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THE SPACE OF CROSS SECTIONS OF A BUNDLE

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ABSTRACT. Let B be a nondiscrete compactum, Y a separable complete metrizable ANR with no isolated point and $p\colon X \to B$ a locally trivial bundle with fiber Y admitting a section. It is proved that the space $\Gamma(X)$ of all cross sections of $p\colon X \to B$ is an l_2-manifold.

0. Introduction. Through the paper, spaces are separable metrizable and maps are continuous. Let $p\colon X \to B$ be a locally trivial bundle with fiber Y, that is, each point $b \in B$ has a neighborhood U and a homeomorphism $\varphi\colon U \times Y \to p^{-1}(U)$ such that $p\varphi = \pi_U$, the projection to U. A map $s\colon B \to X$ is called a *cross section* of $p\colon X \to B$ provided $ps = \mathrm{id}$. The space of all cross sections of $p\colon X \to B$ with compact-open topology is denoted by $\Gamma(X)$. Then $\Gamma(X)$ is a closed subspace of the space $C(B, X)$ of all maps from B into X. If B is compact and d is a compatible metric for X, the topology of $\Gamma(X)$ (and $C(B, X)$) is induced by the sup-metric

$$\hat{d}(f,g) = \sup\{d(f(b), g(b)) \mid b \in B\}.$$

A manifold modeled on Hilbert space l_2 is called an *l_2-manifold.* In this note, we prove the following

MAIN THEOREM. *Let B be a nondiscrete compactum, Y a complete metrizable ANR with no isolated point and $p\colon X \to B$ a locally trivial bundle with fiber Y admitting a section. Then $\Gamma(X)$ is an l_2-manifold.*

For the trivial bundle $\pi_B\colon B \times Y \to B$, the space $\Gamma(B \times Y)$ can be regarded as the space $C(B, Y)$. Thus *the space $C(B, Y)$ is an l_2-manifold if B is a nondiscrete compactum and Y is a complete-metrizable ANR with no isolated point.* This is a generalization of Eells-Geoghegan-Toruńczyk's result [**E**, **Ge**, **To$_1$**].

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1. Preliminaries. Our proof is based on the following:

TORUŃCZYK'S CHARACTERIZATION THEOREM FOR l_2-MANIFOLDS [**To$_2$**] (CF. [**To$_3$**]). *A complete-metrizable ANR X is an l_2-manifold if and only if X has the discrete approximation property, that is, for each map $f\colon \bigoplus_{n\in\mathbf{N}} I^n \to X$ of the free union of n-cells ($n \in \mathbf{N}$) into X and each map $\varepsilon\colon X \to (0,1)$ there is a*

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map $g\colon \bigoplus_{n\in\mathbf{N}} I^n \to X$ *with* $d(f(x), g(x)) < \varepsilon f(x)$ *for all* $x \in \bigoplus_{n\in\mathbf{N}} I^n$ *and* $\{g(I^n) \mid n \in \mathbf{N}\}$ *a discrete family in* X.

In order to verify the discrete approximation property of $\Gamma(X)$, we use the following easy modification of [**DT**, Remark 2] (cf. Proof of [**DT**, Lemma 1]).

LEMMA 1. *Let* $X = (X, d)$ *be a locally path-connected metric space with a tower* $X_1 \subset X_2 \subset \cdots (\subset X)$ *satisfying the following properties*:

(a) *Given a compactum* $A \subset I^n$, *a map* $f\colon I^n \to X$ *with* $f(A) \subset X_i$ *and* $\varepsilon > 0$, *there is a map* $g\colon I^n \to X_j$ *into some* $X_j \supset X_i$ *with* $f|A = g|A$ *and* $\hat{d}(f, g) < \varepsilon$;

(b) *Given* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that any map* $f\colon I^n \to X_i$ *is* ε*-homotopic to a map* $g\colon I^n \to X_j$ *in some* $X_j \supset X_i$ *with* $\operatorname{dist}(f(I^n), g(I^n)) > \delta$.

Then X *has the discrete approximation property.*

By the next lemma, we can treat noncompact ANR like a compact one.

