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THE SPACE OF KÄHLER METRICS

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Abstract

Donaldson conjectured [16] that the space of Kähler metrics is geodesically convex by smooth geodesics and that it is a metric space. Following Donaldson's program, we verify the second part of Donaldson's conjecture completely and verify his first part partially. We also prove that the constant scalar curvature metric is unique in **each** Kähler class if the first Chern class is either strictly negative or 0. Furthermore, if $C_1 \leq 0$, the constant scalar curvature metrics: realizes the global minimum of the Mabuchi K energy functional; thus it provides a new obstruction for the existence of constant curvature metrics: if the infimum of the K energy (taken over all metrics in a fixed Kähler class) is not bounded from below, then there does not exist a constant curvature metric. This extends the work of Mabuchi and Bando [3]: they showed that K energy bounded from below is a necessary condition for the existence of Kähler-Einstein metrics in the first Chern class.

1. Introduction to the problem

1.1 Brief introduction to the classical problems in Kähler geometry

Let V be a Kähler manifold. E. Calabi conjectured in 1954 that any (1,1) form which represents $C_1(V)$ (the first Chern class) is the Ricci form of some Kähler metric on V. Yau [44], in 1976, proved this Calabi conjecture. Aubin [1] and Yau proved independently the existence of a Kähler-Einstein metric on a Kähler manifold with negative first Chern class (also a conjecture of E. Calabi). G. Tian [38], in 1987, proved the existence of Kähler-Einstein metric in a canonical Kähler class on complex surfaces if the first Chern class is positive and the group of

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automorphisms is reductive. For further references on this subject, see [39] and [40]. An important conjecture by Yau [45] relates the existence of Kähler-Einstein metrics to stability in the sense of Hilbert schemes and geometric invariant theory.

Kähler-Einstein metrics can be treated as a special kind of extremal Kähler metric. The question of extremal Kähler metrics was first raised by E. Calabi in his paper [9]: he considered the L^2 norm of curvature as a functional from a given Kähler class; a critical point of this functional is called an "extremal Kähler metric." He showed that any extremal Kähler metric must be symmetric under a maximal compact subgroup of the holomorphic transformation group. Using this structure theorem of Calabi, Marc Levine [30] was able to construct a Kähler surface on which there is no extremal Kähler metric. In 1992, D. Burns and P. de Bartolomeis [8] also produced an example of non-existence of extremal Kähler metrics; their example suggests some new obstruction for the existence of extremal metrics which is related to some borderline semi-stability of hermitian vector bundles. LeBrun [26] also demonstrated that the existence of critical Kähler metrics might be tied up with the stability of corresponding vector bundles. Donaldson [16] thought that Yau's conjecture [45] should extend over to the general extremal Kähler metrics. For further references on the subject of extremal metrics, please; see [27], [26], [19] and references therein.

Futaki [18] in 1983 introduced an analytic invariant for any Kähler manifold with positive first Chern class. The vanishing of this invariant is a necessary condition for the existence of a Kähler-Einstein metric on the manifold. Later, Futaki and Calabi [10] generalized the invariant to any compact Kähler manifold. This generalized Futaki invariant, i.e., Calabi-Futaki invariant, is an analytic obstruction to the existence of a constant scalar curvature metric on a Kähler manifold. In the same paper, Calabi also showed that constant scalar curvature metrics and extremal Kähler metrics with non-constant scalar curvature do not coexist in a single Kähler class.

For uniqueness, the known results are as follows:

1) in the 1950s, E. Calabi showed the uniqueness of Kähler-Einstein metric if $C_1 \leq 0$.

2) in 1987, Mabuchi and S. Bando [3] showed the uniqueness of Kähler-Einstein metric up to holomorphic transformation if the first Chern class is positive. Recently, Tian and X.H. Zhu [43] proved the uniqueness of Kähler-Ricci Soliton with respect to a fixed holomorphic

vector field on any Kähler manifold with positive first Chern class. Although very little was known about the uniqueness of general extremal Kähler metrics, most experts in Kähler geometry expect that the extremal Kähler metric is unique in each Kähler class up to holomorphic transformation. In [12] (also see [11] for further references), we demonstrated two degenerate extremal Kähler metrics in the same Kähler class with different energy levels and different symmetry groups: one example is due to Calabi, the other is due to the author. To my knowledge, it appears that this is the only non-uniqueness example known today.

Main results. Mabuchi ([31])[1] in 1987 defined a Riemannian metric on the space of Kähler metrics, under which it becomes (formally) a non-positive curved infinite dimensional symmetric space. Apparently unaware of Mabuchi's work, Semmes [35] and Donaldson [16] rediscover this same metric again from different angles. In [35], S. Semmes first pointed out that the geodesic equation is a homogeneous complex Monge-Ampère equation on a manifold of one dimension higher. In [16], Donaldson further conjectured that the space is geodesically convex and is a genuine metric space. We prove that it is at least convex by $C^{1,1}$ geodesics,[2] and from which we conclude that the space is indeed a metric space, thus verifying the second part of Donaldson's conjecture. Moreover, this $C^{1,1}$ geodesic realizes the absolute minimum of length over all possible paths connecting the two end points; thus the metric aforementioned is a genuine one. Using these results, we are able to show that the constant curvature metric is unique in each Kähler class if $C_1 < 0$ or $C_1 = 0$. Furthermore, if $C_1 \leq 0$, we show that constant scalar metric (if it exists) realizes the global minimum of K energy, which gives an affirmative answer to a question raised by Gang Tian [42] in this special case. This last statement also extends the work of Mabuchi and Bando [3]: they showed that K energy bounded from below is a necessary condition for the existence of Kähler-Einstein metrics in the first Chern class. Tian [40] showed that in Kähler manifold with positive first Chern class and no non-trivial holomorphic fields, the Kähler-Einstein metric exists if and only if the K energy is proper (he actually uses an equivalent functional instead of the K energy).[3] One would like to ask: Is this still true for constant scalar curvature

[1]Around the same time of Mabuchi's work, J. P. Bourguignon has worked on something similar in a related subject [6].

[2]Here we mean the mixed second derivatives are uniformly bounded. See Theorem 3 in Section 3 for details.

[3]The sufficient part of this result was proved in [14].

metrics?[4]

Organization. In Section 2, we first summarize the different approaches taken by Mabuchi, Semmes and Donaldson in the space of Kähler metrics; then we introduce the Riemannian metric on this infinite dimensional space and prove that it has non-positive sectional curvature in the formal sense. Then we introduce Donaldson's two conjectures and reduce the first conjecture to the existence problem for the complex homogeneous Monge-Ampère equation with Dirichlet boundary data. Readers are alerted that material in Sections 2.3-2.5 is essentially a re-presentation of Donaldson's work [16], included here for the convenience of readers. In Section 3, we prove that this geodesic (CHMA) equation always has a $C^{1,1}$ solution. In Section 4, we prove that a continuous solution to the geodesic (CHMA) equation in some appropriate weak sense is unique. In Section 5, we show that the geodesic distance defined by the length of $C^{1,1}$ geodesics satisfies the triangle inequality. Using this, we prove the space of Kähler metrics is a metric space. In Section 6 we show that the extremal Kähler metric is unique in each Kähler class if either $C_1(V) < 0$ or $C_1(V) = 0$.

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2. Space of Kähler metrics

2.1 Mabuchi and S. Semmes' ideas

Shortly after introducing the K energy, Mabuchi [31] defined a Riemannian metric on the space of Kähler metrics. Besides showing formally it is a locally symmetric space with non-positive sectional curvature, he also pointed out that the K energy is formally convex in this infinite dimensional space (in the sense that the Hessian is semi-positive definite).

[4] Tian informed us that he [41] has conjectured that constant scalar curvature metrics exist if and only if the K energy is proper.

Perhaps, this is his original motivation for introducing such a metric. Unaware of Mabuchi's work, in [35], S. Semmes studied the geometry of solutions of the complex Homogeneous Monge-Ampère equation (CHMA). He observed that in some special domain $\Omega \times D$ where Ω is an n-dimensional domain in C^n and D is a domain in the complex plane, the solution to CHMA is some sort of geodesic equation if the data is rotationally symmetric when restricted to D. He then considered the space of pluri-subharmonic functions in Ω and defined a Riemannian metric in this space according to this geodesic equation. It turns out that this space becomes a non-positively curved (locally) symmetric space in some formal sense. Unlike the real homogeneous Monge-Ampère equation (RHMA) whose solution always has proper geometric meaning, the solution of a CHMA equation doesn't have a preferred geometric interpretation. Without a proper geometry interpretation, it is very hard to work on this subject. Of course, great progress has been made since the famous work of L. Caffarelli, L. Nirenberg and J. Spruck [24] and later their joint work with J. Kohn [23], e.g., L. Lempert [29], E. Bedford and B.A. Taylor [4]; P. Lelong [28] and important work of Krylov [22] and Evans [25]. This is by no means a complete list of papers on complex Monge-Ampère equations since the author is quite new to this important field. For a complete and updated list of references, please; see S. Kolodziej [21]. Donaldson's recent work certainly makes Mabuchi and Semmes's original work more interesting.

2.2 Brief summary of Donaldson's theory on space of Kähler metrics

S. K. Donaldson [16] outlined a strategy to relate this geometry of infinite dimensional space to the existence problems in Kähler geometry. In particular, he explains how one can use this extra structure on the infinite dimensional space to solve the problem of the existence and uniqueness of extremal Kähler metrics. In general, the latter are intractable problems from traditional means. He regards the space of Kähler metrics in a fixed Kähler class as an infinite dimensional symplectic manifold with the automorphism group $\mathrm{SDiff}(V)$ (symplectic diffeomorphism group of V into itself). In [15], he pointed out that scalar curvature is the moment map μ from this infinite dimensional symplectic manifold to the dual space of the Lie algebra of its automorphism group[5] . Thus, to find an extremal Kähler metric in a fixed

[5] The *moment map* point of view here was also observed by A. Fujiki [17].

Kähler class in classical Kähler geometry could be re-interpreted as to find a pre-image of 0 of the moment map μ in this symplectic setting. This acute observation sheds new light into the otherwise intractable problem of the existence of extremal Kähler metrics on a Kähler manifold; at least conceptually, the picture looks much clearer. He then proposed several conjectures whose ultimate resolution will lead to a better understanding of extremal Kähler metrics, and for that matter, better understanding of Kähler geometry as well. The most fundamental one among his conjectures is the so called geodesic conjecture: any two Kähler metrics in the same class are connected by a smooth geodesic. A second conjecture by him is that this space of Kähler metrics is a metric space under this metric. If the geodesic conjecture is true, this second conjecture will be a direct consequence (since this space of Kähler metrics in a fixed Kähler class is non-positively curved in the formal sense.). He went on to show that the uniqueness of extremal Kähler metrics is a consequence of this geodesic conjecture as well.

2.3 Riemannian metrics on the infinite dimensional space.

Consider the space of Kähler potentials in a fixed Kähler class as:

$$\mathcal{H} = \{\varphi \in C^\infty(V) : \omega_\varphi = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi > 0 \text{ on } V\}.$$

We now introduce an L^2 metric in this space (cf. Mabuchi [31]). Clearly, the tangent space $T\mathcal{H}$ is $C^\infty(V)$. Each Kähler potential $\phi \in \mathcal{H}$ defines a measure $d\,\mu_\phi = \frac{1}{n!}\omega_\phi^n$. Now we define a Riemannian metric on the infinite dimensional manifold $\mathcal{H}$ using the L^2 norm provided by this measure. We define the length of any vector $\psi \in T_\varphi\mathcal{H}$ as

$$\|\psi\|_\varphi^2 = \int_V \psi^2 \; d\;\mu_\varphi.$$

For a path $\varphi(t) \in \mathcal{H}(0 \leq t \leq 1)$, the length is given by

$$\int_0^1 \sqrt{\int_V \varphi'(t)^2 d\,\mu_{\varphi(t)}}\; d\,t$$

and the geodesic equation is

$$\varphi(t)'' - \frac{1}{2}|\nabla\varphi'(t)|^2_{\varphi(t)} = 0, \tag{1}$$

where the derivative and norm in the 2nd term of the left-hand side are taken with respect to the metric $\omega_{\varphi(t)}$.

This geodesic equation shows us how to define a connection on the tangent bundle of $\mathcal{H}$. The notation is simplest if one thinks of such a connection as a way of differentiating vector fields along paths. Thus, if $\phi(t)$ is any path in $\mathcal{H}$ and $\psi(t)$ is a field of tangent vectors along the path (that is, a function on $V \times [0,1]$), we define the covariant derivative along the path to be

$$D_t\psi = \frac{\partial \psi}{\partial t} - \frac{1}{2}(\nabla\psi, \nabla\phi')_\phi.$$

This connection is torsion-free because in the canonical "coordinate chart", which represents $\mathcal{H}$ as an open subset of $C^\infty(V)$, the "Christoffel symbol"

$$\Gamma : C^\infty(V) \times C^\infty(V) \to C^\infty(V)$$

at ϕ is just

$$\Gamma(\psi_1, \psi_2) = -\frac{1}{2}(\nabla\psi_1, \nabla\psi_2)_\phi$$

which is symmetric in ψ_1, ψ_2. The connection is metric-compatible because

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\psi\|_\phi^2 &= \frac{d}{dt}\int_V \psi^2 d\mu_\phi \\
&= \int_V \frac{\partial\psi}{\partial t}\psi + \frac{1}{2}\psi^2\Delta(\phi')d\mu_\phi \\
&= \int_V \frac{\partial\psi}{\partial t}\psi - \frac{1}{4}(\nabla(\psi^2), \nabla\phi')_\phi \; d\mu_\phi \\
&= \int_V \left(\frac{\partial\psi}{\partial t} - \frac{1}{2}(\nabla\psi, \nabla\phi')_\phi\right)\psi d\mu_\phi \\
&= \langle D_t\psi, \psi\rangle.
\end{aligned}$$

Here $\triangle$ is a complex Laplacian operator. The main theorem proved in [31] (and later reproved in [35] and [16]) is:

Theorem A. *The Riemannian manifold $\mathcal{H}$ is an infinite dimensional symmetric space; it admits a Levi-Civita connection whose curvature is covariant constant. At a point $\phi \in \mathcal{H}$ the curvature is given by*

$$R_\phi(\delta_1\phi, \delta_2\phi)\delta_3\phi = -\frac{1}{4}\{\{\delta_1\phi, \delta_2\phi\}_\phi, \delta_3\phi\}_\phi,$$

where $\{\ ,\ \}_\phi$ *is the Poisson bracket on* $C^\infty(V)$ *of the symplectic form* ω_ϕ*; and* $\delta_1\phi, \delta_2\phi \in T_\phi\mathcal{H}$.

