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THE SPACE OF p-SUMMABLE SEQUENCES AND ITS NATURAL n-NORM

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We study the space l^p, $1 \leqslant p \leqslant \infty$, and its natural n-norm, which can be viewed as a generalisation of its usual norm. Using a derived norm equivalent to its usual norm, we show that l^p is complete with respect to its natural n-norm. In addition, we also prove a fixed point theorem for l^p as an n-normed space.

1. INTRODUCTION

Let n be a nonnegative integer and X be a real vector space of dimension $d \geqslant n$ (d may be infinite). A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the four properties

(1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(2) $\|x_1, \dots, x_n\|$ is invariant under permutation;

(3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha|\, \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbf{R}$;

(4) $\|x + x', x_2, \dots, x_n\| \leqslant \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$,

is called an *n-norm* on X, and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-normed space.

For instance, any real inner product space $(X, \langle \cdot, \cdot \rangle)$ can be equipped with the standard n-norm

$$\|x_1, \dots, x_n\| := \sqrt{\det\big(\langle x_i, x_j \rangle\big)},$$

which can be interpreted as the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ in X. On $\mathbf{R}^n$, this n-norm can be simplified to

$$\|x_1, \dots, x_n\| = \big|\det(x_{ij})\big|$$

where $x_i = (x_{i1}, \dots, x_{in}) \in \mathbf{R}^n$, $i = 1, \dots, n$.

The theory of 2-normed spaces was first developed by Gähler [**5**] in the mid 1960's, while that of n-normed spaces was studied later by Misiak [**21**]. Related works on n-metric spaces and n-inner product spaces may be found in, for example, [**2, 3, 4, 6, 7, 8, 11, 12**].

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While various aspects of n-normed spaces have been studied extensively (see, for example, [**15, 17, 19, 23, 24**]), there are not many concrete examples that have been studied in depth except the standard ones. Nonstandard examples can be found in, for example, [**14, 20**].

In this note, we shall study the space $l^p, 1 \leqslant p \leqslant \infty$, containing all sequences of real numbers $x = (x_j)$ for which $\sum_j |x_j|^p < \infty$ (or $\sup_j |x_j| < \infty$ when $p = \infty$), and its natural n-norm, which can be regarded as a generalisation of the usual norm $\|x\|_p := \left[\sum_j |x_j|^p\right]^{1/p}$ (or $\|x\|_\infty := \sup_j |x_j|$ when $p = \infty$).

Using a derived norm equivalent to its usual norm, we shall show that l^p is complete with respect to its natural n-norm. In addition, we shall also prove a fixed point theorem for l^p as an n-normed space (see, for example, [**9, 15, 16, 18, 22, 25**] for previous results in this direction).

Throughout this note, we assume that p lies in the interval $1 \leqslant p \leqslant \infty$ unless otherwise stated. All sequences in l^p are indexed by nonnegative integers.

For expository purposes, we shall first discuss l^p and its natural 2-norm, and then generalise the results for all $n \geqslant 2$.

2. l^p and its natural 2-Norm

We already know that l^2, being an inner product space with inner product $\langle x, y\rangle = \sum_j x_j y_j$, can be equipped with the standard 2-norm

$$\|x, y\| := \left[\det \begin{pmatrix} \sum_j x_j^2 & \sum_j x_j y_j \\ \sum_j x_j y_j & \sum_j y_j^2 \end{pmatrix}\right]^{1/2}$$

By properties of determinants and limiting arguments (see [**10**], pp. 109–111), we have

$$\det \begin{pmatrix} \sum_j x_j^2 & \sum_j x_j y_j \\ \sum_j x_j y_j & \sum_j y_j^2 \end{pmatrix} = \sum_j x_j \det \begin{pmatrix} x_j & y_j \\ \sum_k x_k y_k & \sum_k y_k^2 \end{pmatrix}$$
$$= \sum_j \sum_k x_j y_k \det \begin{pmatrix} x_j & y_j \\ x_k & y_k \end{pmatrix}.$$

