal{A}(\pi)$.

To prove Theorem 3.1.3, it remains to show that we can arrange the correct function h_2.

Lemma 3.1.20. *Let* $\underline{h}$ *be a Lutz pair. Then there is a weak deformation retraction* D_t *of* $\mathcal{A}_{h_1}(\pi)$ *into* $\mathcal{A}_{\underline{h}}(\pi)$. *Moreover, we may assume that the deformation is smooth in the deformation parameter* t *and that the restrictions of* $D_t(\alpha)$ *and* α *to the tangent bundles of the pages agree for all* $t \in [0,1]$.

Proof. The proof of this lemma is similar to that of Lemma 3.1.10. However, the difficulties arise at different points.

First of all, if $\mathcal{A}_{h_1}(\pi)$ is empty, so is its subspace $\mathcal{A}_{\underline{h}}(\pi)$. So, let us assume that $\mathcal{A}_{h_1}(\pi)$ is non-empty.

Now, choose a fixed function $\bar{h}_2\colon [0,1] \to \mathbb{R}_0^+$ such that $\bar{h}_2(r) = h_2(2r)$ for $r \leq 1/2$. Given an $\alpha \in \mathcal{A}_{h_1}(\pi)$, we also define a function $\tilde{h}_2$ on $U \cong B \times D^2$ by $\tilde{h}_2 = \iota_{\partial_\varphi}\alpha$.

We would like to deform α into $\mathcal{A}_{\underline{h}}(\pi)$ via a family

$$\alpha_t = \alpha + t\lambda(r)\,\big(\bar{h}_2(r) - \tilde{h}_2\big)d\varphi$$

where λ is a cut-off function.

Unfortunately, unlike in the proof of Lemma 3.1.10, we need to modify $\tilde{h}_2$ first before this ansatz works. Let us defer this preparation to the end of the proof where we see which modification is needed.

We start with the construction of the cut-off function λ. First, choose a smooth monotonously decreasing reference function $\lambda_0\colon \mathbb{R} \to [0,1]$ that is constant of value 1 on $(-\infty, 0]$ and vanishes on $[1,\infty)$. Given this function we set $\lambda(r) = \lambda_0\big(\frac{2}{\delta}\left(r - \frac{1+\delta}{2}\right)\big)$ where $\delta > 0$ is a constant depending continuously on α that we still have to determine.

Inside the set $B \times \bar{B}_{1/2}(0)$, the function λ is constant of value 1; as a result, there, the family α_t is given by

$$\alpha_t = h_1(r/2)\,\alpha_B + \big((1-t)\,\tilde{h}_2 + t\bar{h}_2\big)d\varphi$$

where $\alpha_B = \alpha|_{TB}$. Since the properties of the functions in a Lutz pair are convex, the forms α_t are still standard with respect to h_1 for radius $1/2$. Moreover, $\alpha_0 = \alpha$ and α_1 is standard with respect to the Lutz pair $\underline{h}$ for radius $1/2$ and α_t is constant for $r \leq 1/2$ whenever α is already standard with respect to $\underline{h}$. Thus, it remains only to ensure that the forms α_t are contact forms.

To find a suitable constant δ, we proceed as in the proof of Lemma 3.1.10 and define a parametric version of α_t inside U that does not depend on t and λ. Namely, we set

$$\tilde{\alpha}_s = \alpha + s\big(\bar{h}_2(r) - \tilde{h}_2\big)d\varphi$$

for $s \in [0,1]$.

With this notation at hand, we can write $d\alpha_t$ as

$$d\alpha_t = d\tilde{\alpha}_{t\lambda} + t\lambda'\big(\bar{h}_2 - \tilde{h}_2\big)dr \wedge d\varphi.$$

Consequently, the contact condition reads

$$0 < \alpha_t \wedge (d\alpha_t)^n = \tilde{\alpha}_{t\lambda} \wedge (d\tilde{\alpha}_{t\lambda})^n + nt\lambda'\big(\bar{h}_2 - \tilde{h}_2\big)dr \wedge d\varphi \wedge \alpha \wedge (d\alpha)^{n-1}, \quad (3.8)$$

where we used that the projection of α_t to the pages is independent of t.

Since the forms $\tilde{\alpha}_s$ have the standard form in $B \times \bar{B}_{1/2}(0)$, they are contact forms on this set, *i.e.* $\tilde{\alpha}_s \wedge (d\tilde{\alpha}_s)^n > 0$. Moreover, the form $dr \wedge d\varphi \wedge \alpha \wedge (d\alpha)^{n-1}$ is a positive volume form on this set, too, because α is standard for radius $1/2$. These two properties are open. So, the proof of Corollary A.2 shows that there is a function $\delta\colon \mathcal{A}_{h_1}(\pi) \times [0,1] \to (0, 1/2)$ such that the two properties above hold for $\tilde{\alpha}_s$ and α, respectively, for $r \leq 1/2 + \delta(\alpha, s)$. Taking the minimum over $s \in [0,1]$ yields our choice of δ.

With this choice the first term in (3.8) is positive and the second one has the same sign as the function

$$t\lambda'\big(\bar{h}_2 - \tilde{h}_2\big).$$

Since λ' is non-positive we would be done if $\bar{h}_2 \leq \tilde{h}_2$ on the support of λ'.

So, we precede the deformation above by another deformation

$$\alpha_t^1 = \alpha + t\mu(r)\,\big\|\tilde{h}_2 - \bar{h}_2\big\|_{C^0(U)} d\varphi$$

where μ is a cut-off function defined by $\mu(r) = \lambda_0\big(\frac{2}{\delta}\big(\frac{1+\delta}{2} - r\big)\big)$ inside U. We extend μ to all of M as the constant function with value 1.

Because μ' is non-negative and on its support the form $dr \wedge d\varphi \wedge \alpha \wedge (d\alpha)^{n-1}$ is a positive volume form, the proof of Theorem 2.1.3 shows that $\alpha \wedge (d\alpha)^n$ can only increase if we add $t\mu\big\|\tilde{h}_2 - \bar{h}_2\big\|_{C^0(U)} d\varphi$ to α. Thus, the forms α_t^1 are contact forms.

By the same argument, the forms in the parametric version $\bar{\alpha}_s$ of the subsequent deformation

$$\begin{aligned}
\alpha_t^2 &= \alpha_1^1 + t\lambda\big(\bar{h}_2(r) - \big(\tilde{h}_2 + \mu\big\|\tilde{h}_2 - \bar{h}_2\big\|_{C^0(U)}\big)\big)d\varphi \\
&= \alpha_t + (1 - t\lambda)\,\mu\big\|\tilde{h}_2 - \bar{h}_2\big\|_{C^0(U)} d\varphi
\end{aligned}$$

given by

$$\bar{\alpha}_s = \tilde{\alpha}_s + (1-s)\,\mu\big\|\tilde{h}_2 - \bar{h}_2\big\|_{C^0(U)} d\varphi$$

satisfy $\bar{\alpha}_s \wedge (d\bar{\alpha}_s)^n \geq \tilde{\alpha}_s \wedge (d\tilde{\alpha}_s)^n$. Consequently, our previous choice of δ is still suitable.

Finally, let us take a look at the support of λ'. There, the function μ is constant of value 1. Accordingly, the second term of the right-hand side of (3.8) is non-negative for α_t replaced by α_t^2 because

$$t\lambda'\big(\bar{h}_2 - \big(\tilde{h}_2 + \big\|\tilde{h}_2 - \bar{h}_2\big\|_{C^0(U)}\big)\big) \geq 0.$$

This shows that first applying the deformation α_t^1 and then α_t^2 yields a deformation into $\mathcal{A}_{\underline{h}}(\pi)$. This is a weak deformation retraction since μ vanishes for $r \leq 1/2$ and hence the forms α_t^1 are standard for radius $1/2$ for the same Lutz pair as α.

To achieve smoothness in the parameter we have to concatenate the two deformations as in the proof of Proposition 3.1.8. That the restrictions to the tangent bundles of the pages remain unchanged follows from the fact that we changed α only by adding a multiple of $d\varphi$. $\square$

The lemma above finally concludes the proof of Theorem 3.1.3.

Remark 3.1.21. Since we do not change the restriction to the boundary in Corollary 2.1.6 and Lemma 3.1.20, Remark 3.1.14 about the restricted subspaces also applies in the contact setting. More precisely, all the deformations in the proof of Theorem 3.1.3 change the contact form induced on the binding only by multiplication with a positive constant. Thus, the deformations restrict to the corresponding subspaces in which the induced contact structure on the binding is fixed. Moreover, if we modify the deformations by multiplying the forms with the inverse of this constant at each time of the deformations, we get corresponding deformations of the subspaces in which the induced contact form on the binding is fixed. However, in this case, restrictions to the tangent bundle of the pages do change outside U.

3.1.4. Symplectic Open Books from Contact Open Books

In Section 2.2 we have seen two ways to construct an adapted contact form on $M(P,\Psi)$, given a symplectic open book (P,Ψ,β_0). In this subsection we use Theorem 3.1.3 to show that, up to contactomorphism, every contact manifold (M, ξ) such that ξ is supported by an open book decomposition (B, π) of M can be constructed this way. The corresponding symplectic open book is of the form (P, Ψ, β) where P is diffeomorphic to the closure of the page P_0 of (B, π). More precisely, we prove the following theorem.

Theorem 3.1.22. *Let (B, π) be an open book decomposition of a closed manifold M, $\underline{h}$ a Lutz pair such that h_2 is constant in a neighbourhood of 1, and $U \cong D^2 \times B$ an adapted neighbourhood of the binding B. Furthermore, let α be a contact form adapted to (B, π) and P the subset $P_0 \setminus \big(B \times B_{1/4}(0)\big)$ of the page P_0.*

Then $(M, \ker\alpha)$ is contactomorphic to the result of applying the generalised Thurston-Winkelnkemper construction with the Lutz pair $\underline{h}$ to a symplectic open book (P, Ψ, β) where β coincides with the restriction of α to TP_0 on $P \setminus U$.

Proof. We want to apply Theorem 3.1.3 to obtain a path α_t of adapted contact forms from α to some adapted contact form α_1 that is standard with respect to $\underline{h}$ for radius $1/4$. So we first choose a suitable Lutz pair $\big(\bar{h}_1, \bar{h}_2\big)$ extending $\underline{h}$, *i.e.* a Lutz pair such that $\bar{h}_i(r) = h_i(2r)$, $i = 1, 2$, for all $r \leq 1/2$. Furthermore, given a constant $\epsilon > 0$ such that h_2 is constant on $[1 - 4\epsilon, 1]$, we also demand that $\bar{h}_2$ is constant on $[1/2 - 2\epsilon, 1/2 + 2\epsilon]$.

Now, we apply Theorem 3.1.3 with the Lutz pair $\big(\bar{h}_1, \bar{h}_2\big)$. This yields a path α_t of adapted contact forms from α to some adapted contact form α_1 that is standard with respect to $\big(\bar{h}_1, \bar{h}_2\big)$ for radius $1/2$. Because $\bar{h}_i(r) = h_i(2r)$, $i = 1, 2$, for all $r \leq 1/2$, this implies that α_1 is standard with respect to $\underline{h}$ for radius $1/4$.

By Theorem 1.1.8, the family $\xi_t = \ker\alpha_t$ is covered by an isotopy Φ_t of M that fixes B pointwise. In particular, $(M, \ker\alpha)$ and $(M, \ker\alpha_1)$ are contactomorphic.

Next, we show that $M(B, \pi, \alpha_1)$ is strictly contactomorphic to the result of the generalised Thurston-Winkelnkemper construction applied to the symplectic open book $(P, \Psi, j^*\alpha_1)$ where Ψ is a diffeomorphism of P and j a fixed inclusion of P into M.

To this end, we have to find a suitable vector field transverse to the pages that agrees with the coordinate vector field ∂_φ inside a neighbourhood of $B \times \bar{B}_{1/4}(0)$. Outside this neighbourhood, our candidate for such a vector field is the scaled Reeb vector field $X = fR_{\alpha_1}$ with $f = 1/(\iota_{R_{\alpha_1}} d\varphi)$. This is well defined because $d\alpha_1$ is positively non-degenerate on the tangent bundles of the pages and hence R_{α_1} positively transverse to them.

Thanks to α_1 being standard with respect to $(\bar{h}_1, \bar{h}_2)$ for radius $1/2$ we are able to compute R_{α_1} explicitly inside $B \times \bar{B}_{1/2}(0)$. It is given by

$$R_{\alpha_1} = \frac{\tilde{h}_2'}{\tilde{h}_1\tilde{h}_2' - \tilde{h}_2\tilde{h}_1'} R_{\alpha_B} - \frac{\tilde{h}_1'}{\tilde{h}_1\tilde{h}_2' - \tilde{h}_2\tilde{h}_1'} \partial_\varphi$$

where R_{α_B} is the Reeb vector field to the contact form $\alpha_B = \alpha_1|_{TB}$ on B and $\tilde{h}_i(r) = \bar{h}_{2r}$, $i = 1, 2$. In particular, wherever $\tilde{h}_2$ is constant, we have $R_{\alpha_1} = \bar{h}_2^{-1}\partial_\varphi$. This shows that we can glue the vector field X and ∂_φ along the set $\{r \in (1/4 - \epsilon, 1/4 + \epsilon)\}$. By a slight abuse of notation, we again denote by X the result of gluing these two vector fields.

The vector field X allows us to identify $N = M \setminus \big(B \times B_{1/4}(0)\big)$ with the mapping torus $P(\Psi_{2\pi})$ where Ψ_t is the time-t-flow of X, which exists for all $t \in \mathbb{R}$ because X is parallel to the boundary of N. More precisely, this identification is induced by the map

$$\begin{aligned} \Phi\colon P \times \mathbb{R} &\to N \\ (x, \vartheta) &\mapsto \Psi_\vartheta(j(x))\,, \end{aligned}$$

which descends to a diffeomorphism $P(\Psi_{2\pi}) \to N$ since $\Psi_{2\pi} \circ \Psi_t = \Psi_{t+2\pi}$ for all $t \in \mathbb{R}$.

Pulling back α_1 with Φ yields

$$\Phi^*\alpha_1 = (\Psi_\vartheta \circ j)^*\,\alpha_1 + ((\iota_X\alpha_1) \circ \Phi)\, d\vartheta = j^*\Psi_\vartheta^*\alpha_1 + (f \circ \Phi)\, d\vartheta.$$

We would like to see that this is of the form $j^*\alpha_1 + dh$ for some function h. So, let us examine the dependence of $\Psi_t^*\alpha_1$ on the flow parameter. We have

$$L_X\alpha_1 = d(\iota_X\alpha_1) + \iota_X d\alpha_1 = d\big(f\iota_{R_{\alpha_1}}\alpha_1\big) + f\iota_{R_{\alpha_1}} d\alpha_1 = df\,.$$

With this in mind, we define a function $h\colon P \times \mathbb{R} \to \mathbb{R}$ by $h(x, \vartheta) =$

$\int_0^\vartheta f(\Phi(x,t))\,dt$. Its differential is given by

$$\begin{aligned} dh &= d\left(\int_0^\vartheta f(\Phi(x,t))\,dt\right) \\ &= \int_0^\vartheta d(j^*\Psi_t^* f)\,dt + (f\circ\Phi)\,d\vartheta \\ &= j^*\left(\int_0^\vartheta (\Psi_t^* df)\,dt\right) + (f\circ\Phi)\,d\vartheta \\ &= j^*\left(\int_0^\vartheta (\Psi_t^* L_X\alpha_1)\,dt\right) + (f\circ\Phi)\,d\vartheta \\ &= j^*\left(\Psi_\vartheta^*\alpha_1 - \alpha_1\right) + (f\circ\Phi)\,d\vartheta\,. \end{aligned}$$

Therefore, we have

$$\Phi^*\alpha_1 = j^*\Psi_\vartheta^*\alpha_1 + (f\circ\Phi)\,d\vartheta = j^*\alpha_1 + dh.$$

To see that this form arises from the generalised Thurston-Winkelnkemper construction, we construct a strict contactomorphism between the ordinary mapping torus $(P(\Psi_{2\pi}), \Phi^*\alpha_1)$ and the generalised mapping torus $\big(P_{\hat h}(\Psi_{2\pi}), \beta + d\vartheta\big)$ for some function $\hat h$ and Liouville form β on P.

In this construction, it comes to our help that $\partial_\vartheta h = \iota_{\partial_\vartheta} dh = f\circ\Phi > 0$. Accordingly, the map

$$\begin{aligned} \hat\Phi\colon P\times\mathbb{R} &\to P\times\mathbb{R} \\ (x,\vartheta) &\mapsto (x, h(x,\vartheta)) \end{aligned}$$

is a diffeomorphism. Furthermore, a closer inspection of h shows that

$$\begin{aligned} h\big(\Psi_{2\pi}^{-1}(x), \vartheta + 2\pi\big) &= \int_0^{\vartheta+2\pi} (f\circ\Psi_t)\big((j\circ\Psi_{2\pi}^{-1})(x)\big)\,dt \\ &= \int_0^{\vartheta+2\pi} (f\circ\Psi_{t-2\pi})(j(x))\,dt \\ &= \int_0^{\vartheta} (f\circ\Psi_t)(j(x))\,dt + \int_0^{2\pi} (f\circ\Psi_t)\big((j\circ\Psi_{2\pi}^{-1})(x)\big)\,dt \\ &= h(x,\vartheta) + h\big(\Psi_{2\pi}^{-1}(x), 2\pi\big)\,. \end{aligned}$$

Thus, the map $\hat{\Phi}$ descends to a diffeomorphism from $P(\Psi_{2\pi})$ to $P_{\hat{h}}(\Psi_{2\pi})$ for the function $\hat{h}$ defined by $\hat{h}(x) = h\big(\Psi_{2\pi}^{-1}(x)\,, 2\pi\big)$.

Pulling back $j^*\alpha_1 + d\vartheta$ with $\hat{\Phi}$ yields

$$\hat{\Phi}^*(j^*\alpha_1 + d\vartheta) = j^*\alpha_1 + dh = \Phi^*\alpha_1.$$

Hence, (N, α_1) is strictly contactomorphic to $\big(P_{\hat{h}}(\Psi_{2\pi})\,, j^*\alpha_1 + d\vartheta\big)$.

Since X coincides with ∂_φ in neighbourhood of ∂N, the monodromy $\Psi_{2\pi}$ agrees with the identity on a neighbourhood of ∂P. Consequently, $(P, \Psi_{2\pi})$ is an abstract open book. Furthermore, following the flow of $2\,(j_*)^{-1}\,\partial_r$, we can define a collar neighbourhood $C \cong B \times [1/8, 1/4)$ of ∂P such that $j(x,s) = (x, 2s, 0) \in B \times B_{1/2}(0)$. In these coordinates, $j^*\alpha_1$ is given by

$$j^*\alpha_1 = \bar{h}_1(s)\,\alpha_B,$$

which implies that, inside C, a Liouville vector field to $j^*\alpha_1$ is given by $Y = \bar{h}_1/\bar{h}_1'\,\partial_s$. Because $\bar{h}_1$ is positive and $\bar{h}_1'$ negative, Y points outwards along ∂P. Consequently, $(P, \Psi_{2\pi}, j^*\alpha_1)$ is a symplectic open book.

Denote by $\hat{\alpha}$ the contact form on $M(P, \Psi_{2\pi})$ from the generalised Thurston-Winkelnkemper construction for $(P, \Psi_{2\pi}, j^*\alpha_1)$. Then, by the discussion above, we already know that (N, α_1) is strictly contactomorphic to $(P(\Psi_{2\pi})\,, \hat{\alpha})$.

The other part of M, *i.e.* $B \times \bar{B}_{1/4}(0)$, is diffeomorphic to $B \times D^2$ via the map ψ given by $\psi(x, (r, \varphi)) = (x, (4r, \varphi))$. Pulling back α_1 yields

$$\psi^*\alpha_1 = \psi^*(h_1(4r)\,\alpha_B + h_2(4r)\,d\varphi) = h_1(r)\,\alpha_B + h_2(r)\,d\varphi.$$

Thus, $\big(B \times \bar{B}_{1/4}(0)\,, \alpha_1\big)$ and $\big(B \times D^2, \hat{\alpha}\big)$ are strictly contactomorphic. This concludes the proof of the theorem. $\square$

Combining this with the following result by Giroux and Mohsen [22], we can infer that up to contactomorphism every contact manifold arises from the generalised Thurston-Winkelnkemper construction. More precisely, we can infer that Theorem 2.2.9 holds.

Theorem 3.1.23 (*Cf.* [22, Théorème 10]). *Every contact structure is supported by an open book decomposition whose pages are Weinstein manifolds.*

Unfortunately, like for the stronger result Theorem 2.2.9, so far no detailed proof of this result has been published.

3.2. Manifolds with Boundary

On manifolds with non-empty boundary, many constructions known for closed manifolds fail. This is often connected to the problem that flows of a vector fields do not have to exist globally. Consequently, it is necessary to introduce suitable boundary conditions. In this section, we construct several weak deformation retractions that allow us to strengthen boundary conditions on diffeomorphisms and symplectic forms on a manifold with boundary.

3.2.1. Diffeomorphisms Fixing the Boundary

The most basic boundary condition one can impose on a diffeomorphism Ψ of a manifold W with boundary is that the restriction of Ψ to the boundary ∂W agree with the identity. A stronger, and often more useful, boundary condition is to demand that Ψ have compact support in the interior of P. Given a collar neighbourhood $C \cong (-2,1] \times \partial W$ of the boundary, we can also define the even stronger boundary condition that Ψ coincide with the identity on the fixed collar $C' = (-1,0] \times \partial W$.

Let us denote by $\mathcal{D}_\partial$, $\mathcal{D}$, and $\mathcal{D}_C$ the spaces of those diffeomorphisms of W satisfying these boundary conditions, respectively, in the same order as above.

The aim of this subsection is to show that we can always arrange the stronger boundary conditions using a deformation, provided one of the weaker ones is satisfied. More precisely, we prove the following theorem.

Theorem 3.2.1. *Let P be a manifold with non-empty boundary and $C \cong (-2,1] \times \partial W$ a collar neighbourhood of the boundary. Then there is a weak deformation retraction of $\mathcal{D}_\partial$ into its subspace $\mathcal{D}$ that restricts to a weak deformation retraction of $\mathcal{D}_C$ into $\mathcal{D}$.*

Proof. All three spaces contain the identity map and, hence, are non-empty. So, let $\Psi \in \mathcal{D}_\partial$.

We construct the deformation in two steps. First, we arrange that Ψ coincides with the identity on a small neighbourhood of the boundary and then we extend this neighbourhood to $C' = (-1,0] \times \partial W$.

Because Ψ is a diffeomorphism and its restriction to ∂W agrees with the identity, we know that there is an open neighbourhood $U \subset \Psi^{-1}(C')$ of ∂W in which $\iota_{\Psi_*\partial_s} ds > 0$, where s is the collar coordinate.

The condition that $\iota_{\Psi_* \partial_s} ds|_{\{s\}\times\partial W} > 0$ is an open condition on Ψ, as well as the condition that $\Psi(\{s\} \times \partial W) \subset C'$. Thus, the function $E\colon \mathcal{D}_\partial \to (0,1]$ mapping Ψ to the supremum of all $r \in [0,1)$ such that $[-2r, 0] \times \partial W$ is contained in $\Psi^{-1}(C')$ and $\iota_{\Psi_* \partial_s} ds$ positive on $[-2r, 0] \times \partial W$ is lower semi-continuous. We would like to apply Theorem A.1 to obtain a continuous function $s_0\colon \mathcal{D}_\partial \to (0,1)$ satisfying $0 < s_0 < E$. Then the properties in the definition of E would be satisfied on $[-2s_0(\Psi), 0] \times \partial W$.

The only assumption from Theorem A.1 that remains to be verified is the paracompactness of the space $\mathcal{D}_\partial$. That $\mathcal{D}_\partial$ is paracompact follows from the fact that it is metrizable by [27, Proposition 42.3] or rather an adapted version. Strictly speaking, the methods in [27, Section 42] do not apply if the range is a manifold with boundary. Nevertheless, with minor changes the treatment applies if the maps are prescribed at every point where the maps are allowed to touch the boundary. Since this is the case for $\mathcal{D}_\partial$, we can obtain the function s_0 via Theorem A.1.

Now we are ready to start with the first step of the construction, which we again divide into two steps. In the first step we arrange that Ψ preserves the level sets of the collar coordinate s inside $[-s_0(\Psi), 0] \times \partial W$. Then we use this property to shift the level sets in order to arrange that Ψ agrees with the identity on $[-{}^{s_0(\Psi)}/_2, 0] \times \partial W$.

The idea of the first step is to construct a suitable isotopy Φ_t from the identity to a diffeomorphism that sends $C_{s_0} = [-s_0(\Psi), 0] \times \partial M$ to $\Psi(C_{s_0})$ and then deform Ψ through the family $\Psi_t^1 = \Phi_t^{-1} \circ \Psi$.

To construct the isotopy Φ_t, we have to take a closer look at the map Ψ inside C_{s_0}. Because $\iota_{\Psi_* \partial_s} ds > 0$, the set $\Psi(\{s\} \times \partial W)$ is a graph over ∂W for all $s \in [0, s_0(\Psi)]$. This implies that $\mathrm{pr}_{\partial W} \circ \Psi_s$ is a diffeomorphism of ∂W for the same s, where $\mathrm{pr}_{\partial W}$ is the projection to ∂W in the collar $C \cong (-2, 0] \times \partial W$, and $\Psi_s\colon \partial W \to C$ is the map given by $\Psi_t(p) = \Psi(s, p)$. This diffeomorphism has the property that

$$\mathrm{pr}_{\partial W} \circ \left(\Psi_s \circ (\mathrm{pr}_{\partial W} \circ \Psi_s)^{-1} \right) = \mathrm{id}_{\partial W} .$$

Thus, the smooth strictly monotonously decreasing family of functions

$$\rho_s = \mathrm{pr}_{(-2,0]} \circ \left(\Psi_s \circ (\mathrm{pr}_{\partial W} \circ \Psi_s)^{-1} \right),$$

where $\mathrm{pr}_{(-2,0]}$ is the projection to $(-2, 0]$ in C, measures the height of the graph $\Psi(\{s\} \times \partial W)$ over, or rather under, ∂W.

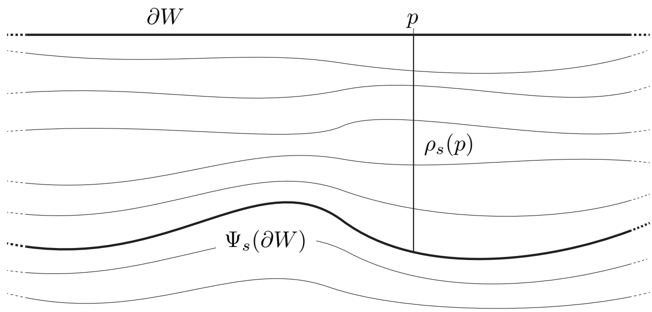

Figure 3.2.: The collar neighbourhood C'

We use the family ρ_s to construct a continuous family $\nu_\Psi \colon (-2,0] \times \partial M \to (-2,0]$ of smooth strictly monotonously increasing functions with $\nu_\Psi(s,p) = \rho_s(p)$ for $s \in [-s_0(\Psi), 0]$ whose restriction to $(-2,-3/2] \times \partial M$ agrees with $\mathrm{pr}_{(-2,0]}$. Moreover, we demand that ν_Ψ be given by $\mathrm{pr}_{(-2,0]}$ whenever $\rho_s \equiv s$ for all $s \in [-s_0(\Psi), 0]$. Such a family can be constructed by interpolating a suitable affine linear function depending continuously on $m := \min \rho_{s_0(\Psi)}$ and $m' := \max_{s\in[-s_0(\Psi),0]} \max \partial_s \rho_s$ on one side with $\mathrm{pr}_{(-2,0]}$ and on the other side with $(p,s) \mapsto \rho_s(p)$, as indicated in Figure 3.3.

Now we are in the position to introduce the isotopy $\Phi_t^1 \colon W \to W$ advertised earlier. On the complement of the collar C, we define Φ_s to agree with the identity and, inside C, we set

$$\Phi_t(s,p) = (t\nu_\Psi(s,p) + (1-t)\, s, p)\,.$$

This depends continuously on Ψ and, since $\partial_s \nu_\Psi > 0$, it is a family of diffeomorphisms.

Moreover, if there is an $\bar{s} \in [0,2)$ such that Ψ agrees with the identity on $(-\bar{s},0] \times \partial W$, then we know that

$$\begin{aligned}\Phi_t(s,p) &= (t\nu_\Psi(s,p) + (1-t)\, s, p) = (t\rho_s(p) + (1-t)\, s, p)\\ &= (ts + (1-t)\, s, p) = (s,p)\end{aligned}$$

for $s \leq \min\{s_0(\Psi), \bar{s}\}$. Hence, the isotopy Φ_t has compact support in the interior of W whenever this is true for Ψ.

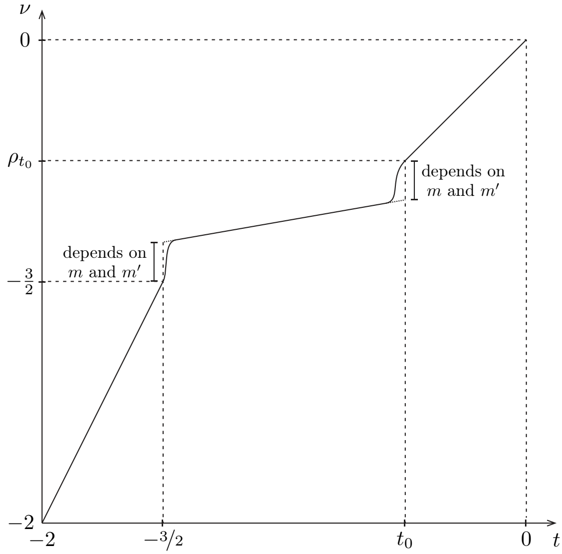

Figure 3.3.: The function ν_Ψ

If moreover Ψ agrees with the identity on $C' = (-1, 0] \times \partial M$, we have $\rho_s(p) = s$ for all $s \leq s_0(\Psi) < 1$. This implies that, in this case, the isotopy Φ_t is constant.

As indicated before, we define the first deformation of Ψ by

$$\Psi_t^1 = \Phi_t^{-1} \circ \Psi.$$

Since Φ_1 sends the sets $\{s\} \times M$ to $\Psi(\{s\} \times M)$ for all $s \geq -s_0(\Psi)$, the map Ψ_1^1 preserves the level sets of the collar coordinate s inside $C_{s_0} = [-s_0(\Psi), 0] \times \partial W$. Moreover, by the discussion above, the isotopy Ψ_t^1 preserves the spaces $\mathcal{D}$ and $\mathcal{D}_C$.

Equipped with the knowledge that Ψ_1^1 preserves the level sets of the collar coordinate, we are able to push the restriction of Ψ_1^1 to the boundary, which is the identity, onto a small neighbourhood of the boundary by displacing the level sets. More precisely, we perform the following construction.

Let $\lambda_{s,r} \colon [-2, 0] \to [-2, 0]$, $(s, r) \in [0, 1] \times (0, 1]$, be a continuous family of smooth monotonously increasing functions that coincide with the identity on $[-2,-r]$ and vanish on $[-{}^{sr}/_{2},0]$. Furthermore, we demand that $\lambda_{0,r}$ be the identity for all $r \in (0,1]$. Then the family Ψ_t^2 with $\Psi_t^2|_{W\setminus C} = \Psi_1^1$ and

$$\Psi_t^2(s,p) = \left(s, \mathrm{pr}_{\partial W}\left(\Psi_1^1\left(\lambda_{t,s_0(\Psi)}(s)\,,p\right)\right)\right)$$

for $(s,p) \in C$ defines a deformation such that $\Psi_1^1|_{[-{}^{s_0(\Psi)}/_2,0]\times\partial W}$ is the identity and $\Psi_0^2 = \Psi_1^1$.

Note that, whenever Ψ_1^1 agrees with the identity on $[\bar{s},0]\times\partial W$ for some $\bar{s} < 0$, this is also true for Ψ_t^2 on $\lambda_{t,s_0(\Psi)}^{-1}([\bar{s},0])\times\partial W$. In particular, if Ψ_1^1 agrees with the identity on all of $C' = (-1,0]\times\partial W$, then this is also true for Ψ_t^2 because $s_0(\Psi) < 1$. This shows that this deformation preserves $\mathcal{D}_\partial$ and $\mathcal{D}_C$.

For our final deformation, we construct a family Φ_t^r of isotopies of W such that Φ_1^r maps $[-{}^{r}/_{2},0]\times\partial W$ to $[-1,0]\times\partial W$. Then, conjugating Ψ_1^2 with $\Phi_t^{s_0(\Psi)}$ yields the final deformation Ψ_t^3 into $\mathcal{D}_C$.

To construct the family Φ_t^r, we first choose a continuous family of smooth strictly monotonously increasing functions $\mu_{t,r}\colon\ [-2,0]\to[-2,0]$, $(t,r)\in[0,1]\times(0,1]$, that agree with the identity on $[-2,-{}^{3}/_{2}]$ and satisfy $\mu_{t,r}(s) = s\left((1-t)+{}^{2t}/_{r}\right)$ for all $t\in[-{}^{r}/_{2},0]$. Moreover, we demand that $\mu_{0,r}$ agree with the identity and that $\mu_{t,r}(-1)$ decrease monotonously with respect to t for all $r\in(0,1]$.

With this family at hand, we define $\Phi_t^r\colon W\to W$ to agree with the identity on $W\setminus C$ and on C to be given by $\Phi_t^r(s,p) = (\mu_{t,r}(s)\,,p)$. Then, for fixed $r\in(0,1]$, the maps Φ_t^r form an isotopy satisfying

$$\Phi_t^r([-{}^{r}/_{2},0]) = [-\,(1-t)\,{}^{r}/_{2}-t,0]\times\partial W.$$

Accordingly, for $\Psi\in\mathcal{D}_\partial$, we know that the map

$$\Psi_t^3 = \Phi_t^{s_0(\Psi)}\circ\Psi_1^2\circ\left(\Phi_s^{s_0(\Psi)}\right)^{-1}$$

agrees with the identity on $[-\,(1-t)\,{}^{s_0(\Psi)}/_{2}-t,0]\times\partial W$, because Ψ_1^2 agrees with the identity on $[-{}^{s_0(\Psi)}/_{2},0]\times\partial W$.

It remains to show that this deformation leaves the space $\mathcal{D}_C$ invariant. For every $\Psi\in\mathcal{D}_C$ we know that $\Psi_1^2 = \Psi$ and that this map agrees with the identity on $[-1,0]\times\partial W$. This implies that Ψ_t^3 agrees with the identity on

$$\Phi_t^{s_0(\Psi)}([-1,0]\times\partial W) = \left[\mu_{t,s_0(\Psi)}(-1)\,,0\right]\times\partial W \supset [-1,0]\times\partial W.$$

Hence, Ψ_t^3 is an element of $\mathcal{D}_C$ for all $t \in [0,1]$.

Concatenating the three deformations we constructed yields a weak deformation retraction of $\mathcal{D}_\partial$ into $\mathcal{D}_C$ that preserves $\mathcal{D}$. $\square$

3.2.2. Symplectic Forms Prescribed on the Boundary

In contact topology we often encounter symplectic manifolds with boundary, sometimes as symplectic cobordisms and sometimes as pages of open books. In both cases certain boundary conditions apply.

Let $\Omega_0^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$ be the space of symplectic forms that endow a manifold W with the structure of a symplectic cobordism from the strict contact manifold $(\partial_- W, \alpha_-)$ to $(\partial_+ W, \alpha_+)$. Then every $\omega \in \Omega_0^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$ has a primitive β in a neighbourhood of ∂W that agrees on $T\partial W_\pm$ with $\alpha_\pm$. Furthermore, all $\omega \in \Omega_0^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$ induce the same orientation on W.

In the construction of contact open books in Subsection 2.2.2 we encountered another kind of boundary conditions. Namely, we demanded that the elements of $\mathcal{B}_\infty(P, \beta_0)$ agree with a given Liouville form β_0 on the boundary including all derivatives. Again, in principle, the Liouville form β_0 need only be defined in a neighbourhood of the boundary ∂P.

Though the boundary condition above is stronger than the first one, in practice one often needs an even stronger condition, namely that the symplectic forms under consideration agree with a given symplectic form on a neighbourhood of the boundary. A notable example is the Moser trick: we need the constructed vector field to vanish in a neighbourhood of the boundary in order to guarantee the global existence of the flow. This can be achieved by restriction to symplectic forms agreeing outside a compact set in the interior.

Motivated by the examples above we introduce the following spaces. Let W be a compact manifold with non-empty boundary and ω_0 a symplectic form defined on a neighbourhood of ∂W. Then we define the spaces $\Omega_0^{\mathrm{S}}(W)$ of symplectic forms that induce the same orientation as ω_0 and whose restriction to $T\partial W$ agrees with that of ω_0, its subspace $\Omega_\infty^{\mathrm{S}}(W)$ consisting of those symplectic forms that agree with ω_0 on ∂W including all derivatives, and the subspace $\Omega_{\mathrm{c}}^{\mathrm{S}}(W)$ of those symplectic forms that agree with ω_0 on a neighbourhood of ∂W.

Furthermore, we introduce the corresponding subspaces $\Omega_i^{\mathrm{ES}}(W)$ and $\Omega_i^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$ of the space $\Omega^{\mathrm{ES}}(W)$ of exact symplectic forms on W and of $\Omega_0^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$, respectively, and the spaces $\mathcal{B}_i(W)$ of Liouville forms satisfying the boundary conditions with ω_0 replaced by a Liouville form β_0 defined on a neighbourhood of ∂W where $i = 0, \infty, \mathrm{c}$.

The goal of this subsection is to show that for each of the triples of spaces given by the different values of i, the spaces given by $i = 0, \infty$ can be deformed into the space given by $i = c$. In particular, the spaces in each triple are homotopy equivalent.

The following theorem provides the first of these deformations; the existence of the remaining ones will follow from a closer examination of the construction of this one.

Theorem 3.2.2. *Let W be a compact manifold with boundary and ω_0 a symplectic form defined in a neighbourhood of the boundary of W. Then there is a weak deformation retraction of $\Omega_0^{\mathrm{S}}(W)$ into $\Omega_{\mathrm{c}}^{\mathrm{S}}(W)$ that preserves $\Omega_\infty^{\mathrm{S}}(W)$.*

Proof. If $\Omega_0^{\mathrm{S}}(W)$ is empty, so are its subspaces $\Omega_\infty^{\mathrm{S}}(W)$ and $\Omega_{\mathrm{c}}^{\mathrm{S}}(W)$. So, let us assume that $\Omega_0^{\mathrm{S}}(W)$ is non-empty.

Fix a collar neighbourhood $C \cong (-2, 0] \times \partial W$ of the boundary of W such that ω_0 is defined on all of C. Now, let $\omega \in \Omega_0^{\mathrm{S}}(W)$.

The idea of the proof is to define the deformation by

$$\tilde{\omega}_t = \omega_0 + (1 - t)\,(\omega - \omega_0) + t\Phi^*(\omega - \omega_0)$$

with $t \in [0, 1]$. Here, $\Phi\colon W \to W$ is a map given by the identity outside of C and by $(s, p) \mapsto (\lambda(s)\,, p)$ inside C, where s is the collar parameter and $\lambda\colon (-2, 0] \to (-2, 0]$ a smooth monotonously increasing function that agrees with the identity on $(-2, -1]$ and vanishes on some neighbourhood of 0.