Recall that in infinite dimensions the usual argument gives the uniqueness of a Levi-Civita [i.e., torsion-free, metric-compatible] connection, but not the existence in general. The formula for the curvature of $\mathcal{H}$ entails that the sectional curvature is non-positive, given by

$$K_\phi(\delta_1\phi, \delta_2\phi) = -\frac{1}{4}\|\{\delta_1\phi, \delta_2\phi\}_\phi\|_\phi^2.$$

Different proofs of this theorem have appeared in [31], [35] and [16]. We will skip the proof here; interested readers are referred to these papers for the proof.

The expression for the curvature tensor in terms of Poisson brackets shows that R is invariant under the action of the symplectic-morphism group. Since the connection on $T\mathcal{H}$ is induced from an SDiff-connection, it follows that R is covariant constant, and hence $\mathcal{H}$ is indeed an infinite-dimensional symmetric space.

2.4 Splitting of $\mathcal{H}$

There is obviously a decomposition of the tangent space:

$$T_\phi\mathcal{H} = \left\{\psi : \int_V \psi d\mu_\phi = 0\right\} \oplus \mathbf{R}.$$

We claim that this corresponds to a Riemannian decomposition

$$\mathcal{H} = \mathcal{H}_0 \times \mathbf{R}.$$

We are interested in seeing this Riemannian splitting more explicitly, partly because we see the appearance of a functional I on the space of Kähler potentials, which is well-known in the literature; see [2], [40] for example. The decomposition of the tangent space of $\mathcal{H}$ gives a 1-form α on $\mathcal{H}$ with

$$\alpha_\phi(\psi) = \int_V \psi d\mu_\phi,$$

and it is straightforward to verify that this 1-form is *closed.* Indeed

$$(d\alpha)_\phi(\psi, \widetilde{\psi}) = \int_V \left(\widetilde{\psi}\Delta\psi - \psi\Delta\widetilde{\psi}\right) = 0.$$

This means that there is a function $I : \mathcal{H} \to \mathbf{R}$ with $I(0) = 0$ and $dI = \alpha$, and it is this function which gives rise to the corresponding Riemannian decomposition. We call a Kähler potential ϕ *normalized* if $I(\phi) = 0$. Then any Kähler metric has a unique normalized potential, and the restriction of our metric on $\mathcal{H}$ to $I^{-1}(0)$ endows the space $\mathcal{H}_0$ of Kähler metrics with a Riemannian structure; this is independent of the choice of the base point ω_0 and clearly makes $\mathcal{H}_0$ into a symmetric space. The functional I can be written more explicitly by integrating α along lines in $\mathcal{H}$ to give the formula

$$I(\phi) = \sum_{p=0}^{n} \frac{1}{(p+1)!(n-p)!} \int_V \omega_0^{n-p} (\partial\overline{\partial}\phi)^p \ \phi.$$

2.5 Donaldson' Conjectures

We will now study the geodesic equation in $\mathcal{H}$ in more detail, and interpret the solutions geometrically. Suppose $\phi_t,\ t \in [0,1]$, is a path in $\mathcal{H}$. We can view this as a function on $V \times [0,1]$ and in turn as a function on $V \times [0,1] \times S^1$, with trivial dependence on the S^1 factor; that is, we define

$$\Phi(v, t, e^{is}) = \phi_t(v).$$

We regard the cylinder $\mathbf{R} = [0,1] \times S^1$ as a Riemann surface with boundary in the standard way—so $t + is$ is a local complex coordinate. Let Ω_0 be the pull-back of ω_0 to $V \times \mathbf{R}$ under the projection map and put $\Omega_\Phi = \Omega_0 + \partial\overline{\partial}\Phi$, a $(1,1)$-form on $V \times \mathbf{R}$. Then we have:

Proposition 1. *The path ϕ_t satisfies the geodesic equation* (1) *if and only if $\Omega_\Phi^{n+1} = 0$ on $V \times \mathbf{R}$.*

Proof. Denote the metrics defined by ω_0, ω_ϕ as g, g'. Then

$$\frac{1}{n!}\omega_\phi^n = \det\, g'; \qquad \frac{1}{n!}\omega_0^n = \det\, g.$$

Thus the geodesic equation is equivalent to the following (if $\det g' \neq 0$)

$$(\phi'' - \frac{1}{2} \mid \nabla\phi' \mid_{g'}^2) \ \det g' = 0,$$

which may be given as

$$\det\begin{pmatrix} & g' & & \begin{pmatrix} \frac{\partial\phi'}{\partial z_1} \\ \frac{\partial\phi'}{\partial z_2} \\ \vdots \\ \frac{\partial\phi'}{\partial z_n} \end{pmatrix} \\ \begin{pmatrix} \frac{\partial\phi'}{\partial \overline{z_1}} & \frac{\partial\phi'}{\partial \overline{z_2}} & \cdots & \frac{\partial\phi'}{\partial \overline{z_n}} \end{pmatrix} & & & \phi'' \end{pmatrix} = 0.$$

Let $w = t + \sqrt{-1}s$. Then $t = Re(w)$. The above equation could be rewritten as

$$\det\begin{pmatrix} & (g + \frac{\partial^2\phi}{\partial z_\alpha \partial \overline{z}_\beta})_{n\,n} & & \begin{pmatrix} \frac{\partial^2\phi}{\partial z_1 \partial \overline{w}} \\ \frac{\partial^2\phi}{\partial z_2 \partial \overline{w}} \\ \vdots \\ \frac{\partial^2\phi}{\partial z_n \partial \overline{w}} \end{pmatrix} \\ \begin{pmatrix} \frac{\partial^2\phi}{\partial \overline{z_1} \partial w} & \frac{\partial^2\phi}{\partial \overline{z_2} \partial w} & \cdots & \frac{\partial^2\phi}{\partial \overline{z_n} \partial w} \end{pmatrix} & & & \frac{\partial^2\phi}{\partial w \partial \overline{w}} \end{pmatrix} = 0.$$

This is just $\Omega_\Phi^{n+1} = 0$. The proposition is then proved. q.e.d.

Given boundary data — a real valued function $\rho \in C^\infty(\partial(V \times \mathbf{R}))$, we consider the set of functions Φ on $V \times \mathbf{R}$ which agree with ρ on the boundary. Then we define the variation of I_ρ on this set by

$$\delta I_\rho = \frac{1}{(n+1)!} \int_{V \times \mathbf{R}} \delta\Phi \; \Omega_\Phi^{n+1},$$

where the variation $\delta\Phi$ vanishes on the boundary by hypothesis. This boundary condition means that we can show easily that this formula defines a functional I_ρ. To prove this, one only needs to show that the second derivatives of I_ρ with respect to two infinitesimal variations $\delta_1\Phi$ and $\delta_2\Phi$ are symmetric. The second derivatives are:

$$\frac{1}{2} \cdot \frac{1}{(n+1)!} \int_V \delta_1\Phi \bigtriangleup \delta_2\Phi \; \Omega_\Phi^{n+1},$$

which are clearly symmetric in $\delta_1\Phi$ and $\delta_2\Phi$. Here $\bigtriangleup$ is the Laplacian operator of Ω_Φ on $V \times \mathbf{R}$.

This functional I_ρ reduces to the energy functional on paths, by an integration by parts, in the case when $\mathbf{R}$ is the cylinder and we restrict to S^1-invariant data. Suppose $\phi(t)(0 \leq t \leq 1)$ is a path in $\mathcal{H}$, and $\delta\phi$ represents the infinestimal variation of ϕ while keeping the value of ϕ fixed when $t = 0, 1$. Thus, the variation of I_ρ in the $\delta\phi$ direction is (follow notation in the proof of the previous proposition):

$$\begin{aligned}\delta I_\rho =& \frac{1}{(n+1)!}\int_{V\times R} \delta\phi\ \Omega_\Phi^{n+1}\\ =& \frac{1}{(n+1)!}\int_{t=0}^1\int_V \delta\phi(\phi'' - \frac{1}{2}\mid \nabla\phi' \mid_{g'}^2)\ \det g' d\,t.\end{aligned}$$

On the other hand, the variation of energy functional along this path is:

$$\delta E = \int_{t=0}^1 \int_V \delta\phi\ (\phi'' - \frac{1}{2}\mid \nabla\phi' \mid_{g'}^2)\ \det g' d\,t,$$

where $E = \int_{t=0}^1 \int_V \phi'(t)^2\ \det g' d\,t$. Thus, in case when $\mathbf{R}$ is the cylinder and we restrict to S^1-invariant data, I_ρ equals to the energy functional on the path up to a constant multiple.

The following is the first conjecture by Donaldson in [16]:

Conjecture 1 (Donaldson). *Let $\mathbb{R}$ be a compact Riemann surface with boundary and $\rho : V \times \partial\mathbf{R} \to \mathbb{R}$ be a function such that $\omega_0 - \sqrt{-1}\,\overline{\partial}\partial\rho$ is a strictly positive $(1,1)$ form on each slice $V \times \{z\}$ for each fixed $z \in \partial\mathbf{R}$. Let $\mathcal{S}_\rho$ be the set of functions Φ on $V \times \mathbf{R}$ equal to ρ over the boundary and such that $\omega_0 - \sqrt{-1}\,\overline{\partial}\partial\Phi$ is strictly positive on every slice $V \times \{w\}, w \in \mathbf{R}$. Then there is a unique solution of the Monge-Ampère equation $(\Omega_0 - \sqrt{-1}\,\overline{\partial}\partial\Phi)^{n+1} = 0$ in $\mathcal{S}_\rho$, and this solution realizes the absolute minimum of the functional I_ρ.*

This question is a version of the Dirichlet problem for the complete degenerate Monge-Ampère equation, a topic around which there is a substantial literature; see [2], [20] for example. Note that regularity questions are very important in this theory, since the equation is not elliptic.

In the case of the geodesic problem, when the functional can be rewritten as the energy of a path; if these infima are strictly positive, for all choices of fixed, distinct, end points, they make $\mathcal{H}$ into a metric space, in the usual fashion. In this connection, Donaldson proposes the following conjecture (after verifying that it will be satisfied by a smooth geodesic):

Conjecture 2 (Donaldson). *If $\phi \in \mathcal{H}_0$ is normalized and $\widetilde{\phi}_t,\ t \in [0,1]$ is any path from 0 to ϕ in $\mathcal{H}$, then*

$$\int_0^1 \int_V \left(\frac{d\widetilde{\phi}}{dt}\right)^2 d\mu_{\widetilde{\phi}_t} dt \geq M^{-1}\left(\max\Big(\int_{\phi>0} \phi d\mu_\phi, -\int_{\phi<0} \phi d\mu_0\Big)\right)^2. \tag{2}$$

The restriction to normalized potentials ϕ is not important since we know that $\mathcal{H}$ splits as a product, and we could immediately write down a corresponding inequality, involving $I(\phi)$, for any $\phi \in \mathcal{H}$. If this conjecture and the geodesic conjecture are proved, then $\mathcal{H}$ is a metric space. We want to use the continuity method to treat this existence problem of geodesics between any two points in $\mathcal{H}$.

3. Existence of $C^{1,1}$ solutions

Let V be a $n-$ dimensional Kähler manifold without boundary, $\mathbf{R}$ be a Riemann surface with boundary. The case we are concerned the most is when $\mathbf{R}$ is a cylinder. Suppose $g = g_{\alpha\overline{\beta}} dz_\alpha d\,\overline{z_\beta} (1 \leq \alpha, \beta \leq n)$ is a given Kähler metric on V. Then $\widetilde{g} = g_{\alpha\overline{\beta}} dz_\alpha d\,\overline{z}_\beta + dw\,\overline{dw}$ is a Kähler metric on $V \times \mathbf{R}$, and $\widetilde{\varphi} = \varphi - |w|^2$. For convenience, we still denote $\widetilde{g}$ as g, and $\widetilde{\varphi}$ as φ when there is no confusion. Also, let $z_{n+1} = w$. Then $z = (z_1, z_2, \ldots, z_n, z_{n+1})$ is a point on $V \times \mathbf{R}$ and $z' = (z_1, z_2, \ldots, z_n)$ is a point in V. Let $\varphi(z) = \varphi(z', w)$ be a function on $V \times \mathbf{R}$ such that $g + \partial_{z'}\overline{\partial_{z'}}\varphi(z', w)$ is a Kähler metric on V for each $w \in \mathbf{R}$. We want to solve the degenerate Monge-Ampère equation:

$$\begin{aligned} \det\left(g + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z}_\beta}\right)_{(n+1)(n+1)} &= 0 \ \text{ in } V \times \mathbf{R}; \\ \text{and } \ \varphi = \varphi_0 \ &\text{in } \ \partial(V \times \mathbf{R}). \end{aligned} \tag{3}$$

We want to use the continuity method to solve this equation. Consider the continuity equation $0 \leq t \leq 1$:

$$\begin{aligned} \det\left(g + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z}_\beta}\right) &= t \det\left(g + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \overline{z}_\beta}\right), \ \text{ in } V \times \mathbf{R}; \\ &\text{and } \ \varphi = \varphi_0 \text{ in } \ \partial(V \times \mathbf{R}). \end{aligned} \tag{4}$$

Suppose φ_0 is a solution to (4) at $t = 1$ such that

$$\sum_{\alpha,\beta=1}^{n+1} \left(g_{\alpha\overline{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \, \partial \overline{z}_\beta} \right) dz_\alpha d\,\overline{z}_\beta$$

is a strictly positive Kähler metric on $V \times \mathbf{R}$[6] . Let

$$f = \det \left(g + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \overline{z}_\beta} \right) (\det \ g)^{-1} > 0.$$