At the same time, we also have

$$\det\begin{pmatrix}\sum_j x_j^2 & \sum_j x_jy_j\\ \sum_j x_jy_j & \sum_j y_j^2\end{pmatrix} = \sum_j y_j \det\begin{pmatrix}\sum_k x_k^2 & \sum_k x_ky_k\\ x_j & y_j\end{pmatrix}$$

$$= \sum_j\sum_k y_jx_k \det\begin{pmatrix}x_k & y_k\\ x_j & y_j\end{pmatrix}$$

$$= \sum_j\sum_k -x_ky_j \det\begin{pmatrix}x_j & y_j\\ x_k & y_k\end{pmatrix}.$$

Hence we obtain

$$2\det\begin{pmatrix}\sum_j x_j^2 & \sum_j x_jy_j\\ \sum_j x_jy_j & \sum_j y_j^2\end{pmatrix} = \sum_j\sum_k (x_jy_k - x_ky_j)\det\begin{pmatrix}x_j & y_j\\ x_k & y_k\end{pmatrix}$$

$$= \sum_j\sum_k \left|\det\begin{pmatrix}x_j & x_k\\ y_j & y_k\end{pmatrix}\right|^2.$$

Therefore, we may rewrite the standard 2-norm on l^2 as

$$\|x, y\| = \left[\frac{1}{2}\sum_j\sum_k\left|\det\begin{pmatrix}x_j & x_k\\ y_j & y_k\end{pmatrix}\right|^2\right]^{1/2}.$$

This looks like the usual norm on l^2 except that now we are taking the square root of half the sum of squares of determinants of 2×2 matrices. Here $\left|\det\begin{pmatrix}x_j & x_k\\ y_j & y_k\end{pmatrix}\right|$ represents the area of the projected parallelogram on the two dimensional subspace spanned by $e_j = (\delta_{jl})$ and $e_k = (\delta_{kl})$.

Inspired by the above observation, let us define the following function $\|\cdot,\cdot\|_p$ on $l^p \times l^p$, $1 \leqslant p < \infty$, by

$$\|x, y\|_p := \left[\frac{1}{2}\sum_j\sum_k\left|\det\begin{pmatrix}x_j & x_k\\ y_j & y_k\end{pmatrix}\right|^p\right]^{1/p}.$$

As in [**14**] and [**20**], define also $\|\cdot,\cdot\|_\infty$ on $l^\infty \times l^\infty$ by

$$\|x, y\|_\infty := \sup_j\sup_k\left|\det\begin{pmatrix}x_j & x_k\\ y_j & y_k\end{pmatrix}\right|.$$

(One might like to interpret the value of $\|\cdot,\cdot\|_p$ in terms of the areas of the 'projected parallelograms' on the subspaces spanned by e_j and e_k, for all j and k, and compare it to the standard case.)

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The following fact tells us that $\|\cdot,\cdot\|_p$ makes sense.

Fact 2.1. *l The inequality*

$$\|x, y\|_p \leqslant 2^{1-(1/p)}\|x\|_p\|y\|_p,$$

holds whenever $x, y \in l^p$.

Proof: Let $1 \leqslant p < \infty$. Then, by the triangle inequality for real numbers and Minkowski's inequality for double series, we have

$$\begin{aligned}
\|x, y\|_p &= \left[\frac{1}{2}\sum_j\sum_k |x_j y_k - x_k y_j|^p\right]^{1/p} \\
&\leqslant \left[\frac{1}{2}\sum_j\sum_k \left[|x_j||y_k| + |x_k||y_j|\right]^p\right]^{1/p} \\
&\leqslant 2^{-1/p}\left[\left[\sum_j\sum_k |x_j|^p|y_k|^p\right]^{1/p} + \left[\sum_j\sum_k |x_k|^p|y_j|^p\right]^{1/p}\right] \\
&= 2^{1-(1/p)}\|x\|_p\|y\|_p,
\end{aligned}$$

whenever $x, y \in l^p$. For $p = \infty$, the inequality

$$\|x, y\|_\infty \leqslant 2\|x\|_\infty\|y\|_\infty$$

can be verified in a similar fashion. ∎

Remark. Of course, for $p = 2$, we have a better inequality

$$\|x, y\|_2 \leqslant \|x\|_2\|y\|_2,$$

which is a special case of Hadamard's inequality (see [**10**, p. 202]). For our purposes, however, the inequality in Fact 2.1 is good enough.