Since the restrictions of ω and ω_0 to $T\partial W$ agree, $\tilde{\omega}_1$ agrees with ω_0 on the neighbourhood on which λ vanishes. Moreover, whenever ω already coincides with ω_0 on the boundary including all derivatives, or even on a neighbourhood of the boundary, this is also true for the family $\tilde{\omega}_t$.

The difficulty of the proof is to find functions λ continuously depending on ω for which $\tilde{\omega}_t$ stays non-degenerate, and hence symplectic, for all $t \in [0, 1]$.

The non-degeneracy of $\tilde{\omega}_t$ is a problem on C only because outside of C the forms $\tilde{\omega}_t$ and ω agree. So we may restrict our attention to C. There, we can decompose $\omega - \omega_0$ as

$$\omega_\Delta = \omega - \omega_0 = \omega_\partial^\Delta + \omega_s^\Delta,$$

where $\omega_s^\Delta = ds \wedge (\iota_{\partial_s}\omega_\Delta)$. Then ω_∂^Δ is a $(-2,0]$-family of forms on ∂W that vanishes for $s = 0$.

In this decomposition, $\tilde{\omega}_t$ can be written as

$$\begin{aligned}
(\tilde{\omega}_t)_{(s,p)} &= (\omega_0 + (1-t)\,\omega_\Delta + t\Phi^*\omega_\Delta)_{(s,p)} \\
&= \omega_{(s,p)} + t\,(\Phi^*\omega_\Delta - \omega_\Delta)_{(s,p)} \\
&= \omega_{(s,p)} + t\left((\omega_\Delta)_{(\lambda(s),p)} - (\omega_\Delta)_{(s,p)} + (\lambda'(s)-1)\left(\omega_s^\Delta\right)_{(\lambda(s),p)}\right) \\
&= \omega_{(s,p)} + t\left(\left(\omega_\Delta^{\lambda(s)}\right)_{(s,p)} + (\lambda'(s)-1)\left(\omega_s^\Delta\right)_{(\lambda(s),p)}\right)
\end{aligned}$$

where we introduced the abbreviation $(\omega_\Delta^r)_{(s,p)} = (\omega_\Delta)_{(r,p)} - (\omega_\Delta)_{(s,p)}$.

After identifying top-dimensional forms on C with functions via the volume form ω_0^n, this allows us to write the condition that $\tilde{\omega}_t^n$ be non-degenerate as

$$\begin{aligned}
0 < (\tilde{\omega}_t^n)_{(s,p)} = tn\,(\lambda'(s)-1)\left(\omega_s^\Delta\right)_{(\lambda(s),p)} \wedge \left(\omega + t\omega_\Delta^{\lambda(s)}\right)_{(s,p)}^{n-1} \\
+ \left(\omega + t\omega_\Delta^{\lambda(s)}\right)_{(s,p)}^{n}.
\end{aligned} \tag{3.9}$$

In order to obtain conditions useful in the construction of λ we replace $\tilde{\omega}_t^n$ by a parametric version $\Omega_{t,q}^r$ not depending on λ. It is given by

$$\left(\Omega_{t,q}^r\right)_{(s,p)} = (\omega + t\omega_\Delta^r)_{(s,p)}^n + tqn\left(\omega_s^\Delta\right)_{(r,p)} \wedge (\omega + t\omega_\Delta^r)_{(s,p)}^{n-1}$$

where $r \in [0,1]$ represents the possible values of λ and $q \in [-1,\infty)$ the possible values of $(\lambda' - 1)$.

For $r = 0 = s$ we have

$$\begin{aligned}
\left(\Omega_{t,q}^0\right)_{(0,p)} &= \omega_{(0,p)}^n + tqn\left(\omega_s^\Delta\right)_{(0,p)} \wedge \omega_{(0,p)}^{n-1} \\
&= (1+tq)\,\omega_{(0,p)}^n + tq\left(n\left(\omega_s^\Delta\right)_{(0,p)} \wedge \omega_{(0,p)}^{n-1} - \left(\omega_0 + \omega_s^\Delta\right)_{(0,p)}^n\right) \\
&= (1+tq)\,\omega_{(0,p)}^n - tq\,(\omega_0)_{(0,p)}^n \\
&= (1+tq)\,\omega_{(0,p)}^n + (1-(1+tq))\,(\omega_0)_{(0,p)}^n.
\end{aligned}$$

Note that by our restrictions on t and q we know that $(1+tq) \geq 0$. Thus, $\left(\Omega_{t,q}^0\right)_{(0,p)} > 0$ if one of the following conditions holds.

- $\omega^n_{(0,p)} \geq (\omega^n_0)_{(0,p)}$
- $(1+tq) < \frac{(\omega_0^n)_{(0,p)}}{(\omega_0^n)_{(0,p)} - (\omega^n)_{(0,p)}}$

Now, let $\mu\colon \mathbb{R}^+ \to (1,2]$ be a smooth function that is constant of value 2 on $[1,\infty)$ and satisfies $\mu(\vartheta) \leq {}^1/_{(1-\vartheta)}$ for $\vartheta \in (0,1)$. This function can be turned into a continuous function on $\Omega_0^{\mathrm{S}}(W)$ by defining its value at $\omega \in \Omega_0^{\mathrm{S}}(W)$ as $\mu(\omega) = \mu(\min({}^{\omega^n}/_{\omega_0^n})|_{\partial W})$. Then, we know that $\left(\Omega^0_{s,q}\right)_{(0,p)} > 0$ for all $p \in \partial W$, $t \in [0,1]$, and $-1 \leq q \leq \mu(\omega) - 1$.

The condition that $\left(\Omega^r_{t,q}\right) > 0$ on $\{s\} \times \partial W$ is open with respect to ω. Moreover, it is satisfied on $\partial W = \{0\} \times \partial W$ for $r = 0$ for all $t \in [0,1]$ and $q \in [-1, \mu(\omega)-1]$, as we have just seen. Consequently, Corollary A.2 provides a function $S_0\colon \Omega_0^{\mathrm{S}}(W) \times [-1,\infty) \times [0,1] \to (0,1)$ such that $\left(\Omega^r_{t,q}\right) > 0$ on $[-S_0(\omega,q,t)\,, 0] \times \partial W$ for all $r \in [0, S_0(\omega,q,t)]$, provided $q \leq \mu(\omega) - 1$. Taking the minimum of S_0 over $q \in [-1, \mu(\omega)-1]$ and $t \in [0,1]$ for fixed ω yields a function $s_0\colon \Omega_0^{\mathrm{S}}(W) \to (0,1)$ independent of q and t with the same properties as S_0. Next, we use this function to construct a suitable function λ.

Let $\lambda_{\rho,\sigma}\colon (-2,0] \to (-2,0]$, $\rho \in [0,1]$, $\sigma \in (0, {}^1/_2)$, be a continuous family of monotonously increasing smooth functions that agree with the identity on $(-2,-\rho]$, vanish on $[-\sigma\rho, 0]$ and satisfy $\lambda'_{r,\sigma} \leq {}^{(1+\sigma)}/_{(1-\sigma)}$. Such functions exist because ${}^{(1+\sigma)}/_{(1-\sigma)} > {}^1/_\sigma$.

Given this family, a suitable function λ can be defined by

$$\lambda = \lambda(\omega) = \lambda_{s_0(\omega), \frac{\mu(\omega)-1}{2(1+\mu(\omega))}}\,.$$

Here, we have

$$0 < \sigma = \frac{1}{2}\frac{\mu(\omega)-1}{(1+\mu(\omega))} < \frac{1}{2}.$$

Because ${}^{(1+\sigma)}/_{(1-\sigma)}$ is strictly monotonously increasing for $\sigma > 0$, we know that

$$\lambda' \leq \frac{1+\sigma}{1-\sigma} < \frac{(\mu(\omega)+1) + (\mu(\omega)-1)}{(\mu(\omega)+1) - (\mu(\omega)-1)} = \mu(\omega)\,.$$

Let us now return to the non-degeneracy of $\tilde{\omega}_t$. Outside the collar C and for $s \leq -s_0(\omega)$ the map Φ agrees with the identity and, hence, $\tilde{\omega}_s$ with ω. Furthermore, we know that $\lambda(-s_0(\omega)) = -s_0(\omega)$ and $\lambda' \in [0, \mu(\omega))$.

This implies that $\lambda(s) \in [-s_0(\omega), 0]$ for all $s \in [-s_0(\omega), 0]$. Thus, by the construction of s_0 and μ, we know that $\tilde{\omega}_t$ is non-degenerate on $[-s_0(\omega), 0] \times \partial W$. This concludes the proof. □

Remark 3.2.3. The deformation constructed above is constant on the space of those symplectic forms that agree with ω_0 on the entire collar C.

Suppose that $\omega_0 = d\beta_0$ on the collar neighbourhood $C \cong (-2, 0] \times \partial W$ from the proof above, and $\omega = d\beta$ in a neighbourhood of ∂W. Then there is a probably smaller collar neighbourhood $C' = (-\epsilon, 0] \times \partial P \subset C$ such that $\omega = d\beta$ on C'. Consequently, on C', the family $\tilde{\omega}_t$ from the proof of Theorem 3.2.2 is given by

$$\begin{aligned}\tilde{\omega}_t &= \omega_0 + (1-t)(\omega - \omega_0) + t\Phi^*(\omega - \omega_0)\\ &= d\beta_0 + (1-t)(d\beta - d\beta_0) + t\Phi^*(d\beta - d\beta_0)\\ &= d(\beta_0 + (1-t)(\beta - \beta_0) + t\Phi^*(\beta - \beta_0)).\end{aligned}$$

This shows that $\tilde{\omega}_t$ is exact on C' with a primitive $\tilde{\beta}_t$ that agrees with β on $T\partial W$. Whenever β agrees with β_0 on $T\partial W$, on ∂W including all derivatives, or even on a neighbourhood of ∂W, this is also true for $\tilde{\beta}_t$. Moreover, under any of these circumstances, $\tilde{\beta}_1$ agrees with β_0 on a neighbourhood of ∂W.

The observation above implies the following four corollaries to Theorem 3.2.2.

Corollary 3.2.4. *If $\omega_0 \in \Omega_0^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$ and β_0 is a corresponding primitive on a neighbourhood of ∂W, then the deformation from Theorem 3.2.2 preserves the spaces $\Omega_0^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$, $\Omega_\infty^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$, and $\Omega_{\mathrm{c}}^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$.*

Corollary 3.2.5. *If $\omega_0 = d\beta_0$, then the deformation from Theorem 3.2.2 preserves the spaces $\Omega_0^{\mathrm{ES}}(W)$, $d(\mathcal{B}_0(W))$, $\Omega_\infty^{\mathrm{ES}}(W)$, $d(\mathcal{B}_\infty(W))$, $\Omega_{\mathrm{c}}^{\mathrm{ES}}(W)$, and $d(\mathcal{B}_{\mathrm{c}}(W))$.*

We would like to obtain a corresponding result for the underlying spaces of Liouville forms. For this we have to put in some additional work.

Corollary 3.2.6. *There is a weak deformation retraction of the space $d^{-1}\big(\Omega_0^{\mathrm{ES}}(W)\big)$ into $d^{-1}\big(\Omega_{\mathrm{c}}^{\mathrm{ES}}(W)\big)$ that preserves $d^{-1}\big(\Omega_\infty^{\mathrm{ES}}(W)\big)$.*

Corollary 3.2.7. *The deformation from Corollary 3.2.6 restricts to a weak deformation retraction of $\mathcal{B}_0(W)$ into $\mathcal{B}_{\mathrm{c}}(W)$ that preserves $\mathcal{B}_\infty(W)$.*

Combining Corollary 3.2.4 and Corollary 3.2.5 yields the following result about exact symplectic cobordisms.

Corollary 3.2.8. *If $\omega_0 \in \Omega_0^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$ and β_0 is a corresponding primitive on a neighbourhood of ∂W, then the deformation from Theorem 3.2.2 preserves the intersection of $\Omega_0^{\mathrm{ES}}(W)$, $d(\mathcal{B}_0(W))$, $\Omega_\infty^{\mathrm{ES}}(W)$, $d(\mathcal{B}_\infty(W))$, $\Omega_{\mathrm{c}}^{\mathrm{ES}}(W)$, and $d(\mathcal{B}_{\mathrm{c}}(W))$ with $\Omega_0^{\mathrm{SC}}(W, \alpha_-, \alpha_+)$.*

Proof. We choose the collar neighbourhood C inside the neighbourhood of ∂W in which the primitive β_0 is defined. Then it follows from the discussion after the proof of Theorem 3.2.2 that the deformation from Theorem 3.2.2 preserves the spaces in the assumption of this corollary. $\square$

The results we obtained in this subsection concern symplectic forms that are prescribed at the boundary of a symplectic manifold W. The methods used to prove them also apply to a different setup.

Remark 3.2.9. Results corresponding to those in this section also hold for spaces of symplectic forms prescribed on a closed hypersurface in the interior of W: to prove this we only have to mirror at the boundary the constructions in the proofs of Theorem 3.2.2 and the discussion following it.

3.2.3. Symplectic Forms and Liouville Forms

The aim of this short subsection is to show that the spaces of exact symplectic forms and the spaces of Liouville forms with the boundary conditions we considered in the last subsection are all homotopy equivalent.

This result follows from the observation that we can use Hodge theory to construct a right-inverse of the exterior differential on $\Omega_\infty^{\mathrm{S}}(W)$. Here, and throughout this subsection, W is a compact manifold with non-empty boundary. Moreover, we fix a Liouville form β_0 on W.

Lemma 3.2.10. *The restriction of the exterior differential d to the space $d^{-1}\big(\Omega_\infty^{\mathrm{ES}}(W)\big)$ has a continuous affine linear right-inverse $\tilde{d}$ that satisfies $\tilde{d}\,(d\beta_0) = \beta_0$. Furthermore, we may assume that the image of $\tilde{d}$ is contained in $\mathcal{B}_0(W)$.*

Proof. If any of the two spaces $(d^{-1}\big(\Omega_\infty^{\mathrm{ES}}(W)\big)$ and $\Omega_\infty^{\mathrm{ES}}(W)$ is empty, so is the other one. So, let us assume that neither of the two spaces is empty.

First, let us subtract $d\beta_0$ from the forms in $\Omega_\infty^{\mathrm{ES}}(W)$. This sends this space to the space of deviations from $d\beta_0$.

In this space, all elements vanish at the boundary of W including all derivatives. Accordingly, we can identify it with the subspace V_W of the space $d\big(\Omega^1\big(W \cup \overline{W}\big)\big)$ of exact 2-forms on the double $W \cup_{\mathrm{id}_{\partial W}} \overline{W}$ of W given by those forms that vanish identically on $\overline{W}$. Here, $\overline{W}$ is a copy of W with reversed orientation.

Now, we endow $W \cup \overline{W}$ with a Riemannian metric. Then Hodge theory tells us that there is a linear right-inverse $\bar{d}$ of d on $d\big(\Omega^1\big(W \cup \overline{W}\big)\big)$. More precisely, by [3, Theorem 4.16] in combination with [3, Remark 4.12] we have the orthogonal splitting

$$\Omega^k\big(W \cup \overline{W}\big) = \ker \Delta_k \oplus \operatorname{im}(d_{k-1}) \oplus \operatorname{im}\big(d^*_{k+1}\big)$$

where the map $d_{k-1}\colon \Omega^{k-1}\big(W \cup \overline{W}\big) \to \Omega^k\big(W \cup \overline{W}\big)$ is the exterior derivative, the map $d^*_{k+1}\colon \Omega^{k+1}\big(W \cup \overline{W}\big) \to \Omega^k\big(W \cup \overline{W}\big)$ the formal L^2-adjoint to d_k, which is given by $d^* = *d*$ where $*$ is the Hodge star operator, and $\Delta_k = d^*_{k+1}d_k + d_{k-1}d^*_k$ the Laplace-Beltrami operator.

Since d_1 vanishes both on $\ker \Delta_1$ and $\operatorname{im}(d_0)$, the space $d\big(\Omega^1\big(W \cup \overline{W}\big)\big)$ is the image of the restriction of d_1 to $\operatorname{im}(d^*_2)$. Moreover, d_1 is one-to-one on $\operatorname{im}(d^*_2)$, because d^* is the formal L^2-adjoint to d.

The two spaces $\operatorname{im}(d^*_2) = \operatorname{im}(\Delta_1) \cap \ker d_1$ and $\operatorname{im}(d_1) = \operatorname{im}(\Delta_2) \cap \ker d^*_2$ are closed subspaces of $\Omega^1\big(W \cup \overline{W}\big)$ and $\Omega^2\big(W \cup \overline{W}\big)$, respectively, as the intersection of closed subspaces: $\ker d_1$ and $\ker d^*_2$ are closed because d_1 and d^*_2 are continuous and $\operatorname{im}(\Delta_1)$ and $\operatorname{im}(\Delta_2)$ are closed by [3, Theorem 3.10]. This implies that $\operatorname{im}(d^*_2)$ and $\operatorname{im}(d_1)$ are Fréchet spaces. Thus, the open mapping theorem (see [42, Section II.5]) applies to the restriction of d_1 to $\operatorname{im}(d^*_2)$. Consequently, the inverse of this map is continuous. This is the desired right-inverse $\bar{d}$ of d_1.

After restricting $\bar{d}$ to the closed subspace V_W, it is still linear and continuous. This allows us to define a linear right-inverse $\hat{d}$ of d on $\Omega_\infty^{\mathrm{ES}}(W)_\Delta$ by first applying $\bar{d}$ and then pulling back with the inclusion of W into $W \cup \overline{W}$.

Since $\hat{d}$ is linear, $\hat{d}(0) = 0$. So, we can define the desired affine-linear right-inverse $\check{d}$ on $\Omega_\infty^{ES}(W)$ by first subtracting $d\beta_0$, then applying $\hat{d}$, and finally adding β_0.

The image of this right-inverse may not be contained in $\mathcal{B}_0(W)$ because we did not control the restriction to $T\partial W$. Fortunately, for every $\omega \in \Omega_\infty^{\mathrm{ES}}(W)$ there is a closed form on W whose restriction to $T\partial W$ agrees with that of $\check{d}\omega - \beta_0$, namely the pullback of $\bar{d}(\omega - d\beta_0)$ to $\overline{W}$.

Subtracting this closed form from the image of $\check{d}$ yields the desired affine linear right-inverse $\tilde{d}$. □

As an immediate consequence, we get the following corollary.

Corollary 3.2.11. *The two restrictions $d\colon d^{-1}\big(\Omega_\infty^{\mathrm{ES}}(W)\big) \to \Omega_\infty^{\mathrm{ES}}(W)$ and $\bar{d}\colon \mathcal{B}_\infty(W) \to \Omega_\infty^{\mathrm{ES}}(W)$ of the exterior differential are homotopy equivalences.*

Proof. We first show that the right-inverse $\tilde{d}$ from Lemma 3.2.10 is a homotopy inverse of d.

If any of the two spaces is empty, so is the other one. So, let us assume that neither of the two spaces is empty.

Since $\tilde{d}$ is a right-inverse we already know that $d \circ \tilde{d} = \mathrm{id}$.

Now, let $\beta \in d^{-1}\big(\Omega_\infty^{\mathrm{ES}}(W)\big)$. Then $d\big((\tilde{d} \circ d)(\beta)\big) = d\beta$ and, hence, also $d\big((1-t)\,(\tilde{d} \circ d)(\beta) + t\beta\big) = d\beta$ for all $t \in [0,1]$. This shows that $(t,\beta) \mapsto (1-t)\,(\tilde{d} \circ d)(\beta) + t\beta$ is a homotopy from $\tilde{d} \circ d$ to the identity.

Next, we show that $D_1 \circ \tilde{d}$ is a homotopy inverse of $\bar{d}$ where D_t is the weak deformation retraction from Corollary 3.2.7.

The weak deformation retraction from Corollary 3.2.5 can be obtained by applying the exterior derivative to D_t, which preserves $\Omega_\infty^{\mathrm{ES}}(W)$. Consequently, $d \circ D_t \circ \tilde{d}$ is a homotopy from the identity on $\Omega_\infty^{\mathrm{ES}}(W)$ to $d \circ D_1 \circ \tilde{d}$.

Because the image of $\tilde{d}$ is contained in $\mathcal{B}_0(W)$, the homotopy from the identity on $d^{-1}\big(\Omega_\infty^{\mathrm{ES}}(W)\big)$ to $\tilde{d} \circ d$ preserves $\mathcal{B}_0(W)$. Accordingly, D_1 and $d_1 \circ \tilde{d} \circ \bar{d}$ are homotopic.

Since D_t is a weak deformation retraction that preserves $\mathcal{B}_\infty(W)$, its restriction to this space is homotopic to the identity. This shows that $D_1 \circ \tilde{d}$ is a homotopy inverse of $\bar{d}$. □

Combining this corollary with the results of the last subsection, we obtain the following theorem.

Theorem 3.2.12. *All the spaces $\Omega_0^{\mathrm{ES}}(W)$, $\Omega_\infty^{\mathrm{ES}}(W)$, $\Omega_{\mathrm{c}}^{\mathrm{ES}}(W)$, $\mathcal{B}_0(W)$, $\mathcal{B}_\infty(W)$, and $\mathcal{B}_{\mathrm{c}}(W)$ are homotopy equivalent, where the homotopy equivalences are given by inclusions and the exterior differential.*

Proof. The homotopy equivalences follow from the fact that the weak deformation retraction from Corollary 3.2.5 can be obtained by applying the exterior derivative to the weak deformation retraction in Corollary 3.2.6. □

3.2.4. Homotopy Sequence for the Space of Symplectomorphisms

It is a well-known fact that, on a closed symplectic manifold (W, ω_0), we can construct isotopies covering cohomologous families ω_t of symplectic forms; see [31, Theorem 3.17]. So it is reasonable to think that, in the case that W is a compact manifold with boundary and ω_0 exact, the map $\pi\colon \mathrm{Diff}(W) \to \Omega^{\mathrm{ES}}(W)$ given by $\pi(\Psi) = \Psi^*\omega_0$ is a quasifibration.[1]

Unfortunately, for manifolds with boundary, the techniques used to prove [31, Theorem 3.17] do not work without restrictions. One problem is that the Moser trick requires the global integrability of certain vector fields, and another one that Hodge theory does not work properly without boundary conditions. Accordingly, we cannot expect the map π above to be a quasifibration, *i.e.* we cannot expect the existence of long exact homotopy sequences

$$\cdots \longrightarrow \pi_k(\mathrm{Symp}(W)) \longrightarrow \pi_k(\mathrm{Diff}(W)) \longrightarrow \pi_k\big(\Omega^{\mathrm{ES}}(W)\big) \longrightarrow \cdots.$$

The aim of this subsection is to show that such a long exact homotopy sequence exists under the mildest boundary condition possible, namely that the diffeomorphisms agree with the identity on the boundary. In this case, we replace π by the corresponding map $\pi_\partial\colon \mathcal{D}_\partial \to \Omega_0^{\mathrm{ES}}(W)$. Moreover, we introduce the map $\pi_{\mathrm{c}}\colon \mathcal{D} \to \Omega_{\mathrm{c}}^{\mathrm{ES}}(W)$ corresponding to the stronger boundary condition that the diffeomorphisms have compact support in the interior of W. The fibre of π_∂ over ω_0 is the space $\mathcal{S}_\partial$ of symplectomorphisms of (W, ω_0) that agree with the the identity on the boundary and the fibre of π_{c} over ω_0 is the space $\mathcal{S}$ of symplectomorphisms with compact support in the interior of W.

Using the results of the previous two subsections we can prove the following theorem.

[1] For the definition of the term *quasifibration*, see Definition B.4.

Theorem 3.2.13. *Let $(W, d\beta_0)$ be a symplectic manifold with boundary. Then there is a long exact homotopy ladder diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k(\mathcal{S}_\partial) \arrow[r] & \pi_k(\mathcal{D}_\partial) \arrow[r] & \pi_k\big(\Omega_0^{\mathrm{ES}}(W)\big) \arrow[r] & \cdots \\
\cdots \arrow[r] & \pi_k(\mathcal{S}) \arrow[r] \arrow[u] & \pi_k(\mathcal{D}) \arrow[r] \arrow[u] & \pi_k\big(\Omega_{\mathrm{c}}^{\mathrm{ES}}(W)\big) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

where the base point of $\Omega_{\mathrm{c}}^{\mathrm{ES}}(W)$ is $d\beta_0$.

Proof. To prove the existence of the ladder diagram we have to verify the two assumptions of Lemma B.8 for the maps π_∂ and π_{c} for all $n \in \mathbb{N}$. We do this for π_∂ only: from the construction it will follow that the same arguments work for π_{c}, as well.

Let $n \in \mathbb{N}_0$. Then the first of the two assumptions of Lemma B.8 is that the diagram

$$\begin{tikzcd}
D^n \times \{0\} \arrow[r, "\Psi"] \arrow[d, hook] & \mathcal{D}_\partial \arrow[d, "\pi_\partial"] \\
(D^n \times I, \partial(D^n \times I)) \arrow[r, "\omega"] \arrow[ur, dashed] & \big(\Omega_0^{\mathrm{ES}}(W), \{d\beta_0\}\big)
\end{tikzcd}$$

can always be completed as indicated up to a homotopy of ω relative $\partial\,(D^n \times I)$.

The construction we use to prove this carries over to the proof that the second assumption from Lemma B.8 is satisfied. So, in preparation, we relax the condition on $\omega|_{D^n\times\{0\}}$ and only assume that $\omega|_{\partial(D^n\times I)} = d\beta$ for a map $\beta\colon \partial\,(D^n \times I) \to \mathcal{B}_{\mathrm{c}}(W)$ such that $\beta(x,t) = \beta_0$, whenever $x \in \partial D^n$ or $t = 1$. Accordingly, we also replace the D^n-family of symplectomorphisms Ψ_x by a D^n-family of diffeomorphisms such that $\omega(x,0) = \Psi_x^* d\beta_0$.

We want to apply the Moser trick. However, to be able to apply it, we first need to guarantee that for fixed $x \in D^n$ the paths $\omega(x,t)$ are induced by smooth paths of Liouville forms constant on a neighbourhood of the boundary.

The first step in arranging this is to deform ω via the weak deformation retraction ϕ_s from Corollary 3.2.5 into a map whose image is contained in $\Omega^{\mathrm{ES}}_{\infty}(W)$. More precisely, we first choose a collar neighbourhood C of ∂P on which the Liouville forms $\beta(x,t)$ agree with β_0; this is possible because β is a map into $\mathcal{B}_{\mathrm{c}}(W)$ and $\partial\,(D^n \times I)$ compact. Then we set $\omega_s = \phi_s \circ \omega$ where ϕ_s is constructed using the collar neighbourhood C. By Remark 3.2.3 this special choice guarantees that ω_s is a deformation relative $\partial\,(D^n \times I)$.

That the image of ω_1 is contained in $\Omega^{\mathrm{ES}}_{\infty}(W)$ enables us to use the right-inverse $\tilde{d}$ of the exterior differential d from Lemma 3.2.10 to obtain a map $\beta_1 \colon D^n \times I \to d^{-1}\big(\Omega^{\mathrm{ES}}_{\infty}(W)\big)$ such that $\omega_1(x,t) = d\beta_1(x,t)$.

Since $\tilde{d}$ is a right-inverse of d, we know that the restriction of β_1 to $\partial\,(D^n \times I)$ differs from β only by addition of a map $\beta_\Delta \colon \partial\,(D^n \times I) \to \Omega^1(W)$ such that $d\beta_\Delta$ vanishes.

We deform β_1 into a map that agrees with β on $\partial\,(D^n \times I)$. For convenience of notation, let us identify $D^n \times [0,1]$ and D^{n+1} via the homeomorphism obtained by following rays originating in $(0, 1/2)$. Then such a deformation is given by

$$\bar{\beta}_s(x,t) = \beta_1(x,t) - s\lambda(\|(x,t)\|)\,\beta_\Delta\left(\frac{(x,t)}{\|x,t\|}\right)$$

where $\lambda\,[0,1] \to [0,1]$ is a smooth monotonously increasing function that vanishes on a neighbourhood of 0 and is constant of value 1 on a neighbourhood of 1.

The corresponding deformation of ω_1, given by $d\bar{\beta}_s$, is constant because we only add closed forms to $\beta_1(x,t)$.

Note that we now know that for every $(x,t) \in \partial\,(D^n \times I)$ the Liouville form $\bar{\beta}_1(x,t)$ agrees with β_0 on the collar neighbourhood C. Thus, the homotopy $\hat{\beta}_s = \psi_s \circ \bar{\beta}_1$, where ψ_s is the weak deformation retraction from Corollary 3.2.7 constructed using C, is a homotopy relative $\partial\,(D^n \times I)$. Accordingly, this is also true for the corresponding deformation of ω_1 given by $d\hat{\beta}_s$.

Since the image of $\hat{\beta}_1$ is contained in $\mathcal{B}_{\mathrm{c}}(W)$, we are nearly in the position to apply the Moser trick. We need only find a homotopy relative $\partial\,(D^n \times I)$ from $\hat{\beta}_1$ to a map that is smooth in the coordinate t on I. Such a homotopy $\tilde{\beta}_s$ exists by the proof of Theorem B.10.

Now we apply the Moser trick to the paths $\beta_{x,t} = \tilde{\beta}_1(x,t)$ for fixed $x \in D^n$.

Let $X_{x,t}$ be the D^n-family of time-dependent vector fields defined by the condition

$$\iota_{X_{x,t}} d\beta_{x,t} = -\frac{d}{dt}\beta_{x,t} =: -\dot{\beta}_{x,t}$$

and denote by $\Phi_{x,t}$ the corresponding flows, which exist since $\dot{\beta}_{x,t}$ vanishes in a neighbourhood of the boundary and accordingly $X_{x,t}$, as well. Then we have

$$\begin{aligned}
\Phi^*_{x,t} d\beta_{x,t} - d\beta_{x,0} &= \int_0^t \frac{d}{dt}\big(\Phi^*_{x,r} d\beta_{x,r}\big)\, dr \\
&= \int_0^t \Phi^*_{x,r}\Big(d\dot{\beta}_{x,r} + L_{X_{x,r}} d\beta_{x,r}\Big)\, dr \\
&= \int_0^t \Phi^*_{x,r}\Big(d\dot{\beta}_{x,r} + d\big(\iota_{X_{x,r}} d\beta_{x,r}\big)\Big)\, dr \\
&= 0.
\end{aligned}$$

This implies that

$$d\beta_{x,t} = \left(\Phi^{-1}_{x,t}\right)^* d\beta_{s,0} = \left(\Phi^{-1}_{x,t}\right)^* d(\Psi^*_x \beta_0) = \left(\Psi_x \circ \Phi^{-1}_{x,t}\right)^* d\beta_0.$$

So, $\left(\Psi_x \circ \Phi^{-1}_{x,t}\right)$ is a lift with the correct initial values.

It remains to consider the second assumption of Lemma B.8. This assumption says that, for every diagram

$$\begin{tikzcd}
(D^n \times \{0\}, \partial D^n \times \{0\}) \arrow[r, "\Psi"] \arrow[d, hook] & (\mathcal{D}_\partial, \mathcal{S}_\partial) \arrow[d, "\pi_\partial"] \\
(D^n \times I, (\partial D^n \times I) \cup D^n \times \{1\}) \arrow[r, "\omega"] \arrow[ur, dashed] & \left(\Omega_0^{\mathrm{ES}}(W), \{d\beta_0\}\right),
\end{tikzcd}$$

we can find a map Ψ' homotopic to Ψ and a map ω' such that the diagram can be completed as indicated after replacing Ψ by Ψ' and ω by ω'.

The map Ψ' can be obtained from the map Ψ via the weak deformation retraction D_s from Theorem 3.2.1, *i.e.* via the homotopy $\Psi_s = D_s \circ \Psi$. The corresponding map ω' is obtained by concatenating the maps $(x,t) \mapsto \Psi^*_{1-t} d\beta_0$ and ω.

Since the image of Ψ_1 is contained in $\mathcal{D}$, that of $\Psi_1^*\beta_0$ is contained in $\mathcal{B}_{\mathrm{c}}(W)$. Thus, our discussion above shows that ω' can be lifted with initial values Ψ_1. This concludes the proof. □

Remark 3.2.14. We can obtain the corresponding diagram for any base point in $\mathcal{D}$, not necessarily contained in $\mathcal{S}$, by first pulling back with this map. However, this changes the base point in $\Omega_{\mathrm{c}}^{\mathrm{ES}}(W)$.

As an immediate consequence of the theorem above, we obtain the following weak homotopy equivalence.

Corollary 3.2.15. *The inclusion of $\mathcal{S}$ into $\mathcal{S}_\partial$ is a weak homotopy equivalence.*

Proof. By Theorem 3.2.1 the inclusion of $\mathcal{D}$ into $\mathcal{D}_\partial$ is a homotopy equivalence, and the inclusion of $\Omega_{\mathrm{c}}^{\mathrm{ES}}(W)$ into $\Omega_0^{\mathrm{ES}}(W)$ is one by Corollary 3.2.5. Consequently, the Five lemma can be applied to the homotopy ladder diagram from Theorem 3.2.13. This proves that the inclusion of $\mathcal{S}$ into $\mathcal{S}_\partial$ is a weak homotopy equivalence.

For a discussion why the Five lemma applies at the level of π_0, see Appendix C. □

3.3. Symplectic Fibrations over S^1

In preparation for Chapter 5, we use this section to present a moderate generalisation of a well-known neighbourhood theorem for *symplectic fibrations* over S^1.

Definition 3.3.1. Let $\pi\colon M \to S^1$ be a smooth fibre bundle whose total space M is a $(2n+1)$-dimensional manifold. Then we say that (M, π, ω) is a **symplectic fibration** over S^1 if ω is a symplectic form on the hyperplane bundle $\ker d\varphi$ with $d\varphi = \pi^* d\theta$.

By construction, the forms ω and $d\varphi$ from the definition of a symplectic fibration satisfy $d\varphi \wedge \omega^n \neq 0$ and $\ker \omega \subset TM = \ker d(d\varphi)$. This shows that a symplectic fibration is a special case of a *stable Hamiltonian structure*.

Definition 3.3.2. Let M be a $(2n+1)$-dimensional manifold. A pair (λ, ω) consisting of a 1-form and a 2-form on M is called a **Hamiltonian structure** if $\lambda \wedge \omega^n$ is a volume form on M. If, moreover, $\ker \omega \subset \ker d\lambda$, then we say that (λ, ω) is **stable**.

In the special case of hypersurfaces symplectically fibred over S^1, we can extend the neighbourhood theorem [9, Lemma 2.3] for stable Hamiltonian structures to not necessarily compact hypersurfaces as follows.

Proposition 3.3.3. *Let M be a hypersurface in the interior of a $(2n+2)$-dimensional symplectic manifold (W, ω) such that M is closed in W together with a map $\pi\colon M \to S^1$ such that $(M, \pi, \omega|_{TM})$ is a symplectic fibration over S^1.*

Then there is a neighbourhood of M in W that is symplectomorphic to $\big((-\epsilon, \epsilon) \times M, \omega|_{TM} + dt \wedge d\varphi\big)$, where t is the coordinate on the interval, $d\varphi = \pi^* d\theta$, and ϵ a smooth positive function on M.

Proof. Let $U = (-\delta, \delta) \times M$ be a tubular neighbourhood of M in W; here, δ is a smooth positive function on M. Endow each level set $\{t\} \times M$ with the form $d\varphi$. Then there is a (possibly smaller) tubular neighbourhood such that $d\varphi \wedge \omega^n|_{T(\{t\}\times M)}$ is a volume form on $\{t\} \times M$ for all t. Without loss of generality, we may assume that this is already the case on U.

Now, let Y be the unique vector field such that $\iota_Y \omega = d\varphi$. Then we have

$$\iota_Y \omega^{n+1}|_{TM} = (n+1)\,(\iota_Y \omega) \wedge \omega^n|_{TM} = (n+1)\,d\varphi \wedge \omega^n|_{TM} \neq 0.$$

Thus, the vector field Y is transverse to M.

Use the flow Ψ_t of this vector field to identify a small neighbourhood of M in U with $(-\epsilon, \epsilon) \times M$. We have

$$\Psi_t^* \omega - \omega = \int_0^t \Psi_s^* (L_Y \omega)\, ds = \int_0^t \Psi_s^* d(\iota_Y \omega)\, ds = \int_0^t \Psi_s^* d(d\varphi)\, ds = 0.$$

Consequently, the pullback of ω to $(-\epsilon, \epsilon) \times M$ is constant in the coordinate t on $(-\epsilon, \epsilon)$.

It remains to show that $\omega|_M = \omega|_{TM} + dt \wedge d\varphi$.

Let X be the unique vector field in $\ker \omega|_{TM}$ that satisfies $\iota_X d\varphi \equiv 1$ and denote by pr_{TM} the projection from $T\left((-\epsilon, \epsilon) \times M\right)|_M$ to TM. Then, both X and ∂_t are contained in $\ker(\omega \circ \mathrm{pr}_{TM})$ and $\omega(\partial_t, X) = d\varphi(X) \equiv 1$. This implies that $\omega = \omega|_{TM} + dt \wedge d\varphi$. □

By the special nature of symplectic fibrations, this neighbourhood theorem can be strengthened further. In order to do so, we need the concept of the *holonomy* of a symplectic fibration.

Definition 3.3.4. Let (M, π, ω) be a symplectic fibration over $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and X the unique vector field in $\ker \omega$ satisfying $\iota_X d\varphi \equiv 1$. Then we say that the time-2π-flow $\Psi_{2\pi}$ of X starting in $\pi^{-1}(0)$ is the **holonomy** of (M, π, ω), provided it exists.

Remark 3.3.5. If the holonomy exists, it is a symplectomorphism of the fibre $F = \pi^{-1}(0)$ endowed with the symplectic form $\omega|_{TF}$.

With this definition at hand, we get the following stronger version of Proposition 3.3.3.

Proposition 3.3.6 (*Cf.* [16, Lemma 2.3] for the 4-dimensional case)**.** *Let M_i, $i = 0, 1$, be hypersurfaces in the interior of symplectic manifolds (W_i, ω_i) such that M_i is closed in W_i. Furthermore, let there be maps $\pi_i \colon M_i \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ such that $(M_i, \pi_i, \omega_i|_{TM_i})$ are symplectic fibrations over S^1. In addition, assume that the two symplectic manifolds $(F_i, \bar{\omega}_i) = \left(\pi_i^{-1}(0), \omega|_{TF_i}\right)$ are symplectomorphic via a symplectomorphism $\Psi \colon F_0 \to F_1$, that the holonomies Ψ^0 and Ψ^1 of $(M_0, \pi_0, \omega_0|_{TM_0})$ and $(M_1, \pi_1, \omega_1|_{TM_1})$ exist, and that Ψ^0 and $\Psi^{-1} \circ \Psi^1 \circ \Psi$ are isotopic as symplectomorphisms of $(F_0, \bar{\omega}_0)$.*

Then M_0 and M_1 have symplectomorphic neighbourhoods in W_0 and W_1.