Then equation (4) can be rewritten in a better form

$$\det \left(g + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z}_\beta} \right) = t \cdot f \cdot \det(g) \ \text{ in } V \times \mathbf{R}; \qquad \text{and } \ \varphi = \varphi_0 \text{ in } \ \partial(V \times \mathbf{R}). \tag{5}$$

Clearly, φ_0 is the unique solution to this equation at $t = 1$. Since the equation is elliptic, this equation can be uniquely solved for t sufficiently closed to 1 (the kernel of the linearized operator is zero for any $t > 0$). Let t_0 be such that (5) has a unique smooth solution for every $t \in (t_0, 1]$. We want to show that $t_0 = 0$ in this section. Observe that equation (5) is elliptic for every $t > 0$. Hence, the solution will be as smooth as the boundary data once we show that 2nd derivatives of φ are uniformly bounded. Let h be a super harmonic function on $V \times \mathbf{R}$ with respect to g such that $\triangle_g h + n + 1 = 0$. and $h = \varphi_0$ in $\partial(V \times \mathbf{R})$. Then for any solution of equation (5) for $t < 1$, we have a C^0 bound on the solution:

Lemma 1. *If φ is a solution of equation* (5) *for* $0 < t < 1$, *then φ has the following a priori C^0 estimate due to the maximum principle:*

$$\varphi_0 \le \varphi \le h, \qquad \text{in } V \times \mathbf{R}.$$

For a C^2 estimate, we follow Yau's famous work on Calabi's conjecture. Essentially, we reduce it to a boundary estimate since we have a C^0 estimate:

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[6]By definition, for any $\varphi_0 \in \mathcal{H}$, $\sum_{\alpha,\beta=1}^{n+1} (g_{\alpha\overline{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \overline{z}_\beta}) dz_\alpha d\,\overline{z}_\beta$ is a strictly positive Kähler metric on each $V-$ slice $V \times \{w\}$. Let Ψ be a strictly convex function of w which vanishes on $\partial\mathbf{R}$. Then for large enough constants m, $\sum_{\alpha,\beta=1}^{n+1} (g_{\alpha\overline{\beta}} + \frac{\partial^2 (\varphi_0 + m\Psi)}{\partial z_\alpha \partial \overline{z}_\beta}) dz_\alpha d\,\overline{z}_\beta$ is a strictly positive Kähler metric on $V \times \mathbf{R}$.

Lemma 2 (Yau). *If φ is a solution of equation* (5) *for $0 < t < 1$, then φ has the following a priori C^2 estimate:*

$$\begin{aligned}
&\triangle'(e^{-C\varphi}(n+1+\triangle\varphi)) \\
&\qquad \geq e^{-C\varphi}(\triangle \ln f - (n+1)^2 \inf_{i\neq l}(R_{i\bar{i}l\bar{l}})) \\
&\qquad\quad - Ce^{-C\varphi}(n+1)(n+1+\triangle\varphi) \\
&\qquad\quad + (C + \inf_{i\neq l}(R_{i\bar{i}l\bar{l}}))e^{-C\varphi}(n+1+\triangle\varphi)^{1+\frac{1}{n}}(tf)^{-1},
\end{aligned}$$

where $C + \inf_{i\neq l}(R_{i\bar{i}l\bar{l}}) > 1$, $\triangle$ is the Laplacian operator with respect to g, while $\triangle'$ is the Laplacian operator with respect to $g' = g + \frac{\partial^2\varphi}{\partial z_\alpha \partial \bar{z}_\beta} d\, z_\alpha \, \overline{d\, z_\beta}$ and $R_{i\bar{i}l\bar{l}}$ is the Riemannian curvature of g.

From the a priori estimate in Lemma 2, either $e^{-C\varphi}(n+1+\triangle\varphi)$ is uniformly bounded in $V \times \mathbf{R}$ or it achieves its maximum value at $\partial(V \times \mathbf{R})$. Lemma 1 asserts that φ is uniformly bounded from above and below, then

Corollary 1. *There exists a constant C which depends only on $(V \times \mathbf{R}, g)$ such that*

$$\max_{V\times\mathbf{R}} (n+1+\triangle\varphi) \leq C(1 + \max_{\partial(V\times\mathbf{R})} (n+1+\triangle\varphi)).$$

Theorem 1. *If φ is a solution of equation* (5) *for $0 < t < 1$, then there exists a constant C which depends only on $(V \times \mathbf{R}, g)$ such that:*

$$\max_{V\times\mathbf{R}} (n+1+\triangle\varphi) \leq C \max_{V\times\mathbf{R}} (|\nabla\varphi|_g^2 + 1). \tag{6}$$

In light of Corollary 1, we only need to prove inequality (6) on the boundary, i.e.,

$$\max_{\partial(V\times\mathbf{R})} (n+1+\triangle\varphi) \leq C \max_{V\times\mathbf{R}} (|\nabla\varphi|_g^2 + 1).$$

We will prove this inequality in the next subsection.

Theorem 2. *If $\varphi_i (i = 1, 2, \dots)$ are solutions of equation* (5) *for $0 < t_i < 1$, and the inequality* (6) *holds uniformly for all these solutions $\{\varphi_i, i \in \mathbf{N}\}$, then there exists a constant C_1 independent of i such that*

$$\max_{V\times\mathbf{R}} (n+1+\triangle\varphi) \leq C \max_{V\times\mathbf{R}} (|\nabla\varphi|_g^2 + 1) < C_1.$$

This is proved via a blowing up argument. We will show this in Subsection 3.2.

Remark 1. By now it is a standard estimate of Monge-Ampère equations, that if

$$\max_{V\times\mathbf{R}} (n+1+\triangle\varphi) \le C \max_{V\times\mathbf{R})} (|\nabla\varphi|_g^2+1) < C_1$$

then equation (5) for $t_1, t_2, \ldots$ is a sequence of uniform elliptic equations. The higher derivative of the solution φ_i has a uniform bound as long as $\liminf_{i\to\infty} t_i > 0$.

Theorem 3. *There exists a $C^{1,1}(V\times\mathbf{R})$ function which solves equation (3) weakly. In other words, for any two points $\varphi_0, \varphi_1 \in \mathcal{H}$, there exists a geodesic path $\varphi(t) : [0,1] \to \overline{\mathcal{H}}$ and a uniform constant C such that the following holds:*

$$0 \le \left(g_{i\bar{j}} \ + \ \frac{\partial^2\varphi}{\partial z_i \partial \overline{z_j}}\right)_{(n+1)(n+1)} \le C\left(\widetilde{g}_{i\bar{j}}\right)_{(n+1)(n+1)}.$$

Here $z_1, z_2 \ldots, z_n$ are local coordinates in V and $t = Re\ (z_{n+1})$. And $\widetilde{g} = g_{\alpha\overline{\beta}} dz_\alpha d\,\overline{z}_\beta + dw\,\overline{dw}$ is a fixed product metric on $V\times\mathbf{R}$.

Following notation in Theorem 2, we want to show that $t_0 = \liminf_{i\to\infty} t_i = 0$. Otherwise, assume $t_0 > 0$. Then equation (5) has a unique smooth solution for $1 \ge t > t_0$. By Theorem 2 we have a uniform upper bound for $\triangle\varphi + (n+1)$ for all $t_i > t_0 > 0$. Then equation (5) implies that $g_i' = g + \frac{\partial^2\varphi_i}{\partial z_\alpha \partial \overline{z}_\beta} d\,z_\alpha\,\overline{d\,z_\beta}$ is bounded uniformly from below by a uniform positive constant (this positive lower bound approaches 0 when $t\to 0$). Thus, from equation (5), we obtain uniform higher derivative estimates for solutions φ_i. Therefore these solutions converge to a regular solution at $t_0 > 0$. Again, since equation (5) at t_0 is an elliptic equation and the kernel of the linearized operator is zero, it can then be solved for any t sufficiently close to t_0. But this contradicts the definition of t_0. Thus $t_0 = 0$. We can choose a subsequence of $t_i \to 0$ such that the φ_i converge weakly in $C^{1,1}(V\times\mathbf{R})$ where Ω is a relative compact subset of $V\times\mathbf{R}$. Again via the maximum principle, we can show this limit is unique and defines a weak solution of equation (3).

3.1 Boundary estimate

We want to estimate $\triangle\varphi$ at any point in the boundary $\partial(V \times \mathbf{R}) = V \times \partial\mathbf{R}$. Let p be a generic point in $\partial(V \times \mathbf{R})$. Now choose a small neighborhood U of p in $V \times \mathbf{R}$ (this will be a half geodesic ball since $p \in \partial(V \times \mathbf{R})$) and a local coordinate chart such that $g_{\alpha\overline{\beta}}(p) = \delta_{\alpha\overline{\beta}}$ and $p = (z = 0)$

$$\frac{1}{2}\delta_{\alpha\overline{\beta}} \leq g_{\alpha\overline{\beta}}(q) \leq 2\delta_{\alpha\overline{\beta}}, \qquad \forall q \in U.$$

Since $\displaystyle\sum_{\alpha,\beta=1}^{n+1}\left(g_{\alpha\overline{\beta}} + \frac{\partial^2\varphi_0}{\partial z_\alpha \partial \overline{z}_\beta}\right) d\, z_\alpha d\, \overline{z}_\beta$ is a positive Kähler metric on $V \times \mathbf{R}$, there exists a constant $\epsilon > 0$ such that

$$g_{\alpha\overline{\beta}} + \frac{\partial^2\varphi_0}{\partial z_\alpha \partial \overline{z}_\beta} > 2\,\epsilon \cdot g_{\alpha\overline{\beta}}, \qquad \text{in } V \times \mathbf{R}.$$

In the neighborhood U of p, we have

$$g_{\alpha\overline{\beta}} + \frac{\partial^2\varphi_0}{\partial z_\alpha \partial \overline{z}_\beta} > \epsilon \cdot \delta_{\alpha\overline{\beta}} \qquad \text{in } V \times \mathbf{R}. \tag{7}$$

We have the trivial estimates in $\partial(V \times \mathbf{R})$:

$$\frac{\partial(\varphi - \varphi_0)}{\partial z_\alpha} = 0, \qquad \frac{\partial^2(\varphi - \varphi_0)}{\partial z_\alpha \partial \overline{z}_\beta} = 0, \qquad \forall\, 1 \leq \alpha, \beta \leq n.$$

In order to estimate $\triangle\varphi = \displaystyle\sum_{\alpha,\beta=1}^{n+1} g^{\alpha\overline{\beta}}\, \frac{\partial^2\varphi}{\partial z_\alpha \partial \overline{z}_\beta}$ in $\partial(V \times \mathbf{R})$, we only need to estimate $\frac{\partial^2(\varphi-\varphi_0)}{\partial z_\alpha \partial \overline{z}_\beta}$ when either α or β is $n+1$. We will estimate $\frac{\partial^2(\varphi-\varphi_0)}{\partial z_\alpha \partial \overline{z}_{n+1}}(\alpha \leq n)$ first, then use equation (5) to derive as estimate for $\frac{\partial^2(\varphi-\varphi_0)}{\partial z_{n+1} \partial \overline{z}_{n+1}}$.

Now we set up some conventions:

$$z_\alpha = x_\alpha + \sqrt{-1}\, y_\alpha, \;\; \forall\, 1 \leq \alpha \leq n; \qquad z_{n+1} = x + \sqrt{-1}\, y,$$

where $\mathbf{R}$ near $\partial\mathbf{R}$ is given by $x \geq 0$.

Lemma 3. *There exists a constant C which depends only on $(V \times \mathbf{R}, g)$ such that*

$$\left|\frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z}_{n+1}}(p)\right| \leq C(\max_{V\times\mathbf{R}}\ |\nabla\varphi|_g + 1).$$

Proof of Theorem 1. At point p, equation (5) reduces to

$$\det\left(\delta_{\alpha\overline{\beta}} + \frac{\partial^2\varphi}{\partial z_\alpha \partial\overline{z}_\beta}\right) = t\cdot f.$$

In other words,

$$\frac{\partial^2\varphi}{\partial z_{n+1}\partial\overline{z}_{n+1}} = t\cdot f - \frac{\partial^2\varphi}{\partial z_\alpha\partial\overline{z}_{n+1}}\cdot\frac{\partial^2\varphi}{\partial\overline{z}_\alpha\partial z_{n+1}}.$$

Lemma 3 then implies that

$$\left|\frac{\partial^2\varphi}{\partial z_{n+1}\partial\overline{z}_{n+1}}\right| \leq C(\max_{V\times\mathbf{R}}|\nabla\varphi|_g^2 + 1).$$

Thus,

$$|\triangle\varphi(p)| = \left|\sum_{\alpha,\beta=1}^{n+1} g^{\alpha\overline{\beta}}\ \frac{\partial^2\varphi}{\partial z_\alpha\partial\overline{z}_\beta}(p)\right| \leq C(\max_{V\times\mathbf{R}}|\nabla\varphi|_g^2+1).$$

Since p is a generic point in $\partial(V\times\mathbf{R})$, Theorem 2 holds true. q.e.d.