Fact 2.2. *The function $\|\cdot,\cdot\|_p$ defines a 2-norm on l^p.*

Proof: We need to check that $\|\cdot,\cdot\|_p$ satisfies the four properties of a 2-norm. First note that the 'if' part of (1), (2) and (3) are obvious. To verify the 'only if' part of (1), suppose that $\|x, y\| = 0$. Then

$$\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix} = 0$$

for all j and k, and so we conclude that x and y are linearly dependent.

It now remains to verify (4). By a property of determinants and the triangle inequality for real numbers, we have

$$\left|\det\begin{pmatrix} x_j + x'_j & x_k + x'_k \\ y_j & y_k \end{pmatrix}\right| \leqslant \left|\det\begin{pmatrix} x_j & x_k \\ y_j & y_k \end{pmatrix}\right| + \left|\det\begin{pmatrix} x'_j & x'_k \\ y_j & y_k \end{pmatrix}\right|.$$

Hence, by Minkowski's inequality for double series, (4) follows and this completes the proof. ∎

As a consequence of Fact 2.2, we have:

Corollary 2.3. *The space l^p, equipped with $\|\cdot,\cdot\|_p$, is a 2-normed space.*

2.1. Completeness Recall that a sequence $x(m)$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to *converge* to some $x \in X$ in the 2-norm whenever

$$\lim_{m\to\infty} \|x(m) - x, y\| = 0$$

for every $y \in X$. Also, $x(m)$ is said to be *Cauchy* with respect to the 2-norm if

$$\lim_{l,m\to\infty} \|x(l) - x(m), y\| = 0$$

for every $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the 2-norm.

From textbooks on functional analysis (see, for example, [**1**, pp. 91–92]), we know that l^p is complete with respect to its usual norm $\|\cdot\|_p$. Our aim now is to show that l^p is complete with respect to its natural 2-norm $\|\cdot,\cdot\|_p$. To do so, we need the following lemma.

Lemma 2.4. *A sequence in l^p is convergent in the 2-norm $\|\cdot,\cdot\|_p$ if and only if it is convergent in the usual norm $\|\cdot\|_p$. Similarly, a sequence in l^p is Cauchy with respect to $\|\cdot,\cdot\|_p$ if and only if it is Cauchy with respect to $\|\cdot\|_p$.*

The 'if' parts of Lemma 2.4 follow immediately from Fact 2.1. To prove the 'only if' parts, we shall invoke a derived norm as previously done in [**13**] and [**14**].

In general, given a 2-normed space $(X, \|\cdot,\cdot\|)$ of dimension $\geqslant 2$, we can choose an arbitrary linearly independent set $\{a_1, a_2\}$ in X and, with respect to $\{a_1, a_2\}$, define a norm $\|\cdot\|_p^*$ on X by

$$\|x\|_p^* := \left[\|x, a_1\|^p + \|x, a_2\|^p\right]^{1/p},$$

for $1 \leqslant p < \infty$, or

$$\|x\|_\infty^* := \sup\left\{\|x, a_1\|, \|x, a_2\|\right\},$$

for $p = \infty$.

For our 2-normed space l^p, we choose, for convenience, $a_1 = (1, 0, 0, \dots)$ and $a_2 = (0, 1, 0, \dots)$, and define $\|\cdot\|_p^*$ with respect to $\{a_1, a_2\}$ as above. Then we have:

Fact 2.5. *The derived norm $\|\cdot\|_p^*$ is equivalent to the usual norm $\|\cdot\|_p$ on l^p. Precisely, we have*

$$\|x\|_p \leqslant \|x\|_p^* \leqslant 2^{1/p}\|x\|_p$$

for all $x \in l^p$. In particular, $\|\cdot\|_\infty^* = \|\cdot\|_\infty$.