Proof. Let X be the unique vector field in $\ker \omega_0|_{TM_0}$ satisfying $\iota_X d\varphi \equiv 1$, and $((-\epsilon, \epsilon) \times M_0, \omega_0|_{TM_0} + dt \wedge d\varphi)$ the neighbourhood from Proposition 3.3.3. By assumption, we know that the time-s-flow Ψ_s^0 of X exists for all $s \in [0, 2\pi]$. Thus, we can define the map

$$\begin{aligned} \Phi^0 \colon\ [0, 2\pi] \times F_0 &\to M_0 \\ (s, x) &\mapsto \Psi_s^0(x)\,. \end{aligned}$$

This map descends to a diffeomorphism from the mapping torus $F_0\left(\Psi^0\right)$ to M_0, which we keep calling Φ^0.

Since $X \in \ker \omega_0|_{TM_0}$ and $\iota_X d\varphi \equiv 1$, we know that

$$L_X \omega_0 = \iota_X d\omega_0 + d(\iota_X \omega_0) = d(d\varphi) = 0.$$

Hence, a symplectomorphism of $\left((-\epsilon_0, \epsilon_1) \times F_0(\Psi_{2\pi})\,, dt \wedge ds \oplus \bar{\omega}_0\right)$ with a tubular neighbourhood of M_0 in $(-\epsilon, \epsilon) \times M_0$ is given by the map $\mathrm{id} \oplus \Phi^0$.

Completely analogous, the hypersurface M_1 has a tubular neighbourhood in W_1 symplectomorphic to $\big((-\epsilon_1,\epsilon_1)\times F_1\big(\Psi^1\big)\,, dt\wedge ds\oplus\bar\omega_1\big)$.

Using the symplectomorphism Ψ between $(F_0,\bar\omega_0)$ and $(F_1,\bar\omega_1)$ on each level, a (possibly smaller) tubular neighbourhood of M_1 is symplectomorphic to $\big((-\tilde\epsilon_1,\tilde\epsilon_1)\times F_0\big(\Psi^{-1}\circ\Psi^1\circ\Psi\big)\,, dt\wedge ds\oplus\bar\omega_0\big)$.

By assumption, we know that there is an isotopy Ψ_t from the identity to $\Psi^{-1}\circ\big(\Psi^1\big)^{-1}\circ\Psi\circ\Psi^0$. As in the proof of Proposition 1.4.4, this induces a diffeomorphism Φ from $F\big(\Psi^0\big)$ to $F\big(\Psi^{-1}\circ\Psi^1\circ\Psi\big)$ given by $\Phi(x,s)=\big(\Psi_{\mu(s)}(x)\,,s\big)$ where $\mu\colon\ [0,2\pi]\to[0,1]$ is a smooth monotonously increasing function that vanishes close to 0 and is constant of value 1 close to 2π.

This diffeomorphism pulls back $\bar\omega_0$ to $\Psi^*_{\mu(s)}\bar\omega_0=\bar\omega_0$ and ds to ds. Consequently, the map $\mathrm{id}\oplus\Phi$ is a symplectomorphism from $(-\epsilon,\epsilon)\times F(\Psi^0)$ to $(-\epsilon,\epsilon)\times F\big(\Psi^{-1}\circ\Psi^1\circ\Psi\big)$, where ϵ is a smooth positive function smaller than both ϵ_0 and $\tilde\epsilon_1$. This concludes the proof. $\square$

Using that the direction of the vector field Y in the proof of Proposition 3.3.3 is determined by the sign of the volume form on M induced by the symplectic fibration, we obtain the following corollary.

Corollary 3.3.7. *Let M_i, $i=0,1$, be boundary components of symplectic manifolds (W_i,ω_i). Furthermore, let there be maps $\pi_i\colon M_i\to S^1=\mathbb{R}/2\pi\mathbb{Z}$ such that $(M_i,\pi_i,\omega_i|_{TM_i})$ are symplectic fibrations over S^1 that induce the boundary orientation on M_0 and the opposite orientation on M_1. In addition, assume that the two symplectic manifolds $(F_i,\bar\omega_i)=\big(\pi_i^{-1}(0)\,,\omega|_{TF_i}\big)$ are symplectomorphic via a symplectomorphism $\Psi\colon F_0\to F_1$, that the holonomies Ψ^0 and Ψ^1 of $(M_0,\pi_0,\omega_0|_{TM_0})$ and $(M_1,\pi_1,\omega_1|_{TM_1})$ exist, and that Ψ^0 and $\Psi^{-1}\circ\Psi^1\circ\Psi$ are isotopic as symplectomorphisms of $(F_0,\bar\omega_0)$.*

Then W_0 and W_1 can be glued along M_0 and M_1.

4. Obstructions for Adapted Contact Forms

In the preceding chapter, we introduced several neighbourhood theorems. In this chapter, we employ these to define two obstructions to homotopies of (pointed) S^n-families of contact forms adapted to the same open book decomposition. The vanishing of both obstructions will be sufficient for the existence of a homotopy.

The first of these two obstructions is the difference of homotopy class of the projections of the two S^n-families to the tangent bundle of the page P_0. If this first obstruction vanishes, we define a second obstruction in a quotient of π_{n+1} of the space $\mathcal{B}(\pi, \alpha_B)$ of induced Liouville forms on the page P_0 inducing the contact form α_B on the binding.

For technical reasons, the second obstruction cannot be written down easily if the induced Liouville forms on the page P_0 do not agree with the base point β_0 of $\mathcal{B}(\pi, \alpha_B)$. However, if they do, then it can be obtained as follows.

Choose a vector field X satisfying $\iota_X d\varphi \equiv 1$ and use its flow Ψ_t to identify the pages of the open book decomposition with the page P_0. Via this flow, the restrictions of an adapted contact form to the tangent bundles of the pages pull back to a path from the Liouville form β induced on P_0 to $\Psi_{2\pi}^*\beta$. Thus, the S^n-families of adapted contact forms define S^n-families β_φ of paths in $\mathcal{B}(\pi, \alpha_B)$ from β_0 to $\Psi_{2\pi}^*\beta_0$ whenever the induced Liouville forms on P_0 agree with β_0. This is true, in particular, for the trivial family, *i.e.* the constant family given by the base point. So, we can define an S^n-family of loops at β_0 in $\mathcal{B}(\pi, \alpha_B)$ by concatenating the paths β_φ with the inverse path of that corresponding to the trivial family. As we will see, this defines an element in $\pi_{n+1}(\mathcal{B}(\pi, \alpha_B))$ independent of the choice of the vector field X.

To construct the second obstruction in a general setting, we prove the existence of several long exact homotopy sequences in Section 4.1 and Section 4.2. Then we combine these in Section 4.3 in order to define the second obstruction. In this section, we also provide conditions under which the second obstruction can be defined in a more satisfying way.

Finally, in Section 4.4 we draw a connection between the second obstruction and the space of diffeomorphisms and symplectomorphisms with compact support in the interior of the page P_0 and use it to provide two examples of adapted contact forms for which the first obstruction vanishes, but not the second. One of these contact forms will be contactomorphic to the base point in the space of adapted contact forms and the other one will not, by reasons presented in [6].

4.1. Induced Form on a Page

By Theorem 3.1.3 and Theorem 3.1.16 we know that the space $\mathcal{A}(\pi)$ of contact forms adapted to an open book decomposition (B, π) of a closed manifold M is homotopy equivalent to its subspace $\mathcal{A}_{\underline{h}}(\pi)$ of those contact forms that are standard for radius $1/2$ for a given Lutz pair $\underline{h} = (h_1, h_2)$, and that the space $\mathcal{B}(\pi)$ of induced Liouville forms on the page P_0 is homotopy equivalent to its subspace $\mathcal{B}_{h_1}(\pi)$ consisting of those induced Liouville forms on P_0 that are standard for distance $1/2$ with respect to h_1.

In this section, we only consider adapted contact forms and induced Liouville forms that induce a fixed contact form α_B on the binding. Using this property, we construct the second obstruction for such special families of adapted contact forms. Then, in the next section, we provide the means to extend the construction to general families of adapted contact forms.

By Remark 3.1.21 and Remark 3.1.18 the homotopy equivalences above also exist for the restricted spaces. More precisely, $\mathcal{A}(\pi, \alpha_B)$ is homotopy equivalent to $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$, and $\mathcal{B}(\pi, \alpha_B)$ to $\mathcal{B}_{h_1}(\pi, \alpha_B)$.

Note that the property to be standard for radius or distance $1/2$ in combination with that to induce α_B on the binding completely fixes the adapted contact forms and adapted Liouville forms on $B_{1/2}(0) \times B \subset U$ where U is the adapted neighbourhood with respect to which the forms are standard. This completely removes all difficulties close to the binding and, thus, allows us to view an adapted contact form with the properties above as a section of a bundle over S^1 with fibres given by $\mathcal{B}_{h_1}(\pi, \alpha_B)$. This is the picture that will lead us in the more precise construction below.

The main result of this section is the following theorem.

Theorem 4.1.1. *Let $\alpha \in \mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ and denote its restriction to the tangent bundle of P_0 by β_0. Then there is a long exact homotopy sequence*

$$\cdots \to \pi_{k+1}(\mathcal{B}_{h_1}(\pi, \alpha_B), \beta_0) \xrightarrow{i_*^k} \pi_k\big(\mathcal{A}_{\underline{h}}(\pi, \alpha_B)\big) \xrightarrow{\mathrm{pr}_*^k}$$
$$\xrightarrow{\mathrm{pr}_*^k} \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B), \beta_0) \xrightarrow{\partial_*^k} \pi_{(k-1)+1}(\mathcal{B}_{h_1}(\pi, \alpha_B), \beta_0) \to \cdots$$

where $(\mathcal{B}_{h_1}(\pi, \alpha_B), \beta_0)$ denotes the space $\mathcal{B}_{h_1}(\pi, \alpha_B)$ with base point β_0 and pr *the projection to TP_0. Both the inclusions i_*^k and the connection maps ∂_*^k are determined by the choice of base point in $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ and the isotopy class of the monodromy of the open book decomposition.*

Given this sequence, the second obstruction to homotopies of S^n-families is the difference in the projections of their preimages under i^n projected to $\pi_{n+1}(\mathcal{B}_{h_1}(\pi, \alpha_B))\,/\mathrm{im}\,\partial_*^{n+1}$.

The proof of the theorem above takes up the remainder of this section.

The first step in the proof is to replace the space $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ by the space $\Omega^1_{h_1}(\pi, \alpha_B)$ of adapted forms standard for radius $1/2$ with respect to h_1. This allows us to neglect the contact condition and is justified by the weak deformation retraction from Theorem 2.1.3 in combination with Corollary 2.1.6.

Now, choose an auxiliary vector field X that agrees with ∂_φ on $B_{1/2}(0) \times B$ and satisfies $\iota_X d\varphi \equiv 1$. Then we denote by $\Omega^1_{h_1,X}(\pi, \alpha_B)$ the subspace of $\Omega^1_{h_1}(\pi, \alpha_B)$ consisting of those forms α satisfying $\iota_X \alpha \equiv 0$. The following lemma shows that this space is homotopy equivalent to $\Omega^1_{h_1}(\pi, \alpha_B)$.

Lemma 4.1.2. *There is a strong deformation retraction from $\Omega^1_{h_1}(\pi, \alpha_B)$ to its subspace $\Omega^1_{h_1,X}(\pi, \alpha_B)$.*

Proof. If $\Omega^1_{h_1}(\pi, \alpha_B)$ is empty, so is $\Omega^1_{h_1,X}(\pi, \alpha_B)$. So, let us assume that $\Omega^1_{h_1,X}(\pi, \alpha_B)$ is non-empty.

Let $\alpha \in \Omega^1_{h_1}(\pi, \alpha_B)$. We already know that $\iota_X \alpha \equiv 0$ on $B_{1/2}(0) \times B$, because α is standard for radius $1/2$. Since, moreover, only the restriction of α to the tangent bundle of the pages is relevant for the question whether α is adapted to the open book decomposition, the path

$$\alpha_t = \alpha - t\,(\iota_X \alpha)\,d\varphi$$

stays inside $\Omega^1_{h_1}(\pi, \alpha_B)$. As a result, this construction defines a strong deformation retraction from $\Omega^1_{h_1}(\pi, \alpha_B)$ to $\Omega^1_{h_1,X}(\pi, \alpha_B)$. □

We can use the flow Ψ_t of X to pull back the forms $\alpha \in \Omega^1_{h_1,X}(\pi, \alpha_B)$. This yields an identification of $\Omega^1_{h_1,X}(\pi, \alpha_B)$ with the space $C^\infty_{\mathbb{R}}(\Psi_{2\pi})$ of smooth paths $\gamma\colon \mathbb{R} \to \mathcal{B}_{h_1}(\pi, \alpha_B)$ satisfying $\Psi^*_{2\pi}\gamma(t-2\pi) = \gamma(t)$ where the projection to the tangent bundle of P_0 corresponds to the evaluation map at $t = 0$.

We would like to identify this space of paths on $\mathbb{R}$ with a space of paths on $[0, 2\pi]$. Because of smoothness issues, we have to perform a deformation before this is possible; it is constructed as follows.

Choose a smooth monotonously increasing function $\mu\colon [0, 2\pi] \to [0, 2\pi]$ that vanishes near 0 and is constant of value 2π close to 2π. Then we define a function $\bar{\mu}\colon \mathbb{R} \to \mathbb{R}$ by $\bar{\mu}(2\pi k + t) = 2\pi k + \mu(t)$ for $t \in [0, 2\pi]$ and $k \in \mathbb{Z}$.

Using this function, we can define a weak deformation retraction of the space $C^\infty_{\mathbb{R}}(\Psi_{2\pi})$ into its subspace $C^\infty_{\mathbb{R},\mathrm{t}}(\Psi_{2\pi})$ consisting of those paths that are constant in a neighbourhood of $2\pi\mathbb{Z}$. It is given by

$$\gamma_s(t) = \gamma((1-s)\,t + s\bar{\mu}(t))\,.$$

The restriction map to $[0, 2\pi]$ is a homeomorphism between $C^\infty_{\mathbb{R},\mathrm{t}}(\Psi_{2\pi})$ and the space $C^\infty_{\mathrm{t}}(\Psi_{2\pi})$ of technical smooth paths $\gamma\colon [0, 2\pi] \to \mathcal{B}_{h_1}(\pi, \alpha_B)$ satisfying $\gamma(2\pi) = \Psi^*_{2\pi}\gamma(0)$. Its inverse is given by sending γ to the path $\bar{\gamma}$ defined by $\bar{\gamma}(2\pi k + t) = \left(\Psi^k_{2\pi}\right)^*\gamma(t)$ for $k \in \mathbb{Z}$ and $t \in [0, 2\pi]$.

Summed up, the discussion above shows the following.

Lemma 4.1.3. *The space $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ is homotopy equivalent to $C^\infty_{\mathrm{t}}(\Psi_{2\pi})$, where the projection to the tangent bundle of the page P_0 corresponds to the evaluation map at* 0.

By the proof of Theorem B.10, every path in $\mathcal{B}_{h_1}(\pi, \alpha)$ can be approximated by a technical smooth path that is homotopic to the original path. Combining this with the lemma above, Proposition 2.2.10, and Lemma 3.1.20 yields the following corollary.

Corollary 4.1.4. *A form $\beta \in \mathcal{B}_{h_1}(\pi, \alpha_B)$ is induced by an adapted contact form $\alpha \in \mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ if and only if the pullback with the monodromy Ψ leaves the path component of β invariant.*

The major advantage of $C^\infty_{\mathrm{t}}(\Psi_{2\pi})$ over $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ is that we can prove the following lemma.

Lemma 4.1.5. *The evaluation map* $\mathrm{ev}_0\colon C^\infty_{\mathrm{t}}(\Psi_{2\pi}) \to \mathcal{B}_{h_1}(\pi, \alpha_B)$ *at* 0 *is a quasifibration.*

Proof. By Theorem B.10, it is sufficient to show that every D^n-family of smooth paths $\gamma_x\colon\ [0,1] \to \mathcal{B}_{h_1}(\pi, \alpha_B)$ constant on $[0, {}^1\!/\!_4] \cup [{}^3\!/\!_4, 1]$ can be lifted with given initial values $g_x \in C^\infty_{\mathrm{t}}(\Psi_{2\pi})$ satisfying $\mathrm{ev}_0(g_x) = \gamma_x(0)$.

So, let γ_x, g_x be families as described above. Since the construction will be continuous both in the paths γ_x and the initial values g_x the dependence on $x \in D^n$ is inessential. Thus, we suppress the index in the remainder of the construction.

The idea is to construct the lift as a concatenation of a path from $\gamma(s)$ to $\gamma(0)$, the path g, and a path back form $\Psi^*_{2\pi}\gamma(0)$ to $\Psi^*_{2\pi}\gamma(s)$. There are two difficulties in realising this idea: we have to construct a suitable technical path from $\gamma(s)$ to $\gamma(0)$ and we have to arrange that we use the path g only on a smaller interval in the interior of $[0, 2\pi]$, even at the start of the lift.

We first deal with the second problem. This is where we use that the path γ is constant on $[0, {}^1\!/\!_4]$; this property allows us to define a lift over said interval by

$$G_s(t) = \begin{cases} \gamma(0) & , t \in \left[0, \frac{8s\pi}{3}\right] \\ g\left(\frac{3}{3-8s}\left(t - \frac{8s\pi}{3}\right)\right) & , t \in \left[\frac{8s\pi}{3}, 2\pi - \frac{8s\pi}{3}\right] \\ \Psi^*_{2\pi}\gamma(0) & , t \in \left[2\pi - \frac{8s\pi}{3}, 2\pi\right] . \end{cases}$$

At $s = {}^1\!/\!_4$ the path above is the concatenation of a constant path, the path g, and another constant path. This is smooth because g is technical. For $s > {}^1\!/\!_4$ we replace the two constant paths by suitable technical paths we construct below.

The natural choice of a path from $\gamma(s)$ to $\gamma(0)$ is the inverse of the restriction of γ to $[0, s]$. However, in general this path is not technical. We repair this by a reparametrisation.

Let $\mu\colon\ [0, 2\pi] \to [0, 1]$ be a technical monotonously increasing smooth function. Then we define a path $\tilde{\gamma}_s\colon\ [0, 2\pi] \to \mathcal{B}_{h_1}(\pi, \alpha_B)$ from $\gamma(s)$ to $\gamma(0)$ by

$$\tilde{\gamma}_s(t) = \gamma(s\,(1 - \mu(t)))\,.$$

With the help of this path we can define a lift of γ over the interval $[1/4, 1]$ as follows.

$$G_s(t) = \begin{cases} \tilde{\gamma}_s(3t) & , t \in \left[0, \frac{2\pi}{3}\right] \\ g\left(3\left(t - \frac{2\pi}{3}\right)\right) & , t \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right] \\ \Psi_{2\pi}^* \tilde{\gamma}_s(6\pi - 3t) & , t \in \left[\frac{4\pi}{3}, 2\pi\right] . \end{cases}$$

Note that for $s \in [0, 1/4]$ the path $\tilde{\gamma}_t^s$ is constant and of value $\gamma(0)$. As a result, the parts of our lift match to a lift G_s over the entire interval $[0, 1]$. □

Let us denote by F the fibre of the evaluation map ev_0 over a form $\beta_0 \in \mathcal{B}_{h_1}(\pi, \alpha_B)$, *i.e.* the space of smooth technical paths from β_0 to $\Psi_{2\pi}^*\beta_0$. Then the lemma above shows in combination with Lemma 4.1.3 that there is the following long exact homotopy sequence.

Corollary 4.1.6. *Let $\alpha \in \mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ and denote its restriction to the tangent bundle of P_0 by β_0. Then there is a long exact homotopy sequence*

$$\longrightarrow \pi_k(F) \xrightarrow{i_*^k} \pi_k\big(\mathcal{A}_{\underline{h}}(\pi, \alpha_B)\big) \xrightarrow{\mathrm{pr}_*^k} \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B), \beta_0) \xrightarrow{\partial_*^k} \pi_{k-1}(F) \longrightarrow$$

where $(\mathcal{B}_{h_1}(\pi, \alpha_B), \beta_0)$ denotes the space $\mathcal{B}_{h_1}(\pi, \alpha_B)$ with base point β_0 and pr *the projection to TP_0.*

We deduce Theorem 4.1.1 from the preceding corollary. To do so, we identify the fibre F with the space $\Omega_{\mathrm{t}}^\infty \mathcal{B}_{h_1}(\pi, \alpha_B)$ of smooth technical loops at the base point β_0 of $\mathcal{B}_{h_1}(\pi, \alpha_B)$. The inclusion of this space into the loop space $\Omega\mathcal{B}_{h_1}(\pi, \alpha_B)$ is a weak homotopy equivalence by Corollary B.16. Moreover, the latter space is well known to have the same homotopy groups as the base space $\mathcal{B}_{h_1}(\pi, \alpha_B)$ but shifted by $+1$; see [5, Corollary VII.6.19].

Lemma 4.1.7. *The fibre F of the evaluation map* ev_0 *over the base point β_0 of $\mathcal{B}_{h_1}(\pi, \alpha_B)$ is homotopy equivalent to the space $\Omega_{\mathrm{t}}^\infty \mathcal{B}_{h_1}(\pi, \alpha_B)$ of smooth technical loops at β_0.*

Proof. Let γ_0 be the base point in F, *i.e.* the path corresponding to the base point of $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$. Denote by γ_0^{-1} the inverse path to γ_0. Then the concatenation of elements $\gamma \in F$ with γ_0^{-1} defines an inclusion j of F into $\Omega_{\mathrm{t}}^{\infty}\mathcal{B}_{h_1}(\pi, \alpha_B)$.

We construct a homotopy inverse of j as follows. Given a technical loop, we first concatenate it with γ_0 and then with γ_0^{-1}. The result is contained in the image of j. Moreover, the map g defined by this construction is homotopic to the identity because the path obtained by concatenating γ_0 and γ_0^{-1} is homotopic to the constant path.

Now that we mapped $\Omega_{\mathrm{t}}^{\infty}\mathcal{B}_{h_1}(\pi, \alpha_B)$ into the image of j, we can project to F by removing the second half of the path. Let us denote this projection by p.

Because the map j is the inverse of the projection p, we see that $j \circ (p \circ g) = g$, which is homotopic to the identity on $\Omega_{\mathrm{t}}^{\infty}\mathcal{B}_{h_1}(\pi, \alpha_B)$, as we have already seen.

It remains to show that the map $(p \circ g) \circ j$ is homotopic to the identity on F. For a given path $\gamma \in F$, the image of this map is obtained by concatenation first with γ_0^{-1} and then with γ_0. So, because the concatenation of γ_0^{-1} and γ_0 is homotopic to the constant map, the map $(p \circ g) \circ j$ is homotopic to the identity on F.

This proves that $(p \circ g)$ is a homotopy inverse of j. $\square$

In combination with Corollary 4.1.6, the lemma above proves the existence part of Theorem 4.1.1. It remains to show that, after the identification of the fibres F with $\Omega_{\mathrm{t}}^{\infty}\mathcal{B}_{h_1}(\pi, \alpha_B)$, the inclusion maps i_*^k and the connection maps ∂_*^k do not depend on the auxiliary vector field X.

The independence of these maps from X is a consequence of the fact that both conditions on X, *i.e.* agreeing with ∂_φ on $B_{1/2}(0) \times B$ and satisfying $\iota_X d\varphi \equiv 1$, are convex. Accordingly, the space of these vector fields is contractible and, hence, so is the space of monodromies they induce.

In the fibres F, however, we cannot say that this induces a homotopy of the paths corresponding to the different vector fields X, because F explicitly depends on the monodromy induced by X. Nevertheless, after our identification of F with $\Omega_{\mathrm{t}}^{\infty}\mathcal{B}_{h_1}(\pi, \alpha_B)$ we can prove the following lemma.

Lemma 4.1.8. *Let $\alpha_x \in \mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ be any continuous family of adapted contact forms restricting to the base point $\beta_0 \in \mathcal{B}_{h_1}(\pi, \alpha_B)$ on TP_0.*

Furthermore, let X_0, X_1 *be two vector fields agreeing with* ∂_φ *on* $B_{1/2}(0) \times B$ *and satisfying* $\iota_{X_i} d\varphi \equiv 1$.

Then the two families of loops $l_x^{X_0}, l_x^{X_1} \in \Omega_{\mathrm{t}}^\infty \mathcal{B}_{h_1}(\pi, \alpha_B)$ *corresponding to* α_x *under the identification using* X_0 *and* X_1*, respectively, are homotopic.*

Proof. To prove the lemma, it is sufficient to construct a homotopy for a single contact form α that depends continuously on α. So, let $\alpha \in \mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ be an adapted contact form restricting to β_0 on the binding and denote by l_{X_0} and l_{X_1} the loops corresponding to α under the identification using X_0 and X_1, respectively.

Up to an inessential reparametrisation, these two loops are given by

$$l_{X_i}(t) = \begin{cases} \left(\Psi^i_{4\pi t}\right)^* (\alpha - (\iota_{X_i}\alpha)\, d\varphi) & , \text{ for } t \in [0, 1/2] \\ \left(\Psi^i_{4\pi(1-t)}\right)^* (\alpha_0 - (\iota_{X_i}\alpha_0)\, d\varphi) & , \text{ for } t \in [1/2, 1] \end{cases}$$

where Ψ^i_t is the flow of X_i and α_0 the base point of $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$.

Denote by X_s the convex interpolation $X_s = (1-s)\, X_0 + sX_1$ from X_0 to X_1 and by Ψ^s_t its time-t-flow. Then the path

$$l_{X_s}(t) = \begin{cases} \left(\Psi^s_{4\pi t}\right)^* (\alpha - (\iota_{X_s}\alpha)\, d\varphi) & , \text{ for } t \in [0, 1/2] \\ \left(\Psi^s_{4\pi(1-t)}\right)^* (\alpha_0 - (\iota_{X_s}\alpha_0)\, d\varphi) & , \text{ for } t \in [1/2, 1] \end{cases}$$

is a homotopy from l_{X_0} to l_{X_1}. □

We immediately get the following corollary.

Corollary 4.1.9. *The inclusion maps* i_*^k *in Theorem 4.1.1 are independent of the choice of the vector field* X *inducing the monodromy of the open book decomposition* (B, π).

Now it remains only to show that the connection maps are independent of the vector field X, as well. To do so, we take a closer look at these maps and write them down explicitly, just as we did for the induced loops above. From this explicit formula the independence from the vector field X will be apparent.

To be able to express the connection maps ∂_*^{k+1} in a concise fashion we use the connection map of the path-loop fibration to write them as maps $\pi_k(\Omega\mathcal{B}_{h_1}(\pi, \alpha_B)) \to \pi_k(\Omega\mathcal{B}_{h_1}(\pi, \alpha_B))$ instead of maps

$\pi_{k+1}(\mathcal{B}_{h_1}(\pi, \alpha_B)) \to \pi_{k+1}(\mathcal{B}_{h_1}(\pi, \alpha_B))$. This is more natural in view of our construction and has the further advantage that loop spaces are H-spaces with respect to concatenation of paths; *cf.* [5, Page 441]. Accordingly, the multiplication in the homotopy groups coincides with concatenation of paths.

Remember that we denote the base point of the fibre F by γ_0. Accordingly, the base point of $\Omega\mathcal{B}_{h_1}(\pi, \alpha_B)$ is given by the homotopically trivial path $\gamma_0 * \gamma_0^{-1}$. With this base point, the lift of a D^n-family of paths γ_x in $\mathcal{B}_{h_1}(\pi, \alpha_B)$ to the path space $P\mathcal{B}_{h_1}(\pi, \alpha_B)$ is given by $\left(\gamma_0 * \gamma_0^{-1}\right) * \gamma_x$.

To describe the image of the connection map of the path-loop fibration with this base point we interpret representatives γ of elements of $\pi_{k+1}\left(\mathcal{B}_{h_1}(\pi, \alpha_B)\right)$ as maps $D^k \times I \to \mathcal{B}_{h_1}(\pi, \alpha_B)$ that evaluate to $\gamma_0(0)$ on $\partial\big(D^k \times I\big)$. Such a map γ can be lifted to a map $D^k \times I \to P\mathcal{B}_{h_1}(\pi, \alpha_B)$ sending (x,t) to the concatenation of $\gamma_0 * \gamma_0^{-1}$ with the restriction of $\gamma(x,\cdot)$ to $[0,t]$. The restriction of this map to $\partial\big(D^k \times I\big)$ represents the image of the connection map of the path-loop fibration. Everywhere except on $D^k \times \{1\}$ this representative evaluates to a possibly reparametrised version of the base point $\left(\gamma_0 * \gamma_0^{-1}\right)$ and for $x \in D^k$ the path at $(x,1)$ is given by $\left(\gamma_0 * \gamma_0^{-1}\right) * \gamma\,(x,\cdot)$. For briefness sake, we identify such a map with the D^k-family of paths given by its restriction to $D^k \times \{1\}$.

Now, after clarifying the identification via the path-loop fibration, we come back to the connection maps ∂_*^{k+1}. The image of each of these maps can be computed analogously to the computation for the path-loop fibration. We see that each representative $\gamma\colon D^k \times I \to \mathcal{B}_{h_1}(\pi, \alpha_B)$ is mapped to the D^k-family $\tilde{\gamma}_x$ of loops in $\mathcal{B}_{h_1}(\pi, \alpha_B)$ given by

$$\begin{aligned}\tilde{\gamma}_x &= \gamma(x,\cdot)^{-1} * \gamma_0 * (\Psi^*\gamma)(x,\cdot) * \gamma_0^{-1} \\ &\simeq \Big(\left(\gamma_0 * \gamma_0^{-1}\right) * \gamma(x,\cdot)^{-1}\Big) * \left(\left(\gamma_0 * \gamma_0^{-1}\right) * \left(\gamma_0 * (\Psi^*\gamma)(x,\cdot) * \gamma_0^{-1}\right)\right)\end{aligned}$$

where $\Psi = \Psi_{2\pi}$ is the monodromy of the open book decomposition.

Let us denote by $\Psi_\#$ the map $\Omega\mathcal{B}_{h_1}(\pi, \alpha_B) \to \Omega\mathcal{B}_{h_1}(\pi, \alpha_B)$ given by sending a loop γ to $\gamma_0 * (\Psi^*\gamma)(x,\cdot) * \gamma_0^{-1}$ and by $\Psi_\#^k$ the maps induced on $\pi_k(\Omega\mathcal{B}_{h_1}(\pi, \alpha_B))$ for $k \in \mathbb{N}$. Then we have proved the following lemma.

Lemma 4.1.10. *For $k \in \mathbb{N}$, the connection map $\partial_*^k\colon \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B)) \to \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B))$ from Theorem 4.1.1 is given by*

$$\partial_*^{k+1}([\gamma_x]) = \left[\gamma_x^{-1}\right] \cdot \Psi_\#^k([\gamma_x])\,.$$

Note that the image $\tilde{\gamma}_x$ of each representative γ_x depends continuously on the base point $\gamma_0 * \gamma_0^{-1}$ and the monodromy Ψ, which, in turn, depend continuously on the vector field X and the base point α_0 of $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$. As a result we get the following lemma.

Lemma 4.1.11. *The connection maps ∂_*^k in Theorem 4.1.1 are independent of the choice of the vector field X inducing the monodromy of the open book decomposition (B, π).*

This lemma finishes our proof of Theorem 4.1.1.

4.2. Induced Form on the Binding

In the last section, we constructed our second obstruction for adapted contact forms that restrict to a given contact form on the binding. If we vary the induced contact form, the construction cannot be performed the same way. This is caused by the fact that though every adapted contact form $\alpha \in \mathcal{A}_{\underline{h}}(\pi)$ induces a path in $\mathcal{B}_{h_1}(\pi)$ not every path in $\mathcal{B}_{h_1}(\pi)$ induces a contact form in $\mathcal{A}_{\underline{h}}(\pi)$: every point in a path in $\mathcal{B}_{h_1}(\pi)$ induced by an $\alpha \in \mathcal{A}_{\underline{h}}(\pi)$ restricts to the same contact from on the binding, because of continuity.

In order to circumvent this problem, we construct homotopy sequences that encapsulate the dependence of the adapted contact forms and induced Liouville forms on the contact form on the binding. Then we use these in Section 4.3 to construct the general second obstruction via a diagram chase.

The main result of this section is the following theorem.

Theorem 4.2.1. *Let α be an adapted contact form, β its restriction to TP_0, and α_B its restriction to TB. Furthermore, let $\underline{h} = (h_1, h_2)$ be a Lutz pair. Then there are long exact homotopy sequences*

$$\cdots \longrightarrow \pi_k\big(\mathcal{A}_{\underline{h}}(\pi, \alpha_B)\big) \longrightarrow \pi_k(\mathcal{A}(\pi)) \longrightarrow \pi_k(\mathcal{A}(B)\,, \alpha_B) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B)) \longrightarrow \pi_k(\mathcal{B}(\pi)) \longrightarrow \pi_k(\mathcal{A}(B)\,, \alpha_B) \longrightarrow \cdots$$

where $(\mathcal{A}(B)\,, \alpha_B)$ denotes the space of contact forms on B with base point α_B.

Since the proofs of the existence of these two sequences are essentially identical, we present only that of the existence of the first one.

As in the proof of Theorem 4.1.1, it turns out that it is convenient first to substitute the spaces involved in the sequences by more suitable spaces homotopy equivalent to them. For technical reasons, we have to do this for all three spaces: we replace the space $\mathcal{A}(B)$ of contact forms on the binding by the space $\tilde{\mathcal{A}}(B)$ of equivalence classes of contact forms on B with respect to multiplication with positive constants, the space $\mathcal{A}(\pi)$ by the space $\tilde{\Omega}^1_L(\pi)$ of pairs (α, r_0) of adapted forms α standard for radius r_0, and $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ by the subspace $\tilde{\Omega}^1_L(\pi, [\alpha_B])$ of $\tilde{\Omega}^1_L(\pi)$ consisting of those forms inducing the element $[\alpha_B] \in \tilde{\mathcal{A}}(B)$ on the binding.

Our claim that the corresponding spaces are homotopy equivalent follows from the results of Section 3.1 and the following two lemmata.

Lemma 4.2.2. *Let α_B be a contact form on B. Then there is a strong deformation retraction from $\tilde{\Omega}^1_L(\pi, [\alpha_B])$ to $\tilde{\Omega}^1_L(\pi, \alpha_B)$.*

Proof. If $\tilde{\Omega}^1_L(\pi, [\alpha_B])$ is empty, so is $\tilde{\Omega}^1_L(\pi, \alpha_B)$. So, let us assume that $\tilde{\Omega}^1_L(\pi, [\alpha_B])$ is non-empty.

Let $(\alpha, r_0) \in \tilde{\Omega}^1_L(\pi, [\alpha_B])$. Then the restriction of α to TB is given by $C\alpha_B$ for some positive number C. Define the deformation retraction by

$$D_t(\alpha, r_0) = \left(\frac{1}{(1-t) + t\, C(\alpha)}\ \alpha, r_0 \right).$$

□

Lemma 4.2.3. *There is a section of the projection* $\mathrm{pr}\colon \mathcal{A}(B) \to \tilde{\mathcal{A}}(B)$ *that is a homotopy equivalence.*

Proof. If $\tilde{\mathcal{A}}(B)$ is empty, so is $\mathcal{A}(B)$. Therefore, let us assume that $\tilde{\mathcal{A}}(B)$ is non-empty.

Choose a Riemannian metric g on TB. This induces a norm on $\Omega^1(B)$ and hence also on its subset $\mathcal{A}(B)$.

We can define a section s of the projection pr by sending $[\alpha_0]$ to the unique $\alpha \in [\alpha_0] \subset \mathcal{A}(B)$ satisfying $\|\alpha\| = 1$.

It remains to show that the identity on $\mathcal{A}(B)$ is homotopic to $s \circ \mathrm{pr}$. A corresponding homotopy is given by

$$\alpha_t = (1-t)\,\alpha + t\, s(\mathrm{pr}(\alpha))\,.$$

□

Owing to the two lemmata above, we need only show that the long exact homotopy sequence exists for our replacements. For these we can prove the following lemma.

Lemma 4.2.4. *The map on* $\tilde{\Omega}^1_L(\pi)$ *assigning to a pair* (α, r_0) *the equivalence class* $[\alpha|_{TB}] \in \tilde{\mathcal{A}}(B)$ *is a quasifibration.*

Proof. By Corollary B.13 it is sufficient to show that every D^n-family of smooth paths $\gamma\colon [0,1] \to \tilde{\mathcal{A}}(B)$ constant on $[0, 1/4] \cup [3/4, 1]$ can be lifted with given initial values $(\alpha_x, (r_0)_x) \in \tilde{\Omega}^1_L(\pi)$ satisfying $[\alpha_x|_{TB}] = \gamma_x(0)$.

Since our construction will be continuous in the paths γ_x and the initial values $(\alpha_x, (r_0)_x) \in \tilde{\Omega}^1_L(\pi)$ the dependence on $x \in D^n$ is inessential. Consequently, we suppress the index throughout the remainder of the proof.

We first define the deformation of r_0. Let $\lambda\colon [0,1] \to [0,1]$ be a monotonously decreasing smooth function that is constant of value 1 on $[0, 1/8]$ and vanishes on $[1/6, 1]$. Then we set

$$r_t = \tfrac{1}{2}\,(1 + \lambda(t))\, r_0$$

for $t \in [0,1]$.

This definition allows us to destroy the standard form of α for radii $r_0/2 < r \leq r_0$ for $t \in [1/6, 1]$ without leaving the space $\tilde{\Omega}^1_L(\pi)$.

Next, we require a suitable family α^B_t in $\mathcal{A}(B)$ representing the path γ, *i.e.* a suitable family satisfying $\left[\alpha^B_t\right] = \gamma(t)$. A first candidate is given by $\alpha^B_t = \|\alpha|_{TB}\|\, s(\gamma(t))$ where s is the section from Lemma 4.2.3 and $\|\cdot\|$ the norm on $\mathcal{A}(B)$ used in this lemma. It has the correct initial value and is smooth.

To be able to lift the family α^B_t, we have to modify it a little bit. Nevertheless, let us defer this modification to the end of the proof where its necessity will become apparent.

Inside the set $B_{r_0}(0) \times B$ we know that α is given by

$$\alpha = h_1(r/r_0)\, \alpha|_{TB}$$

for some Lutz pair $\underline{h} = (h_1, h_2)$. This allows us to define a potential lift α_t by the constant path with value α outside $B_{r_0}(0) \times B$ and inside this set by

$$\alpha_t = h_1(r/r_0)\, \alpha^B_{t\nu(r)}$$

where $\nu(r) = \lambda({}^{r}/_{4r_0})$.

Because λ is constant of value 1 on $[0, {}^{1}/_{8}]$, the forms α_t are standard for radius ${}^{r_0}/_{2}$ and since $\lambda(r)$ vanishes for $r \geq {}^{1}/_{6} < {}^{1}/_{4}$ they agree with α on a neighbourhood of $\{r = r_0\}$. Moreover, the family is constant for $t \leq {}^{1}/_{4}$ since γ and, hence, also α_t^B is constant for these values of t. Consequently, the forms α_t are well defined and standard for radius r_t.