Let D be any constant linear 1st order operator near the boundary (for instance $D = \pm\frac{\partial}{\partial x_\alpha},\ \pm\frac{\partial}{\partial y_\alpha}$ for any $1\leq\alpha\leq n$). Notice D is just defined locally. Define a new operator $\mathcal{L}$ as (ϕ is any test function):

$$\mathcal{L}\phi = \sum_{\alpha,\beta=1}^{n+1} {g'}^{\alpha\overline{\beta}}\frac{\partial^2\phi}{\partial z_\alpha\partial\overline{z}_\beta},$$

where $({g'}^{\alpha\overline{\beta}}) = (g'_{\alpha\overline{\beta}})^{-1} = \left(g_{\alpha\overline{\beta}} + \frac{\partial^2\varphi}{\partial z_\alpha\partial\overline{z}_\beta}\right)^{-1}$. Differentiating both side of equation (5) by D, we get

$$\mathcal{L}\ D\varphi = D\ln f + \sum_{\alpha,\beta=1}^{n+1} {g'}^{\alpha\overline{\beta}} D g_{\alpha\overline{\beta}}.$$

Thus there exists a constant C which depends only on $(V \times \mathbf{R}, g)$ such that

$$\mathcal{L}D(\varphi - \varphi_0) \leq C\left(1 + \sum_{\alpha=1}^{n+1} g'^{\alpha\overline{\alpha}}\right). \tag{8}$$

We will now employ a barrier function of the form

$$\nu = (\varphi - \varphi_0) + s\,(h - \varphi_0) - N \cdot x^2 \tag{9}$$

near the boundary point, and s, N are positive constants to be determined. We may take δ small enough so that x is small in $\Omega_\delta = (V \times \mathbf{R}) \cap B_\delta(0)$. The main essence of the proof is:

Lemma 4. *For N sufficiently large and s, δ sufficiently small, we have*

$$\mathcal{L}\,\nu \leq -\frac{\epsilon}{4}\left(1 + \sum_{\alpha=1}^{n+1} g'^{\alpha\overline{\alpha}}\right) \text{ in } \Omega_\delta, \quad \nu \geq 0 \text{ on } \partial\Omega_\delta.$$

Proof. Since $g_{\alpha\overline{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \overline{z}_\beta} \geq \epsilon\delta_{\alpha\overline{\beta}}$, we have

$$\begin{aligned}\mathcal{L}(\varphi - \varphi_0) &= \sum_{\alpha,\beta=1}^{n+1} g'^{\alpha\overline{\beta}}\left[\left(g_{\alpha\overline{\beta}} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z}_\beta}\right) - \left(g_{\alpha\overline{\beta}} + \frac{\partial^2 \varphi_0}{\partial z_\alpha \partial \overline{z}_\beta}\right)\right] \\ &\leq n + 1 - \epsilon\sum_{\alpha=1}^{n+1} g'^{\alpha\overline{\alpha}}\end{aligned}$$

and

$$\mathcal{L}(h - \varphi_0) \leq C_1\left(1 + \sum_{\alpha=1}^{n+1} g'^{\alpha\overline{\alpha}}\right)$$

for some constant C_1. Furthermore, $\mathcal{L}\,x^2 = 2g'^{(n+1)\overline{n+1}}$. Thus

$$\begin{aligned}\mathcal{L}\nu &= \mathcal{L}(\varphi - \varphi_0) + s \cdot \mathcal{L}(h - \varphi_0) - 2 \cdot N \cdot g'^{(n+1)\overline{n+1}} \\ &\leq n + 1 - \epsilon\sum_{\alpha=1}^{n+1} g'^{\alpha\overline{\alpha}} + sC_1 + sC_1\sum_{\alpha=1}^{n+1} g'^{\alpha\overline{\alpha}} \\ &\quad - 2Ng'^{(n+1)\overline{n+1}}.\end{aligned}$$

Suppose $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n+1}$ are eigenvalues of $(g'_{\alpha\overline{\beta}})_{(n+1)(n+1)}$. Then

$$\sum_{\alpha=1}^{n+1} {g'}^{\alpha\overline{\alpha}} = \sum_{\alpha=1}^{n+1} {\lambda_\alpha}^{-1}, \qquad {g'}^{(n+1)\overline{n+1}} \geq \lambda_n^{-1}.$$

Thus,

$$\begin{aligned}
\frac{\epsilon}{4}\sum_{\alpha=1}^{n+1} {g'}^{\alpha\overline{\alpha}} + N{g'}^{(n+1)\overline{n+1}} &\geq \frac{\epsilon}{4}\sum_{\alpha=1}^{n} {\lambda_\alpha}^{-1} + (N + \frac{\epsilon}{4}){\lambda_{n+1}}^{-1} \\
&\geq (n+1)\frac{\epsilon}{4}N^{\frac{1}{n+1}}(\lambda_1 \cdot \lambda_2 \cdots \lambda_{n+1})^{-\frac{1}{n+1}} \\
&= C_2 N^{\frac{1}{n+1}}.
\end{aligned}$$

Choose N large enough so that

$$-C_2 N^{\frac{1}{n+1}} + (n+1) + sC_1 < -\frac{\epsilon}{4}.$$

Choose s small enough so that $s \cdot C_1 \leq \frac{\epsilon}{4}$. Then

$$\mathcal{L}\nu \leq -\frac{\epsilon}{4}\left(1 + \sum_{\alpha=1}^{n+1} {g'}^{\alpha\overline{\alpha}}\right).$$

From now on we fix N. Observe that $\triangle(h - \varphi_0) < -2\epsilon$. Then there exists a constant C_0 which depends only on g such that $h - \varphi_0 > C_0\, x$ near $\partial(V \times \mathbf{R})$. Choose δ small enough so that

$$s(h - \varphi_0) - Nx^2 \geq (sC_0 - N\delta)x \geq 0.$$

Then $\nu \geq 0$ in $\partial\Omega_\delta$. q.e.d.

Proof of Lemma 3. Let $M = \max(|\nabla\varphi|_g + 1)$. Choose $A \gg B \gg C, C_1$. In addition, choose A, B as big multiples of M. Notice that $|D\varphi| \leq 2M$ in Ω_δ. For δ fixed as in Lemma 4, we have $B\delta^2 - |D(\varphi - \varphi_0)| > 0$. Consider $w = A\,\nu + B\,|z|^2 + D(\varphi - \varphi_0)$. Then $w \geq 0$ in $\partial\Omega_\delta$ and $w(0) = 0$. Moreover,

$$\mathcal{L}w \leq \left(-\frac{\epsilon A}{4} + 2B + C\right)\left(1 + \sum_{\alpha=1}^{n+1} {g'}^{\alpha\overline{\alpha}}\right) < 0.$$

The Maximum Principle implies that $w \geq 0$ in Ω_δ. Since $w(0) = 0$, we have $\frac{\partial w}{\partial x} \geq 0$. In other words,

$$\frac{\partial}{\partial x} D\varphi(0) < C_3 \cdot M$$

for some uniform constant C_3. Since D is any 1st order constant operator near $\partial(V \times \mathbf{R})$, replacing D with $-D$, we get

$$-\frac{\partial}{\partial x} D\varphi(0) < C_3 \cdot M.$$

On the other hand, since $\partial\mathbf{R}$ is given by $x = 0$ in our special case, we then have the trivial estimate:

$$\frac{\partial}{\partial y} D(\varphi - \varphi_0)(0) = 0.$$

Therefore,

$$\left| \frac{\partial}{\partial z_{n+1}} D\varphi(0) \right| < C_3 \cdot M$$

Lemma 3 follows from here directly. q.e.d.

3.2 Blowing up analysis

Lemma 5. *Any bounded weakly sub-harmonic function on the two dimensional plane is a constant.*

This is a standard fact in geometric analysis. We will omit the proof here. Note that the lemma is false if the dimension is no less than 3.

The essence of blowing up analysis is to use a "microscope" to analyze what happens in a small neighborhood via rescaling. Hence it doesn't make any difference what the global structure of the background metric is, or what the metric is. Under rescaling, everything become Euclidean anyway. We might as well view the manifold as a domain in Euclidean space. We will use the variable x to denote the position in $V \times \mathbf{R}$.

Proof of Theorem 2. Suppose $\frac{1}{\epsilon_i} = \max_{V \times \mathbf{R}} |\nabla \varphi_i|_g \to \infty$. We want to draw a contradiction from this statement.

Suppose $|\nabla \varphi_i|_g(x_i) = \frac{1}{\epsilon_i}$. By Theorem 1, we have $\max_{V \times \mathbf{R}} \triangle \varphi_i \leq \dfrac{1}{\epsilon_i^2}$. Choose a convergent subsequence of x_i such that $x_i \to \underline{x}$. Choose a tiny neighborhood $B_\delta(\underline{x})$ of $\underline{x}$ so that $g_{\alpha\overline{\beta}}(\underline{x}) = \delta_{\alpha\overline{\beta}}$ and g is essentially an identity matrix in $B_\delta(\underline{x})$. For simplicity, let us pretend that g is a Euclidean metric in $B_\delta(\underline{x})$. There are two cases to consider: the first case is when $\underline{x} \in \partial(V \times \mathbf{R})$ and the 2nd case is when $\underline{x}$ is in the interior of $V \times \mathbf{R}$.

We define the blowing up sequence as

$$\widetilde{\varphi}_i(x) = \varphi_i(x_i + \epsilon_i x), \forall x \in B_{\frac{\delta}{\epsilon_i}}(0).$$

Then $|\nabla \widetilde{\varphi}_i(0)| = 1$ and

$$\max_{B_{\frac{\delta}{\epsilon_i}}(0)} |\nabla \widetilde{\varphi}_i| \leq 1, \qquad \text{and} \max_{B_{\frac{\delta}{\epsilon_i}}(0)} |\triangle \widetilde{\varphi}_i| \leq C.$$

Observe $\varphi_0 \leq \varphi_i \leq h$ ($\forall i$). Rescale φ_0 and h accordingly:

$$\widetilde{\varphi}_0(x) = \varphi_0(x_i + \epsilon_i x), \qquad \widetilde{h}(x) = h(x_i + \epsilon_i x), \;\; \forall\, x \;\in B_{\frac{\delta}{\epsilon_i}}(0).$$

Thus $\lim_{i\to\infty} \widetilde{\varphi}_0(x) = \varphi_0(\underline{x})$ and $\lim_{i\to\infty} \widetilde{h}(x) = h(\underline{x})$. Moreover,

$$\widetilde{\varphi}_0 \leq \widetilde{\varphi}_i \leq \widetilde{h}, \qquad \forall i = 1, 2, \ldots. \tag{10}$$

There exist a subsequence of $\widetilde{\varphi}_i$ and a limit function $\widetilde{\varphi}$ in C^{n+1} (or half plane in case $\underline{x}$ is in the boundary) such that in any fixed ball $B_l(0)$ (or half ball if $\underline{x}$ is in the boundary) we have $\widetilde{\varphi}_i \to \widetilde{\varphi}$ in $C^{1,\eta}$ in the ball $B_l(0)$ (or half ball) for any $0 < \eta < 1$. This implies

$$|\nabla \widetilde{\varphi}(0)| = 1. \tag{11}$$

In addition, inequality (10) holds in the limit:

$$\varphi_0(\underline{x}) \leq \widetilde{\varphi}(x) \leq h(\underline{x}), \qquad \forall\, x. \tag{12}$$

Case 1. Suppose $\underline{x} \in \partial(V \times \mathbf{R})$. Then $h(\underline{x}) = \varphi_0(\underline{x})$. Inequality (12) implies that $\widetilde{\varphi}$ is a constant function in its domain. In particular, we have $|\nabla \widetilde{\varphi}(x)| \equiv 0$. This contradicts our assertion (11). Thus the theorem is proved in this case.

Case 2. Suppose $\underline{x}$ is in the interior of $V \times \mathbf{R}$. Then $\widetilde{\varphi}(x)$ is a well defined $C^{1,\eta}$ and bounded function in C^{n+1}. We claim that this function is weakly sub-harmonic on any complex line through the origin. If this claim is true, then Lemma 5 says it must be constant for any complex line through the origin. Therefore, the function itself must be a constant as well. Thus $|\nabla \widetilde{\varphi}| \equiv 0$. It again contradicts our assertion (11). Thus the theorem is proved also, provided we can prove this claim.

Without loss of generality, we consider the complex line T to be

$$z_2 = z_3 = \cdots = z_{n+1} = 0.$$

Observe that (near $\underline{x}$) the following holds

$$0 < \left(\delta_{\alpha\overline{\beta}} + \frac{\partial^2 \varphi_i}{\partial z_\alpha \partial \overline{z}_\beta}\right)_{(n+1)(n+1)} < \frac{C}{\epsilon_i^2}(\delta_{\alpha\overline{\beta}})_{(n+1)(n+1)}, \forall\, i.$$

After rescaling, we have

$$0 < \epsilon_i^2 \cdot (\delta_{\alpha\overline{\beta}})_{(n+1)(n+1)} + \left(\frac{\partial^2 \widetilde{\varphi}_i}{\partial z_\alpha \partial \overline{z}_\beta}\right)_{(n+1)(n+1)} < C \cdot (\delta_{\alpha\overline{\beta}})_{(n+1)(n+1)}.$$

Restricting this to a complex line T, we have

$$0 < \epsilon_i^2 + \frac{\partial^2 \widetilde{\varphi}_i}{\partial z_1 \partial \overline{z}_1} < C. \tag{13}$$

Thus one can choose a subsequence of $\widetilde{\varphi}_i$ which converges $C^{1,\eta}(0 < \eta < 1)$ locally in T to some function ψ. Since the convergence is in $C^{1,\eta}$, we have $\psi = \widetilde{\varphi}|_T$; i.e., ψ is the restriction of $\widetilde{\varphi}$ to this complex line T. By taking a weak limit in inequality (13), the $\widetilde{\varphi}_i|_T$ weakly converge to ψ in the $H^{2,p}_{loc}$ topology for any $p > 1$. Hence, ψ is a weakly sub-harmonic function by taking a weak limit in inequality (13). Therefore $\psi = \widetilde{\varphi}|_T$ is a constant by Lemma 5. Our claim is then proved. q.e.d.

4. Uniqueness of weak C^0 geodesic

Notation follows from previous section.

Definition 1. A function φ is *generalized-pluri-subharmonic* in $V \times \mathbf{R}$ if $\displaystyle\sum_{\alpha,\beta=1}^{n+1} \left(g_{\alpha\overline{\beta}} + \frac{\partial^2 \varphi}{\partial z_\alpha \partial \overline{z}_\beta}\right) dz_\alpha d\,\overline{z}_\beta$ defines a strictly positive Kähler metric on $V \times \mathbf{R}$.