LEMMA 2 [**Mi**, LEMMA 2.1]. *Every metric ANR* $Y = (Y, d_0)$ *has a compatible metric* $d \geq d_0$ *with the following property*:

(h) *For each* $\varepsilon > 0$, *there is a* $\delta = \delta(\varepsilon) > 0$ *such that any two* δ*-near maps of an arbitrary space to* $Y = (Y, d)$ *are* ε*-homotopic*

Let $p\colon X \to B$ and $q\colon Z \to B$ be given maps. A map $f\colon Z \to X$ is said to be *fiber-preserving* (f.p.) if $pf = q$. We call $p\colon X \to B$ an *absolute neighborhood fiber extensor* (ANFE) if any f.p. map $f\colon A \to X$ of a closed set in an arbitrary space Z with a map $q\colon Z \to B$ can be extended to an f.p. map $\tilde{f}\colon U \to X$ of a neighborhood of A in Z. A map $p\colon X \to B$ is called an *absolute neighborhood fiber retract* (ANFR) provided whenever X is embedded in a space Z with a map $q\colon Z \to B$ as a closed set with $p = q|X$ there is an f.p. retraction $r\colon U \to X$ of a neighborhood of X in Z onto X. A map $p\colon X \to B$ is an ANFR if and only if p is an ANFE. (See §1 of [**Ya**].) By [**Ya**, 1.2], we have the following

LEMMA 3. *Any locally trivial bundle* $p\colon X \to B$ *with ANR fiber is an ANFR, that is, an ANFE.*

We refer to [**Du**, Chapter XII] for function spaces and to [**Hu**] for ANR's.

2. Proof of main theorem. First we show that $\Gamma(X)$ is an ANR. Let $f\colon A \to \Gamma(X)$ be a map from a closed set in a space Z. Then f induces the map $F\colon A \times B \to X$ with $pF = \pi_B$, the projection. By Lemma 3, F extends to a map $\widetilde{F}\colon W \to X$ from a neighborhood of $A \times B$ in $Z \times B$ with $p\widetilde{F} = \pi_B$. From compactness of B, there is a neighborhood U of A in Z that $U \times B \subset W$. Then $\widetilde{F}|U \times B$ induces the map $\tilde{f}\colon U \to \Gamma(X)$ which is an extension of f. Hence $\Gamma(X)$ is an ANR.

Since Y is complete-metrizable, X is locally complete-metrizable, hence complete-metrizable (cf. [**BP**, Chapter II, Theorem 4.1]). Since B is nondiscrete, B has a cluster point b_∞. From local triviality of p, there is an open neighborhood U of b_∞ in B and a homeomorphism $\varphi\colon \operatorname{cl} U \times Y \to p^{-1}(\operatorname{cl} U)$ such that $p\varphi = \pi_{\operatorname{cl} U}$. Since Y has no isolated point, Y admits a compatible complete metric d_Y such that each component of Y has diam > 1 (cf. [**BP**, Chapter II, Theorem 3.2]). We may assume that (Y, d_Y) has the property (h) in Lemma 2. Let ρ be the product metric on $\operatorname{cl} U \times Y$ defined by a metric for $\operatorname{cl} U$ and d_Y. And let d be a compatible complete metric for X extending the metric on $p^{-1}(\operatorname{cl} U)$ induced from ρ by φ (cf.

[**BP**, Chapter II, Theorem 3.2]). Then the sup-metric $\hat{d}$ on $\Gamma(X)$ is a compatible complete metric.

Let $(b_i)_{i\in\mathbf{N}}$ be a sequence of distinct points in $U\backslash\{b_\infty\}$ which converges to b_∞. For each $i \in \mathbf{N}$, let

$$\Gamma_i(X) = \{s \in \Gamma(X) \mid \pi_Y\varphi^{-1}s(b_j) = \pi_Y\varphi^{-1}s(b_\infty) \text{ for all } j \geq i\}.$$

Thus we have a tower $\Gamma_1(X) \subset \Gamma_2(X) \subset \cdots (\subset \Gamma(X))$. We will show that this tower satisfies the properties (a) and (b) relative to $\hat{d}$. Then $\Gamma(X)$ is an l_2-manifold by Toruńczyk's characterization.