It remains to check whether the forms α_t are adapted, *i.e.* whether the restrictions of $d\alpha_t$ to the tangent bundles of the pages are non-degenerate. Outside $B_{r_0}(0) \times B$, this is evident because, there, α_t agrees with α, which is adapted. Inside $B_{r_0}(0) \times B$, we have

$$\begin{aligned}\tfrac{1}{n}\left(d\alpha_t\right)^n|_{TP_\varphi} &= -\tfrac{1}{r_0} h_1'({}^{r}/_{r_0})\, \alpha^B_{t\nu(r)} \wedge \left(d\alpha^B_{t\nu(r)}\right)^{n-1} \wedge dr \\ &\quad - \tfrac{t}{r_0} h_1({}^{r}/_{r_0})\, \nu'(r)\, \dot\alpha^B_{t\nu(r)} \wedge \left(d\alpha^B_{t\nu(r)}\right)^{n-1} \wedge dr.\end{aligned}$$

The first term is positive since $\alpha_{t\nu(r)}$ is a positive contact form on the binding and $h_1' < 0$. Unfortunately, we do not know whether the top-dimensional form $\dot\alpha^B_{t\nu(r)} \wedge \left(d\alpha^B_{t\nu(r)}\right)^{n-1}$ on the binding is a volume form. So, we have no control over the second term.

Now, we modify the family α_t^B such that $\dot\alpha^B_{t\nu(r)} \wedge \left(d\alpha^B_{t\nu(r)}\right)^{n-1}$ becomes a volume form. A first step towards this goal is to replace this family by the family

$$\tilde\alpha_t^B = e^{Ct}\alpha_t^B$$

for a constant C given by

$$C = C(\gamma) = 1 + \max_{t\in[0,1]} \max_{B} \frac{\left|\dot\alpha_t^B \wedge \left(d\alpha_t^B\right)^{n-1}\right|}{\left|\alpha_t^B \wedge \left(d\alpha_t^B\right)^{n-1}\right|}$$

where we identified top-dimensional forms on B with functions via some volume form.

With this choice we have

$$\dot{\tilde\alpha}_t^B \wedge \left(d\tilde\alpha_t^B\right)^{n-1} = e^{nCt}\left(C\alpha_t^B \wedge \left(d\alpha_t^B\right)^{n-1} + \dot\alpha_t^B \wedge \left(d\alpha_t^B\right)^{n-1}\right) > 0.$$

Unfortunately, this new family is not constant anymore for $t \leq {}^{1}/_{6}$. However, this is necessary to guarantee that α_t is standard for radius r_t. So, we replace this family by a reparametrised version

$$\bar\alpha_t^B = \tilde\alpha^B_{\mu(t)}$$

where $\mu\colon [0,1] \to [0,1]$ is a monotonously increasing smooth function that vanishes on $[0, 1/6]$ and satisfies $\mu(t) = t$ for $t \in [1/4, 1]$.

Then we have
$$\dot{\bar{\alpha}}_t^B = \mu'(t)\,\tilde{\alpha}^B_{\mu(t)} + \dot{\tilde{\alpha}}^B_{\mu(t)}.$$
Because μ' is non-negative this implies that $\dot{\bar{\alpha}}_t^B \wedge \left(d\bar{\alpha}_t^B\right)^{n-1}$ is positive. Moreover, the definition of μ ensures that $\bar{\alpha}_t^B$ is constant for $t \in [0, 1/6]$.

Now, replacing α_t^B in the definition of α_t by $\bar{\alpha}_t^B$ yields the desired lift. □

The preceding lemma concludes the proof of Theorem 4.2.1, because quasifibrations induce long exact homotopy sequences.

Apart from this result, the construction in the proof above also provides means to connect the different types of subspaces of $\mathcal{A}_{\underline{h}}(\pi)$ given by restricting the admissible contact forms induced on the binding. As a first application we can show that it does not matter whether we fix the contact form induced on the binding or the equivalence class with respect to multiplication with a positive constant of a contact form defining it.

Corollary 4.2.5. *Let α_B be a contact form on B. Then there is a weak deformation retraction from $\tilde{\Omega}^1_L(\pi, \ker \alpha_B)$ to $\tilde{\Omega}^1_L(\pi, [\alpha_B])$.*

Proof. If $\tilde{\Omega}^1_L(\pi, \ker \alpha_B)$ is empty, so is $\tilde{\Omega}^1_L(\pi, [\alpha_B])$. So, assume that the space $\tilde{\Omega}^1_L(\pi, \ker \alpha_B)$ is non-empty.

Define a contraction of the space of contact forms defining $\ker \alpha_B$ by
$$(\alpha, t) \mapsto (1 - \mu(t))\,\alpha + \mu(t)\,\alpha_B,$$
where $\mu\colon [0,1] \to [0,1]$ is a smooth function that vanishes on $[0, 1/4]$, is constant of value 1 on $[3/4, 1]$ and is monotonously increasing otherwise. Lift the homotopy given by the projection from $\tilde{\Omega}^1_L(\pi, \ker \alpha_B)$ to the space of contact forms defining $\ker \alpha_B$ followed by this contraction. This defines the desired weak deformation retraction. □

A second application shows that the fibres $\tilde{\Omega}^1_L(\pi, [\alpha_B])$ over the same component of $\tilde{\mathcal{A}}(B)$ are not just weakly homotopy equivalent but homotopy equivalent.

Corollary 4.2.6. *Let α_t^B, $t \in [0,1]$, be a path of contact forms on the binding B. Then the spaces $\tilde{\Omega}^1_L\left(\pi, \left[\alpha_0^B\right]\right)$ and $\tilde{\Omega}^1_L\left(\pi, \left[\alpha_1^B\right]\right)$ are homotopy equivalent.*

Proof. Let us first assume that $\tilde{\Omega}^1_L(\pi, [\alpha_0^B])$ is non-empty.

By the proof of Theorem B.10 the path α_t^B is homotopic to a path from α_0^B to α_1^B that is smooth and constant on $[0, 1/4] \cup [3/4, 1]$. So, we may assume without loss of generality that α_t^B has these properties, as well.

By the proof of Lemma 4.2.4, we can lift the constant $\tilde{\Omega}^1_L(\pi, [\alpha_0^B])$-family of paths $[\alpha_t^B]$ in $\tilde{\mathcal{A}}(B)$ with initial values given by the identity on $\tilde{\Omega}^1_L(\pi, [\alpha_0^B])$. This yields a map Φ into $\tilde{\Omega}^1_L(\pi, [\alpha_1^B])$. Consequently, this space is non-empty, too.

A map Ψ back from $\tilde{\Omega}^1_L(\pi, [\alpha_1^B])$ to $\tilde{\Omega}^1_L(\pi, [\alpha_0^B])$ can be defined by replacing the path α_t^B by the inverse path α_{1-t}^B. We claim that Φ and Ψ are homotopy inverses of each other.

Because of the symmetry of the situation we only prove that $\Psi \circ \Phi$ is homotopic to the identity. Let $(\alpha, r_0) \in \tilde{\Omega}^1_L(\pi, [\alpha_0^B])$. Then the contact form induced on the binding B is given by $E\alpha_0^B$ for some positive number E. This implies that the pair $(\Psi \circ \Phi)(\alpha, r_0)$ is given by $(\tilde{\alpha}, r_0/4)$ where $\tilde{\alpha}$ has the following form.

Outside $B_{r_0}(0) \times B$ it coincides with α, for $r \leq r_0/4$ it is given by

$$\tilde{\alpha} = Ee^{2C} h_1(r/r_0)\, \alpha_0^B,$$

for $r \in [r_0/4, r_0/2]$ by

$$\tilde{\alpha} = Ee^{C} e^{\mu(\nu(2r))C} h_1(r/r_0)\, \alpha^B_{\mu(1-\nu(2r))},$$

and for $r \in [r_0/2, r_0]$ by

$$\tilde{\alpha} = Ee^{\mu(\nu(r))C} h_1(r/r_0)\, \alpha^B_{\mu(\nu(r))}.$$

The original form α is homotopic to $\tilde{\alpha}$ via the family α_t, $t \in [0, 1]$, defined as follows.

Outside $B_{r_0}(0) \times B$, the family is constant and agrees with α, for $r \leq r_0/4$ it is given by

$$\alpha_t = Ee^{2tC} h_1(r/r_0)\, \alpha_0^B,$$

for $r \in [r_0/4, r_0/2]$ by

$$\alpha_t = Ee^{tC} e^{t\mu(\nu(2r))C} h_1(r/r_0)\, \alpha^B_{t\mu(1-\nu(2r))},$$

and for $r \in [r_0/2, r_0]$ by

$$\alpha_t = E e^{t\mu(\nu(r))C} h_1(r/r_0)\, \alpha^B_{t\mu(\nu(r))}.$$

Since α_t stays standard for radius $r_0/4$ for all $t \in [0,1]$, this shows that $\Psi \circ \Phi$ is homotopic to the map $(\alpha, r_0) \mapsto (\alpha, r_0/4)$. This, in turn, is homotopic to the identity via the homotopy $((\alpha, r_0), t) \mapsto (\alpha, (1+3t)\, r_0/4)$. This concludes the proof in the case that $\tilde{\Omega}^1_L\big(\pi, \big[\alpha^B_0\big]\big)$ is non-empty.

Now, suppose $\tilde{\Omega}^1_L\big(\pi, \big[\alpha^B_0\big]\big)$ is empty. Then, by the symmetry of the situation, $\tilde{\Omega}^1_L\big(\pi, \big[\alpha^B_1\big]\big)$ is empty, as well. Hence, the two spaces are homotopy equivalent. □

Combining the two results with Lemma 4.2.2 and the results from Section 3.1 yields the following theorem.

Theorem 4.2.7. *Let $\alpha_0, \alpha_1 \in \mathcal{A}(B)$ be homotopic. Then the three spaces $\mathcal{A}_{\underline{h}}(\pi, \ker \alpha_0)$, $\mathcal{A}_{\underline{h}}(\pi, \alpha_0)$, and $\mathcal{A}_{\underline{h}}(\pi, \alpha_1)$ are homotopy equivalent.*

Remark 4.2.8. The results of this section also apply to the corresponding spaces of induced Liouville forms, by essentially the same proofs.

4.3. Combined Diagram and Consequences

In Section 4.1, we constructed the second obstruction for homotopies of (pointed) S^n- families of adapted contact forms under the assumption that they induce a prescribed contact form on the binding. Here, we combine the long exact sequences of the preceding two sections to define the second obstruction for general (pointed) S^n-families of adapted contact forms.

Our first aim is to combine the two sequences from Theorem 4.2.1 in a long exact homotopy ladder diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k\big(\mathcal{A}_{\underline{h}}(\pi, \alpha_B)\big) \arrow[r, "i_C"] \arrow[d, "p_B"] & \pi_k(\mathcal{A}(\pi)) \arrow[r, "p_C"] \arrow[d, "p"] & \pi_k(\mathcal{A}(B), \alpha_B) \arrow[r, "\partial_C"] \arrow[d, "id"] & \cdots \\
\cdots \arrow[r] & \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B)) \arrow[r, "i_L"] & \pi_k(\mathcal{B}(\pi)) \arrow[r, "p_L"] & \pi_k(\mathcal{A}(B), \alpha_B) \arrow[r, "\partial_L"] & \cdots
\end{tikzcd} \tag{4.1}$$

where the vertical maps are induced by the restriction to the tangent bundle of the page P_0.

The only problem is that the restriction map to TP_0 does not commute with the deformations used in the proof of Theorem 4.2.1 to translate the homotopy sequences obtained into those of the right spaces. Nevertheless, they do commute up to homotopy by the following lemma.

Lemma 4.3.1. *Let $A' \subset A$ and $B' \subset B$ be topological spaces and $\pi\colon A \to B$ a map such that $\pi(A') \subset B'$. Moreover, let D_t^A and D_t^B, $t \in [0,1]$, be deformations of A and B into A' and B', respectively, such that D_1^B is a homotopy equivalence.*

Then the diagram

$$\begin{tikzcd} A \arrow[r, "D_1^A"] \arrow[d, "\pi"] & A' \arrow[d, "\pi"] \\ B \arrow[r, "D_1^B"] & B' \end{tikzcd}$$

commutes up to homotopy.

Proof. We show that the maps $(\pi \circ D_1^A)$ and $(D_1^B \circ \pi)$ are both homotopic to the map $(D_1^B \circ \pi \circ D_1^A)$.

For the map $(D_1^B \circ \pi)$ such a homotopy is given by $(D_1^B \circ \pi \circ D_t^A)$. For the map $(\pi \circ D_1^A)$ the situation is slightly more complicated.

Let $g\colon B' \to B$ be a homotopy inverse of D_1^B. Then the map $(\pi \circ D_1^A)$ is homotopic to the map $(D_1^B \circ g \circ \pi \circ D_1^A)$. This, in turn, is homotopic to $(D_1^B \circ D_1^B \circ g \circ \pi \circ D_1^A)$ via the homotopy $(D_1^B \circ D_t^B \circ g \circ \pi \circ D_1^A)$. Finally, using that g is a homotopy inverse of D_1^B, this is homotopic to $(D_1^B \circ id \circ \pi \circ D_1^A) = (D_1^B \circ \pi \circ D_1^A)$. □

The diagram (4.1) can be combined with the long exact sequence from Theorem 4.1.1 to obtain the braid diagram

$$\begin{tikzcd}[column sep=tiny]
\cdots \arrow[rr, bend left, "p_C"] \arrow[dr, "p"] & & \pi_{k+1}(\mathcal{A}(B)) \arrow[rr, bend left, "\partial_L"] \arrow[dr, "\partial_C"] & & \pi_k(\mathcal{B}_{h_1}(\pi,\alpha_B)) \arrow[rr, bend left, "\partial_B"] \arrow[dr, "i_L"] & & \pi_k(\mathcal{B}_{h_1}(\pi,\alpha_B)) \arrow[rr, bend left, "i"] \arrow[dr, "i_B"] & & \cdots \\
& \pi_{k+1}(\mathcal{B}(\pi)) \arrow[ur, "p_L"] & & \pi_k\big(\mathcal{A}_{\underline{h}}(\pi,\alpha_B)\big) \arrow[ur, "p_B"] \arrow[dr, "i_C"] & & \pi_k(\mathcal{B}(\pi)) \arrow[dr, "p_L"] & & \pi_{k-1}\big(\mathcal{A}_{\underline{h}}(\pi,\alpha_B)\big) \arrow[ur, "i_C"] \arrow[dr, "p_B"] & \\
\cdots \arrow[ur, "i_L"] \arrow[rr, bend right, "\partial_P"'] & & \pi_{k+1}(\mathcal{B}_{h_1}(\pi,\alpha_B)) \arrow[ur, "i_B"] \arrow[rr, bend right, "i"'] & & \pi_k(\mathcal{A}(\pi)) \arrow[ur, "p"] \arrow[rr, bend right, "p_C"'] & & \pi_k(\mathcal{A}(B)) \arrow[ur, "\partial_C"] \arrow[rr, bend right, "\partial_L"'] & & \cdots
\end{tikzcd} \tag{4.2}$$

where the three unbroken strands are exact.

The next step is to prove that the broken braid is exact, as well.

Proposition 4.3.2. *Let $\alpha \in \mathcal{A}_{\underline{h}}(\pi)$ and α_B be the contact form on the binding induced by α. Then the broken strand in the broken braid diagram* (4.2) *is exact.*

Proof. We prove the assertion by a diagram chase.

Let $a \in \ker p^k$. Then $p_C(a) = (p_L \circ p)(a) = 0$. So, there is a $b \in \pi_k\big(\mathcal{A}_{\underline{h}}(\pi, \alpha_B)\big)$ such that $a = i_C(b)$. Consequently, we have $(i_L \circ p_L)(b) = (p \circ i_C)(b) = 0$. This implies that there is a $c \in \pi_{k+1}(\mathcal{A}(B))$ such that $\partial_L(c) = p_L(b)$. Thus, $p_l(b - \partial_C(c)) = 0$. This also makes sense for $k = 0$, because the map ∂_C^1 is induced by an action of $\pi_1(\mathcal{A}(B))$ on $\pi_0\big(\mathcal{A}_{\underline{h}}(\pi, \alpha_B)\big)$; *cf.* the proof of Proposition C.6.

Because $p_L(b - \partial_C(c)) = 0$ there is a $d \in \pi_{k+1}(\mathcal{B}_{h_1}(\pi, \alpha_B))$ such that $i_B(d) = b - \partial_C(c)$. Accordingly, we have

$$i(d) = (i_C \circ i_B)(d) = i_C(b - \partial_C(c)) = a.$$

□

The proposition above allows us to define the second obstruction against homotopies of (pointed) S^n- families of adapted contact forms as the difference of their preimages under i^k projected to $\pi_{k+1}(\mathcal{B}_{h_1}(\pi, \alpha_B)) \,/\, \ker i^k$. By construction, the vanishing of this obstruction guarantees the existence of a homotopy.

The definition of the second obstruction above is somewhat unsatisfying because we do not have a useful description of the kernel of i^k as an image of a map, in contrast to the case of the adapted contact forms with prescribed contact form induced on the binding. However, there are several special situation in which we can obtain such a description by completing the broken braid diagram (4.2).

Lemma 4.3.3. *Let $k \in \mathbb{N}$, $\alpha \in \mathcal{A}_{\underline{h}}(\pi)$, and α_B be the contact form on B induced by α.*

If the map $\big(\partial_C^k \circ p_L^k\big)$ or the map ∂_B^k is trivial, then there is a map ∂^k from $\pi_k(\mathcal{B}(\pi))$ to $\pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B))$ such that (4.2) *still commutes and remains exact after inserting ∂^k.*

Proof. We start with the case that $\left(\partial_C^k \circ p_L^k\right)$ is trivial.

Let $a \in \pi_k(\mathcal{B}(\pi))$. Then $\partial_C(p_L(a)) = 0$. Thus, there is a $b \in \pi_k(\mathcal{A}(\pi))$ such that $p_C(b) = p_L(a)$. Consequently, we have $p_L(a - p(b)) = 0$. This implies that there is a $c \in \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B))$ such that $i_L(c) = a - p(b)$. We define $\partial^k(a) = \partial_B(c)$.

We have to show that this is well defined. So, let $c_\delta \in \ker i_L^k$. Then there is a $d_\delta \in \pi_{k+1}(\mathcal{A}(B))$ such that $\partial_L(d_\delta) = c_\delta$. This implies that

$$\partial_B(c + c_\delta) = \partial_B(c + p_B(\partial_C(d_\delta))) = \partial_B(c)\,.$$

Now, let $b_\Delta \in \ker p_C^k$. Then there is a $c_\Delta \in \pi_k\big(\mathcal{A}_{\underline{h}}(\pi, \alpha_B)\big)$ such that $i_C(c_\Delta) = b_\Delta$. Consequently, we have $i_L(c - p_B(c_\Delta)) = a - p(b + b_\Delta)$ and $\partial_B(c - p_B(c_\Delta)) = \partial_B(c)$.

This shows that ∂^k is well defined. Next, we show that (4.2) still commutes after inserting ∂^k.

Let $a \in \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B))$. Then, $p_L(i_L(a)) = 0$. Hence, we may set $b = 0$ in the construction of $\partial(i_L(a))$. This implies that $\partial(i_L(a)) = \partial_B(a)$.

Now, let $a \in \pi_k(\mathcal{B}(\pi))$. Then $\partial(a) = \partial_B(c)$ for some $c \in \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B))$. Accordingly, $i_B(\partial(a)) = 0 = (\partial_C \circ p_L)(a)$.

It remains to show that the diagram stays exact. To prove this we use a lemma by Wall [5, Lemma IV.6.16]. It states that an (unbroken) braid diagram in which three strands are exact and the fourth one is of order 2 is exact. Here, the condition that the fourth strand is of order 2 means that the composition of each two consecutive maps in this strand is trivial. From its proof, it is clear that it still works for our broken braid diagram, because we already know that the broken strand is exact away from the new map ∂^k we inserted.

Let $a \in \pi_k(\mathcal{A}(\pi))$. Then we may choose $b = a$ in the construction of $\partial(p(a))$ and, hence, also $c = 0$. This shows that $\partial^k \circ p^k$ is trivial. In addition, we have $i \circ \partial = i_C \circ \partial_C \circ p_L = 0$. Thus, the broken strands is of order 2 and, hence, exact by [5, Lemma IV.6.16].

This concludes the proof for the case that $\left(\partial_C^k \circ p_L^k\right)$ is trivial.

Next, we consider the case that ∂_B^k is trivial.

Let $a \in \pi_k(\mathcal{B}(\pi))$. Then we have

$$p_B((\partial_C \circ p_L)(a)) = \partial_L(p_L(a)) = 0.$$

Consequently, there is a unique $b \in \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B))$ such that $i_B(b) = (\partial_C \circ p_L)(a)$.

We set $\partial^k(a) = b$. This is well defined because the only choice made was unique.

By construction, we know that $\left(i_B \circ \partial^k\right) = \left(\partial_C \circ p_L^k\right)$. Moreover, for $a \in \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B))$ we have $(\partial_C \circ p_L)(i_L(a)) = 0$ and, hence, $\partial(i_L(a)) = 0 = \partial_B(a)$.

This shows that (4.2) still commutes after inserting ∂^k. It remains to show that the broken strand stays of order 2 and, hence, exact.

Let $a \in \pi_k(\mathcal{A}(\pi))$. Then

$$(\partial_C \circ p_L \circ p)(a) = (\partial_C \circ p_C)(a) = 0.$$

Consequently, $\partial(p(a)) = 0$.

As in the proof of the first case, we have $i \circ \partial = i_C \circ \partial_C \circ p_L = 0$.

This concludes the proof. □

Piecing together the two parts of the proof above, we obtain the following corollary.

Corollary 4.3.4. *Let $k \in \mathbb{N}_{\geq 2}$, $\alpha \in \mathcal{A}_{\underline{h}}(\pi)$, and α_B be the contact form on B induced by α.*

If the two exact sequences

$$\pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B)) \xrightarrow{i_L} \pi_k(\mathcal{B}(\pi)) \xrightarrow{p_L} \pi_k(\mathcal{A}(B), \alpha_B)$$

and

$$\pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B)) \xrightarrow{\partial_B} \pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B)) \xrightarrow{i_B} \pi_{k-1}\big(\mathcal{A}_{\underline{h}}(\pi, \alpha_B)\big)$$

are split, then there is a map ∂^k from $\pi_k(\mathcal{B}(\pi))$ to $\pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B))$ such that (4.2) *still commutes and remains exact after inserting ∂^k.*

Proof. Because the two sequences are split we have decompositions

$$\pi_k(\mathcal{B}(\pi)) = \ker p_L^k \oplus \ker \partial_L^k$$

and

$$\pi_k(\mathcal{B}_{h_1}(\pi, \alpha_B)) = \ker i_B^k \oplus \ker p_B.$$

On $\ker p_L^k$ the map $\partial_C \circ p_L$ is trivial and the corresponding map ∂^k maps $\ker p_L^k$ into $\ker i_B^{k-1}$.

On $\ker \partial_L^k$ a map ∂^k into $\ker p_B$ is given by ∂_C^k. This corresponds to the map ∂^k constructed in the proof of Lemma 4.3.3 in the case that ∂_B^k is trivial.

The arguments that (4.2) still commutes and remains exact after inserting ∂^k goes through as in the proof of Lemma 4.3.3, separately for both parts of the map. □

As another corollary to Lemma 4.3.3 we obtain a condition under which our second obstruction takes its strongest form, *i.e.* under which we do not have to project to a quotient.

Corollary 4.3.5. *Let $k \in \mathbb{N}_0$. If the map p^{k+1} is onto, then i^k is one-to-one.*

Proof. Since p^{k+1} is onto we can apply the Five Lemma to see that p_B^{k+1} is onto, as well. So ∂_B^{k+1} is trivial.

This allows us to apply Lemma 4.3.3 in order to insert the map ∂^{k+1} into the diagram. Then i^k is one-to-one by exactness. □

A prominent example of a situation in which the assertion above is satisfied is that the monodromy is isotopic to the identity.

Proposition 4.3.6. *If the monodromy is isotopic to the identity with support outside a neighbourhood of the binding, then the maps p^k are onto for all $k \in \mathbb{N}_0$.*

Proof. By choosing our neighbourhood $U \cong D^2 \times B$ of the binding appropriately we may assume that there is an isotopy Φ_t of the identity to the monodromy with support outside $B_{1/2}(0) \times B$.

Because the weak deformation retraction from Theorem 2.1.3 preserves the restrictions to the tangent bundles of pages, it is sufficient to show that the map induced by the restriction map from $\Omega^1(\pi)$ to $\mathcal{B}(\pi)$ is onto. By Theorem 3.1.7, Theorem 3.1.16, and Lemma 4.3.1 we can further substitute this map with the map induced by the restriction map from $\hat{\Omega}^1_{h_1}(\pi)$ to $\mathcal{B}_{h_1}(\pi)$ for any Lutz pair (h_1, h_2).

Let us choose a technical smooth monotonously increasing surjection $\mu\colon [0, 2\pi] \to [0, 1]$. Then we can lift any family $\beta_x \in \mathcal{B}_{h_1}(\pi)$ as the family

$$\alpha_x = \left(\Phi^{-1}_{\mu(\varphi)}\right)^* \beta_x.$$

Accordingly, the maps p^k are onto for all $k \in \mathbb{N}_0$. □

Actually, we proved something more: we constructed a splitting.

Corollary 4.3.7. *If the monodromy is isotopic to the identity with support outside a neighbourhood of the binding, then the exact sequence*

$$\pi_{k+1}(\mathcal{B}_{h_1}(\pi, \alpha_B)) \xrightarrow{i} \pi_k(\mathcal{A}(\pi)) \xrightarrow{p} \pi_k(\mathcal{B}(\pi)) \longrightarrow 0$$

splits for every $k \in \mathbb{N}_0$, where the splitting is only natural if the monodromy is the identity.

Proof. We obtain the splitting from the map $\pi_k(\mathcal{B}(\pi)) \to \pi_k(\mathcal{A}(\pi))$ from the proof of Proposition 4.3.6. This is only natural in the case that the monodromy Ψ is the identity since only then there is a canonical choice for the isotopy from the identity to Ψ employed in the construction. □

Finally, we can obtain a corollary about the situation that the second obstruction vanishes identically, *i.e.* that the map p^k is one-to-one.

Corollary 4.3.8. *Let (P, β) be a Liouville domain with boundary (B, α_B) and $k \in \mathbb{N}_0$. Then the following statements are equivalent.*

1) The map p^k is one-to-one for the open book with pages P and the trivial monodromy.

2) $\pi_{k+1}(\mathcal{B}_{h_1}(\pi, \alpha_B))$ is trivial.

3) The map p^k is one-to-one and the map p^{k+1} is onto for all open books with pages P.

Proof. If the monodromy is trivial, then we know by Proposition 4.3.6 that p^{k+1} is onto. By Corollary 4.3.5, this implies that i^k is one-to-one. Because p^k is one-to-one, we also know that i^k is trivial. This implies that $\pi_{k+1}(\mathcal{B}_{h_1}(\pi, \alpha_B))$ is trivial.

If $\pi_{k+1}(\mathcal{B}_{h_1}(\pi, \alpha_B))$ is trivial, then i^k and ∂_B^{k+1} are trivial. This implies that p^k is one-to-one and, by Lemma 4.3.3, that the diagram can be completed with a map ∂^{k+1}. This map is trivial since $\pi_{k+1}(\mathcal{B}_{h_1}(\pi, \alpha_B))$ is trivial. Consequently, p^{k+1} is onto, by exactness. □

4.4. Connection to Symplectomorphisms of the Pages

In the preceding section, we constructed our second obstruction against homotopies of S^n-families of adapted contact forms. In this section, we connect this obstruction to the spaces of diffeomorphisms and symplectomorphisms of the page P_0 of the open book decomposition (B, π). More precisely, we connect the preimages of the obstruction under the projection $\pi_{n+1}(\mathcal{B}_{h_1}(\pi, \alpha_B)) \to \pi_{n+1}(\mathcal{B}_{h_1}(\pi, \alpha_B)) \,/\, \ker i^n$, where α_B is the contact form induced on the binding B by the base point of $\mathcal{A}(\pi)$, and the map i^n is defined as in (4.2). This connection yields two distinct examples of non-homotopic adapted contact forms on the open book with trivial monodromy and pages symplectomorphic to the unit cotangent bundle $\left(D^*S^2, \lambda_{\text{can}}\right)$ of S^2 endowed with the canonical Liouville form λ_{can}.

Following the general treatment, we specialise to the case $k = 0$ and discuss under which condition we can guarantee that the non-homotopic adapted contact forms are still contactomorphic and under which we cannot. Our two examples each fall into one of these two categories.

To be able to state the connection, we first have to exchange $\mathcal{B}_{h_1}(\pi, \alpha_B)$ with a more suitable space. Denote by P the complement $B_{1/2}(0) \times B \subset U$ in the page P_0, where U is the adapted neighbourhood from the definition of $\mathcal{B}_{h_1}(\pi, \alpha_B)$. Moreover, let $\beta_0 \in \mathcal{B}_{h_1}(\pi, \alpha_B)$. Because forms in $\mathcal{B}_{h_1}(\pi, \alpha_B)$ are completely determined inside $\bar{B}_{1/2}(0) \times B$, we have the following lemma.

Lemma 4.4.1. *The restriction map* res_P *from* $\mathcal{B}_{h_1}(\pi, \alpha_B)$ *to the space* $\mathcal{B}_\infty(P)$ *of Liouville forms on* P *that agree with the restriction of* β_0 *on* ∂P *including all derivatives is a homeomorphism.*

For the space $\mathcal{B}_\infty(P)$ we have the following long exact sequence by Theorem 3.2.13 and Theorem 3.2.12.

$$\cdots \longrightarrow \pi_{k+1}(\mathcal{B}_\infty(P)) \xrightarrow{\partial_S} \pi_k(\mathcal{S}) \xrightarrow{i_S} \pi_k(\mathcal{D}) \xrightarrow{p_S} \pi_k(\mathcal{B}_\infty(P)) \longrightarrow \cdots$$

This sequence shows that there are exactly two sources for non-trivial elements in $\pi_k(\mathcal{B}_\infty(P))$, namely S^k-families of symplectomorphisms that are isotopic to the identity as families of diffeomorphisms but not as families of symplectomorphisms, and S^{k+1}-families of diffeomorphisms that are not homotopic to a family of symplectomorphisms.

At least at the level of π_1, this should not come as a surprise; by the proof of Theorem 3.1.22 we already know that every adapted contact form $\alpha \in \mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ is strictly contactomorphic via a page-preserving diffeomorphism to the result of the generalised Thurston-Winkelnkemper construction applied to a symplectic open book of the form $(P, \Psi, \alpha|_{TP})$.

Our second obstruction now tells us under which conditions families of paths of diffeomorphisms from the identity to a symplectomorphism indeed define homotopically distinct adapted contact forms. In particular, every homotopy class of such families of paths yields a distinct adapted contact form if the monodromy is isotopic to the identity because of Proposition 4.3.6 and Corollary 4.3.5. We use this to obtain our examples.

Our first example is associated to the Dehn-Seidel twist $\tau\colon D^*S^2 \to D^*S^2$ introduced by Seidel in his thesis [35]. The square of the Dehn-Seidel twist is homotopic to the identity as a diffeomorphism, but not as a symplectomorphism; *cf.* [34].

This allows us to construct non-trivial elements of $\pi_1\big(\mathcal{B}_\infty\big(D^*S^2\big)\big)$ as the classes of the loops given by the concatenation of the paths $\Psi_t^*\lambda_{\mathrm{can}}$ and $(1-t)\,\Psi_1^*\lambda_{\mathrm{can}} + t\lambda_{\mathrm{can}}$ where Ψ_t is an isotopy from the identity to τ^{2k} for some $k \in \mathbb{Z} \setminus \{0\}$.

The results of Seidel in [34] are even stronger. They can be summarised as follows.

Theorem 4.4.2 (Seidel)**.** *The space $\mathcal{S}\big(D^*S^2\big)$ is weakly homotopy equivalent to $\mathbb{Z}$, generated by the Dehn-Seidel twist. Moreover, the image of the inclusion $\pi_0\big(\mathcal{S}\big(D^*S^2\big)\big) \to \pi_0\big(\mathcal{D}\big(D^*S^2\big)\big)$ is isomorphic to $\mathbb{Z}_2$.*

This shows that every homotopically non-trivial loop in $\mathcal{D}$ generates a non-trivial element in $\pi_1\big(\mathcal{B}_\infty\big(D^*S^2\big)\big)$. Such a non-trivial loop can be obtained as follows.

We follow parts of the argument by Seidel in [34].

First, we embed the interior of D^*S^2 symplectically into $S^2 \times S^2$, endowed with the standard symplectic form, as the complement of the diagonal Δ.[1] This embedding provides a weak homotopy equivalence of $\mathcal{D}$

[1] An explicit embedding can be found for example in the proof of [13, Corollary 3.2.10].

and the space $\mathcal{D}_2$ of diffeomorphisms of $S^2 \times S^2$ that fix the diagonal and its normal bundle. Let us denote by $\mathcal{D}_1$ the larger space of diffeomorphisms only fixing the diagonal.

Then there is the long exact homotopy sequence

$$\cdots \longrightarrow \pi_2(\mathcal{G}) \longrightarrow \pi_1(\mathcal{D}_2) \longrightarrow \pi_1(\mathcal{D}_1) \longrightarrow \pi_1(\mathcal{G}) \longrightarrow \cdots$$

where $\mathcal{G}$ is the space of sections of the automorphism bundle of the normal bundle $\nu\Delta$ of the diagonal.

The space of sections of the bundle of symplectic automorphisms of $\nu\Delta$ is a deformation retract of $\mathcal{G}$ (*cf.* the proof of [34, Corollary 2]) and weakly homotopy equivalent to $\mathrm{SL}_2(\mathbb{R}) \simeq S^1$, by [34, Lemma 3]. This shows that the non-trivial elements of $\pi_1(\mathcal{D}_2)$ are exactly those generated by homotopically non-trivial loops in $\mathcal{D}_1$ that induce homotopically trivial loops in $\mathcal{G}$.

Explicit examples of such loops are given by the path

$$\begin{aligned} \psi_t^y \colon S^2 \times S^2 &\to S^2 \times S^2 \\ (x,y) &\mapsto \begin{cases} \left(R_y^{4\pi t}(x)\,, y\right) & , \text{ for } t \in [0, 1/2] \\ \left(x, R_x^{-4\pi t}(y)\right) & , \text{ for } t \in [1/2, 1] \end{cases} \end{aligned}$$

and its k-fold concatenation ψ_t^k with itself, for $k \in \mathbb{Z} \setminus \{0\}$. Here, R_y^θ is the rotation in $\mathbb{R}^3$ by the angle θ around the oriented axis determined by y. Furthermore, by concatenating a path with itself a negative number of times we mean the concatenation of the inverse path the corresponding number of times.

On the diagonal, the maps ψ_t^k are trivial, because the points that are rotated agree with the axis. Moreover, the loop induced in $\mathcal{G}$ is trivial: the first half of the path induces the positive generator of $\pi_1(\mathcal{G})$ and the second half the negative one. This can be verified by a straightforward calculation.

Unfortunately, it turns out to be rather tedious to show that the loops ψ_t^k are homotopically distinct for all $k \in \mathbb{Z}$. Therefore, we defer the proof of this fact to the end of this section.

Now, let us come back to a more general setting to discuss the connection of our second obstruction to contactomorphisms. Let Ψ be the monodromy of the open book generated by the rescaled Reeb vector field $\left(\iota_{R_{\alpha_0}} d\varphi\right)^{-1} R_{\alpha_0}$ of the base point α_0 of $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ outside $B_{1/2}(0) \times B \subset U$

and by ∂_φ inside this set, as in the proof of Theorem 3.1.22. Then the identification of $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ with the space $C^\infty_{\mathrm{t}}(\Psi)$ of pathes in $\mathcal{B}_{h_1}(\pi, \alpha_B)$ from some β to $\Psi^*\beta$, which we obtained in Section 4.1, identifies the base point α_0 of $\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$ with the path γ_0 given by

$$\gamma_0(t) = \beta_0 + dh_t,$$

where β_0 is the base point of $\mathcal{B}_{h_1}(\pi, \alpha_B)$ and h_t a strictly monotonously increasing family of positive functions on P_0 constant in a neighbourhood of $\bar{B}_{1/2}(0) \times B$.

Let Ψ_t, $t \in [0, 2\pi]$, be a technical smooth path of diffeomorphisms of P with compact support in the interior of P starting at the identity. By Proposition 1.4.4, this isotopy determines a diffeomorphism Φ from $M(P, \Psi) = M$ to $M\big(P, \Psi \circ \Psi_{2\pi}^{-1}\big)$. Inside $B_{1/2}(0) \times B \subset U$, it agrees with the identity and, on the mapping tori, it is induced by the map

$$\begin{aligned} \Phi\colon P \times [0, 2\pi] &\to P \times [0, 2\pi] \\ (x, \varphi) &\mapsto (\Psi_\varphi(x)\,, \varphi)\,. \end{aligned}$$

The adapted contact form on $M\big(P, \Psi \circ \Psi_{2\pi}^{-1}\big)$ obtained from the symplectic open book $\big(P, \Psi \circ \Psi_{2\pi}^{-1}, \beta_0\big)$, either via the generalised Thurston-Winkelnkemper construction or the construction in Subsection 2.2.2, is given by

$$\alpha_1 = \beta_0 + \delta_\varphi + f\, d\varphi$$

where δ_φ is a $[0, 2\pi]$-family of closed forms on P such that $\delta_0 = 0$ and $\delta_{2\pi} = \big(\Psi \circ \Psi_{2\pi}^{-1}\big)^* \beta_0 - \beta_0$, and f a function on $M\big(P, \Psi \circ \Psi_{2\pi}^{-1}\big)$. Hence, the path of Liouville forms in $\mathcal{B}_\infty(P)$ from β_0 to $\Psi^*\beta_0$ associated to the pullback of this form α_1 is given by

$$\beta_\varphi = \Psi_\varphi^* \beta_0 + \Psi_\varphi^* \delta_\varphi.$$

This implies that the corresponding loop $\beta_\varphi * \gamma_0^{-1}$ is homotopic to the path obtained from the concatenation of the paths $\Psi_{2\pi t}^* \beta_0$ and $(1-t)\, \Psi_{2\pi}^* \beta_0 + t\beta_0$.

Since this is exactly the loop in $\mathcal{B}_\infty(P)$ induced by the isotopy Ψ_t, we proved the following theorem.

Theorem 4.4.3. *Let (P, Ψ, β_0) be a symplectic open book and α a contact form adapted to the natural open book decomposition of $M(P, \Psi)$ that* *induces* β_0 *on the page* P_0*. Furthermore, let* Ψ_φ *be an isotopy with compact support inside the interior of* $P \subset P_0$*, starting at the identity, that induces the second obstruction against a homotopy from* α *to the natural base point of* $\mathcal{A}(\pi)$*.*

Then the contact manifold $(M(P, \Psi), \ker\alpha)$ *is contactomorphic to the manifold* $M\big(P, \Psi \circ \Psi_{2\pi}^{-1}, \beta_0\big)$ *obtained via the generalised Thurston-Winkelnkemper construction from the symplectic open book* $\big(P, \Psi \circ \Psi_{2\pi}^{-1}, \beta_0\big)$*.*

Remark 4.4.4. The natural base point of $\mathcal{A}(\pi)$ on $M(P, \Psi)$ is the adapted contact form obtained via the generalised Thurston-Winkelnkemper construction from the symplectic open book (P, Ψ, β_0).