Definition 2. A continuous function φ in $V \times \mathbf{R}$ is a weak C^0 solution to the degenerate Monge-Ampère equation (3) with prescribed boundary data φ_0 if the following statement is true: $\forall\, \epsilon > 0$, there exists a pluri subharmonic function $\widetilde{\varphi}$ in $V \times \mathbf{R}$ such that $|\varphi - \widetilde{\varphi}| < \epsilon$ and $\widetilde{\varphi}$ solves equation (5) with some positive function $0 < f < \epsilon$ at $t = 1,$ and with the same boundary data φ_0.

Clearly, the solution we obtain through the continuity method is a weak C^0 solution of equation (3).

Theorem 4. *Suppose φ_1, φ_2 are two C^0 weak solutions to the degenerate Monge-Ampère equation with prescribed boundary conditions h_1, h_2. Then*

$$\max_{V \times \mathbf{R}} |\varphi_1 - \varphi_2| \leq \max_{\partial(V \times \mathbf{R})} |h_1 - h_2|.$$

Corollary 2. *The solution to the degenerate Monge-Ampère equation is unique as soon as the boundary data is fixed.*

Proof. Suppose ϕ_1, ϕ_2 are two approximate *generalized-plurisubharmonic* solutions of φ_1, φ_2 in the sense of Definition 2. More precisely,

$$\det\left(g + \frac{\partial^2 \phi_i}{\partial z_\alpha \partial \overline{z}_\beta}\right) = f_i \cdot \det(g) > 0 \;\text{ in } V \times \mathbf{R};$$
$$\text{and } \;\phi_i = h_i \text{ in } \;\partial(V \times \mathbf{R}), \quad i = 1, 2$$

such that $\max\limits_{V \times \mathbf{R}}(\,|\varphi_1 - \phi_1| + f_1)$ and $\max\limits_{V \times \mathbf{R}}(\,|\varphi_2 - \phi_2| + f_2)$ can be made as small as we wanted.

$\forall \epsilon > 0$, we want to show

$$\max_{V \times \mathbf{R}} (\varphi_1 - \varphi_2) \leq \max_{V \times \mathbf{R}} (h_1 - h_2) + 2\epsilon.$$

Choose f_1 such that $0 < f_1 < \epsilon$ and $\max\limits_{V \times \mathbf{R}} |\varphi_1 - \phi_1| < \epsilon$. Choose f_2 such that $0 < f_2 \leq \frac{1}{2} \min\limits_{V \times \mathbf{R}} f_1 < \epsilon$ and $\max\limits_{V \times \mathbf{R}} |\varphi_2 - \phi_2| < \epsilon$. Then ϕ_1 is a sub-solution to ϕ_2 (thus $\phi_1 < \phi_2$) if $h_1 = h_2$. In general, we have

$$\max_{V \times \mathbf{R}} (\phi_1 - \phi_2) \leq \max_{\partial(V \times \mathbf{R})} (h_1 - h_2).$$

Thus

$$\begin{aligned}\max_{V\times\mathbf{R}}(\varphi_1-\varphi_2) &= \max_{V\times\mathbf{R}}(\varphi_1-\phi_1)+\max_{V\times\mathbf{R}}(\phi_1-\phi_2)+\max_{V\times\mathbf{R}}(\phi_2-\varphi_2)\\ &\le \epsilon+\max_{\partial(V\times\mathbf{R})}(h_1-h_2)+\epsilon\\ &= \max_{\partial(V\times\mathbf{R})}(h_1-h_2)+2\epsilon.\end{aligned}$$

Change the role of φ_1 and φ_2, we obtain

$$\max_{V\times\mathbf{R}}(\varphi_2-\varphi_1)\le\max_{\partial(V\times\mathbf{R})}(h_2-h_1)+2\epsilon.$$

Hence

$$\max_{V\times\mathbf{R}}|\varphi_1-\varphi_2|\le\max_{\partial(V\times\mathbf{R})}|h_1-h_2|+2\epsilon.$$

Let $\epsilon\to 0$, we obtain the desired result. q.e.d.

5. The space of Kähler metric is a metric space—Triangle inequality

In this section, we want to prove that the space of Kähler metric is a metric space, and the $C^{1,1}$ geodesic between any two points realizes the global minimal length over all possible paths. To prove this claim, one inevitably needs to take derivatives of lengths for a family of $C^{1,1}$ geodesics. However, the length for a $C^{1,1}$ geodesic is just barely defined (the integrand is in L^p space). In general, one cannot take derivatives. Therefore, we must find ways to circumvent this difficulty.

Definition 3. A path $\varphi(t)(0<t<1)$ in the space of Kähler metrics is a convex path if $\varphi(t)$ is a *generalized-pluri-subharmonic* function in $V\times(I\times S^1)$ (cf. Definition 1).

Suppose $\mathrm{vol}(t)(0\le t\le 1)$ is a family of strictly positive volume form in V such that

$$\int_V \mathrm{vol}(t)=\int_V \det g.$$

The notion of ϵ-approximate geodesic is defined with respect to such a volume form:

Definition 4. A convex path $\varphi(t)$ in the space of Kähler metrics is called an ϵ-approximate geodesic if the following holds:

$$\left(\varphi'' - |\nabla\varphi'|^2_{g(t)}\right) \det g(t) = \epsilon \cdot \mathrm{vol}(t),$$

where $g(t)_{\alpha\overline{\beta}} = g_{\alpha\overline{\beta}} + \frac{\partial^2\varphi}{\partial z_\alpha \partial \bar{z}_\beta}$ $(1 \le \alpha, \beta \le n)$.

Remark 2. The definition is really independent of these volume forms since we only care what happens when ϵ is really small. For convenience, sometimes we choose $\mathrm{vol}(t) \equiv \det g$ (a volume form independent of t).

Lemma 6. *Suppose $\varphi(t)(0 \le t \le 1)$ is an ϵ-approximate geodesic. Define the energy element as $E(t) = \int_V \varphi'(t)^2 d\, g(t)$. Then*

$$\max_t \left|\frac{d\,E}{d\,t}\right| \le 2\,\epsilon \cdot \max_{V\times I}\, |\varphi'(t)| \cdot M,$$

where $M = \int_V \det g$ is the total volume of V which depends only on the Kähler class.

Proof.

$$\begin{aligned}\left|\frac{d\,E}{d\,t}\right| &= \left|\int_V (2\varphi''\varphi' + {\varphi'}^2 \triangle_{g(t)}\varphi')\; d\,g(t)\right| \\ &= 2\,\left|\int_V \varphi'(\varphi'' - \frac{1}{2}|\nabla\varphi'|^2_{g(t)}) \det g(t)\right| \\ &= 2\,\left|\int_V \varphi'\,\epsilon\;\mathrm{vol}(t)\right| \le 2\,\epsilon \cdot \max_{V\times I}\,|\varphi'(t)| \cdot M. \qquad \text{q.e.d.}.\end{aligned}$$

Proposition 2. *Suppose $\varphi(t)$ is a $C^{1,1}$ geodesic in $\mathcal{H}$ from 0 to φ and $I(\varphi) = 0$. Then the following inequality holds*

$$\int_0^1 \sqrt{\int_V {\varphi'}^2 d\mu_{\varphi_t}}\, dt \ge M^{-1}\left(\max\Big(\int_{\varphi>0}\varphi d\mu_\varphi, -\int_{\varphi<0}\varphi d\mu_0\Big)\right).$$

In other words, the length of any $C^{1,1}$ geodesic is strictly positive.

Proof. As in Definition 4, suppose $\varphi(\epsilon, t)$ is a ϵ-approximate geodesic between 0 and φ. (We will drop the dependence of ϵ in this proof since no confusion shall arise from this omission). First of all, from the definition of ϵ-approximate geodesic, we have

$$\varphi'' - \frac{1}{2}|\nabla\varphi'|^2_{g(t)} > 0.$$

In particular, we have $\varphi''(t) \geq 0$. Thus

$$\varphi'(0) \leq \varphi \leq \varphi'(1). \tag{14}$$

Consider $f(t) = I(t\varphi), t \in [0,1]$. Then $f'(t) = \int_V \varphi \, d\mu_{t\varphi}$ and

$$f''(t) = \int_V \varphi \bigtriangleup_{g(t\varphi)} \varphi \, d\mu_{t\varphi} \leq 0.$$

Thus, we have $f'(0) \geq \frac{f(1)-f(0)}{1-0} \geq f'(1)$. In other words, we have

$$\int_V \varphi \; d \; \mu_0 \geq I(\varphi) \geq \int_V \varphi \; d \; \mu_\varphi.$$

Since we assume $I(\varphi) = 0$, and φ is not identically zero, it must take both positive and negative values. Then the length (or energy) of the geodesic is given by

$$E = \int_V {\varphi'}^2 d\mu_{\varphi_t},$$

for any $t \in [0,1]$. In particular, taking $t = 1$,

$$\sqrt{E(1)} \geq M^{-1/2} \int_V |\varphi'(1)| d\mu_\varphi > M^{-1/2} \int_{\varphi'(1)>0} \varphi'(1) d\mu_\varphi,$$

where M is the volume of V (which is of course the same for all metrics in $\mathcal{H}$). It follows from inequality (14) that

$$\int_{\varphi'(1)>0} \varphi' \; d\mu_\varphi \geq \int_{\varphi>0} \varphi \; d\mu_\varphi,$$

where the last term is strictly positive by the remarks above, and depends only on φ and not on the geodesic. A similar argument gives

$$\sqrt{E(0)} > -M^{-1/2} \int_{\varphi<0} \varphi \; d\mu_0.$$

The previous lemma implies that for any $t_1, t_2 \in [0,1]$, we have

$$|E(t_1) - E(t_2)| < C \cdot \epsilon$$

for some constant C independent of ϵ. Thus

$$\sqrt{E(t)} \geq M^{-1/2} \max\left(\int_{\varphi>0} \varphi d\mu_\varphi, -\int_{\varphi<0} \varphi d\mu_0\right) - C \cdot \epsilon.$$

Now integrate from $t = 0$ to 1 and let $\epsilon \to 0$. Then

$$\int_0^1 \sqrt{\int_V {\varphi'}^2 d\mu_\varphi}\, d\,t \geq M^{-1/2} \max\left(\int_{\varphi>0} \varphi d\mu_\varphi, -\int_{\varphi<0} \varphi d\mu_0\right).$$

This proposition is proved. q.e.d.

Remark 3. This proposition verifies Donaldson's second conjecture. Unfortunately, it does not imply $\mathcal{H}$ is a metric space automatically since the geodesic is not sufficiently differentiable. On the other hand, one can easily verify that the $C^{1,1}$ geodesic minimizes length over all possible convex curves between the two end points. To show that it minimizes length over all possible curves, not just convex ones, we need to prove that the triangle inequality is satisfied by the geodesic distance (see Definition 5 below).

Definition 5. Let φ_1, φ_2 be two distinct points in the space of metrics. According to Theorem 3 and Corollary 2, there exists a unique geodesic connecting these two points. Define the geodesic distance as the length of this geodesic, denoted it by $d(\varphi_1, \varphi_2)$.

Theorem 5. *Suppose $C : \varphi(s) : [0,1] \to \mathcal{H}$ is a smooth curve in $\mathcal{H}$. Suppose p is a base point of $\mathcal{H}$. For any s, the geodesic distance from p to $\varphi(s)$ is no greater than the sum of the geodesic distance from p to $\varphi(0)$ and the length from $\varphi(0)$ to $\varphi(s)$ along this curve C. In particular, if $C : \varphi(s) : [0,1] \to \mathcal{H}$ is a geodesic, then the geodesic distance satisfies*

$$d(0, \varphi(1)) \leq d(0, \varphi(0)) + d(\varphi(0), \varphi(1)).$$

Lemma 7 (Geodesic approximation lemma). *Suppose $C_i : \varphi_i(s) : [0,1] \to \mathcal{H} (i = 1,2)$ are two smooth curves in $\mathcal{H}$. For ϵ_0 small enough, there exist two parameter smooth families of curves $C(s,\epsilon) : \phi(t,s,\epsilon) : [0,1] \times [0,1] \times (0,\epsilon_0] (0 \leq t, s \leq 1, 0 < \epsilon \leq \epsilon_0)$ such that the following properties hold:*

1. *For any fixed s and $\epsilon, C(s,\epsilon)$ is an ϵ-approximate geodesic from $\varphi_1(s)$ to $\varphi_2(s)$. More precisely, $\phi(z,t,s,\epsilon)$ solves the corresponding Monge-Ampère equation:*

$$\begin{aligned} &\det\left(g+\frac{\partial^2\phi}{\partial z_\alpha \partial \overline{z}_\beta}\right)=\epsilon\cdot\det(g) \ \text{ in } V\times\mathbf{R}; \\ &\text{and } \ \phi(z',0,s,\epsilon)=\varphi_1(z',s), \ \phi(z',1,s,\epsilon)=\varphi_2(z',s). \end{aligned} \tag{15}$$

   Here we follow notations in Section 3, and $z_{n+1}=t+\sqrt{-1}\,\theta$ where the dependence of ϕ on θ is trivial.

2. *There exists a uniform constant C (which depends only on φ_1,φ_2) such that*

$$|\phi|+\left|\frac{\partial\phi}{\partial s}\right|+\left|\frac{\partial\phi}{\partial t}\right|<C;\qquad 0\le\frac{\partial^2\phi}{\partial t^2}<C,\qquad \frac{\partial^2\phi}{\partial s^2}<C.$$

3. *For fixed s, let $\epsilon\to 0$: the convex curve $C(s,\epsilon)$ converges to the unique geodesic between $\varphi_1(s)$ and $\varphi_2(s)$ in the weak $C^{1,1}$ topology.*

4. *Define the energy element along $C(s,\epsilon)$ by*

$$E(t,s,\epsilon)=\int_V\left|\frac{\partial\phi}{\partial t}\right|^2 d\,g(t,s,\epsilon),$$

   where $g(t,s,\epsilon)$ is the corresponding Kähler metric define by $\phi(t,s,\epsilon)$. Then there exists a uniform constant C such that

$$\max_{t,s}\left|\frac{\partial\,E}{\partial\,t}\right|\le\epsilon\cdot C\cdot M.$$

   In other words, the energy/length element converges to a constant along each convex curve if $\epsilon\to 0$.