PROOF: Let $1 \leqslant p < \infty$. For every $x \in l^p$, we compute

$$\|x, a_1\|_p^p = \sum_{j \neq 1} |x_j|^p$$

and

$$\|x, a_2\|_p^p = \sum_{j \neq 2} |x_j|^p,$$

whence

$$\|x\|_p^* = \left[|x_1|^p + |x_2|^p + 2\sum_{j \geqslant 3} |x_j|^p\right]^{1/p}.$$

We therefore see that

$$\|x\|_p \leqslant \|x\|_p^* \leqslant 2^{1/p}\|x\|_p,$$

that is, $\|\cdot\|_p^*$ and $\|\cdot\|_p$ are equivalent. The proof for $p = \infty$ is similar. □

REMARK. Fact 2.5 tells us in particular that $\|\cdot\|_p$ is dominated by $\|\cdot\|_p^*$. As we shall see below, this is what we actually need to prove Lemma 2.4.

PROOF OF LEMMA 2.4: Suppose that $x(m)$ converges to some $x \in l^p$ in the 2-norm $\|\cdot,\cdot\|_p$. With respect to $a_1 = (1, 0, 0, \dots)$ and $a_2 = (0, 1, 0, \dots)$, define $\|\cdot\|_p^*$ as before. Then, since $\lim_{m\to\infty} \|x(m) - x, a_1\|_p = 0$ and $\lim_{m\to\infty} \|x(m) - x, a_2\|_p = 0$, we have $\lim_{m\to\infty} \|x(m) - x\|_p^* = 0$, that is, $x(m)$ converges to x in $\|\cdot\|_p^*$. But $\|\cdot\|_p$ is dominated by $\|\cdot\|_p^*$, and so we conclude that $x(m)$ also converges to x in $\|\cdot\|_p$.

As mentioned before, the converse follows immediately from Fact 2.1. The following diagram summarises the proof of the first part of the lemma:

convergence in $\|\cdot,\cdot\|_p$

$\nearrow$ $\searrow$

convergence in $\|\cdot\|_p$ $\longleftarrow$ convergence in $\|\cdot\|_p^*$

The second part of the lemma can be proved in a similar fashion: one only needs to replace the expressions 'convergent to x' with 'Cauchy' and '$x(m) - x$' with '$x(l) - x(m)$'. □

Now we come to the main result.

THEOREM 2.6. *The space l^p is complete with respect to the 2-norm $\|\cdot,\cdot\|_p$.*

PROOF: Let $x(m)$ be Cauchy in l^p with respect to $\|\cdot,\cdot\|_p$. Then, by Lemma 2.4, $x(m)$ is Cauchy with respect to the usual norm $\|\cdot\|_p$. But we know that l^p is complete with respect to $\|\cdot\|_p$, and so $x(m)$ must converge to some $x \in X$ in $\|\cdot\|_p$. By another application of Lemma 2.4, $x(m)$ also converges to x in $\|\cdot,\cdot\|_p$. This shows that l^p is complete with respect to the 2-norm $\|\cdot,\cdot\|_p$. □

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3. l^p AND ITS NATURAL n-NORM

By using properties of determinants and limiting arguments as before, we can write the standard n-norm on l^2 as

$$\|x_1,\dots,x_n\| := \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\left|\det(x_{ij_k})\right|^2\right]^{1/2}$$

Now, for $1 \leqslant p < \infty$, define the following function $\|\cdot,\dots,\cdot\|_p$ on $l^p \times \cdots \times l^p$ (n factors) by

$$\|x_1,\dots,x_n\|_p := \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\left|\det(x_{ij_k})\right|^p\right]^{1/p}$$

For $p = \infty$, define $\|\cdot,\dots,\cdot\|_\infty$ on $l^\infty \times \cdots \times l^\infty$ (n factors) by

$$\|x_1,\dots,x_n\|_\infty := \sup_{j_1}\dots\sup_{j_n}\left|\det(x_{ij_k})\right|,$$

as in [**20**].

Then we have the following facts, which are just generalisations of Facts 2.1 and 2.2 (and so we leave the proofs to the reader). Note that the factor $n!$ appearing below is the number of terms in the expansion of $\det(x_{ij_k})$.