(a) Let $A \subset I^n$ be a compactum, $f\colon I^n \to \Gamma(X)$ a map with $f(A) \subset \Gamma_i(X)$ and $\varepsilon > 0$. Then f induces the map $F\colon I^n \times B \to X$ such that $pF = \pi_B$ and

$$\pi_Y\varphi^{-1}F(a, b_k) = \pi_Y\varphi^{-1}F(a, b_\infty) \quad \text{for all } a \in A \text{ and } k \geq i.$$

Let $\delta = \delta(\varepsilon) > 0$ in the property (h) for (Y, d_Y). Since $I^n \times B$ is compact, $\pi_Y\varphi^{-1}F$ is uniformly continuous. Thus we can choose $j \geq i$ so that

$$d_Y(\pi_Y\varphi^{-1}F(z, b_k), \pi_Y\varphi^{-1}F(z, b_\infty)) < \delta \quad \text{for all } z \in I^n \text{ and } k \geq j.$$

By using (h) and the Homotopy Extension Theorem [**Hu**, Chapter IV, Theorem 1.2], we can obtain a map $g'\colon I^n \times \operatorname{cl} U \to Y$ such that

$$\begin{aligned}
&g'|A \times \operatorname{cl} U \cup I^n \times \operatorname{bd} U = \pi_Y\varphi^{-1}F|A \times \operatorname{cl} U \cup I^n \times \operatorname{bd} U,\\
&g'(z, b_k) = \pi_Y\varphi^{-1}F(z, b_\infty) \quad \text{for all } z \in I^n \text{ and } j \leq k \leq \infty,\\
&g' \text{ and } \pi_Y\varphi^{-1}F|I^n \times \operatorname{cl} U \text{ are } \varepsilon\text{-homotopic.}
\end{aligned}$$

Define a map $G\colon I^n \times B \to X$ with $pG = \pi_B$ as follows:

$$G(z, b) = \begin{cases} \varphi(b, g'(z, b)) & \text{if } b \in \operatorname{cl} U,\\ F(z, b) & \text{otherwise.} \end{cases}$$

Then G induces the map $g\colon I^n \to \Gamma_j(X)$ such that $g|A = f|A$ and for each $z \in I^n$,

$$\begin{aligned}
\hat{d}(g(z), f(z)) &= \sup\{d(g(z)(b), f(z)(b)) \mid b \in \operatorname{cl} U\}\\
&= \sup\{d_Y(\pi_Y\varphi^{-1}G(z, b), \pi_Y\varphi^{-1}F(z, b)) \mid b \in \operatorname{cl} U\}\\
&= \sup\{d_Y(g'(z, b), \pi_Y\varphi^{-1}F(z, b)) \mid b \in \operatorname{cl} U\} < \varepsilon.
\end{aligned}$$

(b) For each $\varepsilon > 0$, let $\delta = \delta(\varepsilon) > 0$ in (h) for (Y, d_Y). We may assume $\delta < 1$. Let $f\colon I^n \to \Gamma_i(X)$ be a map. Then f induces the map $F\colon I^n \times B \to X$ such that $pF = \pi_B$ and

$$\pi_Y\varphi^{-1}F(z, b_k) = \pi_Y\varphi^{-1}F(z, b_\infty) \quad \text{for all } z \in I^n \text{ and } k \geq i.$$

From compactness, there exist $y_1, \cdots, y_m \in Y$ such that

$$\pi_Y\varphi^{-1}F(I^n \times B) \subset B(y_1, \delta/3) \cup \cdots \cup B(y_m, \delta/3),$$

where $B(y, r) = \{x \in Y \mid d_Y(x, y) < r\}$. For each $j = 1, \ldots, m$, choose a point $z_j \in B(y_j, \delta)\backslash B(y_j, 2\delta/3)$. (Since each component of Y had diam > 1, we can choose such a point.) Using the Homotopy Extension Theorem and (h), we have a map $g_j\colon Y \to Y$ such that $g_j(B(y_j, \delta/3)) = z_j$, $g_j|Y\backslash B(y_j, \delta) = \mathrm{id}$ and g_j is ε-homotopic to id. Then it follows

$$(*) \qquad \max\{d_Y(y, g_j(y)) \mid j = 1, \ldots, m\} > \delta/3 \text{ for each } y \in \pi_Y\varphi^{-1}F(I^n \times \operatorname{cl} U).$$