Proof. Everything follows from Theorems 3, 4 and Lemma 6 except the bound on $|\frac{\partial\phi}{\partial s}|$ and an upper bound on $\frac{\partial^2\phi}{\partial s^2}$ which follow from the maximum principle directly since

$$\mathcal{L}\left(\frac{\partial\phi}{\partial s}\right)=0$$

and

$$\mathcal{L}\left(\frac{\partial^2\phi}{\partial s^2}\right)=tr_{g'}\left\{\text{Hess}\frac{\partial\phi}{\partial s}\cdot\text{Hess}\frac{\partial\phi}{\partial s}\right\}\ge 0. \qquad \text{q.e.d.}$$

Proof of Theorem 5. Apply the geodesic approximation lemma in the special case that $\varphi_1(s) \equiv p$. We follow notations in the previous lemma. For ϵ_0 small enough, there exist two parameter smooth families of curves $C(s,\epsilon) : \phi(t,s,\epsilon) : [0,1]\times[0,1]\times(0,\epsilon_0](0 \leq t,s \leq 1, 0 < \epsilon \leq \epsilon_0)$ such that

$$\det\left(g + \frac{\partial^2 \phi}{\partial z_\alpha \partial \overline{z}_\beta}\right) = \epsilon \cdot \det(g), \quad \text{in } V \times \mathbf{R};$$
$$\text{and } \ \phi(z',0,s,\epsilon) = 0, \ \phi(z',1,s,\epsilon) = \varphi(z',s).$$

Denote the length of the curve $\phi(t,s,\epsilon)$ from p to $\varphi(s)$ by $L(s,\epsilon)$, denote the geodesic distance between p and $\varphi(s)$ by $L(s)$, and denote the length from $\varphi(0)$ to $\varphi(s)$ along curve C by $l(s)$. Clearly,

$$l(s) = \int_0^s \sqrt{\int_V |\frac{\partial \varphi}{\partial \tau}|^2 d\, g(\tau)}\, d\,\tau,$$

where $g(\tau)$ is the Kähler metric defined by $\varphi(\tau)$, and

$$L(s,\epsilon) = \int_0^1 \sqrt{E(t,s,\epsilon)}\, d\,t = \int_0^1 \sqrt{\int_V |\frac{\partial \phi}{\partial t}|^2 d\, g(t,s,\epsilon)}\, d\,t,$$

and $\lim_{\epsilon\to 0} L(s,\epsilon) = L(s)$. Define $F(s,\epsilon) = L(s,\epsilon) + l(s)$ and $F(s) = L(s) + l(s)$. What we need to prove is : $F(1) \geq F(0)$. This will be done if we can show that $F'(s) \geq 0, \ \forall\ s \ \in [0,1]$. The last statement would be straightforward if the deformation of geodesics is C^1. Since we do not have that, we need to take derivatives of $F(s,\epsilon)$ for $\epsilon > 0$ instead. Notice $\frac{\partial \phi}{\partial s} = 0$ at $t = 0$ in the following deduction:

$$
\begin{aligned}
\frac{d\,L(s,\epsilon)}{d\,s} &= \int_0^1 \frac{1}{2}\; E(t,s,\epsilon)^{-\frac{1}{2}} \int_V \left(2\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial t\partial s}\right.\\
&\qquad\qquad\qquad\qquad \left. + \left(\frac{\partial\phi}{\partial t}\right)^2 \triangle_{g(t,s,\epsilon)}\frac{\partial\phi}{\partial s}\right) dg(t,s,\epsilon)\; d\,t\\
&= \int_0^1 \; E(t,s,\epsilon)^{-\frac{1}{2}}\left\{\frac{\partial}{\partial t}\;\left(\int_V \frac{\partial\phi}{\partial t}\;\frac{\partial\phi}{\partial s}\; d\,g(t,s,\epsilon)\right)\right.\\
&\qquad\qquad \left. - \int_V \frac{\partial\phi}{\partial s}\;\left(\frac{\partial^2\phi}{\partial t^2} - \frac{1}{2}\left|\nabla\frac{\partial\phi}{\partial t}\right|^2\right) d\,g(t,s,\epsilon)\right\} d\,t\\
&= \left.\left\{E(t,s,\epsilon)^{-\frac{1}{2}}\;\int_V \frac{\partial\phi}{\partial t}\;\frac{\partial\phi}{\partial s}\; d\,g(t,s,\epsilon))\right\}\right|_0^1\\
&\quad - \int_0^1 \left\{E(t,s,\epsilon)^{-\frac{1}{2}}\int_V \frac{\partial\phi}{\partial s}\;\left(\frac{\partial^2\phi}{\partial t^2} - \frac{1}{2}\left|\nabla\frac{\partial\phi}{\partial t}\right|^2\right) d\,g(t,s,\epsilon)\right\} d\,t\\
&\quad + \int_0^1 \left\{E(t,s,\epsilon)^{-\frac{3}{2}}\;\int_V \frac{\partial\phi}{\partial t}\;\frac{\partial\phi}{\partial s}\; d\,g(t,s,\epsilon)\right.\\
&\qquad\qquad \left. \cdot\int_V \frac{\partial\phi}{\partial t}\;\left(\frac{\partial^2\phi}{\partial t^2} - \frac{1}{2}\left|\nabla\frac{\partial\phi}{\partial t}\right|^2\right) d\,g(t,s,\epsilon)\right\} d\,t\\
&= \int_V \frac{\partial\phi(1,s,\epsilon)}{\partial t}\;\frac{d\,\varphi}{d\,s}\; d\,g(s)\cdot\left\{\int_V \left|\frac{\partial\phi(1,s,\epsilon)}{\partial t}\right|^2 d\,g(s)\right\}^{-\frac{1}{2}}\\
&\quad - \int_0^1 \left\{E(t,s,\epsilon)^{-\frac{1}{2}}\;\int_V \frac{\partial\phi}{\partial s}\;\epsilon\cdot\det g\right\}\; d\,t\\
&\quad + \int_0^1 \left\{E(t,s,\epsilon)^{-\frac{3}{2}}\;\int_V \frac{\partial\phi}{\partial t}\;\frac{\partial\phi}{\partial s}\; d\,g(t,s,\epsilon)\cdot\int_V \;\frac{\partial\phi}{\partial t}\;\epsilon\cdot\det g\right\} d\,t.
\end{aligned}
$$

Observe that by the Schwartz inequality, we have

$$
\begin{aligned}
\frac{d\,l(s)}{d\,s} &= \sqrt{\int_V \left|\frac{\partial\varphi}{\partial s}\right|^2 d\;g(s)}\\
&\geq\; -\;\int_V \frac{\partial\phi(1,s,\epsilon)}{\partial t}\;\frac{d\,\varphi}{d\,s}\; d\,g(s)\cdot\left\{\int_V \left|\frac{\partial\phi(1,s,\epsilon)}{\partial t}\right|^2 d\,g(s)\right\}^{-\frac{1}{2}}.
\end{aligned}
$$

Observe that $F(s,\epsilon) = L(s,\epsilon) + l(s)$. Thus

$$\begin{aligned}\frac{d\,F(s,\epsilon)}{d\,s} \geq & -\int_0^1 \left\{ E(t,s,\epsilon)^{-\frac{1}{2}} \int_V \frac{\partial\phi}{\partial s}\,\epsilon\cdot\det g\right\} d\,t \\ & + \int_0^1 \left\{ E(t,s,\epsilon)^{-\frac{3}{2}} \int_V \frac{\partial\phi}{\partial t}\,\frac{\partial\phi}{\partial s}\,d\,g(t,s,\epsilon) \right. \\ & \qquad \left. \cdot \int_V \frac{\partial\phi}{\partial t}\,\epsilon\cdot\det \right\} d\,t.\end{aligned}$$

Integrating from 0 to $s \in (0,1]$, we obtain

$$\begin{aligned}F(s,\epsilon) - F(0,\epsilon) \geq & -\int_0^s \int_0^1 \left\{ E(t,\tau,\epsilon)^{-\frac{1}{2}} \int_V \frac{\partial\phi}{\partial \tau}\,\epsilon\cdot\det g\right\} d\,t\,d\,\tau \\ & + \int_0^s \int_0^1 \left\{ E(t,\tau,\epsilon)^{-\frac{3}{2}} \int_V \frac{\partial\phi}{\partial t}\,\frac{\partial\phi}{\partial \tau}\,d\,g(t,\tau,\epsilon) \right. \\ & \qquad \left. \cdot \int_V \frac{\partial\phi}{\partial t}\,\epsilon\cdot\det g \right\} d\,t\,d\,\tau \\ \geq & -C\epsilon\end{aligned}$$

for some large constant C depending only on $(V \times \mathbf{R}, g)$ and the initial curve $C : \varphi(s) : [0,1] \to \mathcal{H}$. Now taking the limit as $\epsilon \to 0$, we have $F(s) \geq F(0)$. In other words, the geodesic distance from p to $\varphi(s)$ is no greater than the sum of the geodesic distance from p to $\varphi(0)$ and the length from $\varphi(0)$ to $\varphi(s)$ along this curve C. q.e.d.

Corollary 3. *The geodesic distance between any two metrics realizes the absolute minimum of the lengths over all possible paths.*

Proof. For any smooth curve $C : \varphi(s) : [0,1] \to \mathcal{H}$, we want to show that the geodesic distance between the two end points $\varphi(0)$ and $\varphi(1)$ is no greater than the length of C. However, this follows directly from Theorem 5 by taking $p = \varphi(1)$ and $s = 1$. q.e.d.

Theorem 6. *For any two Kähler potentials φ_1, φ_2, the minimal length $d(\varphi_1, \varphi_2)$ over all possible paths which connect these two Kähler potentials is strictly positive, as long as $\varphi_1 \neq \varphi_2$. In other words, $(\mathcal{H}, d)$ is a metric space. Moreover, the distance function is at least C^1.*

Proof. Immediately from Corollary 3 and Proposition 2, we see that $(\mathcal{H}, d)$ is a metric space. Now we want to prove the differentiability of the distance function. From the proof of Theorem 5, we have

$$
\begin{aligned}
\frac{d\,L(s,\epsilon)}{d\,s} = \int_V \frac{\partial\phi(1,s,\epsilon)}{\partial t}\,\frac{d\,\varphi}{d\,s}\,d\,g(s)& \\
\cdot\left\{\int_V \left|\frac{\partial\phi(1,s,\epsilon)}{\partial t}\right|^2 d\,g(s)\right\}^{-\frac{1}{2}}& \\
-\int_0^1 \left\{E(t,s,\epsilon)^{-\frac{1}{2}}\int_V \frac{\partial\phi}{\partial s}\,\epsilon\cdot\det g\right\}\,d\,t& \\
+\int_0^1 \left\{E(t,s,\epsilon)^{-\frac{3}{2}}\int_V \frac{\partial\phi}{\partial t}\,\frac{\partial\phi}{\partial s}\,d\,g(t,s,\epsilon)\right.& \\
\left.\cdot\int_V \frac{\partial\phi}{\partial t}\,\epsilon\cdot\det g\right\}\,d\,t.&
\end{aligned}
$$

Integrating this from s_1 to s_2 and dividing it by $s_2 - s_1$, gives

$$
\begin{aligned}
&\left|\frac{L(s_2,\epsilon)-L(s_1,\epsilon)}{s_2-s_1} - \frac{1}{s_2-s_1}\int_{s_1}^{s_2}\int_V \frac{\partial\phi(1,s,\epsilon)}{\partial t}\,\frac{d\,\varphi}{d\,s}\,d\,g(s)\right. \\
&\qquad\qquad\left.\cdot\left\{\int_V \left|\frac{\partial\phi(1,s,\epsilon)}{\partial t}\right|^2 d\,g(s)\right\}^{-\frac{1}{2}} d\,s\right| \\
&\le \frac{1}{s_2-s_1}\int_{s_1}^{s_2}\int_0^1 \left\{E(t,s,\epsilon)^{-\frac{1}{2}}\int_V \left|\frac{\partial\phi}{\partial s}\right|\,\epsilon\cdot\det g\right\}\,d\,t\,d\,s \\
&\quad + \frac{1}{s_2-s_1}\int_{s_1}^{s_2}\int_0^1 \left\{E(t,s,\epsilon)^{-\frac{3}{2}}\int_V \left|\frac{\partial\phi}{\partial t}\right|\,\left|\frac{\partial\phi}{\partial s}\right|\,d\,g(t,s,\epsilon)\right. \\
&\qquad\qquad\left.\cdot\int_V \left|\frac{\partial\phi}{\partial t}\right|\,\epsilon\cdot\det g\right\}\,d\,t\,d\,s \\
&\le C\epsilon.
\end{aligned}
$$

Let $\epsilon \to 0$, and $s_2 \to s_1$. Then we have

$$\begin{aligned}\lim_{s_2\to s_1}\frac{L(s_2)-L(s_1)}{s_2-s_1} &= \lim_{s_2\to s_1}\frac{1}{s_2-s_1}\int_{s_1}^{s_2}\int_V \frac{\partial\phi(1,s)}{\partial t}\,\frac{d\,\varphi}{d\,s}\,d\,g(s)\\ &\qquad\cdot\left\{\int_V\left|\frac{\partial\phi(1,s)}{\partial t}\right|^2 d\,g(s)\right\}^{-\frac12} d\,s\\ &= \int_V \frac{\partial\phi(1,s)}{\partial t}\,\frac{d\,\varphi}{d\,s}\,d\,g(s)\\ &\qquad\cdot\left\{\int_V\left|\frac{\partial\phi(1,s)}{\partial t}\right|^2 d\,g(s)\right\}^{-\frac12}.\end{aligned}$$

The distance function L is therefore a differentiable function. q.e.d.

6. Application: Uniqueness of extremal Kähler metrics if $C_1(V) < 0$ and $C_1(V) = 0$

In this section, we want to show that if $C_1(V) < 0$, or if $C_1(V) = 0$, then the extremal Kähler metric is unique in any Kähler class. Furthermore, if $C_1(V) \leq 0$, the extremal Kähler metric (if it exists) realizes the global minimum of the K energy functional in any Kähler class, thus giving an affirmative answer to a question raised by Tian Gang in this special case.