FACT 3.1. *The inequality*

$$\|x_1,\dots,x_n\|_p \leqslant (n!)^{1-(1/p)}\|x_1\|_p\dots\|x_n\|_p,$$

holds whenever $x_1,\dots,x_n \in l^p$.

FACT 3.2. *The function $\|\cdot,\dots,\cdot\|_p$ defines an n-norm on l^p.*

COROLLARY 3.3. *The space l^p, equipped with $\|\cdot,\dots,\cdot\|_p$, is an n-normed space.*

3.1. COMPLETENESS As in the case $n = 2$, a sequence $x(m)$ in a n-normed space $(X, \|\cdot,\dots,\cdot\|)$ is said to *converge* to some $x \in X$ in the n-norm whenever

$$\lim_{m\to\infty}\left\|x(m) - x, x_2,\dots,x_n\right\| = 0$$

for every $x_2,\dots,x_n \in X$. Also, $x(m)$ is said to be *Cauchy* with respect to the n-norm if

$$\lim_{l,m\to\infty}\left\|x(l) - x(m), x_2,\dots,x_n\right\| = 0$$

for every $x_2,\dots,x_n \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the n-norm.

The following is a generalisation of Lemma 2.4.

LEMMA 3.4. *A sequence in l^p is convergent in the n-norm $\|\cdot,\dots,\cdot\|_p$ if and only if it is convergent in the usual norm $\|\cdot\|_p$. Similarly, a sequence in l^p is Cauchy with respect to $\|\cdot,\dots,\cdot\|_p$ if and only if it is Cauchy with respect to $\|\cdot\|_p$.*

As before, the 'if' parts of Lemma 3.4 are obvious and the 'only if' parts can be proved by using a derived norm, defined with respect to the set $\{a_1, \dots, a_n\}$, where $a_i = (\delta_{ij})$, $i = 1, \dots, n$, by

$$\|x\|_p^* := \left[\sum_{\{i_2,\dots,i_n\} \subseteq \{1,\dots,n\}} \|x, a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{1/p}$$

if $1 \leqslant p < \infty$, or

$$\|x\|_\infty^* := \sup_{\{i_2,\dots,i_n\} \subseteq \{1,\dots,n\}} \|x, a_{i_2}, \dots, a_{i_n}\|_\infty$$

if $p = \infty$.

Indeed, one may observe that $\|x\|_p^*$ defines a norm on l^p (see [**6**] for previous results for $p = 1$, [**12**] for $p = 2$, and [**14**] for $p = \infty$). Further, we have:

Fact 3.5. *The derived norm $\|\cdot\|_p^*$ is equivalent to the usual norm $\|\cdot\|_p$ on l^p. Precisely, we have*

$$\|x\|_p \leqslant \|x\|_p^* \leqslant n^{1/p} \|x\|_p$$

for all $x \in l^p$. In particular, $\|\cdot\|_\infty^* = \|\cdot\|_\infty$.

Proof: As usual, we shall only give the proof for $1 \leqslant p < \infty$ and leave that for $p = \infty$ to the reader.

For every $x \in l^p$, we compute

$$\|x, a_2, a_3, \dots, a_n\|_p^p = |x_1|^p + \sum_{j \geqslant n+1} |x_j|^p.$$

Similarly

$$\|x, a_1, a_3, \dots, a_n\|_p^p = |x_2|^p + \sum_{j \geqslant n+1} |x_j|^p$$

$$\vdots$$

$$\|x, a_1, a_2, \dots, a_{n-1}\|_p^p = |x_n|^p + \sum_{j \geqslant n+1} |x_j|^p.$$

Hence we obtain

$$\|x\|_p^* = \left[|x_1|^p + \dots + |x_n|^p + n \sum_{j \geqslant n+1} |x_j|^p \right]^{1/p}.$$

It therefore follows that

$$\|x\|_p \leqslant \|x\|_p^* \leqslant n^{1/p} \|x\|_p,$$

that is, $\|\cdot\|_p^*$ and $\|\cdot\|_p$ are equivalent. ∎

As a generalisation of Theorem 2.6, we have

Theorem 3.6. *The space l^p is complete with respect to the n-norm $\|\cdot, \dots, \cdot\|_p$.*

3.2. A FIXED POINT THEOREM We shall now use the derived norm to prove the following fixed point theorem for the n-normed space $(l^p, \|\cdot, \dots, \cdot\|_p)$.