For the special case that Ψ_φ is a loop of diffeomorphisms we obtain the following corollary.

Corollary 4.4.5. *Whenever the second obstruction against a homotopy of two contact forms* α_0 *and* α_1 *adapted to an open book decomposition of a manifold* M *is defined and induced by a loop of diffeomorphisms of the pages, the contact manifolds* (M, α_0) *and* (M, α_1) *are contactomorphic.*

Applied to our second example, *i.e.* to the loops ψ_t^k, this yields the following result.

Corollary 4.4.6. *On* $M\big(D^*S^2, \mathrm{id}\big)$ *there are infinitely many non-homotopic adapted contact forms* α_k*,* $k \in \mathbb{N}$*, all adapted to the same open book and all inducing the canonical Liouville form* λ_{can} *on a fixed page such that the contact manifolds* $\big(M\big(D^*S^2, \mathrm{id}\big), \ker\alpha_k\big)$ *are contactomorphic.*

On the other hand, applying Theorem 4.4.3 to our first example, *i.e.* to the adapted contact forms associated to isotopies from the identity to the even powers τ^{2k} of the Dehn-Seidel twist, shows that the corresponding contact manifolds are contactomorphic to $M\big(D^*S^2, \tau^{-2k}, \lambda_{\mathrm{can}}\big)$.

As reasoned in [6], the results form [40] and [20] show that the contact manifolds $M\big(D^*S^2, \tau^{-2k}, \lambda_{\mathrm{can}}\big)$ are not contactomorphic for $k \leq 0$. This proves the following corollary to Theorem 4.4.3.

Corollary 4.4.7. *On* $M\big(D^*S^2, \mathrm{id}\big)$ *there are infinitely many adapted contact forms* α_k*,* $k \in \mathbb{N}$*, all adapted to the same open book and all inducing the canonical Liouville form* λ_{can} *on a fixed page such that the contact manifolds* $\big(M\big(D^*S^2, \mathrm{id}\big), \ker\alpha_k\big)$ *are pairwise not contactomorphic.*

The two corollaries above show that in higher dimension the space of adapted contact forms is much more complicated than in dimension 3 and that there is no such close connection to general contact forms as in said dimension.

However, our treatment only concerned the question whether contactomorphic adapted contact forms on the same manifold are homotopic as adapted forms. It still remains the harder question whether contact forms adapted to the same open book decomposition that are homotopic as general contact forms are automatically homotopic as adapted contact forms. More generally, we can ask the following.

Question 4.4.8. *Given $k \in \mathbb{N}_0$, is the map $i\colon \pi_k(\mathcal{A}(\pi)) \to \pi_k(\mathcal{A}(M))$ induced by the inclusion one-to-one?*

Since this question involves paths of contact forms that leave the space $\mathcal{A}(\pi)$ of adapted contact forms, the methods of this thesis cannot be used to answer this question in the affirmative.

Now, it remains to show that the loops ψ_t^k are indeed homotopically distinct.

Lemma 4.4.9. *The loop ψ_t generates a free subgroup of $\pi_1(\mathcal{D}_1)$.*

To prove this lemma, we need a well-known fact from topology. Denote by $\operatorname{Map}_*(X, Y)$ the space of pointed maps from a pointed topological space X to a pointed topological space Y, and write ΣX for the reduced suspension of X, *i.e.* for the quotient space $\Sigma X = (X \times I) \,/\, (X \times \partial I \cup \{*\} \times I)$, where $*$ is the base point of X. Then the following holds.

Lemma 4.4.10 (*Cf.* [23, Page 395])**.** *Let X and Y be pointed Hausdorff spaces, and let X be compact. Then the space $\operatorname{Map}_*(\Sigma X, Y)$ is homeomorphic to $\operatorname{Map}_*(X, \Omega Y)$, where the base point of ΩY is the constant path at the base point.*

In particular, $\operatorname{Map}_*\big(S^{n+k}, Y\big)$ is homeomorphic to $\operatorname{Map}_*\big(S^n, \Omega^k Y\big)$ for all $k, n \in \mathbb{N}_0$ where $\Omega^k Y$ is the k-fold loop space of Y.

Proof. By the exponential law [5, Theorem VII.2.5], we know that the space $Y^{X \times I}$ of maps from $X \times I$ to Y is homeomorphic to the space $\big(Y^I\big)^X$ of maps from X into the space Y^I of paths in Y via the map Ψ that send $f\colon X \times T \to Y$ to $x \mapsto f(x, \cdot)$.

The space $\mathrm{Map}_*(\Sigma X, Y)$ is in one-to-one correspondence to the subspace A of $Y^{X\times I}$ consisting of the maps that send $X \times \partial I \cup \{*_X\} \times I$ to the base point $*_Y$ of Y, where $*_X$ is the base point of X. Because the set $X \times \partial I \cup \{*_X\} \times I$ is compact, this correspondence is a homeomorphism with respect to the compact-open topology.

The homeomorphism Ψ maps A exactly to the subspace of $\left(Y^I\right)^X$ consisting of those maps that send the base point $*_X$ to the constant path at $*_Y$ and for which the image of every $x \in X$ is a path from $*_Y$ to $*_Y$. In other words, the image is the space $\mathrm{Map}_*(X, \Omega Y)$.

It remains to prove the addendum. This is an immediate consequence of the fact that $\Sigma S^n = S^{n+1}$. □

Apart from this lemma, we also need Thom-Pontryagin theory.

Theorem 4.4.11 (See [5, Theorem II.16.1])**.** *Let $k, n \in \mathbb{N}_0$, and denote by $*_n$ and $*_{n+k}$ the base point of S^n and S^{n+k}, respectively. Then the following map from $\pi_{n+k}(S^n)$ into the cobordism classes of framed k-dimensional closed submanifolds of $\mathbb{R}^{n+k} = S^{n+k} \setminus \{*_{n+k}\}$ is an isomorphism.*

Given a class $a \in \pi_{n+k}(S^n)$, choose a representative f that is smooth everywhere save maybe at $f^{-1}(*_n)$. Then the map sends the class a to the cobordism class of the preimage of any regular value p of f except $*_{n+k}$, where the framing is determined by the preimage of a small disk around p.

This theorem will help us to show that a certain homotopy invariant of ψ_t^k takes pairwise different values for $k \in \mathbb{Z}$.

Proof of Lemma 4.4.9. On the space $\mathcal{D}_1$ of diffeomorphisms of $S^2 \times S^2$ that fix the diagonal Δ we can define a map into the space $\mathrm{Map}_*\left(S^2, S^2\right)$ as follows. Denote by $*$ the base point of S^2 and by i_2 the inclusion of $\{*\} \times S^2$ into $S^2 \times S^2$. Then we send diffeomorphisms $\Psi \in \mathcal{D}_1$ to the map $\mathrm{pr}_2 \circ \Psi \circ i_2$, where pr_2 is the projection to the second factor of $S^2 \times S^2$.

We claim that the image of ψ_t under this map is a free generator of $\pi_1\left(\mathrm{Map}_*\left(S^2, S^2\right), \mathrm{id}\right)$.

By Lemma 4.1.7, the space $\mathrm{Map}_*\left(S^2, S^2\right)$ is homeomorphic to the double loop space $\Omega^2 S^2$. Let us denote the image of the identity of S^2 under the corresponding homeomorphism by $f_x\colon I^2 \to S^2$. This is our base point for $\Omega^2 S^2$. It is given by the composition of the identity on

I^2 and the projection to S^2, which we interpret as I^2 with collapsed boundary. The image g_t of the projection of ψ_t under the homeomorphism can be described as follows. Let us again interpret S^2 as I^2 with collapsed boundary. Then g_t can be lifted to a map $\tilde{g}_t\colon I^2 \to I^2$. After further identifying I^2 with the unit disc $D^2 \subset \mathbb{C}$ via the homeomorphism obtained by following rays from the centre of I^2, the map $\tilde{g}_t$ is given by multiplication with $e^{-2\pi i t}$, up to an inessential reparametrisation with respect to t.

Let us deform $\Omega^2 S^2$ into the space of technical loops in the space of technical loops in S^2 via a path of reparametrisations

$$(x, y) \mapsto \big((1-t)\,x + t\mu(x)\,, (1-t)\,y + t\mu(y)\big)$$

where $\mu\colon I \to I$ is a smooth monotonously increasing function that vanishes close to 0, is constant of value 1 close to 1, and agrees with the identity on $[1/4, 3/4]$. This path of reparametrisations induces a weak deformation retraction, whose time-1-map D_1 is a homotopy equivalence. Let us denote by f_+^μ the image of the base point f_+ under D_1.

Now denote by f_-^μ the inverse loop of f_+^μ, *i.e.* the map given by $f_-^\mu(x, y) = f_+^\mu(1-x, y)$. Then concatenation (in the first component) with f_-^μ send f_+^μ to the component of the constant path: a homotopy from the identity to $f_+^\mu * f_-^\mu$ is given by

$$\begin{aligned} H_t\colon I^2 &\to I^2 \\ (x,y) &\mapsto \begin{cases} f_+(\mu(2\mu(t)\,x)\,, y) & , \text{ for } x \in [0, 1/2] \\ f_+(\mu(2\mu(t)(1-x))\,, y) & , \text{ for } x \in [1/2, 1]\,. \end{cases} \end{aligned}$$

Since concatenation with a loop is a homotopy equivalence in $\Omega^2 S^2$, this shows that $\pi_1\big(\Omega^2 S^2, f_+\big)$ and $\pi_1\big(\Omega^2 S^2, c\big)$ are isomorphic via the map Φ that sends loops $g_t\colon I^2 \to S^2$ to the loop

$$\Phi(g)_t(x,y) = \begin{cases} H_{4t}(x,y) & , \text{ for } t \in [0, 1/4] \\ g_{\mu(2t-1/2)}(2\mu(x)\,, \mu(y)) & , \text{ for } t \in [1/4, 3/4] \text{ and } x \in [0, 1/2] \\ f_+^\mu(2\,(1-x)\,, y) & , \text{ for } t \in [1/4, 3/4] \text{ and } x \in [1/2, 1] \\ H_{4(1-t)}(x,y) & , \text{ for } t \in [3/4, 1]\,, \end{cases}$$

where c is the constant loop at the base point $*$.

By the long exact sequence of the path loop fibration, the fundamental group of $\Omega^2 S^2$ with the base point c is isomorphic to $\pi_3\big(S^2\big) \cong \mathbb{Z}$, which is generated by the Hopf fibration $h\colon S^3 \to S^2$. The identification is given by interpreting S^3 as I^3 with collapsed boundary and loops in $\Omega^2 S^2$ as paths $I \to \Omega^2 S^2$ that start and end in the base point, *i.e.* as maps $I^3 \to S^2$ that map the boundary to $*$.

Our goal is to show that the loop ψ_t is mapped to a map $S^3 \to S^2$ homotopic to the Hopf fibration.

By construction, the map $g\colon I^3 \to S^2$ corresponding to ψ_t is smooth save maybe at $g^{-1}(*)$. Moreover, the point $-*$ is a regular value of this map. Its preimage is a single unknot of Seifert framing 1: the preimage consists of the two arcs $\gamma_i\colon [1/4, 3/4] \to I^3$, $i = 1, 2$, given by $\gamma_1(t) = (1/4, 1/2, t)$ and $\gamma_2(t) = (3/4, 1/2, 3/4 - t)$ connected by one arc each in the sets $\{t \in [0, 1/4]\}$ and $\{t \in [3/4, 1]\}$. The framings of γ_2 and of the connecting arcs are trivial; the framing of γ_1 is given by the path $\tilde\gamma_1$ defined by

$$\tilde\gamma_1 = \gamma_1(t) + (\epsilon e^{2\pi i\mu(t-1/4)}, 0),$$

where we identified I^2 with the unit disc $D^2 \subset \mathbb{C}$, and $\epsilon > 0$ is a small constant. This shows that the Seifert framing is 1.

By Theorem 4.4.11, it remains only to show that the Hopf fibration corresponds to the same cobordism class of framed links in $\mathbb{R}^3$. To show this, let us choose the base point $*$ of S^2 to be the point $(0, 0, 1) \in \mathbb{R}^3$.

Interpreting S^3 as the unit-sphere in $\mathbb{C}^2$, the Hopf fibration h is the smooth map given by

$$h\Big(\sqrt{1-r^2}e^{i\varphi_1}, re^{i\varphi_2}\Big) = \Big(r\sqrt{1-r^2}e^{i(\varphi_1-\varphi_2)}, 2r^2 - 1\Big).$$

The preimage of the regular value $(0, 0, -1)$ is the knot $\gamma(\varphi) = \big(e^{i\varphi}, 0\big)$. A Seifert surface of this knot is given by

$$\Sigma = \Big\{\Big(\sqrt{1-r^2}e^{i\varphi}, -r\Big) \mid r \in [0,1], \varphi \in \mathbb{R}\Big\},$$

and a knot $\tilde\gamma$ determining the framing by $\tilde\gamma(\varphi) = \big(\sqrt{1-\epsilon^2}e^{i\varphi}, \epsilon e^{i\varphi}\big)$ for a small $\epsilon > 0$.

This shows that γ has Seifert framing 1. Thus, Theorem 4.4.11 tells us that the Hopf fibration and the map g corresponding to ψ_t are homotopic. This proves that the image g_t of ψ_t in $\mathrm{Map}_*\big(S^2, S^2\big)$ is a free generator of

$\pi_1\big(\mathrm{Map}_*\big(S^2, S^2\big)\,, \mathrm{id}\big)$. Consequently, the loop ψ_t in $\mathcal{D}_1$ generates a free subgroup of $\pi_1(\mathcal{D}_1)$. $\square$

5. Closed Reeb Orbits

The aim of this chapter is to present a generalised version of our joint results with Geiges and Zehmisch in [12]. In particular, we present the proof that every contact manifold supported by an open book decomposition whose binding can be embedded into a subcritical Stein manifold as a hypersurface of restricted contact type contains a contractible Reeb orbit. In addition, we prove that the same conclusion can be made if the binding is supported by an open book decomposition whose monodromy is trivial. Moreover, we show that it is sufficient for these properties to hold for the lowest level of a *tower of open book decompositions* supporting the contact manifold. Finally, we also show that the strong Weinstein conjecture holds on every contact manifold supported by an open book decomposition whose binding is **planar**, *i.e.* whose binding is supported by an open book decomposition whose pages are diffeomorphic to S^2 with a finite number of discs removed.

Our proof relies on the study of *(pseudo)holomorphic curves* on special symplectic manifolds. Accordingly, in Section 5.1, we provide a brief overview of the properties of *(pseudo)holomorphic curves* we need in the remainder of the chapter. Then, in Section 5.2, we define a generalised version of a **cap**, *i.e.* of a symplectic cobordism of a contact manifold to the empty set. Moreover, we start the construction of our special symplectic manifolds by presenting a construction by which we obtain *generalised caps* for a contact manifold from *generalised caps* of the binding of an open book decomposition supporting the contact structure. Following this, we construct in Section 5.3 *generalised caps* for contact manifolds with the properties mentioned above that contain 'nice' symplectic spheres; these *generalised caps* are the main building blocks of the symplectic manifolds on which we study holomorphic curves. Finally, in Section 5.4, we use the existence of the 'nice' spheres to prove that a certain space of holomorphic spheres is non-empty. This then leads to the existence of nullhomologous Reeb links and contractible Reeb orbits on the contact manifold we started with.

5.1. Holomorphic Curves

In this section, we present the basic properties of *holomorphic curves* we need in Section 5.3 and Section 5.4.

Definition 5.1.1. Let (W_0, J_0) and (W_1, J_1) be two almost complex manifolds. We say that a smooth map $u\colon W_0 \to W_1$ is **holomorphic** if it satisfies

$$u_* \circ J_0 = J_1 \circ u_*. \tag{5.1}$$

If W_0 is a Riemann surface, then we call u a **holomorphic curve**.

One can also define holomorphic curves of class C^l or $W^{k,p}$ with $l, k \in \mathbb{N}$ and $p > 2$ by replacing the demand on u to be smooth by the corresponding condition. In the second case this means that we demand that all coordinate representations be in $W^{k,p}_{\mathrm{loc}}$.

Remark 5.1.2. Condition (5.1) is equivalent to $\bar{\partial}_{J_0,J_1}(u) = 0$. Here, the 1-form

$$\bar{\partial}_{J_0,J_1}(u) := \frac{1}{2}\,(\, du + J \circ\, du \circ j) \in \Omega^{0,1}(W_0, u^*TW_1)$$

with values in the vector bundle u^*TW_1 is the complex antilinear part of the differential du with respect to the almost complex structure J_1.

Remark 5.1.3. If W_0 is a Riemann surface, then J_0 is automatically integrable; *cf.* [31, Theorem 4.16]. In this case, we denote J_0 by j and $\bar{\partial}_{j,J_1}$ by $\bar{\partial}_J$.

Closed holomorphic curves, *i.e.* holomorphic curve $u\colon \Sigma \to W$ where Σ is a closed Riemann surface, have a number of nice properties. One of these is the fact that they are at least as regular as the almost complex structure J on W.

Theorem 5.1.4 (*Cf.* [32, Theorem B.4.1, Remark B.4.3])**.** *Let W be a smooth $(2n)$-dimensional manifold endowed with an almost complex structure J of class C^l, $l \geq 1$, and Σ a closed Riemann surface. Then every J-holomorphic curve $u\colon \Sigma \to W$ of class $W^{1,p}$ with $p > 2$ is of class $W^{l+1,p}$. If $l = \infty$, then u is smooth.*

In our setup in Section 5.4, the theorem above will ensure that all holomorphic curves are smooth. Lower regularities will only appear implicitly in a technical argument regarding transversality.

Let the almost complex structure J be ω-compatible for some symplectic form ω on W. Then closed holomorphic curves have the second useful property that it is a topological property of the class they represent in $H_2(W)$ whether they are constant. To see this, we have to introduce the *energy* of a holomorphic curve.

Definition 5.1.5. Let J be an ω-compatible almost complex structure on a symplectic manifold (W, ω) and (Σ, j) a Riemann surface endowed with a volume form dvol. Then the **energy** of a smooth map $u\colon \Sigma \to W$ is defined as

$$E(u) = \tfrac{1}{2} \left\|du\right\|^2_{J,L^2} = \tfrac{1}{2} \int_\Sigma |du|^2_J \,\mathrm{dvol}$$

where $|du|_J$ is the norm of the linear map $du(z) : T_z\Sigma \to T_{u(z)}W$ defined by

$$|du|_J = \sup_{T_z\Sigma\setminus\{0\}} |\zeta|^{-1} \sqrt{|du(\zeta)|^2 + |du(j(\zeta))|^2}.$$

Here, the norms are induced by the Riemannian metric $\mathrm{dvol}(\cdot, j\cdot)$ on Σ and the Riemannian metric $\omega(\cdot, J\cdot)$ on W.

For holomorphic curves, we can compute this energy as follows.

Lemma 5.1.6. *Let $u\colon \Sigma \to W$ be a holomorphic curve. Then*

$$E(u) = \int_\Sigma u^*\omega.$$

In particular, if Σ is closed, we have

$$E(u) = \omega([u])$$

where $[u]$ is the class in $H_2(W)$ represented by u.

We immediately get our advertised property.

Corollary 5.1.7. *Let Σ be a closed Riemann surface and J an ω-compatible almost complex structure on (W, ω). Then a holomorphic curve $u\colon \Sigma \to W$ is constant if and only if $\omega([u]) = 0$.*

A special situation in which we will use this is that the holomorphic curve is contained in an exact subset of W.

Corollary 5.1.8. *If Σ is closed and there is a neighbourhood of the image of u on which the symplectic form ω is exact, then u is constant.*

Proof. By Stokes's theorem we have

$$\omega([u]) = \int_\Sigma u^*\omega = \int_\Sigma u^* d\beta = \int_\Sigma d(u^*\beta) = \int_{\partial\Sigma} u^*\beta = 0.$$

□

The corollary above is especially interesting in conjunction with the existence of a plurisubharmonic function, because closed holomorphic curves are forced to be contained in a level set of such a function; this is a consequence of the following lemma.

Lemma 5.1.9 (See [32, Lemma 9.2.9])**.** *Let Σ be an open subset of $\mathbb{C}$, (W,ω) a symplectic manifold endowed with an ω-compatible almost complex structure J and $u\colon \Sigma \to W$ a holomorphic curve of class C^2. Furthermore, let $\psi\colon W \to \mathbb{R}$ be a smooth function whose restriction to a neighbourhood of the image of u is plurisubharmonic.*

Then $\psi \circ u$ is subharmonic.

The property advertised above is obtained as a corollary.

Corollary 5.1.10. *Let Σ be a closed connected Riemann surface, (W,ω) a symplectic manifold endowed with an ω-compatible almost complex structure J and $u\colon \Sigma \to W$ a holomorphic curve of class C^2. Furthermore, let $\psi\colon W \to \mathbb{R}$ be a plurisubharmonic function.*

Then the image of u is contained in a level set of ψ.

Proof. By Lemma 5.1.9, the function $\psi \circ u$ is subharmonic. Because Σ is compact, this function attains its maximum. Since Σ has no boundary, this maximum must be attained in the interior. Thus, the maximum principle for subharmonic functions (see [24, Lemma 2.1.1]) asserts that $\psi \circ u$ is constant. Consequently, the image of u is contained in a level set of ψ. □

The last property of general holomorphic curves we will need is *positivity of intersection*, *i.e.* the property that every intersection with a complex hypersurface is positive, provided the holomorphic curve is not contained in the hypersurface.

Proposition 5.1.11 (Positivity of intersection; see [10, Proposition 7.1]). *Let $u\colon (\Sigma, j) \to (W, J)$ be a compact holomorphic curve and $H \subset W$ a compact complex hypersurface such that $u^{-1}(H) \cap \partial\Sigma = \emptyset = u^{-1}(\partial H) \cap \Sigma$ and $u(\Sigma) \not\subset H$. Then the subset $u^{-1}(H)$ is finite and the intersection number $\delta(u, H)$ of u and H is given by*

$$\delta(u, H) = \sum_{z \in u^{-1}(H)} \delta(u, H; z)\,.$$

At every intersection point z, u is tangent to H of order $l \geq 0$ and the local intersection number $\delta(u, H; z)$ satisfies

$$\delta(u, H; z) = l + 1.$$

In particular, each local intersection number is positive.

For holomorphic analysis, the most interesting holomorphic curves are the *simple* ones.

Definition 5.1.12. We say that a holomorphic curve $u\colon \Sigma \to W$ is **multiply-covered** if there is a compact Riemann surface Σ', a holomorphic curve $u'\colon \Sigma' \to W$, and a holomorphic branched covering $\phi\colon \Sigma \to \Sigma'$ with $\deg(\phi) > 1$ such that

$$u = u' \circ \phi.$$

If u is not multiply-covered, we say that u is **simple**..

Simple holomorphic curves have the following useful property.

Proposition 5.1.13 (*Cf.* [32, Corollary 2.5.3]). *Let Σ_i, $i = 0, 1$, be closed Riemann surfaces and $u_i\colon \Sigma_1 \to (W, J)$ holomorphic curves where J is at least of class C^2 and u_0 is simple. Then $u_0(\Sigma_0) = u_1(\Sigma_1)$ if and only if there is a holomorphic branched covering $\phi\colon \Sigma_1 \to \Sigma_0$ such that*

$$u_1 = u_0 \circ \phi.$$

5.2. Generalised Caps

In this section, we introduce a generalised notion of a cap of a contact manifold (M, ξ). Following this, we provide a construction by which we obtain new *generalised caps* from *generalised caps* of the binding of an open book decomposition supporting ξ.

Definition 5.2.1. A **generalised cap** of a contact manifold (M, ξ) is a symplectic manifold (C, ω_C) whose boundary is concave and given by (M, ξ) together with an ω_C-compatible almost complex structure J_C and an exhausting plurisubharmonic function ψ_C that is constant in a neighbourhood of the boundary and everywhere where the choice of J_C is generic.

Remark 5.2.2. By Corollary 5.1.10, the exhausting plurisubharmonic function ψ ensures that closed holomorphic curves with respect to J can only escape to infinity in whole.

Remark 5.2.3. Every cap is a generalised cap: since a cap is compact, we may use any constant function as the plurisubharmonic function ψ.

Our main tool in the construction of new generalised caps is the following theorem. It is a higher dimensional analogue of [15, Theorem 1.1], which was the crucial part of Eliashberg's construction of symplectic caps for weak symplectic fillings of 3-dimensional contact manifolds. Our proof is close in spirit to that of Eliashberg.

In this theorem and the remainder of this section, $(M, \xi = \ker \alpha)$ will be a closed contact manifold and (B, π) an open book decomposition of M to which α is adapted. We denote the page of the open book decomposition by P and the restriction of α to TB by α_B.

Theorem 5.2.4 (See [12, Theorem 4.1])**.** *Let (C, ω_C) be a symplectic manifold whose boundary is concave and agrees with (B, α_B).*

Then there is a symplectic manifold (W, ω) with boundary $\bar{M} \sqcup N$, where (M, α) is a concave boundary component, and N a fibre bundle over S^1 with fibre $F = P \cup C$ such that ω restricts to symplectic form on each fibre. Moreover, the holonomy of the symplectic fibration $N \to S^1$ is the identity on the subset $C \subset F$ of the fibre.

In the theorem above, neither C nor W is assumed to be compact, and in our applications in the following two sections they will not be.

Proof. Topologically, the definition of W is very simple. Let $U_0 \cong B \times \bar{B}_1(0)$ be an adapted neighbourhood of the binding B and denote by $B \times D^2$ its subset $B \times \bar{B}_{1/2}(0)$. Then define W as

$$W = ([-2, 0] \times M) \cup_{B \times D^2} \left(C \times D^2\right)$$

with smooth corners. Here, we think of $B \times D^2$ both as a subset of $M \times \{0\}$ and as $\partial C \times D^2$.

Symplectically, we want to think of $[-2, 0] \times M$ as a part of the symplectisation of M. Accordingly, we equip it with the symplectic form $d(e^t\alpha)$, where t is the coordinate on the interval. On $C \times D^2$, we use the symplectic form $\tilde{\omega}_C = \omega_C + f'(r)\, dr \wedge d\varphi$ where (ρ, φ) are polar coordinates on D^2 and $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a smooth function with the following properties:

(f-i) $f(\rho) = \rho^2$ near $\rho = 0$,

(f-ii) $f' > 0$ for $\rho > 0$,

(f-iii) $f(\rho) = \rho$ near $\rho = 1$.

The first property guarantees the smoothness of $\tilde{\omega}_C$ at $\rho = 0$, the second property ensures that $\tilde{\omega}_C$ is symplectic, and the third property will be convenient in the construction of almost complex structures in Corollary 5.2.8 and Section 5.3.

For the gluing, we work in the neighbourhood

$$U = [-1, 0] \times B \times D^2 \subset [-2, 0] \times M$$

with polar coordinates (r, φ) on D^2 obtained by rescaling the radial coordinate on the adapted neighbourhood U_0. By Theorem 3.1.3 in combination with Theorem 1.1.8 and Example 1.3.5, we may assume that on $B \times D^2$ the form α is given by $\alpha = h_1(r)\, \alpha_B + h_2(r)\, d\varphi$ for a Lutz pair (h_1, h_2) of our choice, up to multiplication of α_B with a positive constant. For notational convenience, we choose a Lutz pair satisfying

(h-i) $h_1(r) = e^{-r^2}$ and $h_2(r) = r^2$ near $r = 0$,

(h-ii) $h_1(r) = e^{-r}$ and $h_2(r) \equiv 1$ near $r = 1$.

Because gluing along parts of the boundary and smoothing of corners are not well defined in the presence of a symplectic form, we construct a model that contains a symplectic copy of $C \times D^2$ and of U, and such that the identification of $[-2, 0] \times M$ and this model along U realises the topological construction.

By the assumption of the theorem, we may glue $(-\infty, 0] \times B$ and C along $B = B \times \{0\}$ to obtain a symplectic manifold. Our model is the product of this manifold with D^2:

$$(W_0, \omega_0) = \big(\big((-\infty, 0] \times B, d\big(e^t \alpha_B\big)\big) \cup_B (C, \omega_C)\big) \times \big(D^2, f'(\rho)\, d\rho \wedge d\varphi\big)\,.$$

A schematic picture of this model is given in Figure 5.1. There, the left part of the horizontal axis represents $(-\infty, 0] \times B$, the right part C. The vertical axis represents the radial coordinate on D^2, so that a 'realistic' picture is given by rotating the figure around the horizontal axis.

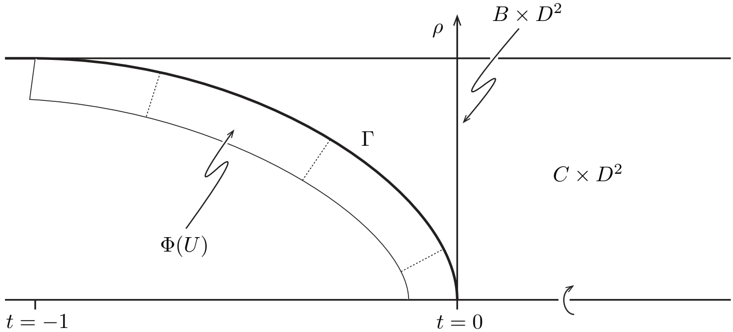

Figure 5.1.: The model (W_0, ω_0) for the symplectic gluing.

We claim that we can find a symplectomorphic copy $\Phi(U)$ of U inside this model as indicated in Figure 5.1. The dotted lines represent flow lines of the Liouville vector field $\Phi_* \partial_s$, and the hypersurface Γ in the model is a strictly contactomorphic copy of $\big(B \times D^2, \alpha\big)$.

The trick is to think of a neighbourhood of $B \times D^2$ in $[-2, 0] \times M$ not as a neighbourhood to the left of the horizontal axis in the model, which would cause the gluing to produce a corner, but as a neighbourhood under the hypersurface Γ, which connects smoothly with the curve $\{\rho = 1\}$ at $t = -1$. We can then glue the part to the right of Γ in a smooth and symplectic fashion to $[-2, 0] \times M$.

Define a primitive β of ω_0 on $(-\infty, 0] \times B \times D^2$ by

$$\beta = e^t \alpha_B + f(\rho)\, d\varphi.$$

The corresponding Liouville vector field Y is given by

$$Y = \partial_t + \frac{f(\rho)}{f'(\rho)} \partial_\rho .$$

Now define

$$\Phi \colon U = [-1, 0] \times B \times D^2 \to (-\infty, 0] \times B \times D^2 \subset W_0$$

by

$$\Phi(s, b, r, \varphi) = \bigl(s + \ln(h_1(r)) \, , b, f^{-1}(e^s h_2(r)) \, , \varphi\bigr) \, .$$

Lemma 5.2.5. *The map Φ is a symplectic embedding with $\Phi^*\beta = e^s\alpha$.*

Proof. Near $r = 0$, we have $f^{-1}(e^s h_1(r)) = e^{s/2} r$ by (f-i) and (h-i), which shows that Φ is smooth.

In order to see that Φ is injective, assume that we have

$$\Phi\left(s_1, b, r_1, \varphi\right) = \Phi\left(s_2, b, r_2, \varphi\right) .$$

By looking at the first and third component of the image, we see that

$$e^{s_1} h_1(r_1) = e^{s_2} h_1(r_2) \quad \text{and} \quad e^{s_1} h_2(r_1) = e^{s_2} h_2(r_2) \, ,$$

which implies that

$$\frac{h_2}{h_1}(r_1) = \frac{h_2}{h_1}(r_2) \, .$$

By the contact condition we know that $\left({}^{h_1}\!/_{h_2}\right)' = \left({}^{h_2' h_1 - h_1' h_2}\right)\!/_{h_1^2} < 0$. Consequently, we have $r_1 = r_2$ and, hence, also $s_1 = s_2$.

To see that Φ is symplectic, we compute

$$\begin{aligned} \Phi^*\beta &= e^{s + \ln(h_1(r))} \alpha_B + f\bigl(f^{-1}(e^s h_2(r))\bigr) \, d\varphi \\ &= e^s \left(h_1(r) \, \alpha_B + h_2(r) \, d\varphi\right) \\ &= e^s \alpha . \end{aligned}$$

Accordingly, Φ is symplectic and, thus, an immersion. $\square$

Because $\Phi^*\beta = e^s\alpha$, we know that the hypersurface Γ in the model, which is given by

$$\Gamma = \Phi\bigl(\{0\} \times B \times D^2\bigr) \, ,$$

is transverse to the Liouville vector field $\Phi_*\partial_s = Y$, and that $\iota_Y \omega_0 = \beta|_{T\Gamma}$ pulls back to α under the embedding Φ.

Lemma 5.2.6. *The hypersurface* $\Gamma \subset (-\infty, 0] \times B \times D^2$ *can be described as a graph*

$$\Gamma = \big((t, b, \rho, \varphi) \in (-\infty, 0] \times B \times D^2 \mid t \in [-1, 0]\,, \rho = g(t)\big)$$

with

$$g(t) = f^{-1}\big(h_2\big(h_1^{-1}(e^t)\big)\big)\,.$$

Proof. The hypersurface Γ is given by the points (t, b, ρ, φ) with

$$t = \ln(h_1(r)) \quad \text{and} \quad \rho = f^{-1}(h_2(r))\,.$$

This clearly translates into the form in the lemma. □

Note that $g' \leq 0$. Close to $t = 0$, we have $g(t) = \sqrt{t}$ by (f-i) and (h-i). This shows that Γ looks like a 'paraboloid' near $t = 0$. In particular, this verifies again that Γ is smooth.

Near $t = -1$, we have $g \equiv 1$ by (f-iii) and (h-ii). This means that Γ coincides with $(-\infty, 0] \times B \times \partial D$ close to $t = -1$. Therefore, the part of W_0 to the right of Γ can be glued to $[-2, 0] \times M$ along $\{0\} \times B \times D^2$, resulting in a symplectic manifold (W, ω). Its boundary is the disjoint union of $\{-2\} \times M$ and

$$N = \big(M \setminus \big(B \times \operatorname{Int}\big(D^2\big)\big)\big) \cup_{B \times S^1} \big((([-1, 0] \times B) \cup C) \times S^1\big)\,.$$

The manifold N fibres over S^1 in an obvious way, with fibres given by

$$F = \big(P \setminus \big(B \times \operatorname{Int}\big(D^2\big)\big)\big) \cup ([-1, 0] \times B) \cup C,$$

which topologically is just $P \cup C$. The restriction of ω to F is given by $d\alpha$ on the first part, by $d(e^t \alpha_B)$ on the second, and by ω_C on the third. So the fibre is indeed symplectic. Finally, the holonomy of the symplectic fibration

$$\big(C \times S^1, \omega_C\big) \to S^1$$

is obviously the identity. This completed the proof of Theorem 5.2.4. □

Remark 5.2.7. After I presented this result in our *Arbeitsgemeinschaft*, an alternative proof was found by Klukas [26].

If the symplectic manifold C in the theorem above is a generalised cap of the binding, then we can complete the manifold W to obtain a generalised cap of M.

Corollary 5.2.8. *Let (C_B, ω_B) be a generalised cap of (B, α_B). Then there is a generalised cap (C, ω_C) of (M, α) into which $(C_B, K\,\omega_B)$ embeds symplectically for some positive constant K.*

Proof. We start with the symplectic manifold (W, ω) from Theorem 5.2.4 constructed using (C_B, ω_B). To this, we attach the half-symplectisation of the boundary component N given by

$$(\mathrm{end}_N, \omega_N) = \left(\mathbb{R}_0^+ \times N, \omega|_N + ds \wedge d\varphi\right).$$

Here, s is the coordinate on $\mathbb{R}_0^+$ and $d\varphi$ the angular differential associated to the symplectic fibration on N. By construction, the symplectic fibration on N induces the boundary orientation with respect to W and the opposite one with respect to end_N. Thus, Corollary 3.3.7 tells us that we can indeed glue end_N to W along N. The result is our new generalised cap (C, ω_C).

It remains to find an almost complex structure J_C and an exhausting plurisubharmonic function ψ_C on C. Let (J_B, ψ_B) be the corresponding data for C_B, and write C_∞^B for the part of C_B where J_B is not generic. Then we define J_C as follows:

(J_C-1) On $C_\infty^B \times D^2$, we choose J_C to be a split almost complex structure $J_B \oplus j$ where $j(\partial_\rho) = \partial_\varphi$ for ρ close to 1. (This is justified by (f-iii) in the proof of Theorem 5.2.4.)

(J_C-2) On each fibre F of the symplectic fibration on N, we choose J_C to restrict to an $\omega_C|_{TF}$-compatible almost complex structure J_F that agrees with J_B on $C_\infty^B \subset F$ and is generic otherwise. (Here, we use that the holonomy is trivial on C_B.)

(J_C-3) On end_N, we demand that $J_C(\partial_s) = X$, where X is the unique vector field in $\ker \omega|_{TN}$ satisfying $\iota_X d\varphi \equiv 1$.

(J_C-4) On the remainder of C, we choose J_C to be generic.

For later reference, let us denote by C_∞ the set $\mathrm{end}_N \cup \left(C_B \times D^2\right)$, where we made a non-generic choice of J_C. Moreover, let us write $C_\infty^B \times \mathbb{C}$ for the set $\left(C_\infty^B \times D^2\right) \cup \left(\mathbb{R}_0^+ \times C_\infty^B \times S^1\right)$.

Now, we define a function $\bar{\psi}_B$ on $C_B \times D^2$ by $\bar{\psi}_B(x, \rho, \varphi) = \psi_B(x)$ and extend it to all of C by the constant value of ψ_B close to the boundary of C_B. This function is plurisubharmonic, and ∂_s and X are contained in the kernel of the corresponding symmetric bilinear form $g_{\bar{\psi}_B}$.

Now define a function $\psi\colon \mathrm{end}_N \to \mathbb{R}_0^+$ by

$$\psi(s,x) = s^3 e^{-1/s^2}.$$

Its exterior derivative is given by

$$d\psi = \left(3s^2 e^{-1/s^2} + 2e^{-1/s^2}\right) ds$$

and its second derivative with respect to s by

$$\partial_s^2 \psi = \left(6s + \tfrac{6}{s} + \tfrac{4}{s^3}\right) e^{-1/s^2}.$$

We can smoothly extended ψ to all of C as the constant function with value 0. Moreover, we have

$$\begin{aligned} -d(d\psi \circ J_C) &= -d((\partial_s \psi)\, ds \circ J_C) \\ &= d((\partial_s \psi)\, d\varphi) \\ &= \left(\partial_s^2 \psi\right) ds \wedge d\varphi. \end{aligned}$$

Thus, ψ is plurisubharmonic.

We define ψ_C as the sum of ψ and $\bar{\psi}_B$. Because ψ and $\bar{\psi}_B$ grow in complementary directions, ψ_C is plurisubharmonic. Moreover, it is exhausting since its sublevel sets are bounded in the direction of s because of ψ and in the fibres because of $\bar{\psi}_B$. Finally, at the boundary of C, both ψ and $\bar{\psi}_B$ are constant and, thus, ψ_C, as well. Since $\omega_C = d(e^t\alpha)$ close to $\{-2\} \times M$, this shows that $\left(C, e^2\omega_C\right)$ is a generalised cap for (M,α). □

The construction above has the following useful property, which will become important in the next two sections.