6.1 Uniqueness of c.s.c metric when $C_1(V) = 0$ and the lower bound of K energy for $C_1(V) \leq 0$

We should now introduce an important operator— the Lichernowicz operator $\mathcal{D}$. For any function h, $\mathcal{D}h = h_{,\alpha\beta}dz^\alpha \otimes dz^\beta$. If $\mathcal{D}h = 0$, then $\uparrow \overline{\partial} h = g^{\alpha\overline{\beta}}\frac{\partial h}{\partial \overline{\beta}}\frac{\partial}{\partial z_\alpha}$ is a holomorphic vector field. Now let us introduce the K energy. Like I_ρ, I, it is again defined by its derivatives and one should check it is well defined by verifying the second derivatives is symmetric (we will leave this to the reader). Let R be the scalar curvature of the metric $g = g_0 + \sqrt{-1}\partial\overline{\partial}\varphi$ and $\underline{R}$ be the average scalar curvature in the cohomology class. Let $\psi \in T_\varphi\mathcal{H}$. Then the variation of K energy of g in the direction ψ is:

$$\delta_\psi E = -\int_V (R - \underline{R})\cdot\psi\ \det g.$$

Along any smooth geodesic $\varphi(t) \in \mathcal{H}$, S. Donaldson shows

$$\frac{d^2 E}{d\,t^2} = \int_V |\mathcal{D}\varphi'(t)|_g^2 \ \det g.$$

Using this, Donaldson shows that the constant curvature metric is unique in each Kähler class if the smooth geodesic conjecture is true. Now we want to prove the uniqueness of constant curvature metrics in each Kähler class when $C_1(V) < 0$ or $C_1(V) = 0$, despite the fact we have not proved the smooth geodesic conjecture yet.

Theorem 7. *If either $C_1(V) < 0$ or $C_1(V) = 0$, then the constant curvature metric (if it exists) in any Kähler class must be unique.*

Proof. Notation follows from Section 5. Suppose $\varphi(t)$ is an ϵ-approximate geodesic. Then

$$\det g \left(\varphi'' - \frac{1}{2} |\nabla \varphi'|_g^2 \right) = \epsilon \cdot \det h,$$

where h is a given metric in the Kähler class such that $\mathrm{Ric}(h) < -ch$ if $C_1(V) < 0$ and $\mathrm{Ric}(h) \equiv 0$ if $C_1(V) = 0$. Let $f = \varphi'' - \frac{1}{2}|\nabla \varphi'|_g^2 \geq 0$. Then

$$\nabla \ln \frac{\det g}{\det h} = - \nabla \ln\, f \tag{16}$$

and

$$\frac{d}{d\,t}\left(\int_V \varphi'(t) \det g \right) = \int_V \ f \cdot \det g = \epsilon \cdot \int_V \ \det h. \tag{17}$$

Let E denote the K energy functional. Then

$$\frac{dE}{d\,t} = - \int_V \ (R - \underline{R}) \cdot \psi \ \det g.$$

A direct calculation yields

$$\begin{aligned} \frac{d^2 E}{d\,t^2} &= \int_V |\mathcal{D}\varphi'(t)|_g^2 \ \det g \\ &\quad - \int_V (\varphi'' - \frac{1}{2}|\nabla \varphi'|_g^2) \cdot R \ \det g + \epsilon \cdot \underline{R} \cdot \int_V \ \det h, \end{aligned} \tag{18}$$

where we already use Equation (17). Now the second term on the right-hand side of the above equation is:

$$
\begin{aligned}
-\int_V R\cdot f\ \det g &= \int_V \Delta_g \ln\det g \cdot f \cdot \det g \\
&= \int_V \Delta_g \frac{\ln\det g}{\ln\ \det h} \cdot f \cdot \det g + \int_V \Delta_g \ln\det h \cdot f \cdot \det g \\
&= -\int_V \nabla_g \frac{\ln\det g}{\ln\ \det h} \cdot \nabla \ln f\ \det g\ - \int_V tr_g(\mathrm{Ric}(h))\ f\ \det g \\
&= \int_V |\nabla f|_g^2 \frac{1}{f} \det g - \int_V tr_g(\mathrm{Ric}(h))\ f\ \det g.
\end{aligned}
$$

Integrating from $t = 0$ to 1 gives

$$
\begin{aligned}
\int_{V\times I} |\mathcal{D}\varphi'|_g^2\ \det g\, d\,t + \int_{V\times I} \frac{|\nabla f|^2}{f}\ \det g\, d\,t \\
- \int_{V\times I} tr_g(\mathrm{Ric}(h))\ f\ \det g\, d\,t \\
= \left.\frac{dE}{d\,t}\right|_0^1 - \epsilon \underline{R} \cdot \int_V \det h\, d\,t.
\end{aligned}
\tag{19}
$$

If $\varphi(0)$ and $\varphi(1)$ are both constant scalar curvature metrics, then $\frac{dE}{d\,t}\ |_0^1 = 0$ and

$$
\begin{aligned}
\int_{V\times I} |\mathcal{D}\varphi'|_g^2\ \det g\, d\,t + \int_{V\times I} \frac{|\nabla f|_g^2}{f}\ \det g\, d\,t \\
- \int_{V\times I} tr_g(\mathrm{Ric}(h))\ f\ \det g\, d\,t \\
= -\epsilon \underline{R} \cdot \int_V \det h\, d\,t.
\end{aligned}
\tag{20}
$$

Observing that $f \det\ g = \epsilon \cdot \det h$, then

$$
\begin{aligned}
\int_{V\times I} \frac{|\mathcal{D}\varphi'|_g^2}{f}\ \det h + \int_{V\times I} |\nabla \ln f|_g^2\ \det h - \int_{V\times I} tr_g(\mathrm{Ric}(h))\ \det h \\
= -\underline{R} \int_V \det h.
\end{aligned}
$$

If $C_1(V) = 0$, then $\underline{R} = 0$. Consequently

$$
\int_{V\times I} \frac{|\mathcal{D}\varphi'|_g^2}{f}\ \det h d\,t + \int_{V\times I} |\nabla \ln f|_g^2\ \det h\, d\,t = 0.
$$

This easily implies that $\mathcal{D}\varphi(t) \equiv 0$ and $\uparrow \overline{\partial}\varphi'(t)$ is a holomorphic vector field. Since $C_1 = 0$, the only holomorphic vector field is constant vector field. Thus $\varphi'(t)$ is constant in the V direction. In other words, $\varphi'(t)$ is a functional of t only. Hence, there exists at most one constant scalar curvature metric in each Kähler class when $C_1 = 0$. We postpone the proof of the case $C_1 < 0$ to the next subsection.

Theorem 8. *If $C_1(V) \leq 0$, then a constant scalar curvature metric, if it exists, realizes the global minimum of the K energy functional in each Kähler class. In other words, if K energy doesn't have a lower bound, then there exists no constant curvature metric in that cohomology class.*

Proof. Suppose $\varphi_0 \in \mathcal{H}$ is a metric of constant curvature. Then

$$\frac{dE}{d\,t}|_{\varphi_0} = -\int_V (R - \underline{R}) \cdot \psi \ \det g = 0.$$

For any metric $\varphi(1)$, let $\varphi(t)(0 \leq t \leq 1)$ be a path in $\mathcal{H}$ which connects $\varphi(0)$ and $\varphi(1)$. In addition, let us assume this is an ϵ-approximate geodesic where $\epsilon > 0$ may be chosen arbitrarily small. From equation (17), we have

$$\begin{aligned}
\frac{d^2 E}{d\,t^2} &= \int_V |\mathcal{D}\varphi'(t)|_g^2 \ \det g - \int_V (\varphi'' - \frac{1}{2}|\nabla\varphi'|_g^2) \cdot R \ \det g \\
&\quad + \epsilon \cdot \underline{R} \cdot \int_V \ \det h \\
&= \int_V |\mathcal{D}\varphi'(t)|_g^2 \ \det g + \int_V |\nabla f|_g^2 \frac{1}{f} \det g \\
&\quad - \int_V tr_g(\mathrm{Ric}(h)) \ f \ \det g + \epsilon \cdot \underline{R} \cdot \int_V \ \det h \\
&> -C\epsilon.
\end{aligned}$$

The last inequality holds since the average of the scalar curvature is a topological invariant. Thus

$$E(t) - E(0) \geq -C\epsilon \frac{t^2}{2}, \qquad \forall t \in [0, 1].$$

In particular, this holds for $t = 1$

$$E(\varphi(1)) - E(\varphi(0)) = E(1) - E(0) \geq -\frac{C \cdot \epsilon}{2}.$$

Let $\epsilon \to 0$, we have

$$E(\varphi(1)) \geq E(\varphi(0)).$$

The theorem is proved since $\varphi(1)$ is arbitrary.

6.2 Uniqueness of c.s.c. metric when $C_1 < 0$

Now we turn our attention to the case $C_1 < 0$. By our initial assumption, $\mathrm{Ric}(h) < -ch$ for some positive constant $c > 0$. Thus

$$\begin{aligned}\int_{V\times I} \frac{|\mathcal{D}\varphi'|_g^2}{f} \det h + \int_{V\times I} |\nabla \ln f|_g^2\, det\, h + c\cdot \int_{V\times I} tr_g(h) \det h \\ \leq C(= -\underline{R}\int_V \det h).\end{aligned} \tag{21}$$

We want to show that in the limit as $\epsilon \to 0$, we still have $\mathcal{D}\varphi'(t) = 0$ in some weak sense. Let us first get an integral estimate on $f^{\frac{q}{2-q}} (1 < q < 2)$ with respect to the measure $\det h\, d\,t$:

$$\begin{aligned}\int_{V\times I} f^{\frac{q}{2-q}} \det h\, d\,t &\leq C \cdot \int_{V\times I} f \det h\, d\,t \\ &\leq C \cdot \int_{V\times I} \left\{ f \cdot \frac{\det g}{\det h}\right\}^{\frac{1}{n}} \cdot \left\{\frac{det\, h}{\det g}\right\}^{\frac{1}{n}} \det h\, d\,t \\ &\leq \epsilon^{\frac{1}{n}} \int_{V\times I} \left\{\frac{\det h}{\det g}\right\}^{\frac{1}{n}} \det h\, d\,t \\ &\leq C\cdot \epsilon^{\frac{1}{n}} \int_{V\times I} tr_g(h) \det h\, d\,t \to 0.\end{aligned}$$

Let $X = \uparrow \overline{\partial}\varphi'(t) = g^{\alpha\overline{\beta}} \frac{\partial \varphi'}{\partial \overline{z_\beta}} \frac{\partial}{\partial z_\alpha}$. We want to show that X is uniformly in L^2 with respect to the measure $h + d\,t^2$.

$$\begin{aligned}\textstyle\int_{V\times I} |X|_h^2 \det h\, d\,t &= \int_{V\times I} \sum_{\alpha,\beta} h_{\alpha\overline{\beta}} X^\alpha \overline{X^\beta} \det h\, d\,t \\ &= \int_{V\times I} \sum_{\alpha,\beta,\gamma,\delta} h_{\alpha\overline{\beta}} g^{\alpha\overline{\gamma}} \frac{\partial\varphi'}{\partial \overline{z_\gamma}} \overline{\left\{ g^{\beta\overline{\delta}} \frac{\partial \varphi'}{\partial \overline{z_\delta}}\right\}} \det h\, d\,t \\ &= \int_{V\times I} \sum_{\alpha,\beta,\gamma,\delta} h_{\alpha\overline{\beta}} g^{\alpha\overline{\gamma}} g^{\delta\overline{\beta}} \frac{\partial\varphi'}{\partial\overline{z_\gamma}} \frac{\partial \varphi'}{\partial z_\delta} \det h\, d\,t \\ &\leq \int_{V\times I} tr_g(h) |\nabla\varphi'|_g^2 \det h \\ &\leq C\cdot \int_{V\times I} tr_g(h) \det h\, d\,t \leq C.\end{aligned}$$

The second to last inequality holds since $f = \varphi'' - \frac{1}{2}|\nabla\varphi'|_g^2 \geq 0$ and $\varphi'' < C$. Thus $X \in L^2(V \times I)$ has a uniform upper bound for the L^2 norm.