Again using the Homotopy Extension Theorem, we can obtain a map $g'\colon I^n \times \operatorname{cl} U \to Y$ such that

$$\begin{aligned}
&g'|I^n \times \operatorname{bd} U = \pi_Y \varphi^{-1} F|I^n \times \operatorname{bd} U,\\
&g'(z, b_k) = \pi_Y \varphi^{-1} F(z, b_\infty) \quad \text{if } i+m < k \le \infty,\\
&g'(z, b_{i+j}) = g_j \pi_Y \varphi^{-1} F(z, b_{i+j}) \quad \text{if } j = 1, \dots, m,\\
&g' \text{ is } \varepsilon\text{-homotopic to } \pi_Y \varphi^{-1} F|I^n \times \operatorname{cl} U \operatorname{rel.} I^n \times \operatorname{bd} U.
\end{aligned}$$

As in the proof of (a), define a map $G\colon I^n \times B \to X$ with $pG = \pi_B$ by using the above g'. It is easy to see that G is f.p. ε-homotopic to F. Then G induces the map $g\colon I^n \to \Gamma_{i+m+1}(X)$ which is ε-homotopic to f. We show $\operatorname{dist}(f(I^n), g(I^n)) \ge \delta/6$. Suppose that $\hat{d}(f(z), g(z')) < \delta/6$ for some $z, z' \in I^n$. Then

$$\begin{aligned}
d_Y(\pi_Y \varphi^{-1} F(z, b_\infty), \pi_Y \varphi^{-1} F(z', b_\infty)) &= d_Y(\pi_Y \varphi^{-1} F(z, b_\infty), g'(z', b_\infty))\\
&= d(F(z, b_\infty), G(z', b_\infty)) \le \hat{d}(f(z), g(z')) < \delta/6.
\end{aligned}$$

And for each $j = 1, \dots, m$,

$$\begin{aligned}
&d_Y(\pi_Y \varphi^{-1} F(z, b_\infty), g_j \pi_Y \varphi^{-1} F(z', b_\infty))\\
&\quad = d_Y(\pi_Y \varphi^{-1} F(z, b_{i+j}), g_j \pi_Y \varphi^{-1} F(z', b_{i+j}))\\
&\quad = d_Y(\pi_Y \varphi^{-1} F(z, b_{i+j}), g'(z', b_{i+j})) = d(F(z, b_{i+j}), G(z', b_{i+j}))\\
&\quad \le \hat{d}(f(z), g(z')) < \delta/6.
\end{aligned}$$

Hence for each $j = 1, \dots, m$,

$$d_Y(\pi_Y \varphi^{-1} F(z', b_\infty), g_j \pi_Y \varphi^{-1} F(z', b_\infty)) < \delta/3.$$

This contradicts $(*)$. Therefore $d(f(z), g(z')) \ge \delta/6$ for any $z, z' \in I^n$, that is, $\operatorname{dist}(f(I^n), g(I^n)) \ge \delta/6$. The proof is completed. □

3. The space of fiber-preserving maps. Let $p\colon X \to B$ and $q\colon Z \to B$ be given maps. By $C_B(Z, X)$, we denote the space of f.p. maps from Z to X with compact-open topology. The following is a generalization of Main Theorem which can be proved directly by the same method but it follows from Main Theorem as a corollary.