Corollary 5.2.9. *Let (C_B, ω_B) be a generalised cap of (B, α_B) and H_B a complex symplectic hypersurface in C_B disjoint from the boundary. Then the generalised cap (C, ω_C) of (M, α) from Corollary 5.2.8 contains a complex symplectic hypersurface H disjoint from the boundary.*

Moreover, if $\omega_B|_{C_B \setminus H_B}$ has a primitive that agrees with α_B on B, then there is a primitive of ω_C on $C \setminus H$ that agrees with α on M.

Proof. Because H_B is a symplectic hypersurface of C_B, the set

$$H = H_B \times \mathbb{C} = H_B \times D^2 \cup \mathbb{R}_0^+ \times H_B \times S^1$$

is a symplectic hypersurface of C. Moreover, by the construction of the almost complex structure J_C, the hypersurface H is complex because H_B is complex. Furthermore, H is obviously disjoint from the boundary of C.

Now, let $\omega_B|_{C_B \setminus H_B}$ have a primitive that agrees with α_B on B. On the complement of H in C, we construct a Liouville vector field of ω_C as follows.

Inside the model (W_0, ω_0) from Theorem 5.2.4, we extend the Liouville vector field $Y = \partial_t + {}^{f(\rho)}/_{f'(\rho)}\partial_\rho$ on $(-\infty, 0] \times B \times D^2$ over $(C_B \setminus H_B) \times D^2$ by the Liouville vector field $Y_B + {}^{f(\rho)}/_{f'(\rho)}\partial_\rho$, where Y_B is the Liouville vector field to the primitive of ω_{C_B} on $C_B \setminus H_B$. Since the primitive agrees with α_B on the boundary, the two Liouville vector fields connect smoothly. Let us denote the projection of the extended Liouville vector field Y to the fibres by Y_F.

By the arguments in the proof of Theorem 5.2.4, Y agrees with the Liouville vector field $\partial_{\tilde{t}}$ on $[-2, 0] \times M$ under the identification via the symplectic embedding Φ. Consequently, we can extend Y over $[-2, 0] \times M$ by $\partial_{\tilde{t}}$. Note that this guarantees the correct primitive on $\{-2\} \times M$.

It remains to show that the Liouville vector Y can be extended to $\mathrm{end}_N \setminus H$.

In order to do this, let us take a look at the coordinate s used in the gluing of the two parts W and end_N of C. It is an extension of the coordinate s on end_N to a neighbourhood of the positive boundary of W defined using the flow of the vector field ∂_s with $\iota_{\partial_s}\omega = d\varphi$. This shows that, on a neighbourhood of $((-\infty, 0] \times B \cup (C_B \setminus H_B)) \times S^1$ in W_0, the coordinate s is given by $s = \rho - 1$. Accordingly, we can extend Y by $Y_W = Y_F + (s+1)\,\partial_s$ to the subset $\big(\mathbb{R}_0^+ \times (N \setminus M)\big) \setminus H$ of end_N.

To further extend Y over the remaining part of $\mathrm{end}_N \setminus H$, we extend the symplectic embedding Φ from the proof of Theorem 5.2.4 to a symplectomorphism of $[-\epsilon, 0] \times M$ with a one-sided neighbourhood V of $M = \big(M \setminus \big(B \times D^2\big)\big) \cup \Gamma$ in

$$W_1 = \Big(\big(M \setminus \big(B \times \mathrm{Int}\big(D^2\big)\big)\big) \times D^2\Big) \cup \Big(W_0 \setminus \big((-\infty, -1) \times B \times D^2\big)\Big)$$

for some $\epsilon > 0$.

As a first step, we identify $M \setminus \big(B \times \mathrm{Int}(D^2)\big)$ with the mapping torus $\tilde{P}(\Psi)$ using the flow of a scaled version of the Reeb vector field to α, as we have done in the proof of Theorem 3.1.22; here, Ψ is the time-2π-flow of the scaled Reeb vector field, and $\tilde{P} = P \setminus \big(B \times \mathrm{Int}(D^2)\big)$. Then the contact form α decomposes as

$$\alpha = \beta + dh = \beta + \sigma_h + h_\varphi\, d\varphi,$$

where β is a Liouville form on $\tilde{P}$ and h a function on $[0, 2\pi] \times P$ such that $h_\varphi = \iota_{\partial_\varphi} dh > 0$. Furthermore, the function h_φ agrees with h_2 inside $B \times D^2$, and $\sigma_h = dh - h_\varphi\, d\varphi$ is a form on $[0, 2\pi] \times P$ with the three properties that it vanishes in a neighbourhood of $[0, 2\pi] \times (P \cap (B \times D^2))$, that its restrictions to the pages $\tilde{P}_\varphi = \{\varphi\} \times \tilde{P}$ are exact forms, and that $\beta + \sigma_h$ descends to a smooth form on the mapping torus $\tilde{P}(\Psi)$.

Now denote by Y_φ the Liouville vector field on $\tilde{P}_\varphi$ to the primitive $\beta + \sigma_h|_{\tilde{P}_\varphi}$ of $d\beta$, and define a vector field $\tilde{Y}$ on $M \setminus (B \times \mathrm{Int}(D^2))$ by $\tilde{Y}|_{\tilde{P}_\varphi} = Y_\varphi$. This vector field is smooth, because $\beta + \sigma_h$ is a smooth form on the mapping torus $\tilde{P}(\Psi)$. Extend it to $\tilde{P}(\Psi) \cup ([-1, 0] \times B)$ by the Liouville vector field ∂_t to $e^t \alpha_B$, and denote the flow of the resulting vector field by $\tilde{\Psi}_t$.

Using this flow, we define the map

$$\begin{aligned} \tilde{\Phi}\colon\ [-\epsilon, 0] \times \tilde{P}(\Psi) &\to [-\bar{\epsilon}, 0] \times \Big(\tilde{P}(\Psi) \cup \big([-1, 0] \times B \times S^1\big)\Big) \\ (\tilde{t}, p, \varphi) &\mapsto \Big(\big(e^{\tilde{t}} - 1\big) h_\varphi\big(\tilde{\Psi}_{\tilde{t}}(p)\big), \tilde{\Psi}_{\tilde{t}}(p)\, , \varphi\Big), \end{aligned}$$

where ϵ and $\bar{\epsilon}$ are small positive numbers.

Note that we can use $\tilde{\Phi}$ to extend Φ over $[-\epsilon, 0] \times \tilde{P}(\Psi)$: by (h-ii) in the proof of Theorem 5.2.4, close to the boundary of $B \times D^2$, the map Φ is given by $\Phi\big(\tilde{t}, b, r, \varphi\big) = \big(\tilde{t} - r, b, e^{\tilde{t}}, \varphi\big)$ for sufficiently large $\tilde{t} \le 0$. This agrees with $\tilde{\Phi}$ because, near the boundary of $B \times D^2$, the coordinate s is given by $s = \rho - 1$, we have $h_\varphi = h_2 \equiv 1$, and the Liouville vector fields on the pages to the Liouville form $\beta + \sigma_h = h_1(r)\, \alpha_B = e^{-r} \alpha_B$ are given by $-\partial_r$, which leads to the identification of t with $-r$ in the gluing of $\tilde{P}$ with $[-1, 0] \times B$.

Now, let $\tilde{\Phi}\big(\tilde{t}, p, \varphi\big) = \tilde{\Phi}\big(\tilde{t}', p', \varphi'\big)$. Then we know that $\tilde{\Psi}_{\tilde{t}}(p) = \tilde{\Psi}_{\tilde{t}'}(p')$ and that $\varphi' = \varphi$. Looking at the first component of the image, this implies that $\tilde{t} = \tilde{t}'$. Consequently, we also have $p = p'$, because the flow of a vector field for a fixed time is injective. This proves that $\tilde{\Phi}$ is one-to-one.

Next, we show that $\tilde{\Phi}$ pulls back the primitive $\beta + dh + s\,d\varphi$ of the symplectic form $d\beta + ds \wedge d\varphi$ on the neighbourhood used in the gluing of W and end_N to the Liouville form $e^{\tilde{t}}\alpha$ on $[-\epsilon, 0] \times M$. Indeed, since $\tilde{Y}$ is a Liouville vector field on the fibres to the Liouville form $\beta + \sigma_h$, we have

$$\begin{aligned}
\tilde{\Phi}^*(\beta + dh + s\,d\varphi) &= \tilde{\Phi}^*(\beta + \sigma_h + (s + h_\varphi)\,d\varphi) \\
&= \tilde{\Psi}^*_{\tilde{t}}(\beta + \sigma_h) + \Big(h_\varphi + \big(e^{\tilde{t}} - 1\big)h_\varphi\Big)d\varphi \\
&= e^{\tilde{t}}(\beta + \sigma_h + h_\varphi\,d\varphi) \\
&= e^{\tilde{t}}\alpha.
\end{aligned}$$

This shows that $\tilde{\Phi}$ is symplectic and, hence, an embedding. Moreover, we see that the differential of $\tilde{\Phi}$ sends $\partial_{\tilde{t}}$ to the Liouville vector field $Y_P = \tilde{Y} + (s + h_\varphi)\,\partial_s$ to the Liouville form $\beta + dh + s\,d\varphi$.

Since $h_\varphi = h_2 \equiv 1$ close to the boundary of $B \times D^2$ by (h-ii) in the proof of Theorem 5.2.4, this shows that we can extend Y over $\mathrm{end}_N \setminus H$ by the vector field we obtain by gluing Y_P and Y_W along $\mathbb{R}_0^+ \times \{-1\} \times B \times S^1$. Accordingly, there is a global primitive of $e^2\omega_C$ on $C \setminus H$ that restricts to α on the boundary $\{-2\} \times M$. $\square$

A second useful property of the construction above, which will also play a role in Section 5.4, is the following.

Lemma 5.2.10. *Every closed holomorphic curve in the generalised cap (C, ω_C) from Corollary 5.2.8 is either disjoint from end_N or contained in one of the fibres F.*

Proof. Let Σ be a closed Riemann surface, $u\colon \Sigma \to C$ a holomorphic curve, and F one of the fibres in end_N. We claim that the intersection number of u and F is 0.

Let us first explain why this intersection number is well defined although F and C might not be compact. Since Σ is compact, both the function $\psi \circ u$ and the function $\bar{\psi}_B \circ u$ are bounded from above, where ψ and $\bar{\psi}_B$ are the plurisubharmonic functions from the proof of Corollary 5.2.8; ψ increases in the direction of the coordinate s in the half-symplectisation end_N and is constant on the fibres; $\bar{\psi}_B$ increases inside the fibres and is constant in the direction of the coordinate s.

Let $c_\psi > \psi(F)$ and $c_{\bar{\psi}}$ be regular values of ψ and $\bar{\psi}_B$, respectively, such that the image of u is contained in the interior of the intersection $\hat{C}$ of the sublevel sets $\psi^{-1}((-\infty, c_\psi])$ and $\bar{\psi}_B^{-1}\big((-\infty, c_{\bar{\psi}}]\big)$; the regular values exist by Sard's theorem, see [5, Theorem 6.2]. After smoothing corners, $\hat{C}$ is a smooth compact manifold with boundary. Moreover, the intersection $\hat{F}$ of F with $\hat{C}$ is a smooth compact submanifold of $\hat{C}$ whose boundary is contained in $\partial\hat{C}$. This implies that $\hat{F}$ and u have a well-defined intersection number in $\hat{C}$. Since this intersection number does not depend on the choice of the regular values c_ψ and $c_{\bar{\psi}}$ or the precise way how we smoothen corners, we obtain a well-defined intersection number of F and u.

Now, we show that this intersection number is 0. If F does not intersect the image of u, then this is also true for $\hat{F}$. Consequently, the intersection number is 0. If F does intersect the image of u, we can make it disjoint from this image by pushing it sufficiently far along the s-direction. This translates into the same procedure for $\hat{F}$, proving our claim that F and u always have intersection number 0.

Suppose that u intersects end_N. Then it also intersects one of the fibres. These are complex hypersurfaces by the construction of the almost complex structure J_C on C. So, positivity of intersection (Proposition 5.1.11) tells us that the image of u is contained in a fibre since the intersection number is 0. □

To distinguish generalised caps constructed via Corollary 5.2.8 from general generalised caps, we introduce the following notion.

Definition 5.2.11. We say that a generalised cap if of **order** $k \in \mathbb{N}_0$ if it is constructed by k times first applying Corollary 5.2.8 and then attaching a Liouville cobordism. We call the embedded generalised caps obtained by applying the construction above $l \in \{1, \ldots, k\}$ times the **levels** of the generalised cap.

Remark 5.2.12. A generalised cap of order $k > 0$ is also of order $k - 1$.

We introduce the following corresponding concept for open book decompositions.

Definition 5.2.13. We say that (B_i, π_i), $i = 1, \ldots, k$, is a **tower of open book decompositions** of height k of a closed manifold M if (B_k, π_k) is an open book decomposition of M and for each $i > 1$ there is a cobordism W_i from B_i to a manifold with the open book decomposition (B_{i-1}, π_{i-1}). The **lowest level** of the tower is B_1, if $k > 0$, and M if $k = 0$.

A contact structure ξ on M is said to be **supported** by a tower of open book decompositions of height k if (B_k, π_k) supports ξ and the cobordisms W_i are Liouville cobordisms whose convex boundary is supported by (B_{i-1}, π_{i-1}).

Remark 5.2.14. In principle, we could allow the cobordisms W_i to be general symplectic cobordisms. However, for our applications in Section 5.4, the definition above is more convenient.

Using the new language above, we can combine Corollary 5.2.8 and Corollary 5.2.9 as follows.

Corollary 5.2.15. *Let (M, ξ) be supported by a tower of open book decompositions of height $k \in \mathbb{N}_0$ such that the lowest level has a generalised cap (C_0, ω_0). Then (M, ξ) has a generalised cap (C, ω_C) of order k into which $(C_0, K\,\omega_0)$ embeds symplectically for some positive constant K.*

Moreover, if the generalised cap of the lowest level contains a complex symplectic hypersurface H_0 disjoint from the boundary, then there is a corresponding complex symplectic hypersurface H in C. Furthermore, if there is a primitive of ω_0 on $C_0 \setminus H_0$ that agrees on the boundary with the contact form on the lowest level, then $\omega_C|_{C\setminus H}$ has a primitive that agrees with a contact form defining ξ on the boundary, which can be chosen freely.

5.3. Symplectic Spheres in Generalised Caps

In this section, we construct generalised caps for contact manifolds with one of three special properties. These generalised caps contain both a complex hypersurface H, whose complement is exact in two of the cases, and a region U_∞ fibred by holomorphic spheres. This is the cornerstone of our proofs in the next section.

Our main result reads as follows.

Theorem 5.3.1. *Let $(M, \xi = \ker\alpha)$ be a closed contact manifold that is supported by a tower of open book decompositions of height $k \in \mathbb{N}_0$ such that there is a Liouville cobordism from the lowest level to a contact manifold (L, α_L) with one of the following three properties:*

(L-h) *L is at least 3-dimensional and embeds into a subcritical Stein manifold as a hypersurface of restricted contact type.*

(L-m) *L is supported by an open book decomposition whose monodromy is trivial.*

(L-p) *L is planar.*

Then M has a generalised cap (C, ω_C) of order k that contains a subset U_∞ fibred by holomorphic symplectic spheres. Moreover, this generalised cap contains a complex hypersurface H, and in the cases (L-h) and (L-m), there is a primitive β_C of ω_C that agrees with α on TM on the complement of H.

Remark 5.3.2. In the case (L-m), we refer to the monodromy induced by a vector field X obtained from a scaled Reeb vector field as in the proof of Theorem 3.1.22. This is equivalent to saying that (L, α_B) is the contact open book $M(P, \mathrm{id}, \alpha|_{TP})$ obtained from the symplectic open book $(P, \mathrm{id}, \alpha|_{TP})$ via the generalised Thurston-Winkelnkemper construction.

In regard of Corollary 5.2.15, it is sufficient to prove Theorem 5.3.1 for the case $k = 0$. We start with the case (L-h); the construction we use is basically taken from [18].

Lemma 5.3.3. *Let (M, α) be an at least 3-dimensional closed contact manifold that embeds into a subcritical Stein manifold (W, J_W) as a hypersurface of restricted contact type. Then M has a generalised cap (C_h, ω_h) containing a subset U_∞^h fibred by holomorphic symplectic spheres. Moreover, this generalised cap contains a complex hypersurface H_h on whose complement there is a primitive β_h of ω_h agreeing with α on TM.*

Proof. We construct the generalised cap separately for each component of M. Hence, we may as well assume that M is connected.

Since (W, J_W) is subcritical, Theorem 1.3.13 tells us, in combination with Theorem 1.3.16, that (W, J_W) is symplectomorphic to a split Stein manifold. Consequently, we may assume that M embeds into a complex manifold $(V \times \mathbb{C}, J_V \oplus i)$ where (V, J_V) is Stein. On this manifold, we have the strictly plurisubharmonic function $\psi = \psi_V + |z|^2/4$ where ψ_V is the strictly plurisubharmonic function on V.

Let us take a closer look at our hypersurface M. Following the proof of [18, Corollary 4.2], we see that M separates $V \times \mathbb{C}$ into a bounded and an unbounded component because $V \times \mathbb{C}$ has trivial homology in codimension 1. Moreover, since M is a hypersurface of restricted contact type, there is a global Liouville vector field Y on $V \times \mathbb{C}$ transverse to M. We claim that it points into the closure C_0 of the unbounded component of $(V \times \mathbb{C}) \setminus M$.

Take a regular value c of ψ such that M is contained in the interior of the sublevel set $\psi^{-1}((-\infty, c])$, and denote by A_h the intersection of C_0 with $\psi^{-1}((-\infty, c])$. Now, assume that Y pointed outwards along the boundary of C_0. Then A_h would be an exact filling of the disjoint union of the level set $S_h = \psi^{-1}(c)$ of the plurisubharmonic function ψ on $V \times \mathbb{C}$ and the non-empty contact manifold M. However, this is impossible by [18, Theorem 3.4]. Consequently, Y points into C_0 and A_h is an exact symplectic cobordism from M to S_h.

Now choose a constant $R > 0$ such that S_h is contained in $V \times B_R(0)$ and, hence, also A_h. Consequently, A_h is contained in the compactified manifold $V \times \mathbb{C}P^1$, where $\mathbb{C}P^1$ is endowed with the Fubini-Study form of total volume πR^2. We choose our generalised cap C_h to be given by the closure of the component of $\left(V \times \mathbb{C}P^1\right) \setminus M$ containing $\{\infty\}$. This manifold can be decomposed into A_h and the closure of the component of $\left(V \times \mathbb{C}P^1\right) \setminus S_h$ containing $\{\infty\}$, which we call C_∞^h.

Note that, after removing the complex hypersurface $H_h = V \times \{\infty\}$ from C_h, the Liouville vector field Y is globally defined. Thus, there is a primitive of the symplectic form on the complement of H_h that agrees with α on TM.

On C_∞^h, we choose the almost complex structure J_h on C_h to agree with the product structure $J_V \oplus i$ and, on the complement, we choose it generically. With this choice, H_h is a complex hypersurface. Moreover, the set $U_\infty^h = \psi_V^{-1}([c, \infty))$, is fibred by the holomorphic symplectic spheres $\{p\} \times \mathbb{C}P^1$.

It remains to show that there is an exhausting plurisubharmonic function ψ_h on C_h that is constant in a neighbourhood of M. The function ψ_V is exhausting and plurisubharmonic, but not constant in a neighbourhood of M. So, our goal is to cut off ψ_V in a suitable way.

By possible adding a constant to ψ_V, we may assume that $c = 0$. Let $f\colon \mathbb{R} \to \mathbb{R}_0^+$ be defined by

$$f(t) = t^3 e^{-1/t^2}$$

on $\mathbb{R}^+$ and by 0, otherwise.

We define ψ_h to be given by $f \circ \psi_V$. On the complement of $\psi_V^{-1}([c,\infty))$, this function vanishes identically. Consequently, we only have to show that it is plurisubharmonic on $\psi_V^{-1}([c,\infty))$. From the proof of Corollary 5.2.8, we already know that the first two derivatives of f are non-negative. (There, f was denoted by ψ.) This implies that

$$\begin{aligned}
-d(d\psi_h \circ J_h)(v, J_h v) &= -d((f' \circ \psi_V)\, d\psi_V \circ J_h)(v, J_h v) \\
&= -(f' \circ \psi_V)\, d(d\psi_V \circ J_h)(v, J_h v) \\
&\quad - (f'' \circ \psi_V)\,(d\psi_V \wedge (d\psi_V \circ J_h))(v, J_h v) \\
&= -(f' \circ \psi_V)\, d(d\psi_V \circ J_h)(v, J_h v) \\
&\quad + (f'' \circ \psi_V)\Big(d\psi_V(v)^2 + d\psi_V(J_h v)^2\Big) \\
&\geq 0.
\end{aligned}$$

Thus, ψ_h is plurisubharmonic. Accordingly, C_h is a generalised cap with the desired properties. □

Next, we deal with the case (L-m).

Lemma 5.3.4. *Let (M,α) be a contact manifold supported by an open book decomposition (B,π) with trivial monodromy. Then M has a generalised cap (C_m,ω_m) that contains a subset U_∞^m fibred by holomorphic symplectic spheres. Moreover, this generalised cap contains a complex hypersurface H_m on whose complement there is a primitive β_m of ω_m that agrees with α on TM.*

Proof. A generalised cap (C_B,ω_B) of the binding B is given by the half symplectisation $\big(\mathbb{R}_0^+ \times B, d(e^t\alpha_B)\big)$ where $\alpha_B = \alpha|_{TB}$. The almost complex structure J_B on C_B is given by any almost complex structure that sends ∂_t to the Reeb vector field R_{α_B} to α_B, leaves $\xi_B = \ker\alpha_B$ invariant, and whose restriction to ξ_B is $d\alpha_B$-compatible; the corresponding plurisubharmonic function ψ_B is given by $f(e^t - 1 + \epsilon)$ where f is the function from the end of the proof of Lemma 5.3.3 and ϵ a positive constant. This function is plurisubharmonic because e^t is strictly plurisubharmonic:

$$-d\big(d(e^t) \circ J_B\big) = -d\big(e^t\, dt \circ J_B\big) = d\big(e^t\alpha_B\big) = \omega_B.$$

Now, we use Theorem 5.2.4 with this generalised cap to obtain a symplectic manifold (W,ω) whose boundary is the disjoint union of M and the trivial symplectic fibration $(N, \omega|_{TN}) = \big(P \times S^1, d\alpha|_{TP}\big)$, where P is the page of the open book decomposition (B, π). Instead of proceeding as in Corollary 5.2.8 and gluing in the half-symplectisation end_N, we use the triviality of the symplectic fibration and glue in $\big(\mathbb{C}P^1 \setminus D^2\big) \times N$. Here, we interpret $\mathbb{C}P^1$ as the compactification of $B_R(0) \subset \mathbb{C}$ for some $R > 1$ with the standard volume form; it is endowed with the Fubini-Study form of total volume πR^2. A schematic picture of the resulting symplectic manifold (C_m, ω_m) is provided in Figure 5.2 below.

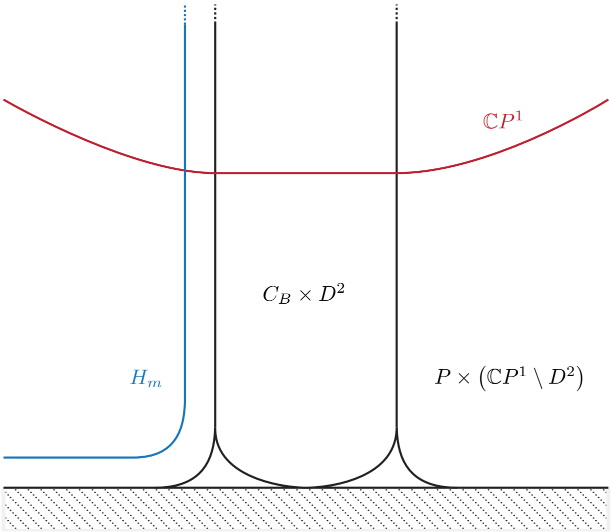

Figure 5.2.: The generalised cap (C_m, ω_m).

We choose the almost complex structure J_m on C_m as follows:

(J_m-1) On $C_B \times D^2$, we choose J_m to be a split almost complex structure $J_B \oplus j$ where $j(\partial_\rho) = \partial_\varphi$ for ρ close to 1. (This is justified by (f-iii) in the proof of Theorem 5.2.4.)

(J_m-2) On $P \times \big(\mathbb{C}P^1 \setminus D^2\big)$, we choose J_m to be a split almost complex structure $J_P \oplus i$ where i is the standard complex structure on $\mathbb{C}P^1$ and J_P an almost complex structure that agrees with J_B on $C_B \subset F$ and is generic otherwise.

(J_m-3) On the remainder of C_m, we choose J_m to be generic.

With this choice, the set

$$U_\infty^m = C_B \times \mathbb{C}P^1 = \big(C_B \times D^2\big) \cup \big(C_B \times \big(\mathbb{C}P^1 \setminus D^2\big)\big)$$

is fibred by the holomorphic symplectic spheres $\{p\} \times \mathbb{C}P^1$ with $p \in C_B$. Moreover, the hypersurface $H_m = P \times \{\infty\}$ is a complex hypersurface on whose complement there is a primitive of ω_m that agrees with $e^{-2}\alpha$ on $T(\{-2\} \times M)$ by the same arguments as in the proof of Corollary 5.2.9. In addition, the function ψ_m given by $f(e^t - 1)$ is an exhausting plurisubharmonic function after extending it to $C \setminus U_\infty^m$ by 0. Consequently, $\big(C_m, e^2\omega_m\big)$ is a generalised cap with the desired properties.

For later reference, let us denote by C_∞^m the set $U_\infty^m \cup \big(P \times \big(\mathbb{C}P^1 \setminus D^2\big)\big)$, where we made a non-generic choice of the almost complex structure J_m. $\square$

It remains to cover the case (L-p). In principle, we could try to use the cap for planar contact manifolds constructed by Etnyre [16]. However, it turns out to be more convenient to alter his construction.

Lemma 5.3.5. *Let (M, α) be a contact manifold supported by an open book decomposition (B, π) with planar pages $P = S^2 \setminus \big(\bigcup_{l=1}^n D_l^2\big)$. Then M has a generalised cap (C_p, ω_p) that contains a subset U_∞^p fibred by holomorphic symplectic spheres. Moreover, this cap contains a complex hypersurface H_p.*

Proof. Since S^2 and $\mathbb{C}P^1$ are diffeomorphic via an orientation preserving diffeomorphism, we can identify the pages with $\mathbb{C}P^1 \setminus \big(\bigcup_{l=1}^n D_l^2\big)$, where $\infty \in D_1^2$.

Because the restriction of $d\alpha$ to the pages is a volume form, it differs only by multiplication with a positive function from the restriction of the Fubini-Study form ω_{FS} on $\mathbb{C}P^1$. Consequently, there is a positive function λ on $\mathbb{C}P^1$ such that $\lambda\omega_{\mathrm{FS}}$ extends $d\alpha$. This implies that a cap (C_B, ω_B) of the binding B is given by $\big(\bigcup_{l=1}^n D_l^2, \lambda\omega_{\mathrm{FS}}\big)$.

To this cap, we apply Corollary 5.2.8. A schematic picture of the resulting generalised cap (C_p, ω_p) is given in Figure 5.3.

We claim that there is a symplectomorphism Φ from the half-symplectisation end_N of the symplectic fibration N with fibres $\mathbb{C}P^1$ to the product symplectic manifold $\big(\mathbb{R}_0^+ \times \mathbb{C}P^1 \times S^1, \lambda\,\omega_{\mathrm{FS}} \oplus ds \wedge d\varphi\big)$ that leaves $\mathbb{R}_0^+ \times$

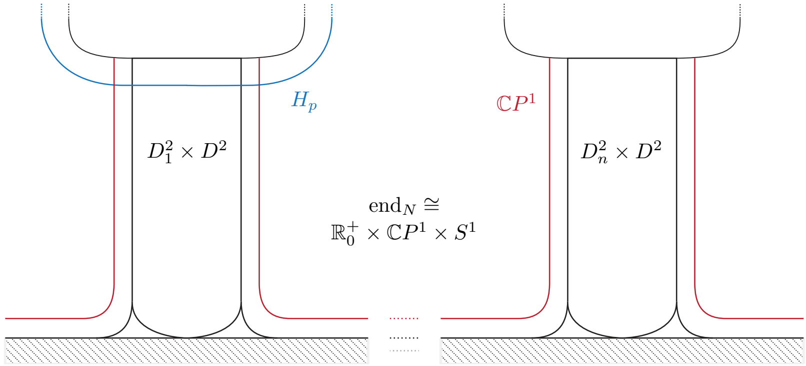

Figure 5.3.: The generalised cap (C_p, ω_p).

$D_1^2 \times S^1$ pointwise fixed. By the proof of Proposition 3.3.6, it is sufficient to show that the holonomy of the symplectic fibration on N is isotopic to the identity through symplectomorphisms that fix D_1^2 pointwise.

We know, by construction, that the holonomy Ψ fixes a neighbourhood of D_1^2. Accordingly, it is contained in the space $\mathcal{S}$ of symplectomorphisms of $\left(D = \mathbb{C}P^1 \setminus D_1^2, \lambda\omega_{\mathrm{FS}}\right)$ which agree with the identity a neighbourhood of the boundary. By Theorem 3.2.13, there is a long exact homotopy sequence

$$\cdots \longrightarrow \pi_k(\mathcal{S}) \xrightarrow{i} \pi_k(\mathcal{D}) \longrightarrow \pi_k\left(\Omega_{\mathrm{c}}^{\mathrm{ES}}(D)\right) \longrightarrow \cdots.$$

Here, $\mathcal{D}$ is the space of diffeomorphisms of D that agree with the identity in a neighbourhood of the boundary, and $\Omega_{\mathrm{c}}^{\mathrm{ES}}(D)$ is the space of exact symplectic forms that agree with $\lambda\omega_{\mathrm{FS}}$ on a neighbourhood of the boundary.

Since D is contractible, all symplectic forms on D are exact. Consequently, the space $\Omega_{\mathrm{c}}^{\mathrm{ES}}(D)$ consists of the products of $\lambda\omega_{\mathrm{FS}}$ with positive functions with compact support in the interior of D. This shows that $\Omega_{\mathrm{c}}^{\mathrm{ES}}(D)$ is contractible and, hence, the inclusion i a weak homotopy equivalence.

By [36, Theorem B], the space $\mathcal{D}$ is contractible. Thus, $\mathcal{S}$ is weakly contractible; in particular, it is connected. This proves the existence of our desired symplectomorphism Φ.

That end_N is symplectomorphic to $\mathbb{R}_0^+ \times \mathbb{C}P^1 \times S^1$ via a diffeomorphism that fixes $\mathbb{R}_0^+ \times D_1^2 \times S^1$ allows us to define the almost complex structure J_p on C_p as follows:

(J_p-1) On end_N, we choose J_p to be given by the split complex structure $i \oplus j$ where i is the standard complex structure on $\mathbb{C}P^1$ and j the complex structure on $\mathbb{R}_0 \times S^1$ satisfying $j(\partial_s) = \partial_\varphi$.

(J_p-2) On $D_1^2 \times \mathrm{Int}(D^2)$, we choose J_m to be a split almost complex structure $J_D \oplus j$ where j in a complex structure on D^2 that satisfies $j(\partial_\rho) = \partial_\varphi$ for ρ near 1. (This is justified by (f-iii) in the proof of Theorem 5.2.4 and the fact that the symplectomorphism Φ fixes $\mathbb{R}_0^+ \times D_1^2 \times S^1$ pointwise.)

(J_p-3) On the remainder of C_p, we choose J_p to be generic.

With this choice, the set $U_\infty^p = \mathrm{end}_N$ is fibred by the holomorphic symplectic spheres $\{s\} \times \mathbb{C}P^1 \times \{\varphi\}$. Moreover, the symplectic hypersurface

$$H_p = \{\infty\} \times \mathbb{C} = \big(\{\infty\} \times D^2\big) \cup \big(\mathbb{R}_0^+ \times \{\infty\} \times S^1\big),$$

is complex. Unfortunately, the complement of H is only exact if $n = 1$, *i.e.* if we only glued in one disc.

Finally, the function ψ_p on $\mathrm{end}_N \cong \mathbb{R}_0^+ \times \mathbb{C}P^1 \times S^1$ given by $f(s)$, with the function f from the end of the proof of Lemma 5.3.3, is an exhausting plurisubharmonic function that extends to the rest of C_p by 0. Consequently, (C_p, ω_p) is a generalised cap with the desired properties.

For later reference, let us denote by C_∞^p the set $U_\infty^p \cup \big(D_1^2 \times D^2\big)$, where we made a non-generic choice of the almost complex structure J_p. □

The three preceding lemmata, together with Corollary 5.2.15, prove Theorem 5.3.1.

In the remainder of this section, we examine the holomorphic spheres in the generalised caps we have constructed. The lemma below is an extension of parts of [18, Lemma 5.2] to our additional two setups.

Lemma 5.3.6. *Let $u\colon \mathbb{C}P^1 \to C_\infty^i$, with $i = h, m, p$, be a non-constant holomorphic sphere. Then the image of u is contained in U_∞^i and u is a holomorphic branched covering of one of the spheres by which U_∞^i is fibred.*

Proof. We claim that in all three cases the set $C^i_\infty \setminus U^i_\infty$ is exact.

In the case that $i = h$, we know that $C^h_\infty \setminus U^h_\infty$ is a subset of $V \times \mathbb{C}P^1$ endowed with the product symplectic structure and the product complex structure; the symplectic structure on V is the exact symplectic form ω_{ψ_V} induced by the strictly plurisubharmonic function ψ_V and that on $\mathbb{C}P^1$ is given by a scaled Fubini-Study form.

On $C^h_\infty \setminus U^h_\infty$, we have $\psi_V < c$, where c is the regular value of the plurisubharmonic function $\psi = \psi_V + {|z|^2}/4$ on $V \times \mathbb{C}$ such that the level set $\psi^{-1}(c)$ is the boundary of C^h_∞. Thus, the set $V \times \{0\}$ does not intersect $C^h_\infty \setminus U^h_\infty \subset V \times \mathbb{C}P^1$. Since $\mathbb{C}P^1 \setminus \{0\}$ is contractible, this shows that $C^h_\infty \setminus U^h_\infty \subset V \times \mathbb{C}P^1$ is exact.

In the case that $i = m$, the set $C^m_\infty \setminus U^m_\infty$ is given by $(P \setminus C_B) \times \big(\mathbb{C}P^1 \setminus D^2\big)$ and, hence, exact.

In the case that $i = p$, the set $C^p_\infty \setminus U^p_\infty$ is given by $D^2_1 \times \mathrm{Int}\big(D^2\big)$, which is exact, as well.

By Corollary 5.1.8, this implies that the image of u has to intersect U^i_∞. Next, we show that this already implies that the image is contained in U^i_∞ and that it is a holomorphic branched covering of one of the spheres by which U^i_∞ is fibred.

We start again with the case that $i = h$. The product complex structure ensures that the curve u splits as (u_1, u_2), where u_1 is a holomorphic curve on V and u_2 a holomorphic map $\mathbb{C}P^1 \to \mathbb{C}P^1$. Since V is exact, Corollary 5.1.8 asserts that u_1 is constant. Thus, the image of u must be contained in or disjoint from $U^h_\infty = \psi_V^{-1}([c, \infty)) \times \mathbb{C}P^1$. Since the latter is impossible, as we have already seen, the image of u is contained in one of the spheres $\{p\} \times \mathbb{C}P^1$ inside U^h_∞.

Because u is not constant its image must agree with the holomorphic sphere $\{p\} \times \mathbb{C}P^1$. Thus, Proposition 5.1.13 shows that u is a holomorphic branched covering of this sphere.

The case in which $i = m$ is very similar. This time, the product complex structure ensures that the curve u splits as (u_1, u_2), where u_1 is a holomorphic curve on P and u_2 a holomorphic map $\mathbb{C}P^1 \to \mathbb{C}P^1$. Since the pages P are exact, Corollary 5.1.8 asserts that u_1 is constant. Thus, the image of u must be contained in or disjoint from $U^m_\infty = C_B \times \mathbb{C}P^1$. Since the latter is impossible, the image of u is contained in one of the spheres $\{p\} \times \mathbb{C}P^1$ inside U^m_∞.

Because of the same reason as in the preceding case, u is a holomorphic branched covering of $\{p\} \times \mathbb{C}P^1$.

The case in which $i = p$ is somewhat different. Here, we use that we chose the complex structure on $U^p_\infty = \mathrm{end}_N$ essentially as in Corollary 5.2.8. Thus, we can apply Lemma 5.2.10 to see that the image of u must be contained in end_N because it cannot be disjoint from it. This lemma even shows that the image of u must be contained in one of the fibres, which are the copies of $\mathbb{C}P^1$ by which U^p_∞ is fibred. Since u is not constant, it follows, as in the preceding cases, that u is a holomorphic branched covering of this copy of $\mathbb{C}P^1$. □

5.4. From Symplectic Spheres to Closed Reeb Orbits

The aim of this section is to prove the existence of nullhomologous Reeb links and contractible Reeb orbits on contact manifolds supported by a tower of open book decompositions with one of three special properties. The first of our two main results is the following theorem.

Theorem 5.4.1. *Let $(M, \xi = \ker\alpha)$ be a closed $(2n-1)$-dimensional contact manifold that is Liouville cobordant to a contact manifold (M', ξ') supported by a tower of open book decompositions of height $k \in \mathbb{N}_0$ with the property that there is a Liouville cobordism from the lowest level to a contact manifold (L, α_L) with one of the following two properties:*

(L-h) *L is at least 3-dimensional and embeds into a subcritical Stein manifold as a hypersurface of restricted contact type.*

(L-m) *L is supported by an open book decomposition whose monodromy is trivial.*

Then the Reeb vector field R_α to α has a contractible orbit.

Our second main result is the following theorem about contact manifolds supported by an open book decomposition whose binding is planar.

Theorem 5.4.2. *Let $(M, \xi = \ker\alpha)$ be a closed contact manifold that is symplectically cobordant to a contact manifold (M', ξ') supported by an open book decomposition (B, π) with the property that there is a symplectic cobordism from B to a contact manifold (L, α_L) that is planar.*

Then the strong Weinstein conjecture holds for (M, ξ)*,* i.e. *the orbits to the Reeb vector field to every contact form defining* ξ *contain a nullhomologous link.*

For reference in the proof let us call this theorem case (L-p).

Before we start a joint proof of the two theorems above in the next subsection, let us see which known results we recover or strengthen in the special case that M itself is Liouville cobordant to L, *i.e.* in the case $k = 0$.

In case (L-h), we recover the result from [18] that every at least 3-dimensional contact manifold that is Liouville cobordant to a contact manifold embedding into a subcritical Stein manifold as a hypersurface of restricted contact type contains a contractible Reeb orbit, with essentially the same proof.

Note that case (L-m) reduces to case (L-h) whenever the pages are Stein. So, this case, in a sense, provides an extension of case (L-h) and, hence, of the corresponding result from [18].

In case (L-p), we recover the result of Abbas, Cieliebak, and Hofer [1] that every planar contact manifold satisfies the strong Weinstein conjecture.