Consider $|D\varphi'|_g$ as a function in $L^2(V \times I)$. First of all, it has a weak limit in $L^2(V \times I)$; secondly, its $L^q (1 < q < 2)$ norm tends to 0 as $\epsilon \to 0$.

$$\begin{aligned}
\int_{V\times I} |\mathcal{D}\varphi'|_g{}^q \ \det h \, d\,t &= \cdot \int_{V\times I} \frac{|\mathcal{D}\varphi'|_g{}^q}{f^l} \cdot f^l \ \det h \, d\,t \\
&\leq \cdot \left(\int_{V\times I} \frac{|\mathcal{D}\varphi'|_g{}^{sq}}{f^{ls}} \ \det h \, d\,t \right)^{\frac{1}{s}} \\
&\quad \cdot \left(\int_{V\times I} f^{l\tau} \ \det h \, d\,t \right)^{\frac{1}{\tau}} \\
&\quad \text{(where } \frac{1}{s} + \frac{1}{\tau} = 1).
\end{aligned}$$

Now l is some number which we should choose appropriately:

$$ls = 1; \qquad qs = 2; \qquad \frac{1}{s} + \frac{1}{\tau} = 1.$$

Since for any $q < 2$, we have

$$s = \frac{2}{q}; \qquad l = \frac{q}{2}; \qquad \tau = \frac{2}{2-q},$$

the above inequality reduces to

$$\begin{aligned}
&\int_{V\times I} |\mathcal{D}\varphi'|_g{}^q \ \det h \, d\,t \\
&\quad \leq C \cdot \left(\int_{V\times I} \frac{|\mathcal{D}\varphi'|_g{}^2}{f} \ \det h \, d\,t \right)^{\frac{2}{q}} \cdot \left(\int_{V\times I} f^{\frac{q}{2-q}} \ \det h \, d\,t \right)^{\frac{(2-q)}{2}} \to 0.
\end{aligned}$$

For any vector $Y \in T^{1,0}(V \times I)$ (i.e., $Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial z_i}$ where $z_1, z_2, \ldots, z_n$ are all of the coordinate functions in a local chart in V. We use $\frac{\partial Y}{\partial \overline{z}}$ to denote the vector valued (0,1) form $\sum_{i,j=1}^n \frac{\partial Y^i}{\partial \overline{z_j}} \frac{\partial}{\partial z_i} \otimes d\,\overline{z_j}$.) For a scalar function ψ in $V \times I$, let $\frac{\partial \psi}{\partial \overline{z}} = \sum_{j=1}^n \frac{\partial \psi}{\partial \overline{z_j}} d\,\overline{z_j}$. Now the norms of $\frac{\partial Y}{\partial \overline{z}}$ and

$\frac{\partial\psi}{\partial\bar{z}}$ in terms of the metric h are:

$$\left|\frac{\partial Y}{\partial\bar{z}}\right|_h^2 = \sum_{\alpha,\beta,r,\delta=1}^{n} h_{\alpha\bar{r}} h^{\overline{\beta}\delta} \frac{\partial Y^\alpha}{\partial \overline{z_\beta}} \overline{\left(\frac{\partial Y^r}{\partial \overline{z_\delta}}\right)} \tag{22}$$

and

$$\left|\frac{\partial\psi}{\partial\bar{z}}\right|_h^2 = \sum_{\alpha,\beta=1}^{n} h^{\alpha\overline{\beta}} \frac{\partial\psi}{\partial z_\alpha}\frac{\partial\psi}{\partial\overline{z_\beta}}. \tag{23}$$

We claim the following inequality holds (for some uniform constant C):

$$\left|\frac{\partial X}{\partial\bar{z}}\right|_h \leq \sqrt{\sum_{\alpha,\beta,r,\delta=1}^{n} h_{\alpha\bar{r}} h^{\overline{\beta}\delta} \frac{\partial X^\alpha}{\partial \overline{z_\beta}} \overline{\left(\frac{\partial X^r}{\partial \overline{z_\delta}}\right)}} \leq C\,\sqrt{tr_g(h)}\,|D\varphi'|_g. \tag{24}$$

This could be proved by choosing preferred coordinates, where $h_{i\bar{j}} = \delta_{i\bar{j}}(1 \leq i,j \leq n)$ while $g_{i\bar{j}} = \lambda_i\delta_{i\bar{j}}(1 \leq i,j \leq n)$ at an arbitrary point O. Here λ_i are eigenvalues of the metric g in terms of the metric h. These λ_i's are uniformly bounded from above since g is so. We want to verify the above inequality at this point O.

$$\begin{aligned}
\left|\frac{\partial X}{\partial\bar{z}}\right|_h^2 &= \sum_{\alpha,\beta,a,b=1}^{n} \frac{\partial X^\alpha}{\partial z_{\overline{\beta}}}\frac{\partial X^{\overline{a}}}{\partial z_b} h_{\alpha\overline{a}} h^{\overline{\beta}\,b} \\
&= \sum_{\alpha,\beta,a,b,c,d=1}^{n} h_{\alpha\overline{a}} h^{\overline{\beta}\,b} g^{\alpha\bar{c}} \varphi'_{,\bar{c}\overline{\beta}}\varphi'_{,d\,b} g^{\overline{a}d} \\
&= \sum_{\alpha,\beta=1}^{n} \delta_{\alpha\overline{a}}\delta^{\overline{\beta}\,b}\frac{1}{\lambda_\alpha}\delta^{\alpha\bar{c}}\varphi'_{,\bar{c}\overline{\beta}}\varphi'_{,d\,b}\frac{1}{\lambda_a}\delta^{\overline{a}d} \\
&= \sum_{\alpha,\beta=1}^{n} \frac{1}{\lambda_\alpha^2}\varphi'_{,\overline{\alpha}\overline{\beta}}\varphi'_{,\alpha\,\beta} \\
&\leq \left(\sum_{\alpha,\beta=1}^{n}\frac{\lambda_\beta}{\lambda_\alpha}\right)\sum_{\alpha,\beta=1}^{n}\frac{1}{\lambda_\alpha\lambda_\beta}\varphi'_{,\overline{\alpha}\overline{\beta}}\varphi'_{,\alpha\,\beta} \\
&\leq C\cdot tr_g(h)\cdot|D\varphi'|_g^2.
\end{aligned}$$

Here C is a uniform constant. From inequality (21) and the fact that g

is bounded from above, it follows that

$$\int_{V\times I}\left|\nabla\log\frac{\det g}{\det h}\right|_h^2 = \int_{V\times I} \mid \nabla\log f \mid_h^2$$
$$\leq C\cdot\int_{V\times I} \mid \nabla\log f \mid_g^2 \det g\, d\,t \leq C,$$

and

$$\int_{V\times I}\left(\frac{\det h}{\det g}\right)^{\frac{1}{n}}\ \det h \leq \int_{V\times I} tr_g(h)\ \det h \leq C.$$

From now on, all of the norms, inner products, and integrations are taken with respect to the the metric $h + d\,t^2$ unless otherwise specified. Now define a new vector field Y by

$$Y = X\cdot\frac{\det g}{\det h}.$$

Then

$$|Y|_h = |X|_h\frac{\det g}{\det h} \leq C.$$

In other words, Y has a uniform L^∞ bound. This implies that $Y\cdot\partial\ln\frac{\det g}{\det h}/\partial\overline{z}$ has a uniform L^q bound for any $1 < q < 2$. Moreover, for any $1 < q < 2$, we have

$$\int_{V\times I}\left|\frac{\partial Y}{\partial\overline{z}} - Y\cdot\frac{\partial\ln\frac{\det g}{\det h}}{\partial\overline{z}}\right|_h^q = \int_{V\times I}\left(\left|\frac{\partial X}{\partial\overline{z}}\right|_h\frac{\det g}{\det h}\right)^q$$
$$\leq \int_{V\times I}\left(\sqrt{tr_g(h)}\frac{\det g}{\det h}\right)^q\ |D\varphi'|_g^q$$
$$\leq \int_{V\times I} C\ |D\varphi'|_g^q \to 0.$$

This immediately implies that $\frac{\partial Y}{\partial\overline{z}}$ is uniformly bounded in L^q for any $1 < q < 2$.

Now, all of these quantities, $X, Y, \frac{\partial Y}{\partial\overline{z}},$ and $\frac{\det g}{\det h},\ldots$ are geometric quantities which depend on ϵ. Since their respective Soblev norms are uniformly controlled, we can take weak limits of these quantities in some appropriate sense. Denote the corresponding weak limits (when $\epsilon\to 0$)

as $X, Y, \frac{\det g}{\det h}, \ldots$. Then $X(\epsilon) \rightharpoonup X$ weakly in $L^2(V \times I)$, $Y(\epsilon) \rightharpoonup Y$ weakly in $L^\infty(V \times I)$ and $\frac{\det g}{\det h}(\epsilon) \rightharpoonup \frac{\det g}{\det h}$ weakly in $L^\infty(V \times I), \ldots$.

Consider $u = \ln \frac{\det h}{\det g}$. For simplicity, assume $u > 0$ (otherwise $u > -c$ for some positive constant). Then the following two equations hold in the limit

$$\frac{\partial Y}{\partial \overline{z}} + Y \cdot \frac{\partial u}{\partial \overline{z}} = 0, \qquad \text{and} \qquad Y = X\, e^{-u}$$

in the sense of $L^q(V \times I)$ for any $1 < q < 2$. Moreover, we have the following estimates:

$$\int_{V \times I} e^{\frac{1}{n} u} \leq C; \qquad \int_{V \times I} \left| \frac{\partial u}{\partial \overline{z}} \right|^2 \leq C; \qquad \text{and} \int_{V \times I} |X|^2 \leq C.$$

Now define a new sequence of vectors $X_{,k} (k = 1, 2, \ldots)$ by $X_{,k} = Y \displaystyle\sum_{i=0}^{k} \frac{u^i}{i!}$. This is well defined since u is in $L^p(V \times I)$ for any $p > 1$. Then

$$\begin{aligned} |X_{,k}| &= |Y| \sum_{i=0}^{k} \frac{u^i}{i!} \\ &\leq (|X| e^{-u}) e^{u} \leq |X|. \end{aligned}$$

Equality holds in the last inequality whenever $e^{-u} \neq 0$. Thus

$$\int_{V \times I} |X_{,k}|^2 \leq \int_{V \times I} |X|^2 \leq C.$$

By definition, it is clear $\|X_{,k}\|_{L^2(V \times I)} \leq \|X_{,m}\|_{L^2(V \times I)}$ whenever $k \leq m$. Thus, there exists a positive number $A \leq \|X\|_{L^2(V \times I)}$ such that $\lim_{k \to \infty} \|X_{,k}\|_{L^2(V \times I)} = A$. For $m > k$, we have

$$\begin{aligned} \|X_{,m}\|^2_{L^2(V \times I)} &= \int_{V \times I} |X_{,m}|^2 \\ &= \int_{V \times I} |Y|^2 (\sum_{i=0}^{m} \frac{u^i}{i!})^2 \\ &\geq \int_{V \times I} |Y|^2 \left((\sum_{i=0}^{k} \frac{u^i}{i!})^2 + (\sum_{i=k+1}^{m} \frac{u^i}{i!})^2 \right) \\ &= \int_{V \times I} (|X_{,k}|^2 + |X_{,m} - X_{,k}|^2) \\ &= \|X_{,k}\|^2_{L^2(V \times I)} + \|X_{,m} - X_{,k}\|^2_{L^2(V \times I)}. \end{aligned}$$

Taking limits as $m, k \to \infty$, we have $\|X_{,m} - X_{,k}\|^2_{L^2(V\times I)} \to 0$. Thus $X_{,k}$ $(k = 1, 2, \dots)$ is a Cauchy sequence in $L^2(V \times I)$ and there exists a strong limit $X_{,\infty}$ in $L^2(V \times I)$. By definition, we know that $X_{,\infty} = X$ almost everywhere in $V \times I$ [7]. We want to show that $X_{,\infty}$ is weakly holomorphic in the V direction. A straightforward calculation yields

$$\begin{aligned}\frac{\partial X_{,k}}{\partial \overline{z}} &= \frac{\partial Y}{\partial \overline{z}} \sum_{i=0}^{k} \frac{u^i}{i!} + Y \frac{\partial}{\partial \overline{z}} \left(\sum_{i=0}^{k} \frac{u^i}{i!} \right) \\ &= -Y \frac{\partial u}{\partial \overline{z}} \sum_{i=0}^{k} \frac{u^i}{i!} + Y \left(\sum_{i=0}^{k-1} \frac{u^i}{i!} \right) \frac{\partial u}{\partial \overline{z}} \\ &= -(X_{,k} - X_{,k-1}) \frac{\partial u}{\partial \overline{z}}.\end{aligned}$$

We want to show that $\frac{\partial X_{,\infty}}{\partial \overline{z}} = 0$ in the sense of distributions. We just need to show it in any open set $U \times I$ where U is a coordinate chart in V. Let $(z_1, z_2, \dots, z_n)$ be coordinate variables in U. Then, for any vector valued smooth function $\psi = (\psi^1, \psi^2, \dots, \psi^n)$ which vanishes on $\partial(U \times I)$, and for any $1 \le j \le n$. We have

$$\begin{aligned}\left| \int_{V\times I} X_{,k} \cdot \frac{\partial \overline{\psi}}{\partial \overline{z_j}} \right| &= \left| - \int_{V\times I} \frac{\partial X_{,k}}{\partial \overline{z_j}} \cdot \overline{\psi} \right| \\ &= \left| \int_{V\times I} (X_{,k} - X_{,k-1}) \frac{\partial u}{\partial \overline{z_j}} \overline{\psi} \right| \\ &\le C \cdot \|X_{,k} - X_{,k-1}\|_{L^2(V\times I)} \cdot \sqrt{\int_{V\times I} |\nabla u|^2} \\ &\le C \|X_{,k} - X_{,k-1}\|_{L^2(V\times I)}.\end{aligned}$$

Taking the limit as $k \to \infty$ yields

$$\int_{V\times I} X_{,\infty} \cdot \frac{\partial \overline{\psi}}{\partial \overline{z_j}} = 0, \qquad \text{for any } j = 1, 2, \dots, n,$$

and for any smooth vector valued function $\psi = (\psi^1, \psi^2, \dots, \psi^n)$ which vanishes on $\partial(U \times I)$. Thus, $X_{,\infty}$ is a weak holomorphic vector field in the V direction for almost all t. Now recall that

$$\int_{V\times I} |X_{,\infty}|_h^2 \,\det h \, dt < C.$$

[7] It is easy to prove that $X_{,\infty} = X$ in the sense of $L^q(V \times I)$ for any $(1 < q < 2)$.

This implies that $X_{,\infty}$ is in $L^2(V \times \{t\})$ for almost all $t \in [0,1]$. Since $X_{,\infty}$ is weakly holomorphic in $V \times \{t\}$ for all t, $X_{,\infty}$ must be holomorphic for those t where $X_{,\infty}$ is in $L^2(V \times \{t\})$. However, there is no holomorphic vector field on V since $C_1 < 0$. Thus $X_{,\infty} \equiv 0$ for all of those t where $X_{,\infty}$ is in $L^2(V \times \{t\})$. This shows that $X_{,\infty} = 0$ in $V \times I$. Thus $X = 0$ since $X = X_{,\infty}$ in the sense of $L^q(V \times I)$ for any $1 < q < 2$. Recall

$$\frac{\partial \varphi'(t)}{\partial z_\alpha} = \sum_{\beta=1}^{n} g_{\alpha\overline{\beta}} X^{\overline{\beta}} = \sum_{\beta=1}^{n} g_{\alpha\overline{\beta}} {X_{,\infty}}^{\overline{\beta}} = 0.$$

In other words, $\varphi'(t)$ is trivial in the V direction and it is a function of t only for all $t \in [0,1]$. Therefore, $\varphi(0)$ and $\varphi(1)$ differ only by a constant in the V direction, and hence represent the same metric in each Kähler class.

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