COROLLARY. *Let $q\colon Z \to B$ be a map of a nondiscrete compactum Z, Y a complete-metrizable ANR with no isolated point and $p\colon X \to B$ a locally trivial bundle with fiber Y. Then $C_B(Z, X)$ is an l_2-manifold if $C_B(Z, X) \ne \emptyset$.*

PROOF. Consider the fiber-product $X \times_B Z = \{(x, z) \in X \times Z \mid p(x) = q(z)\}$ of X and Z over B. Let $\pi_X\colon X \times Z \to X$ and $\pi_Z\colon X \times Z \to Z$ denote the projections. Then $\pi_Z|X \times_B Z$ is a locally trivial bundle with fiber Y because so is $p\colon X \to B$. Let $\Gamma(X \times_B Z)$ be the space of cross-sections of this bundle. Define a map $\theta\colon \Gamma(X \times_B Z) \to C_B(Z, X)$ by $\theta(s) = \pi_X \circ s$. Then it is easy to see that θ is a homeomorphism. The result follows from Main Theorem. □

As mentioned in Lemma 3, any locally trivial bundle with ANR fiber is an ANFR. Besides a Hurewicz fibration between ANR's and a proper strongly regular map onto a finite-dimensional space with ANR fibers are also ANFR's (cf. [**Ya**, 1.2]). We have the f.p. Homotopy Extension Theorem [**Ya**, 1.1 (iv)] and the f.p. version of Lemma 2. By the same method as Main Theorem, we can prove the following theorem which is analogous to the above corollary but in a little different setting.

THEOREM. *Let $q\colon Z \to B$ be a map of a compactum and $p\colon X \to B$ an ANFR such that X is complete-metrizable. Then $C_B(Z,X)$ is an l_2-manifold if $C_B(Z,X) \neq \emptyset$, $q^{-1}(b)$ is nondiscrete and $p^{-1}(b)$ has no isolated point for some $b \in B$.*

Because of similarity, the proof is left to the reader. (We can also prove the relative version of this theorem, i.e., the f.p. version of [**To₁**, Theorem 5.5].) Here we give the proof of the f.p. version of Lemma 2.

LEMMA 2′. *Let $p\colon X \to B$ be an ANFR and d_0 a metric for X. Then X has a compatible metric $d \geq d_0$ with the following property*

(h′) *For each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that any two δ-near f.p. maps of an arbitrary space Z with a map $q\colon Z \to B$ to $X = (X, d)$ are f.p. ε-homotopic.*

PROOF. Since $X = (X, d_0)$ can be isometrically embedded in a normed linear space E as a closed set (cf. [**BP**, Chapter II, Corollary, 1.1]), we have an f.p. closed embedding $i\colon X \to B \times E$ such that $\pi_E i$ is an isometry, where $\pi_E\colon B \times E \to E$ is the projection. Identify X with $i(X) \subset B \times E$ and $p = \pi_E|$. Let d_1 be the product metric on $B \times E$. Then $d_1 \geq d_0$ on X. Since $p\colon X \to B$ is an ANFR, there is an f.p. retraction $r\colon G \to X$ of a neighborhood G of X in $B \times E$. For a subset $S \subset B \times E$, we denote

$$\mathrm{conv}_B S = \bigcup \{\{b\} \times \mathrm{conv}\, \pi_E(S \cap \{b\} \times E) \mid b \in B\},$$

where $\mathrm{conv}\, A$ denotes the convex hull of $A \subset E$. For each $x \in X$ and each neighborhood V of x in X, there is a basic open set $U = U_1 \times U_2$ in $B \times E$ such that $\pi_E(U) = U_2$ is convex and $x \in U \subset r^{-1}(V)$. Let $W = U \cap X$. Then $\mathrm{conv}_B W \subset U \subset G$ and $r(\mathrm{conv}_B W) \subset r(U) \subset V$. By the same way as [**Mi**, Lemma 2.1], we have a compatible metric $d \geq d_1$ on X with the following property:

(c) To every $\varepsilon > 0$ corresponds a $\delta = \delta(\varepsilon) > 0$ such that if $S \subset X$ with $\mathrm{diam}\, S < \delta$ then $\mathrm{conv}_B S \subset G$ and $\mathrm{diam}\, r(\mathrm{conv}_B S) < \varepsilon$.

By standard arguments, (h′) follows from (c). (Cf. [**Hu**, Chapter IV, Theorem 1.1].) □

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