Finally, we have the following corollary, based on a construction by Klukas [26].

Corollary 5.4.3. *Let* (P, Ψ, β) *be a symplectic open book. Then the Reeb vector field of every contact form defining the contact structure of the contact open book* $M(P, \Psi, \beta)$ *or the contact open book* $M\big(P, \Psi^{-1}, \beta\big)$ *has a contractible orbit.*

Proof. In [26], Klukas constructed a Liouville cobordism from the disjoint union $M(P, \Psi, \beta) \sqcup M\big(P, \Psi^{-1}, \beta\big)$ to the contact open book $M(P, \mathrm{id}, \beta)$. Thus, we may apply Theorem 5.4.1 (L-m) to obtain contractible orbits for all contact forms defining the contact structure on the disjoint union $M(P, \Psi, \beta) \sqcup M\big(P, \Psi^{-1}, \beta\big)$. If there is a contact form on one of the two contact manifolds whose Reeb vector field does not have a contractible orbit, then the Reeb vector fields of the contact forms defining the contact structure on the other contact manifold must all have contractible orbits. □

Together with Giroux's result that every contact manifold arises from the generalised Thurston-Winkelnkemper construction (Theorem 2.2.9), this corollary shows that at least half of all contact manifolds have contractible Reeb orbits.

5.4.1. Setup

In this subsection, we construct the special symplectic manifold $(\tilde{W}, \tilde{\omega})$ on which we study holomorphic curves.

Let (W, ω) be the Liouville cobordism from (M, ξ) to (M', ξ'). By Example 1.3.5, we may assume, without loss of generality, that the concave boundary of W is given by the strict contact manifold (M, α).

At this part of the boundary, we attach a negative half-symplectisation of (M, α), *i.e.* we attach the manifold

$$(\mathrm{end}_-, \omega_-) = \left(\mathbb{R}_0^- \times M, d\left(e^t \alpha\right)\right).$$

At the convex boundary, we attach the generalised cap (C, ω_C) of M' from Theorem 5.3.1, which is of order k. The result is the symplectic manifold $(\tilde{W}, \tilde{\omega})$.

Now, we choose an $\tilde{\omega}$-compatible almost complex structure J on $\tilde{W}$ as follows:

(J-1) On the generalised cap C, we choose J to agree with the almost complex structure J_C on C.

(J-2) On end_-, we choose J to be an $\mathbb{R}_0^-$-invariant almost complex structure that sends ∂_t to the Reeb vector field R_α to α, preserves $\xi = \ker \alpha$, and restricts to a $d\alpha|_\xi$-compatible complex structure on ξ.

(J-3) On the remainder of $\tilde{W}$, we choose J to be generic.

By this choice, the symplectic spheres in $U_\infty^i \times \mathbb{C}^k \subset C$ are holomorphic, where $i = h, m, p$, depending on the case in Theorem 5.4.1 or Theorem 5.4.2 we are dealing with. Each of these spheres intersects the symplectic hypersurface H in C in exactly one point and positively transversely. Consequently, every holomorphic sphere representing the same homology class $A \in H_2\left(\tilde{W}; \mathbb{Z}\right)$ as these spheres has intersection number 1 with H. (The intersection number is well defined by the same arguments as for the intersection numbers in the proof of Lemma 5.2.10.)

By positivity of intersection (Proposition 5.1.11), this implies that all such holomorphic spheres are simple, because H is complex.

General holomorphic spheres in $\tilde{W}$ have the following properties, analogous to those in [18, Lemma 5.2].

Lemma 5.4.4. *Let $u\colon \mathbb{C}P^1 \to \tilde{W}$ be a non-constant holomorphic sphere.*

(*i*) *If $u\big(\mathbb{C}P^1\big) \cap \mathrm{end}_l \neq \emptyset$, then $u\big(\mathbb{C}P^1\big) \subset \mathrm{end}_l$, where $\mathrm{end}_l = \mathrm{end}_{N_l} \times \mathbb{C}^{k-l}$ with the symplectic fibration N_l in the lth level of C.*

(*ii*) *If $u\big(\mathbb{C}P^1\big) \subset C^i_\infty \times \mathbb{C}^k$, then $u\big(\mathbb{C}P^1\big) \subset U^i_\infty \times \mathbb{C}^k$ and u is of the form $z \mapsto (v(z), p_1, \ldots, p_k)$ where $p_1, \ldots, p_k \in \mathbb{C}$ are points and v is a holomorphic branched covering of one of the spheres by which U^i_∞ is fibred.*

(*iii*) *If $u\big(\mathbb{C}P^1\big) \cap \big(U^p_\infty \times \mathbb{C}^k\big) \neq \emptyset$, then $u\big(\mathbb{C}P^1\big) \subset U^p_\infty \times \mathbb{C}^k$.*

(*iv*) *If $u\big(\mathbb{C}P^1\big) \cap \big(\mathrm{Int}\big(U^i_\infty\big) \times \mathbb{C}^k\big) \neq \emptyset$, then $u\big(\mathbb{C}P^1\big) \subset U^i_\infty \times \mathbb{C}^k$ and u is one of the spheres in (ii).*

(*v*) *If $i = h, m$, then $u\big(\mathbb{C}P^1\big) \cap H \neq \emptyset$.*

Proof. (i) Since the almost complex structure on $\mathrm{end}_l = \mathrm{end}_{N_l} \times \mathbb{C}^{k-l}$ is split, the symplectic hypersurfaces $\tilde{F}_l = F_l \times \mathbb{C}^{k-l}$ are complex; here, F_l is the fibre in the lth level of C. Now we can reason as in the proof of Lemma 5.2.10 to show that u is either disjoint from every complex hypersurface $\tilde{F}_l$ or contained in one. Because end_l is fibred by these hypersurfaces, the image of u must be contained in end_l whenever it intersects this set.

(ii) On $C^i_\infty \times \mathbb{C}^k$, the almost complex structure is split. Accordingly, u decomposes into holomorphic spheres $u_0, \ldots, u_k$ on the factors. Because $\mathbb{C}$ is contractible and, hence, exact, Corollary 5.1.8 tells us that the spheres $u_1, \ldots, u_k$ are constant. The sphere u_0 is of the desired form by Lemma 5.3.6.

(iii) This case is completely analogous to (i). We only have to exchange the fibres F_l by the complex hypersurfaces in the cap C_0 of the lowest level given by the holomorphic spheres by which $U^p_\infty = \mathrm{end}_N$ is fibred.

(iv) We already know from (iii) that the the first part of the assertion is true whenever $i = p$.

Now, let $i = h$. Then the image of u is contained in a level set of the plurisubharmonic function ψ_h from the proof of Lemma 5.3.3. Since the interior of $U^h_\infty \times \mathbb{C}^k$ is exactly the preimage of $(0, \infty)$ under ψ_h, the image of u must be contained in this set.

Next, we consider the case that $i = m$. The treatment is analogous to that for $i = h$. We only have to replace the function ψ_h by the plurisubharmonic function ψ_m from the proof of Lemma 5.3.4.

Finally, whenever the image of u is contained in $U^i_\infty \times \mathbb{C}^k$, then it is contained in $C^i_\infty \times \mathbb{C}^k$. So, (ii) tells us that u is of the desired form.

(v) If $i = h, m$, the complement of the symplectic hypersurface H is exact, because the primitive on $C \setminus H$ can be glued with that on the Liouville cobordism W and the half-symplectisation end_-. Consequently, Corollary 5.1.8 shows that no non-constant holomorphic sphere can be contained in $\tilde{W} \setminus H$. □

The lemma above has an immediate consequence on the intersection number of holomorphic spheres with the symplectic hypersurface H in the cases (L-h) and (L-m).

Corollary 5.4.5. *If $i = h, m$, then the intersection number of H and every non-constant holomorphic sphere in $\tilde{W}$ is at least* 1.

Proof. By Lemma 5.4.4 (v), every non-constant holomorphic sphere in $\tilde{W}$ has to intersect H. Moreover, non of these spheres can be contained in H because it is an exact symplectic hypersurface of $\tilde{W}$.

Now the statement immediately follows from positivity of intersection (Proposition 5.1.11), because H is complex. □

5.4.2. Moduli Space of Holomorphic Spheres

Choose one of the spheres in the interior of $U^i_\infty \times \left(D^2\right)^k$ and call it S.

Write $\tilde{\mathcal{M}}$ for the moduli space of all holomorphic spheres $u \colon \mathbb{C}P^1 \to \tilde{W}$ that represent the homology class $A = [S]$ and whose image is not contained in the union of the sets end_l. We exclude these spheres to avoid problems with transversality. By Lemma 5.4.4 (i), the exclusion of such spheres even implies that the image of every $u \in \tilde{\mathcal{M}}$ is contained in the complement of the sets end_l.

We claim that $\tilde{\mathcal{M}}$ is a smooth manifold.

Proposition 5.4.6. *The moduli space $\tilde{\mathcal{M}}$ is a smooth manifold of dimension $(2n + 4)$.*

Before we prove this, we would like to present the basic idea used in [32] to prove this for generic almost complex structures.

Let $\mathcal{B} \subset C^\infty\big(\mathbb{C}P^1, \tilde{W}\big)$ be the space of all smooth maps that represent the homology class A. Furthermore, write $\mathcal{E} \to \mathcal{B}$ be the infinite dimensional vector bundle over $\mathcal{B}$ with fibres

$$\mathcal{E}_u = \Omega^{0,1}\big(\mathbb{C}P^1, u^*T\tilde{W}\big).$$

Then the map $\mathcal{S}\colon \mathcal{B} \to \mathcal{E}$ defined by

$$\mathcal{S}(u) = \big(u, \bar{\partial}_J u\big)$$

is a section of $\mathcal{E}$ whose intersection with the zero section is precisely the space of holomorphic curves representing the class A.

Now, the idea is to use the implicit function theorem to prove that the zero section is a smooth manifold. In order to do so, one has to show that the **vertical differential** D_u, given by the projection of the differential

$$\mathcal{S}_*(u) : T_u\mathcal{B} \to T_{(u,0)}\mathcal{E} \cong T_u\mathcal{B} \oplus \mathcal{E}_u$$

of $\mathcal{S}$ to $\mathcal{E}_u$, is onto.

Definition 5.4.7. An almost complex structure J is called **regular** for spheres with respect to a homology class $A \in H_2\big(\tilde{W}; \mathbb{Z}\big)$ if the vertical different D_u is onto at every holomorphic sphere representing A.

In truth, the implicit function theorem does not work here because $C^\infty\big(\mathbb{C}P^1, \tilde{W}\big)$ is no tamed Fréchet manifold. So, one has to substitute this space by more suitable ones and then find a procedure to regain the desired result. In [32], the path is taken through the Banach spaces $W^{k,p}\big(\mathbb{C}P^1, \tilde{W}\big)$, $p > 2$, of maps of lower regularity. Consequently, the definition of regularity of J is altered accordingly.

Proof of Proposition 5.4.6. If we can show that J is regular, then [32, Theorem 3.1.5 (i)] asserts that $\tilde{\mathcal{W}}$ is a smooth manifold of dimension $(2n + 2c_1(S))$ because we know that all holomorphic spheres representing $A = [S]$ are simple. Since a neighbourhood of S splits as $S \times (B_\epsilon(0))^{k+1}$ with a split almost complex structure $i \oplus J'$, the Chern number $c_1(S)$ is given by the Euler characteristic of S, *i.e.* by 2. This proves our claim, provided we can show that J is regular.

By [32, Theorem 3.1.5 (ii)], a generic almost complex structure is regular. Because of [32, Remark 3.2.3], the vertical differential D_u can still be assumed to be onto at a holomorphic sphere u if its image intersects an open subset on which J is chosen generically. So, it remains to show that D_u is onto for spheres contained in the sets where we prescribed J.

Since end_- is exact, the image of no non-constant curve can be contained therein. Because we excluded all holomorphic spheres contained in the sets end_l, it remains only to consider spheres contained in $C^i_\infty \times (\mathrm{Int}\,(D^2))^k$. By Lemma 5.4.4 (ii), these spheres cover one of the spheres by which $U^i_\infty \times (\mathrm{Int}\,(D^2))^k$ is fibred holomorphically. Since we know that all spheres in $\tilde{\mathcal{M}}$ are simple, the covering must be of degree 1.

Because $U^i_\infty \times \big(\mathrm{Int}\,\big(D^2\big)\big)^k$ is a complex product with one factor given by the sphere S, [32, Corrolary 3.3.5] tells us that D_u is onto for the holomorphic sphere contained in $C^i_\infty \times \big(\mathrm{Int}\,\big(D^2\big)\big)^k$. This concludes the proof that J is regular and, hence, also the proof of Proposition 5.4.6. □

Since all spheres in $\tilde{\mathcal{M}}$ are simple, the quotient space

$$\mathcal{M} = \tilde{\mathcal{M}} \times_{\mathrm{Aut}(\mathbb{C}P^1)} \mathbb{C}P^1$$

is a smooth manifold of dimension $2n$.

The action of a Möbius transformation $\phi \in \mathrm{Aut}\big(\mathbb{C}P^1\big)$ is given by

$$\phi \cdot (u, z) = \big(u \circ \phi, \phi^{-1}(z)\big)\,.$$

This implies that the evaluation map on $\tilde{\mathcal{M}} \times \mathbb{C}P^1$ descends to a well-defined evaluation map on $\mathcal{M}$ given by

$$\begin{aligned} \mathrm{ev}\colon \mathcal{M} &\to \tilde{W} \\ [u, z] &\mapsto u(z)\,. \end{aligned}$$

5.4.3. Spheres Intersecting a Curve

Let $\gamma\colon \mathbb{R} \to \tilde{W}$ be a properly embedded curve that is transverse to the spheres by which $U^i_\infty \times \big(D^2\big)^k$ is fibred and whose image is contained in the complement of the sets end_l. Furthermore, let $\gamma(0)$ be contained in the sphere S and $\gamma(\mathbb{R} \setminus [-1, 1]) \subset \mathrm{end}_-$.

Proposition 5.4.8. *The space $\mathcal{M}_\gamma = \mathrm{ev}^{-1}(\gamma)$ is a smooth 1-dimensional manifold.*

Proof. The statement of [32, Theorem 3.4.1] tells us that $\mathcal{M}_\gamma$ is a smooth manifold of dimension $2n + 2c_1(S) - 4 - \operatorname{codim}\gamma = 1$, provided the evaluation map is transverse to γ.

Moreover, it tell us that this would be true if J was generic. By [32, Remark 3.4.8], the evaluation map can still be assumed to be transverse to γ at every $[u, z]$ such that the image of u intersects an open set where we chose J generically.

As we have seen in the proof of Proposition 5.4.6, the only holomorphic spheres in $\widetilde{W}$ for which this is not the case are those inside $U^i_\infty \times \left(\operatorname{Int}\left(D^2\right)\right)^k$. These split into holomorphic spheres $u_1, \ldots, u_{k+1}$ on the factors. Since $\operatorname{Int}\left(D^2\right)$ is contractible and, hence, exact, all these spheres but u_1 are constant. Because $U^i_\infty \times \left(\operatorname{Int}\left(D^2\right)\right)^k$ is fibred by these spheres and γ is transverse to them, the evaluation map ev is transverse to γ on these spheres, too. $\square$

Note that, by Lemma 5.4.4 (iv), for every point $x \in \gamma \cap \operatorname{Int}\bigl(U^i_\infty \times (D^2)^k\bigr)$ there is exactly one sphere $[u, z] \in \mathcal{M}_\gamma$ such that $\mathrm{ev}([u, z]) = x$. This implies that $\mathcal{M}_\gamma$ is not closed. Since it does not have boundary, it must be non-compact.

Moreover, the non-compactness cannot be caused by spheres escaping through C: the holomorphic spheres must be contained in a level set of the exhausting plurisubharmonic function ψ_C on C, after extending it to W and end$_-$ by its constant value close to the boundary of C. Furthermore, this level set cannot belong to a value bigger then the maximum of $\psi_C \circ \gamma$, which exists because only a compact subset of γ intersects C.

5.4.4. Conclusion of the Argument

Let us first assume that the contact form α is **non-degenerate**, *i.e.* that the linearised Poincaré return map along every closed orbit of the Reeb vector field R_α, including multiples, does not have an eigenvalue 1.

By the non-compactness of $\mathcal{M}_\gamma$, there is a sequence in $\mathcal{M}_\gamma$ without any convergent subsequences. The compactness result from symplectic field theory [4, Theorem 10.2] tells us that there still is a subsequence that converges (in the sense of this paper) to a holomorphic *building* of height $k_- \mid 1$ and total energy at most $E(S)$, because the spheres in the sequence intersect only a compact subset of C. We want to show that $k_- > 0$. Since the subsequence does not converge in $\mathcal{M}_\gamma \subset \mathcal{M}$, this is equivalent to showing that no sequence in its preimage in $\tilde{\mathcal{M}} \times \mathbb{C}P^1$ *Gromov-converges* (see [32, Definition 5.5.1]) to a *stable map* (see [32, Definition 5.1.1]) in $\tilde{W}$ consisting of more then one sphere.

This is the point where we have to take different routs in the proof of Theorem 5.4.1 and Theorem 5.4.2, *i.e.* in the cases (L-h) and (L-m), and the case (L-p).

The cases (L-h) and (L-m)

By Corollary 5.4.5, the intersection number of H with each holomorphic sphere in $\tilde{W}$ is at least 1. Because the intersection number of H with the sphere S is 1, this implies that there cannot be any stable map in $\tilde{W}$ that represents the homology class of S and consists of more then one sphere or a multiply-covered one. Thus, we know that $k_- > 0$.

Lemma 5.4.9. *The building to which the subsequence converges contains a finite energy plane with a positive puncture.*

Proof. Every holomorphic building of height $k_- \mid 1$ with $k_- > 0$ contains at least two finite energy planes.

We claim that only one of these two finite energy planes can intersect the hypersurface H. Since the holomorphic planes are properly embedded with the puncture going to $-\infty$ in end_-, we can apply the same reasoning as in Corollary 5.4.5 for holomorphic spheres to show that the intersection number of H and each plane that intersects it must be at least 1. Because the intersection number of H with S is 1, this implies that only one plane can intersect H.

Since the complement of H in $\tilde{W}$ is exact, Stokes's theorem implies that the *Hofer energy* of any holomorphic plane in $\tilde{W}$ with a negative puncture would be negative; *cf.* [4, Lemma 5.16]. For the same reason, there cannot be a finite energy plane with a negative puncture in one of the lower levels of the building, which are all copies of the symplectisation $(\mathbb{R} \times M, d(e^t\alpha))$ of M. This shows that the second finite energy plane must have a positive puncture. $\square$

Because every sphere in our subsequence can only intersect a compact subset of the generalised cap C, the finite energy plane with a positive puncture must be contained in one of the lower levels $(\mathbb{R} \times M, d(e^t\alpha))$ of the holomorphic building.

By [4, Proposition 5.8], the holomorphic plane, seen as a punctured holomorphic sphere $\mathbb{C}P^1 \setminus \{\infty\}$, is asymptotic to a cylinder over a periodic orbit γ_∞ in M. By construction of the almost complex structure J on $\mathbb{R} \times M$ and the fact that the puncture is positive, at each point, $\dot{\gamma}_\infty$ is a positive multiple of $J\partial_t = R_\alpha$. Consequently, the orbit γ_∞ is a closed orbit of the Reeb vector field R_α, after reparametrisation.

Let us write the finite energy plane as $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times M$. Because $\tilde{u}$ is asymptotic to the cylinder over γ_∞, the loops γ_r given by $\gamma_r(s) = u\big(re^{is}\big)$ converge to γ_∞. Thus, $t \mapsto \gamma_{\tan(2(1-t)/\pi)}$ is a contraction of γ_∞. This concludes the proof of Theorem 5.4.1 in the non-degenerate case.

The case (L-p)

The case (L-p) is considerably more complicated because the complement of the complex hypersurface H is not exact. Fortunately, for dimensional reasons, the symplectic manifold $\tilde{W}$ is **semi-positive**, *i.e.* every class $A \in \pi_2\big(\tilde{W}\big)$ with $\tilde{\omega}(A) > 0$ and $c_1(A) \geq 3 - \frac{1}{2} \dim \tilde{W}$ satisfies $c_1(A) \geq 0$. This allows us to follow the treatment in [18, Section 6.3].

Let $[u_m, z_m]$ be a sequence in $\mathcal{M}_\gamma$ such that (u_m, z_m) Gromov-converges to a stable map $\big(\{u_\alpha\}_{\alpha \in T}, \{\alpha_1, z\}\big)$ modelled on a tree T where $u_{\alpha_1}(z) \in \gamma$. Let $e(T)$ be the number of edges of T. If we can show that $e(T) = 0$, then $[u_m, z_m]$ converges in $\mathcal{M}_\gamma$. This would imply that every non-converging sequence in $\mathcal{M}_\gamma$ converges to a holomorphic building of height $k_- \mid 1$ with $k_- > 0$.

Proposition 6.1.2 from [32] tell us that there is a simple stable map $\big(\{v_\beta\}_{\beta \in T'}, \{\beta_1, z'\}\big)$, *i.e.* a stable map such that the spheres v_β are simple and such that the images of different non-constant spheres are distinct, with the following properties:

(v-i) $$\bigcup_{\alpha \in T} u_\alpha\big(\mathbb{C}P^1\big) = \bigcup_{\beta \in T'} v_\beta\big(\mathbb{C}P^1\big).$$

(v-ii) There are positive constants m_β such that $[S] = \sum_{\beta \in T'} m_\beta \, [v_\beta]$.

(v-iii) $v_{\beta_1}(z') = u_{\alpha_1}(z)$.

Because S has intersection number 1 with the complex hypersurface H, there must be a unique $\beta_0 \in T'$ such that v_{β_0} is a non-constant holomorphic sphere intersecting H. We claim that this is the only vertex in T'.

Lemma 5.4.10 (See [18, Lemma 6.3])**.** *The tree T' consists of exactly one vertex,* i.e. $e(T') = 0$.

Proof. We have to consider two cases. First, let us assume that there is a $\beta_U \in T'$ such that v_{β_U} is non-constant and its image intersects $U_\infty^p \times \mathbb{C}$. Then Lemma 5.4.4 (iii) tells us that v_{β_U} is a branched covering of one of the spheres by which $U_\infty^p \times \mathbb{C}$ is fibred. Since v_{β_U} is simple, the covering is trivial. So, the sphere v_{β_U} represents the class $[S]$.

The total energy of the stable map $\left(\{v_\beta\}_{\beta\in T'}, \{\beta_1, z'\}\right)$ satisfies

$$E(\{v_\beta\}) = \sum_{\beta\in T'} E(v_\beta) = \sum_{\beta\in T'} \tilde{\omega}([v_\beta]) \leq \sum_{\beta\in T'} m_\beta\tilde{\omega}([v_\beta]) = E(S)$$

because of Lemma 5.1.6 and the fact that the energy of each non-constant holomorphic sphere is positive. This implies that the energy of all spheres but v_{β_U} must be 0. Consequently, they have to be constant. Since, by the stability condition in the definition of a stable map (see [32, Definition 5.1.1]), a stable map with one marked point cannot contain constant spheres as long as it contains only one non-constant sphere, T' must consist of only one component.

Next, let us consider the case that the image of v_β is disjoint from $U_\infty^p \times \mathbb{C}$ for all $\beta \in T'$. Then the image of each of these maps intersects an open subset of $\tilde{W}$ where the almost complex structure J is chosen generically. Thus, we can use the transversality arguments from [32].

Let us denote by

$$\mathcal{M}^*([v_\beta]\,; J) = \tilde{\mathcal{M}}^*([v_\beta]\,; J) \times_{\mathrm{Aut}(\mathbb{C}P^1)} \mathbb{C}P^1$$

the moduli space of simple unparametrised holomorphic spheres representing the class $[v_\beta]$ whose images are disjoint from $U_\infty^p \times \mathbb{C}$ and the sets end_l. After possibly choosing our almost complex structure more generic, [32, Theorem 3.1.5] tells us that these moduli spaces are non-empty smooth manifolds of dimension $(6 + 2c_1([v_\beta]) - 6) = 2c_1([v_\beta])$ for all $\beta \in T'$ for which v_β is not constant. In particular, we have $c_1([v_\beta]) \geq 0$ for all $\beta \in T'$.

Now, write $\mathcal{M}^*_{T'}(\{[v_\beta]\}\,;J)$ for the moduli space of simple unparametrised stable maps modelled on T' with one marked point that represent the classes $[v_\beta]$ and whose images are disjoint from $U^p_\infty \times \mathbb{C}$ and the sets end_l. Then [32, Theorem 6.2.6] shows that, after possibly refining our choice of J, this moduli space has dimension

$$\begin{aligned}\dim \mathcal{M}^*_{T'}(\{[v_\beta]\}\,;J) &= 6 + 2c_1\left(\sum_{\beta\in T'}[v_\beta]\right) + 2 - 6 - 2e(T') \\ &= 2 + 2\sum_{\beta\in T'} c_1([v_\beta]) - 2e(T')\,.\end{aligned}$$

On $\mathcal{M}^*_{T'}(\{[v_\beta]\}\,;J)$, we have the evaluation map

$$\begin{aligned}\mathrm{ev}_1\colon \mathcal{M}^*_{T'}(\{[v_\beta]\}\,;J) &\to \tilde{W} \\ [\{w_\beta\}\,,\{\beta_1,z'\}] &\mapsto w_{\beta_1}(z')\,.\end{aligned}$$

After possibly further refining our choice of J, this map is transverse to $\gamma\setminus(U^p_\infty\times\mathbb{C})$ by the proof of [32, Theorem 6.3.1] in combination with [32, Remark 3.4.8]. Accordingly, $\mathrm{ev}_1^{-1}(\gamma)$ is a smooth non-empty manifold of dimension

$$\begin{aligned}\dim \mathrm{ev}_1^{-1}(\gamma) &= 2 + 2\sum_{\beta\in T'} c_1([v_\beta]) - 2e(T') - \mathrm{codim}(\gamma) \\ &= 2\sum_{\beta\in T'} c_1([v_\beta]) - 2e(T') - 3 \\ &\le 2\sum_{\beta\in T'} m_\beta c_1([v_\beta]) - 2e(T') - 3 \\ &= 2c_1(S) - 2e(T') - 3 \\ &= 1 - 2e(T')\,.\end{aligned}$$

This shows that $e(T') = 0$. □

The only remaining possibility for T to consist of more then one vertex is that several non-constant spheres u_α have the same image as v_{β_0}. However, this is impossible because each of them would have to intersect the complex hypersurface H. Then positivity of intersection

(Proposition 5.1.11) would imply that the intersection number of the class $[S]$ with the class $[H]$ was greater then 1, which is not true.

The discussion above shows that T also consists of only a single vertex. Consequently, every sequence $[u_m, z_m]$ in $\mathcal{M}_\gamma$ Gromov-converging to a stable map in $\tilde{W}$ must converge in $\mathcal{M}_\gamma$. Accordingly, our non-convergent sequence converges to a building of height $k_- \mid 1$ with $k_- > 0$.

In the lowest level, a holomorphic building of height $k_- \mid 1$ with $k_- > 0$ consists of punctured holomorphic spheres whose punctures are all positive. Let $\tilde{u} = (a, u) : \mathbb{C}P^1 \setminus Z \to \mathbb{R} \times M$ be such a sphere in our building where $Z = \{z_1, \ldots, z_n\} \subset \mathbb{C}$ is the finite set of punctures.

As in the cases (L-h) and (L-m), we know by [4, Proposition 5.8] that, at each puncture z_k, the map $\tilde{u}$ is asymptotic to a holomorphic cylinder over an orbit γ_0^k of the Reeb vector field R_α to α.

Via the homotopies $\gamma_t^k(s) = u\big(z_k + t\epsilon e^{is}\big)$, these orbits are homotopic to the curves γ^k given by $\gamma^k(s) = u\big(z_k + \epsilon e^{is}\big)$; here, the constant $\epsilon > 0$ is chosen such that the disks $\big\{z_k + re^{is} \mid s \in \mathbb{R}, r \in [0, \epsilon]\big\}$ do not intersect. This shows that the link γ_0^k, $k = 1, \ldots, n$, is nullhomologous, because the link γ^k, $k = 1, \ldots, n$, is the boundary of $u|_{\mathbb{C}P^1 \setminus \left(\bigcup_{k=1}^n B_\epsilon(z_k)\right)}$.

This concludes the proof of Theorem 5.4.2 in the non-degenerate case.

Degenerate case

Now, let α be degenerate. Then we can follow the last paragraph of [1] to recover the result from the non-degenerate case.

Let λ_n, $n \in \mathbb{N}$, be a sequence of positive functions on M converging in C^∞ to the constant function of value 1 with the property that the contact forms $\lambda_n \alpha$ are non-degenerate. Then, for all $n \in \mathbb{N}$, there are orbits γ_n^l, $l = 1, \ldots, k_n$, of the Reeb vector field to $\lambda_n \alpha$ that constitute a nullhomologous link.

We claim that the contact area

$$A_n^l = \mathcal{A}_n\big(\gamma_n^l\big) = \int \big(\gamma_n^l\big)^* \lambda_n \alpha$$

of these orbits is both bounded away from 0 by a constant $\delta > 0$ and bounded from above by a constant m. We start with the lower bound.

By Darboux's theorem (Theorem 1.1.3), every point $p \in M$ has a neighbourhood U_p with coordinates $z, x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$ in which

α is given by $\alpha|_{U_p} = dz + \sum x_i\, dy_i$. Without loss of generality, we may assume that these neighbourhoods are of the form $(-\epsilon_p, \epsilon_p) \times B_{2\epsilon_p}(0)$ for a positive constant ϵ_p. Since M is compact, it is covered by finitely many of the sets $V_p = (-\epsilon_p, \epsilon_p) \times B_{\epsilon_p}(0) \subset U_p$. Let ϵ be the minimum of the corresponding constants ϵ_{p_n}.

Inside U_p, the Reeb vector field R_α is given by ∂_z. Hence, every orbit $\tilde{\gamma}$ of R_α must have contact area at least 2ϵ. Moreover, after a C^∞-perturbation of α, the orbits still do not close inside U_p, and those intersecting V_p still leave U_p through $\{-\epsilon_p, \epsilon_p\} \times B_{2\epsilon_p}(0)$, because the Reeb vector field smoothly depends on the contact form. Consequently, the action of all orbits is still bounded from below by $\delta = 2\epsilon$.

Next, let us prove that the contact area is bounded from above. Since the contact area of every orbit is positive, it is sufficient to obtain a uniform upper bound for the total contact area for each $n \in \mathbb{N}$, *i.e.* for the sum $\sum_{l=1}^{k_n} A_n^l$ of the contact areas of the orbits γ_n^l, $l = 1, \dots, k_n$.

Let S_n be the sphere in the proof of the existence of the link γ_n^l, $l = 1, \dots, k_n$. Then [4, Lemma 5.16] tells us that the total contact area for this n is bounded from above by the total energy of the holomorphic building in the corresponding proof, which itself is bounded from above by the energy of the sphere S_n. Using Example 1.3.5, we can arrange our construction in a way such that the energy of the spheres S_n is uniformly bounded from above by $m = 2 \max |\lambda_n - 1|\, E(S)$, where S is the sphere in the construction corresponding to α. Because of this bound of the total contact area, the contact area of every individual orbit is bounded, as well.

The combination of the upper bound for the total contact area and the lower bound for the individual contact areas shows that there can be at most m/δ orbits in each of the links. Thus, after descending to a subsequence, we may assume that the number k_n of orbits in each link does not depend on n.

After reparametrisation, we may further assume that the orbits γ_n^l are all defined on $[0, 1]$ and satisfy $\iota_{\dot{\gamma}_n} \lambda_n \alpha \equiv A_n^l$. Because the sequence A_n^l is bounded and bounded away from 0 for every fixed l, after descending to a subsequence, it converges to a positive constant A^l.

Next, we obtain a uniform C^2-bound for the sequences γ_n^l. Choose any Riemannian metric on M. Because the Reeb vector field is smooth in the contact form, we know that the difference of the Reeb vector fields $R_{\lambda_n \alpha}$ and R_α and its derivative are bounded uniformly. Since, moreover, the sequences A_n^l are bounded, the sequences $\dot\gamma_n^l = A_n^l R_{\lambda_n\alpha}$ have a uniform C^1-bound. This implies that the sequences γ_n^l have a uniform C^2-bound.

Because M is compact, the uniform C^2-bound allows us to apply the Arzelà-Ascoli theorem (see [25, Theorem 7.21]) to obtain a subsequence of each sequence γ_n^l, which we keep calling γ_n^l, that converges in C^1 to a closed curve γ^l. As the limit of a nullhomologous link, the link γ^l is still nullhomologous. Moreover, if for a fixed $l \in \{1, \dots, k\}$ the orbits γ_n^l are all contractible, then so is γ^l.

The orbits γ^l, furthermore, satisfy

$$\iota_{\dot\gamma^l}\alpha = \lim_{n\to\infty} \iota_{\dot\gamma_n^l}\alpha = \lim_{n\to\infty} \tfrac{A_n^l}{\lambda_n} \equiv A^l$$

and

$$\iota_{\dot\gamma^l} d\alpha = \lim_{n\to\infty} \iota_{\dot\gamma_n^l} d\alpha = \lim_{n\to\infty} -\tfrac{1}{\lambda_n}\iota_{\dot\gamma_n^l}\,(d\lambda_n \wedge \alpha) \equiv 0,$$

where we used the uniform bound for $\dot\gamma_n^l$ in the last equation. Consequently, the limits γ^l are orbits of R_α.

This completes the proof of Theorem 5.4.1 and Theorem 5.4.2.

A. The Between Theorem

In this appendix, we would like to remind the reader of a simple yet quite useful theorem from general topology, the Between Theorem.

Theorem A.1 (The Between Theorem; see [25, Problem 5.X])**.** *Let f and g be a lower and an upper semi-continuous real-valued function on a paracompact space X, respectively, such that $g(x) < f(x)$ for every $x \in X$. Then there is a continuous real-valued function h satisfying $g(x) < h(x) < f(x)$ for every $x \in X$.*

Proof. For each $x \in X$, choose a number p_x such that $g(x) < p_x < f(x)$. By the semi-continuity of f and g there is a neighbourhood U_x of x such that $g(y) < p_x < f(y)$ for all $y \in U_x$. Now, choose a partition of unity $\{\lambda_x\}_{x\in X}$ subordinate to the covering of X by the sets U_x. Then $h(x) = \sum_{y\in X} p_y \lambda_y(x)$ is the desired continuous function. □

At first glance, this theorem does not look very impressive, but in this thesis it plays a major role in the construction of various weak deformation retractions. Our setup usually looks as follows.

We consider some compact smooth manifold M which contains a submanifold B with trivial normal bundle, *e.g.* the binding of an open book or the boundary of the manifold. There, we restrict our attention to a neighbourhood $U \cong D^n \times B$ of B. Depending on the situation we either choose polar coordinates on D^n or a collar coordinate in the case in which B is the boundary of M.

Now, let there be a vector bundle $\pi\colon V \to B$ and denote by $\pi'\colon D^n \times V \to U$ an extension over U. Then we are interested in non-empty paracompact spaces X with maps ψ into the space of smooth sections of π', *e.g.* the space of forms α adapted to an open book with the restriction map to the families of forms $\alpha|_{T(\{x\}\times B)}$, $x \in D^2$. The Between Theorem yields the following.

Corollary A.2. *In the setup above, let O be an open subset of the smooth section of π such that $\psi(x)\,|_B \in O$ for all $x \in X$.*

Then there is a continuous function $\epsilon\colon X \to (0,1)$ such that the restriction of $\psi(x)$ to $\{y\} \times B$ is contained in O for all $y \in \bar{B}_{\epsilon(x)}(0) \subset D^n$.

Proof. As the upper semi-continuous function g from the Between Theorem we choose the constant zero-function. It remains to construct a lower semi-continuous function $f\colon X \to (0,1]$ such that $\psi(x)\,|_{\{y\}\times B} \in O$ for all $y \in \bar{B}_{f(x)}(0)$. Then the Between Theorem shows that the desired function exists.

Since the set O is open and the restriction morphism a continuous map, we may choose f to be given by

$$f(x) = \sup\left\{r \in [0,1] \mid \psi(x)\,|_{\{y\}\times B} \in O \text{ for all } y \in \bar{B}_r(0)\right\}.$$

$\square$

B. Quasifibrations

Throughout this thesis we encounter several instances of maps $\pi\colon E \to B$ that seem to be *fibrations* but for which we are not quite able to prove this because we do not know of all paths in B how to lift them. The aim of this appendix is to show that these maps still induce long exact sequences in homotopy.

To set us on a firm ground, we remind the reader of the definition of a *fibration*.

Definition B.1. We say that a map $\pi\colon E \to B$ has the **homotopy lifting property** for a topological space W if the following commutative diagram can always be completed as indicated.

$$\begin{tikzcd} W \times \{0\} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "\pi"] \\ W \times I \arrow[r, "G"] \arrow[ur, dashed, "\tilde{g}"] & B \end{tikzcd}$$

We say that the map π is a **Hurewicz fibration** or **fibre space** if it has the homotopy lifting property for all topological spaces. If the map π is surjective and has the homotopy lifting property for D^n for all $n \in \mathbb{N}_0$, we call it a **(Serre) fibration**.

One of the most interesting properties of fibrations is that they induce long exact sequences in homotopy.

Theorem B.2 (See [5, Theorem VII.6.7])**.** *Let $\pi\colon E \to B$ be a fibration and denote by F the fibre $\pi^{-1}(*)$ over the base point $*$ of B. Then there is a long exact sequence*

$$\cdots \longrightarrow \pi_{k+1}(B) \xrightarrow{\partial_*} \pi_k(F) \xrightarrow{i_*} \pi_k(E) \xrightarrow{\pi_*} \pi_k(B) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(E) \xrightarrow{\pi_*} \pi_1(B) \xrightarrow{\partial_*} \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{\pi_*} \pi_0(B)$$

where $i\colon F \hookrightarrow E$ is the natural inclusion.

Remark B.3. At the level of π_0, the sequence above is only exact as a sequence of pointed spaces. For a definition of this term, see Appendix C.

Taking a closer look at the proof of Theorem B.2, as given in [5], we realise that we do not really need a proper fibration: it is sufficient that $\pi\colon (E,F) \to (B,*)$ be a weak homotopy equivalence of pairs, *i.e.* that the induced maps $\pi_*\colon \pi_n(E,F) \to \pi_n(B,*)$ are isomorphisms for all $n \in \mathbb{N}$. This motivates the following generalisation of a fibration.

Definition B.4. Let $\pi\colon E \to B$ be a map and B_E the union of those components of B that intersect the image of π. Then we say that π is a **quasifibration** if, for every $b \in \pi(E)$, the map $\pi\colon \big(E, \pi^{-1}(b)\big) \to (B_E, b)$ induces maps $\pi_*^k\colon \pi_k\big(E, \pi^{-1}(b)\big) \to \pi_k(B_E, b)$ in homotopy that are isomorphisms for all $k \in \mathbb{N}$.

Remark B.5. This definition deviates from the original definition by Dold and Thom [11] in two points: first, we do not demand a quasifibration π to be onto and, second, we forgo demanding that the induced map $\pi_0\colon \pi_0(E,F) \to \pi_0(B,*)$ be an isomorphism; here, $\pi_0(E,F)$ is the topological quotient $\pi_0(E)\,/\pi_0(F)$.