THEOREM 3.7. (Fixed point theorem) *Let T be a self-mapping of l^p such that*

$$\|Tx - Tx', x_2, \dots, x_n\|_p \leqslant C\|x - x', x_2, \dots, x_n\|_p$$

for all $x, x', x_2, \dots, x_n$ in X and some constant $C \in (0, 1)$, that is, T is contractive with respect to $\|\cdot, \dots, \cdot\|_p$. Then T has a unique fixed point in X.

Before we prove the theorem, note that l^p is complete with respect to the derived norm $\|\cdot\|_p^*$. Indeed, if $x(m)$ is Cauchy with respect to $\|\cdot\|_p^*$, then by Fact 3.5 it is also Cauchy with respect to $\|\cdot\|_p$ and hence, since l^p is complete with respect to $\|\cdot\|_p$, it must converge to some $x \in l^p$. By Fact 3.1, we conclude that $x(m)$ converges to x in $\|\cdot, \cdot\|_p$ and, eventually, in $\|\cdot\|_p^*$.

PROOF OF THEOREM 3.7: If we can show that T is also contractive with respect to the derived norm $\|\cdot\|_p^*$, defined with respect to the set $\{a_1, \dots, a_n\}$ as before, then we are done (for we have just seen that l^p is complete with respect to $\|\cdot\|_p^*$). But this is easy since, by hypothesis, we have

$$\|Tx - Tx', a_{i_2}, \dots, a_{i_n}\|_p \leqslant C\|x - x', a_{i_2}, \dots, a_{i_n}\|_p$$

for all $x, x' \in l^p$ and $\{i_2, \dots, i_n\} \subseteq \{1, \dots, n\}$, whence

$$\|Tx - Tx'\|_p^* \leqslant C\|x - x'\|_p^*$$

for all $x, x' \in l^p$ (with the same C), that is, T is contractive with respect to $\|\cdot\|_p^*$. $\square$

4. CONCLUDING REMARKS

The n-norms $\|\cdot, \dots, \cdot\|_p$ can be defined analogously on $\mathbf{R}^d$ with $d \geqslant n$. However, they are all equivalent here and we already know what happens with the standard or finite-dimensional case in general (see [**13**] and [**14**]).

As the reader will realise, our results also extend to $L^p(X)$ spaces, where X is a measure space with at least n disjoint subsets of positive measure. Recall that $L^p(X)$ is the space of equivalence classes (modulo equivalence almost everywhere) of functions such that $\int_X |f(x)|^p d\mu(x) < \infty$ (if $1 \leqslant p < \infty$) or $\sup_{x \in X} |f(x)| < \infty$ (if $p = \infty$). Indeed, one may define $\|\cdot, \dots, \cdot\|_p$ on $L^p(X) \times \dots \times L^p(X)$ (n factors) by

$$\|f_1, \dots, f_n\|_p := \left[\frac{1}{n!}\int_X \dots \int_X \left|\det(f_i(x_j))\right|^p dx_1 \dots dx_n\right]^{1/p}$$

if $1 \leqslant p < \infty$, or

$$\|f_1, \dots, f_n\|_\infty := \sup_{x_1 \in X} \dots \sup_{x_n \in X} \left|\det(f_i(x_j))\right|$$

if $p = \infty$, and check that this function defines an n-norm on $L^p(X)$. Clearly the analogues of Fact 3.1, Fact 3.2, Corollary 3.3 and the 'if' parts of Lemma 3.4 hold. The remaining results may be verified by using a derived norm defined with respect to $\{\chi_{A_1}, \ldots, \chi_{A_n}\}$, where $A_1, \ldots, A_n$ are disjoint sets of positive measure. The key is to show that the usual norm on $L^p(X)$ is dominated by this derived norm.

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