Remark B.6. A quasifibration need not be onto on B_E, in contrast to a fibration. The prime example for this is the inclusion of a point into a disc.

The concept of a quasifibration is truly a generalisation of a fibration because of the following theorem.

Theorem B.7 (*Cf.* [5, Theorem VII.6.6])**.** *Every fibration is a quasifibration.*

This statement was already given by Dold and Thom in [11] for their version of a quasifibration, though without proper proof. In fact, it does not hold for their definition because of a problem at the level of π_0.

Let us now present conditions that guarantee that a map is a quasifibration. The proof of Theorem B.7 can be modified to obtain the following.

Lemma B.8. *Let $\pi\colon E \to B$ be a map, $b \in \pi(E)$, $F = \pi^{-1}(b)$, $n \in \mathbb{N}$, and B_E the union of those components of B that intersect the image of*

π. Then the induced map $\pi_*^n\colon \pi_n(E,F) \to \pi_n(B_E,b)$ is an isomorphism, provided the following two conditions hold for every choice of base point $e \in F$ of E.

(1) Every map $G\colon D^{n-1} \times I \to B$ such that $G\big(\partial(D^{n-1} \times I)\big) = \{b\}$ is homotopic relative $\partial\big(D^{n-1} \times I\big)$ to a map that can be lifted with initial condition e.

(2) For every pair of maps $G\colon D^n \times I \to B$ and $g\colon D^n \to E$ such that $G(\partial D^n \times I \cup D^n \times \{1\}) = \{b\}$ and $G|_{D^n \times \{0\}} = \pi \circ g$ there is a pair of maps G' and g' such that

- *g' represents the same class as g in $\pi_n(E,F)$,*
- *$G'|_{D^n \times \{0\}} = \pi \circ g'$,*
- *$G'(\partial(D^n \times I)) = \{b\}$, and*
- *G' can be lifted with initial condition g'.*

Before we prove this lemma, let us state which consequences it has concerning quasifibrations.

Corollary B.9. *Let $\pi\colon E \to B$ be a map, $b \in \pi(E)$, and B_E the union of those components of B that intersect the image of π. If the assumptions of Lemma B.8 are satisfied for all $b \in \pi(E)$ and $n \in \mathbb{N}$, then π is a quasifibration.*

Proof of Lemma B.8. We first prove the surjectivity of the induced map $\pi_*^n\colon \pi_n(E,F) \to \pi_n(B,b)$. So, let $G\colon S^n \to B$ be a map of pointed spaces.

Because S^n is the quotient $\big(D^{n-1} \times I\big)/\partial\big(D^{n-1} \times I\big)$, the map G induces a map on $D^{n-1} \times I$ with the property that the boundary is mapped to b. By a slight abuse of notation, we call this map G, as well.

Now, let $g\colon D^{n-1} \times \{0\} \to E$ be the constant map with value e. Then, by condition (1), there is a map G' homotopic to G relative $\partial\big(D^{n-1} \times I\big)$ that can be lifted to a map $\tilde{g}\colon D^{n-1} \times I \to E$ which maps $D^{n-1} \times \{0\}$ to e. In particular, this map maps $\partial\big(D^{n-1} \times I\big)$ to F. Hence, it represents an element of $\pi_n(E,F)$ that is mapped by π_*^n to $[G'] = [G] \in \pi_n(B_E,b)$.

This proves the surjectivity of the map π_*^n.

The proof of the injectivity of the map π_*^n turns out to be somewhat more involved. We start with two maps of pointed spaces $f_i\colon D^n \to E$, $i = 0,1$, that map the boundary to F. Moreover, let $H\colon D^n \times I \to B$ be a homotopy relative ∂D^n from $(\pi \circ f_0)$ to $(\pi \circ f_1)$.

Our goal is to lift some homotopy H' relative ∂D^n, which probably differs from H, to a homotopy $\tilde{H}$ relative $*$ from f_0 to f_1, where $* \in \partial D^n$ is the base point of D^n. To do this, we need some preparation.

Let Ψ be a homeomorphism of $D^n \times I$ such that $(D^n \times \partial I) \cup (\{*\} \times I)$ is mapped to $D^n \times \{0\}$ and denote by S the image of this set. Furthermore, let $r\colon D^n \times \{0\} \to S$ be the time-1-map of a strong deformation retraction of $D^n \times \{0\}$ to S with the property that $r((D^n \times \{0\}) \setminus S) \subset \partial S$.

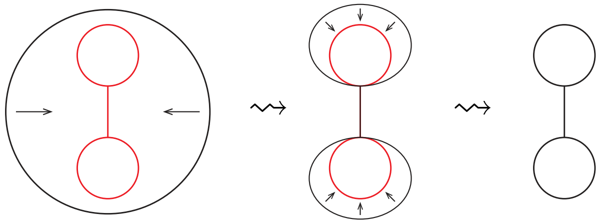

Figure B.1.: Strong deformation retraction from $D^n \times \{0\}$ to S

We can define a map f on S by $f_i \circ \Psi^{-1}$ on $\Psi(D^n \times \{i\})$ and the constant map with value e on $\Psi(\{*\} \times I)$. Then we set $g = f \circ r$ and $G = H \circ \Psi^{-1}$. This pair of maps satisfies the assumption of condition (2). Accordingly, there is a pair of maps G' and g' such that G' can be lifted with initial condition g' to a map $\tilde{g}\colon D^n \times I \to E$, and a homotopy $H_g\colon D^n \times I \to E$ relative base point from g to g' with the property that $H_g(\partial D^n \times I) \subset F$.

We concatenate the homotopy H_g and the map $\tilde{g}$ and call the result $\tilde{g}'$. Then $\tilde{g}'$ coincides with g on $D^n \times \{0\}$ and maps $(\partial D^n \times I) \cup (D^n \times \{1\})$ to F. So $\tilde{H} = \tilde{g}' \circ \Psi$ is a homotopy relative base point from f_0 to f_1 with the property that $\tilde{H}(\partial D^n \times I) \subset F$.

This proves the injectivity of the map π_*^n. $\square$

Up to now, we considered rather abstract conditions. Now, we apply these to a more concrete setup.

Theorem B.10. *Let M be a compact smooth manifold, $V \to M$ a smooth vector bundle, and B the intersection of a convex and an open subset of the space $\Gamma^\infty(V)$ of smooth sections of V. Furthermore, let $\pi\colon E \to B$ be* *a map of pointed spaces for which the diagram*

$$\begin{tikzcd} D^n \times \{0\} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "\pi"] \\ D^n \times I \arrow[r, "G"] \arrow[ur, dashed, "\tilde{g}"] & B \end{tikzcd}$$

can always be completed as indicated, provided the map G is smooth in the factor I and constant on $\{x\} \times (I \setminus (1/4, 3/4))$ for all $x \in D^n$.

Then π is a quasifibration whose image contains the components of B that it intersects.

Proof. By Corollary B.9, it is sufficient to verify the two conditions from Lemma B.8. We deal with them simultaneously.

Let $n \in \mathbb{N}_0$ and $G\colon D^n \times I \to B$ be a map that maps $\partial D^n \times I$ to the base point b of B. We claim that there is a map $\tilde{G}$ homotopic to G relative $\partial(D^n \times I)$ such that $\tilde{G}$ is smooth in the factor I and constant on $\{x\} \times (I \setminus (1/4, 3/4))$ for all $x \in D^n$. This proves that the two conditions from Lemma B.8 hold. Moreover, it proves that the image of π contains the components of B that it intersects.

We prove the existence of the map $\tilde{G}$ in two steps: first, we smoothen in the factor I relative ∂I and, then, we reparametrise.

To smoothen G relative $D^n \times \partial I$, we use a smooth partition of unity $\{\lambda_\alpha\}$ on I, *i.e.* we replace G by the map

$$\begin{aligned} G'\colon D^n \times I &\to \Gamma^\infty(\tilde{V}) \\ (x,t) &\mapsto \sum_\alpha \lambda_\alpha(t)\, H(x, t_\alpha) \end{aligned}$$

where t_α is a point in the support of λ_α.

Since we only take convex combinations of elements in B, the image of G' is contained in the convex set defining B. Consequently, to see that G' is a map to B, we only have to verify that it is contained in the open set defining B, as well. This is the case for a sufficiently fine partition of unity because $\Gamma^\infty(\tilde{V})$ is locally convex, G continuous, and D^n compact.

To be a bit more precise, the support of λ_α is chosen such that for all $x \in D^n$ the points $G(x,t)$ with $t \in \operatorname{supp}(\lambda_\alpha)$ are contained in a convex open neighbourhood of $G(x,t)$ that itself is contained in the open set defining B. This does not only show that G' is a map to B, but also that this is the case for every convex combination of G and G'. Accordingly, G and G' are homotopic relative $D^n \times \partial I$.

Finally, note that, whenever the restriction of G to $\{x\} \times I$ is constant for some $x \in D^n$, the restriction of any convex combination of G and G' to this set is constant, too, with the same value as G. Since G maps $\partial D^n \times I$ to the base point b, this shows that G and G' are homotopic relative $\partial (D^n \times I)$.

It remains to change G' such that it is constant on $\{x\} \times (I \setminus (1/4, 3/4))$ for all $x \in D^n$. To do this, we choose a smooth monotonously increasing function $\mu\colon I \to I$ that vanishes on $[0, 1/4]$ and is constant of value 1 on $[3/4, 1]$. Then the map

$$\begin{aligned} H\colon D^n \times I \times I &\to B \\ ((x,t), s) &\mapsto G'(x, (1-s)\,t + s\mu(t)) \end{aligned}$$

is a homotopy relative $\partial (D^n \times I)$ from G' to a map $\tilde{G}$ that is smooth in the factor I and constant on $\{x\} \times (I \setminus (1/4, 3/4))$ for all $x \in D^n$. This concludes the proof. □

So far, the base space B has to be a subset of the sections of a vector bundle. By a result of Dold and Thom [11, Satz 2.2], we can relax this condition.

Definition B.11. Let $\pi\colon E \to B$ be a map. Then we call a subset $U \subset B$ **distinguished** if the restriction $\pi_U : \pi^{-1}(U) \to U$ of π is a quasifibration.

Theorem B.12 (*Cf.* [11, Satz 2.2])**.** *Let $\pi\colon E \to B$ be a map and $\mathcal{U} = \{U_\alpha\}$ an open covering of B such that each U_α is distinguished and every non-empty intersection $U_\alpha \cap U_\beta$ contains a set $U_\gamma \in \mathcal{U}$.*

Then π is a quasifibration.

Proof. We have to show that the proof by Dold and Thom still works for our less restrictive definition of a quasifibration. We only explain why no problems in the proof arise and do not repeat it here.

The only two points in the proof where the injectivity of the induced maps $(\pi|_U)_*^0 : \pi_0\big(\pi^{-1}(U), \pi^{-1}(b)\big) \to \pi_0(U, b)$ is used are the proof of the injectivity of π_*^0 and that of the surjectivity of π_*^1.

To show that π is a quasifibration in our sense, we do not have to show that π_*^0 is injective. Furthermore, we do not really need the injectivity of $(\pi|_U)^0_*$ in the proof of the surjectivity of π^1_*: it is only used in a special application of [11, Hilfssatz 2.6] in [11, Hilfssatz 2.7]. In the special case of π_0, [11, Hilfssatz 2.6] is only applied to maps of the form $h\colon \{*\} \to \big(\pi^{-1}(U)\,, \pi^{-1}(V)\,, y\big)$ given by $h(*) = y$, where y is the base point of $\pi^{-1}(U)$. Then the injectivity of the maps $(\pi|_U)^0_*$ is used to guarantee that there is a homotopy from h to the map to the base point, *i.e.* to the map h itself. Since the constant homotopy always satisfies the demands, we do not really need the injectivity of the maps $(\pi|_U)^0_*$. □

The theorem above shows that it is sufficient in Theorem B.10 that B satisfy the assumptions locally.

Corollary B.13. *Let M be a compact smooth manifold and B a space that locally has the structure of the intersection of a convex and an open subset of the space of smooth sections of a vector bundle over M. Furthermore, let $\pi\colon E \to B$ be a map of pointed spaces for which the diagram*

$$\begin{tikzcd} D^n \times \{0\} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "\pi"] \\ D^n \times I \arrow[r, "G"] \arrow[ur, dashed, "\tilde{g}"] & B \end{tikzcd}$$

can always be completed as indicated, provided the map G is smooth in the factor I and constant on $\{x\} \times (I \setminus (1/4, 3/4))$ for all $x \in D^n$.

Then π is a quasifibration whose image contains the components of B that it intersects.

Remark B.14. The assumptions on B are satisfied by open subsets of the space of smooth sections σ of a fibre bundle that are fixed on a closed subset A of the base space such that $\sigma|_{B\setminus A}$ does not intersect the boundary of the fibres, *e.g.* the space of diffeomorphisms of a compact manifold with boundary that agree with the identity on the boundary; *cf.* [27, Chapter IX].

Example B.15. The space $\Xi(M)$ of contact structures on a closed manifold M is an open subset of the smooth sections of the Grassmannian of oriented hyperplanes in TM. This Grassmannian is a fibre bundle with fibres diffeomorphic to the Grassmannian of hyperplanes of $\mathbb{R}^n$. Moreover, the Gray stability theorem (Theorem 1.1.4) provides lifts for smooth families of contact structures. Thus, Corollary B.13 shows, in combination with the remark above, that the map $\pi\colon \mathrm{Diff}(M) \to \Xi(M)$ given by $\pi(\Psi) = \Psi_*\xi_0$ is a quasifibration.

A useful application of Corollary B.13 is the following result about spaces of smooth technical loops.

Corollary B.16. *If B satisfies the assumptions on the corresponding space in Corollary B.13, then the inclusion of the space $\Omega_{\mathrm{t}}^\infty B$ of smooth technical loops at the base point b of B into the space ΩB of continuous loops at the same point is a weak homotopy equivalence.*

Proof. By the usual path-loop fibration and Corollary B.13, we get the following long exact ladder diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{k+1}(B) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_k(\Omega_t^\infty B) \arrow[r] \arrow[d] & \pi_k(P_t^\infty B) \arrow[r] \arrow[d, "i_*"] & \pi_k(B) \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\
\cdots \arrow[r] & \pi_{k+1}(B) \arrow[r] & \pi_k(\Omega B) \arrow[r] & \pi_k(PB) \arrow[r] & \pi_k(B) \arrow[r] & \cdots
\end{tikzcd}$$

Both the space PB of continuous paths starting in b and its subspace $P_{\mathrm{t}}^\infty B$ consisting of the technical smooth paths are contractible. Hence, the Five Lemma shows that i is a weak homotopy equivalence. That the Five Lemma applies to this sequence at the level of π_0, we explain in Appendix C. $\square$

C. Homotopy Sequences and the Five Lemma

The goal of this appendix is to show to which extent the Five Lemma applies to long exact homotopy sequences induced by a quasifibration. The results seem to be standard knowledge, but they are nowhere to be found readily. We start by discussing sufficient conditions that apply to general exact sequences of pointed spaces. Then we apply our findings to the long exact sequence induced by a quasifibration.

To set the discussion on a firm ground, we recall the definition of an exact sequence of pointed spaces.

Definition C.1. Let A, B, and C be pointed spaces and denote their base points by $*$. Furthermore, let $f\colon A \to B$ and $g\colon B \to C$ be maps of pointed spaces, *i.e.* maps sending the base point to the base point. Then we say that the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is **exact** if $g^{-1}(*) = f(A)$.

As can be seen easily, exactness of a sequence

$$\{*\} \to A \xrightarrow{f} B \xrightarrow{g} C \to \{*\}$$

for a single base point does only imply surjectivity of g but not injectivity of f. If, however, the sequence is exact for all possible consistent choices of base points, then f is one-to-one.

This short discussion shows that, in general, we need to impose assumptions for all base points in the Five Lemma.

For the remainder of the discussion for general exact sequences of pointed spaces, we consider the following commutative diagram of pointed spaces with exact rows.

$$\begin{tikzcd} A_1 \arrow[r, "\partial_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "\partial_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\partial_3"] \arrow[d, "\varphi_3"] & A_4 \arrow[r, "\partial_4"] \arrow[d, "\varphi_4"] & A_5 \arrow[d, "\varphi_5"] \\ B_1 \arrow[r, "\tilde{\partial}_1"] & B_2 \arrow[r, "\tilde{\partial}_2"] & B_3 \arrow[r, "\tilde{\partial}_3"] & B_4 \arrow[r, "\tilde{\partial}_4"] & B_5 \end{tikzcd} \tag{C.1}$$

We start with the part of the Five Lemma concerning injectivity of the map φ_3.

Proposition C.2. *Let the map φ_3 be fixed and assume that for every choice of base point in A_3 there is a diagram like* (C.1) *with the additional property that φ_1 is onto and that φ_2 and φ_4 are one-to-one. Then φ_3 is one-to-one.*

Proof. Let $a, a' \in A_3$ such that $\varphi_3(a) = \varphi_3(a')$. We may choose a' to be the base point of A_3. Then the same diagram chase as in the proof of the usual Five Lemma shows that a has to be the base point, as well. □

Remark C.3. In the case that φ_3 is a group homomorphism, it is sufficient that the sequence in the proposition above exist for the case that the base point is the identity element of A_3: the usual proof of the Five Lemma applies in this setup.

The situation regarding surjectivity is more complicated.

Proposition C.4. *Let the maps φ_3, φ_4, ∂_3, and $\tilde{\partial}_3$ be fixed. Furthermore, assume that for every choice of base point in A_3 there is a diagram like* (C.1) *with the additional property that φ_2 and φ_4 are onto and that φ_5 is one-to-one. Then φ_3 is onto.*

Proof. Let $b \in B_3$. Then there is an $a_4 \in A_4$ such that $\varphi_4(a_4) = \partial_3(b)$. Because $\tilde{\partial}_4\big(\tilde{\partial}_3(b)\big)$ is the base point and the map φ_5 is one-to-one, the map ∂_4 maps a_4 to the base point. Hence, there is an $a_3 \in A_3$ such that $a_4 = \partial_3(a_3)$. Choose a_3 to be the base point of A_3.

Then $\tilde{\partial}_3(b)$ is the corresponding base point of B_4. As a result, there is a $b_2 \in B_2$ such that $\tilde{\partial}_2(b_2) = b$. Since φ_2 is onto, this implies that there is an $a_2 \in A_2$ such that $\varphi_3(\partial_2(a_2)) = b$. □

Remark C.5. In the case that φ_3 is a group homomorphisms and ∂_3 and $\tilde{\partial}_3$ are induced by actions of A_3 on A_4 and of B_3 on B_4, respectively, it is sufficient that the sequence in the proposition above exist for the case that the base point is the identity element of A_3: the usual proof of the Five Lemma applies in this setup.

Now, let there be a commutative diagram

$$\begin{tikzcd} F_b \arrow[r, hook, "i"] \arrow[d, "h_{F_b}"] & E \arrow[r, "p"] \arrow[d, "h_E"] & B \arrow[d, "h_B"] \\ F'_{b'} \arrow[r, hook, "\tilde{i}"] & E' \arrow[r, "\tilde{p}"] & B' \end{tikzcd}$$

where the rows are quasifibrations with fibres $F_b = p^{-1}(b)$ and $F'_{b'} = \tilde{p}^{-1}(b')$ over the base points b and b' of B and B', respectively. Then, for every base point in E, there is the following long exact homotopy ladder diagram; *cf.* [11].

$$\begin{tikzcd} \cdots \arrow[r] & \pi_1(E) \arrow[r, "p^1_*"] \arrow[d, "(h_E)^1_*"'] & \pi_1(B) \arrow[r, "\partial^1_*"] \arrow[d, "(h_B)^1_*"'] & \pi_0(F_b) \arrow[r, "i^0_*"] \arrow[d, "(h_{F_b})^0_*"'] & \pi_0(E) \arrow[r, "p^0_*"] \arrow[d, "(h_E)^0_*"'] & \pi_0(B) \arrow[d, "(h_B)^0_*"'] \\ \cdots \arrow[r] & \pi_1(E') \arrow[r, "\tilde{p}^1_*"] & \pi_1(B') \arrow[r, "\tilde{\partial}^1_*"] & \pi_0(F'_{b'}) \arrow[r, "\tilde{i}^0_*"] & \pi_0(E') \arrow[r, "\tilde{p}^0_*"] & \pi_0(B') \end{tikzcd} \tag{C.2}$$

A priori it is not clear that the Five Lemma may be applied to this ladder diagram whenever its assumptions are satisfied for all base points in E: at the level of π_0 the ladder diagram is just a ladder diagram of pointed spaces and the maps ∂^1_* and $\tilde{\partial}^1_*$ do depend on the base point. Nevertheless, it can be applied.

Proposition C.6. *The Five Lemma applies to the diagram* (C.2) *whenever the assumptions of the (usual) Five Lemma hold for all choices of the base points of E.*

Proof. To all maps left of those shown in (C.2) and to the map $(h_E)^1_*$ the Five Lemma applies as usual since the corresponding maps are group homomorphisms, which allows us to apply Remark C.3 and Remark C.5. Consequently, we only have to take a closer look at the maps to the right of $(h_E)^1_*$.

Since the diagram ends at $\pi_0(B)$ and $\pi_0(B')$ the assumptions of the Five Lemma can never be satisfied for $(h_B)^0_*$. Moreover, this is also true for $(h_E)^0_*$ for the part of the Five Lemma about surjectivity.

The part about injectivity, however, applies to $(h_E)^0_*$ because this map does not depend on the base point in E. Thus, we may invoke Proposition C.2. Analogously, the map $(h_{F_b})^0_*$ does not depend on the base point in F_b and, hence, the part of the Five Lemma concerning injectivity may be applied to this map, as well. Here, the situation is even better as the maps $(h_E)^0_*$, i^0_*, and $\tilde{i}^0_*$ also do not depend on the base point in F_b. Consequently, the part about surjectivity can be applied by Proposition C.4.

It remains to discuss the map $(h_B)^1_*$. The part about injectivity applies because $(h_B)^1_*$ is a group homomorphism and because of Remark C.3. The part about surjectivity is more complicated: by the construction of the ladder diagram (C.2) the base point of $\pi_1(B)$ always has to be the identity element. So we have to verify the assumptions of Remark C.5.

We show that, for a quasifibration $p\colon E \to B$ with fibres F_b the maps $\partial^e_*\colon \pi_1(B,b) \to \pi_0(F_b)$ are induced by a right action of $\pi_1(B,b)$ on $\pi_0(F_b)$, where $e \in F_b$ is the base point of E.

The canonical candidate for this action is given by

$$\begin{aligned} \rho\colon \pi_1(B,b) \times \pi_0(F_b) &\to \pi_0(F_b) \\ (a,[x]) &\mapsto \partial^x_*(a)\,. \end{aligned}$$

We have to verify two things: ρ is well defined, and it is an action.

Let x and $\tilde{x}$ be two points in the same path component of F_b and $a \in \pi_1(B,b)$. Then $\partial^x_*(a)$ is given by the component of the end point of a path $\gamma^x_a\colon I \to E$ representing $(p_*)^{-1}(a) \in \pi_1(E,F_b,x)$, *i.e.* a path such that $\gamma^x_a(0) = x$, $\gamma^x_a(1) \in F_b$, and $[p \circ \gamma^x_a] = a$. Now, let γ be a path from x' to x in F_b. Then the concatenation $\tilde{\gamma} = \gamma * \gamma^x_a$ satisfies $\tilde{\gamma}(0) = x'$, $\tilde{\gamma}(1) \in F_b$, and $[p \circ \tilde{\gamma}] = a$, *i.e.* $\tilde{\gamma}$ represents $(p_*)^{-1}(a) \in \pi_1(E,F_b,x')$. Since $p^1_*\colon \pi_1(E,F_b,x') \to \pi_1(B,b)$ is an isomorphism, such a path is unique up to a homotopy that fixes the start point and varies the end point only in F_b. Consequently, we have $\partial^{x'}_*(a) = [\tilde{\gamma}(1)] = \partial^x_*(a)$. This shows that ρ is well defined.

It remains to show that ρ is an action. So, let $a_0, a_1 \in \pi_1(B)$ and $x \in F_b$. Let $\gamma^x_{a_0}$ be a path representing $(p_*)^{-1}(a_0) \in \pi_1(E,F_b,x)$ and write y for $\gamma^x_{a_0}(1)$. Furthermore, let $\gamma^y_{a_1}$ be a path representing $(p_*)^{-1}(a_1) \in \pi_1(E,F_b,y)$. Then the concatenation $\tilde{\gamma} = \gamma^x_{a_0} * \gamma^y_{a_1}$ satisfies $[p \circ \tilde{\gamma}] = \left[p \circ \gamma^x_{a_0}\right]\left[p \circ \gamma^y_{a_1}\right] = a_0 a_1$. Accordingly, the path $\tilde{\gamma}$ represents

$(p_*)^{-1}(a_0 a_1) \in \pi_1(E, F_b, x)$. This shows that $\rho(a_0 a_1, x) = \rho(a_1, y) = \rho(a_1, \rho(a_0, x))$, *i.e.* that ρ is a right-action. This concludes the proof. $\square$

As an application of the proposition above, we prove that the weak homotopy type of the fibres of a quasifibration with path connected base does not depend on the base point.

Theorem C.7 (See [11, Satz 1.10]). *Let $p\colon E \to B$ be a quasifibration over a connected base space B. Then the weak homotopy type of the fibres $F_b = p^{-1}(b)$ with $b \in p(E)$ does not depend on the point b.*

Proof. The idea of the proof is to replace the quasifibration p by a Hurewicz fibration $p'\colon E' \to B$ for which we can show that all fibres are homotopy equivalent. This Hurewicz fibration is constructed as follows.

Denote the space of continuous paths $\gamma\colon I \to B$ by P. Then we define the total space E' to be the subset of $E \times P$ given by those pairs (y, γ) such that $\gamma(0) = p(y)$ and the fibration by $p'(y, \gamma) = \gamma(1)$. This is a Hurewicz fibration because, for a family $\bar{\gamma}_i$ of paths in B over some index set I and a family (y_i, γ_i) in E' with $\gamma_i(1) = \bar{\gamma}(0)$, a lift is given by the family $\big(y_i, \gamma_i * \bar{\gamma}_i|_{[0,t]}\big)$.

Define a map $h\colon E \hookrightarrow E'$ given by $h(y) = \big(y, c_{p(y)}\big)$ where $c_{p(y)}$ is the constant path at $p(y)$. Then h is an inclusion of E into E' that induces the identity on the base space. In other words, the diagram

$$\begin{tikzcd}
F_b \arrow[r, hook, "i"] \arrow[d, "h_{F_b}"] & E \arrow[r, "p"] \arrow[d, "h"] & B \arrow[d, "\mathrm{id}"] \\
F'_b \arrow[r, hook, "\tilde{i}"] & E' \arrow[r, "p'"] & B
\end{tikzcd}$$

commutes where h_{F_B} is the restriction of h to the fibre F_b of p over the base point b of B.

Consequently, we get a long exact ladder diagram in homotopy for the two quasifibrations. Moreover, the map h is a homotopy equivalence because E' can be deformed into $h(E)$ by the deformation retraction given by $((y, \gamma), s) \mapsto (y, t \mapsto \gamma((1-s)\,t))$. Hence, h induces isomorphisms between the homotopy groups of E and E'. Accordingly, we can apply Proposition C.6 to this ladder. This shows that the map h_{F_b} is a weak homotopy equivalence between the fibres F_b and F'_b.

It remains to show that the fibres of the Hurewicz fibration p' are all homotopy equivalent. We may either refer to the fact that this is always true for Hurewicz fibrations with a connected base space (see [23, Proposition 4.61]) or prove it directly for this special Hurewicz fibration. We do the latter.

Let $x, y \in B$ and $\gamma\colon I \to B$ be a path from x to y. Then we define a map $L_\gamma\colon F'_x \to F'_y$ by $L_\gamma(y, \tilde{\gamma}) = (y, \tilde{\gamma} * \gamma)$. A homotopy inverse of this map is given by $L_{\gamma^{-1}}$ where γ^{-1} is the inverse path to γ. This is the case since both $\gamma * \gamma^{-1}$ and $\gamma^{-1} * \gamma$ are homotopic to the constant path relative to the endpoints. This concludes the proof. $\square$

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List of Symbols

(M,ξ)	contact manifold 1
(M,α)	strict contact manifold 1
$\Xi(M)$	space of contact structures on M 1
$\mathcal{A}(M)$	space of contact forms on M 1
R_α	Reeb vector field to α 2
(W,ω)	symplectic manifold 5
X_H	Hamilton vector field to H 6
$\overline{M}$	the manifold M with reversed orientation 9
(P,Ψ)	abstract open book 14
$P(\Psi)$	mapping torus of Ψ 14
$M(P,\Psi)$	manifold associated to the abstract open book (P,Ψ) 14
(B,π)	open book decomposition of a closed manifold 14
P_φ	page $\pi^{-1}(\varphi)$ of an open book decomposition (B,π) 14
$\Omega^1(\pi)$	space of forms adapted to (B,π) 20
$\mathcal{A}(\pi)$	space of contact forms adapted to (B,π) 20
$\Omega^1(\pi,\alpha_B)$	space of forms α adapted to (B,π) with $\alpha\vert_{TB}=\alpha_B$ 20
$\Omega^1(\pi,\xi_B)$	space of forms α adapted to (B,π) with $\ker\alpha\vert_{TB}=\xi_B$ 20
$\mathcal{A}(\pi,\alpha_B)$	space of contact forms α adapted to (B,π) with $\alpha\vert_{TB}=\alpha_B$ 20
$\mathcal{A}(\pi,\xi_B)$	space of contact forms α adapted to (B,π) with $\ker\alpha\vert_{TB}=\xi_B$ 20

$\mathcal{B}(\pi)$	space of induced Liouville forms on the page P_0 20
$\mathcal{B}(\pi, \alpha)$	space of induced Liouville forms β on the page P_0 with $\beta\|_{TB} = \alpha$ 20
$\Xi(\pi)$	space of contact structures supported by (B, π) 28
$\Xi(\pi, \xi_B)$	space of contact structures ξ supported by (B, π) with $\xi \cap TB = \xi_B$ 28
(P, Ψ, β)	symplectic open book 32
$P_h(\Psi)$	generalised mapping torus of Ψ with respect to the function h 32
$\underline{h}$	Lutz pair $\underline{h} = (h_1, h_2)$ 33
$\alpha_{\underline{h}, \alpha_B}$	contact form on $B \times D^2$ associated to the Lutz pair $\underline{h}$ and the contact form α_B on B 33
$M(P, \Psi, \beta)$	contact manifold obtained from the exact symplectic open book (P, Ψ, β) via the generalised Thurston-Winkelnkemper construction 35
$\mathcal{B}_\infty(P, \beta_0)$	space of Liouville forms on P that agree with β_0 on ∂P including all derivatives 36
$C^\infty_r(P)$	space of all smooth non-negative functions f on P with regular level set $f^{-1}(0) = \partial P$ 38
$\mathrm{i}\mathcal{B}(P)$	space of Liouville forms on the interior of P such that for every $f \in C^\infty_r(P)$ the form $f\beta$ extends smoothly to ∂P and, there, induces a contact structure 38
$\mathrm{i}\mathcal{B}(P, \xi)$	space of Liouville forms on the interior of P such that for every $f \in C^\infty_r(P)$ the form $f\beta$ extends smoothly to ∂P and, there, induces the contact structure ξ 38
(P, ω, ξ)	ideal Liouville domain 38
$\mathcal{B}_f(P)$	space of Liouville forms that provide P with the structure of a Liouville domain 44
$\mathcal{B}_f(P, \alpha)$	space of Liouville forms on P whose restriction to $T\partial P$ is given by the contact form α 44
$\mathcal{A}_{\underline{h}}(\pi)$	space of contact forms adapted to (B, π) standard with respect to $\underline{h}$ for radius $1/2$ 52

$\mathcal{A}_{\underline{h}}(\pi, \xi_B)$	space of contact forms α adapted to (B, π) with $\ker \alpha\|_{TB} = \xi_B$ that are standard with respect to $\underline{h}$ for radius $1/2$ 52
$\mathcal{A}_{\underline{h}}(\pi, \alpha_B)$	space of contact forms α adapted to (B, π) with $\alpha\|_{TB} = \alpha_B$ that are standard with respect to $\underline{h}$ for radius $1/2$ 52
$\Omega^1_{h_1}(\pi)$	space of forms adapted to (B, π) standard with respect to h_1 for radius $1/2$ 54
$\Omega^1_{h_1}(\pi, \xi_B)$	space of forms α adapted to (B, π) with $\ker \alpha\|_{TB} = \xi_B$ that are standard with respect to h_1 for radius $1/2$ 54
$\Omega^1_{h_1}(\pi, \alpha_B)$	space of forms α adapted to (B, π) with $\alpha\|_{TB} = \alpha_B$ that are standard with respect to h_1 for radius $1/2$ 54
$\Omega^1_L(\pi)$	space of forms adapted to (B, π) that are standard 54
$\tilde{\Omega}^1_L(\pi)$	space of pairs (α, R) such that α is adapted to (B, π) and standard for radius $R \in (0, 1/2]$ 62
$\tilde{\Omega}^1_{h_1}(\pi)$	space of pairs (α, R) such that α is adapted to (B, π) and standard with respect to h_1 for radius $R \in (0, 1/2]$ 63
$\hat{\Omega}^1_{h_1}(\pi)$	space of forms adapted to (B, π) standard with respect to h_1 for some radius smaller or equal to $1/2$ 66
$\mathcal{B}_{h_1}(\pi)$	space of Liouville forms on the page P_0 standard with respect to h_1 70
$\mathcal{B}_{h_1}(\pi, \xi_B)$	space of induced Liouville forms β on the page P_0 with $\ker \beta\|_{TB} = \xi_B$ that are standard with respect to h_1 70
$\mathcal{B}_{h_1}(\pi, \alpha_B)$	space of induced Liouville forms β on the page P_0 with $\beta\|_{TB} = \alpha_B$ that are standard with respect to h_1 70
$\mathcal{B}_0(\pi)$	space of induced Liouville forms β on the page P_0 with $\beta\|_B = \beta\|_{TB}$ 70
$\bar{\Omega}^1_{h_1}(\pi)$	space of forms α with a decomposition $\alpha = \alpha_0 + f\, d\varphi$ where $\alpha_0 \in \Omega^1_{h_1}(\pi)$ and f is a suitable function 72
$\mathcal{A}_{h_1}(\pi)$	space of contact forms adapted to (B, π) standard with respect to h_1 for radius $1/2$ 73

$\mathcal{D}_\partial$	space of diffeomeorphisms of a manifold W with boundary that agree with the identity on ∂W 81
$\mathcal{D}$	space of diffeomeorphisms of a manifold W with boundary that have compact support in the interior of W 81
$\mathcal{D}_C$	space of diffeomeorphisms of a manifold W with boundary that agree with the identity on a fixed collar neighbourhood of ∂W 81
$\Omega_0^{\mathrm{SC}}(W)$	space $\Omega_0^{\mathrm{SC}}(W,\alpha_-,\alpha_+)$ of symplectic forms endowing W with the structure of a symplectic cobordism from $(\partial_- W,\alpha_-)$ to $(\partial_+ W,\alpha_+)$ 86
$\Omega_0^{\mathrm{S}}(W)$	space of symplectic forms on W inducing the same orientation as ω_0 and agreeing with ω_0 on $T\partial W$ 86
$\Omega_\infty^{\mathrm{S}}(W)$	space of symplectic forms on W agreeing with ω_0 on ∂W including all derivatives 86
$\Omega_{\mathrm{c}}^{\mathrm{S}}(W)$	space of symplectic forms on W agreeing with ω_0 on a neighbourhood of ∂W 86
$\Omega_i^{\mathrm{ES}}(W)$	intersection of $\Omega^{\mathrm{ES}}(W)$ with $\Omega_i^{\mathrm{S}}(W)$ where $i=0,\infty,\mathrm{c}$ 86
$\Omega_i^{\mathrm{SC}}(W)$	intersection $\Omega_i^{\mathrm{SC}}(W,\alpha_-,\alpha_+)$ of the space $\Omega_i^{\mathrm{S}}(W)$ with $\Omega_0^{\mathrm{SC}}(W,\alpha_-,\alpha_+)$ where $i=0,\infty,\mathrm{c}$ 86
$\Omega^{\mathrm{ES}}(W)$	space of exact symplectic forms on W 86
$\mathcal{B}_0(W)$	space of Liouville forms on W inducing the same orientation as β_0 and agreeing with β_0 on $T\partial W$ 86
$\mathcal{B}_{\mathrm{c}}(W)$	space of Liouville forms on W agreeing with β_0 on a neighbourhood of ∂W 86
$\mathcal{S}_\partial$	space of symplectomorphisms of a symplectic manifold (W,ω_0) with boundary that agree with the identity on ∂W 94
$\mathcal{S}$	space of symplectomorphisms of a symplectic manifold (W,ω_0) with boundary that have compact support in the interior of W 94

$\Omega^1_{h_1,X}(\pi,\alpha_B)$	space of forms α adapted to (B,π) that are standard with respect to h_1 for radius $1/2$, induce the contact form α_B on the binding, and satisfy $\iota_X\alpha \equiv 0$ 105
$C^\infty_{\mathbb{R}}(\Psi_{2\pi})$	space of paths $\gamma\colon \mathbb{R} \to \mathcal{B}_{h_1}(\pi,\alpha_B)$ with the property that $\Psi^*_{2\pi}\gamma(t-2\pi) = \gamma(t)$ 106
$C^\infty_{\mathbb{R},\mathrm{t}}(\Psi_{2\pi})$	space of paths $\gamma\colon \mathbb{R} \to \mathcal{B}_{h_1}(\pi,\alpha_B)$ that are constant in a neighbourhood of $2\pi\mathbb{Z}$ and satisfy $\Psi^*_{2\pi}\gamma(t-2\pi) = \gamma(t)$ 106
$C^\infty_{\mathrm{t}}(\Psi_{2\pi})$	space of technical paths $\gamma\colon\ [0,2\pi] \to \mathcal{B}_{h_1}(\pi,\alpha_B)$ satisfying $\Psi^*_{2\pi}\gamma(0) = \gamma(2\pi)$ 106
$\Omega^\infty_{\mathrm{t}}B$	space of technical smooth loops at the base point of B 108
ΩB	space of loops at the base point of B 108
PB	space of paths starting at the base point of B 111
$\mathrm{Map}_*(X,Y)$	space of pointed maps from X to Y 130
ΣX	reduced suspension of X 130
Y^X	space of maps from X to Y 130
$E(u)$	energy of a smooth map u 137
end_N	positive half-symplectsation of a symplectic fibration N 145
C^a_∞	part of a genearlised cap C_a where the almost complex structure is not generic 145
U^a_∞	part of a genearlised cap C_a fibred by holomorphic spheres 151
end_-	negative half-symplectsation of a strict contact manifold (M,α) 162
end_l	$\mathrm{end}_{N_l} \times \mathbb{C}^{k-l}$ 163

Index

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Selbstständigkeitserklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist, sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Hansjörg Geiges betreut worden.

Eine Teilpublikation erfolgte in [12] zusammen mit Prof. Hansjörg Geiges und Dr. Kai Zehmisch.