The Specification of Oscillator Characteristics from Measurements Made in the Frequency Domain

R. VESSOT, L. MUELLER, AND J. VANIER

Abstract—A cross-correlation technique for measuring the very short-term (milliseconds to seconds) properties of stable oscillators is described. Time-dependent functions representing signals from two separate oscillators are led to a function multiplier where the instantaneous product of the functions is made. The oscillators are either set to a given phase relation or allowed a small relative drift so that a slow beat frequency is observed. Short-term fluctuations superimposed upon the slow beat signal from the multiplier output will represent the instantaneous phase difference between the oscillators when the inputs are in quadrature. When the inputs are in and out of phase, the fluctuations represent amplitude fluctuations. The time averaging function is determined by a filter having a rectangular pass band from nearly zero frequency to a cutoff frequency ν_c.

The mean square frequency deviation measured in a bandwidth ω_c is obtained by differentiating, filtering, squaring, and averaging the signal from the function multiplier data being taken when the input signals are in quadrature. Mean square averages of amplitude and phase averaged over various bandwidths ω_c may be obtained by bypassing the differentiator. Sample data from measurements on hydrogen masers are presented, and the effect of thermal noise is seen to be the major factor limiting the short-term frequency stability of the signals.

I. The Effect of the Measuring System Bandwidth on the Characterization of Oscillator Performance

ONE OF THE IMPORTANT problems in measuring and describing the properties of oscillators in the time domain is that data are obtained from apparatus that has a bounded frequency range. The representation of the properties of an oscillator over a given averaging time τ implies the use of a Fourier transform of a sharply rectangular time response function or "window," giving a result in the frequency domain that has unbounded frequency limits [1]. The effect of the bandwidth limitations of the measuring equipment depends on the nature of the instabilities present in the oscillator. In certain types of oscillators, as will be shown later, the observed instabilities depend very strongly on the added noise due to the first stages of the measuring apparatus. In general, the signal from an oscillator, which will be considered here to be a given device with a pair of output terminals, is accompanied by thermal noise within a bandwidth defined by the circuitry of the device. If there is no a priori knowledge about the bandwidth and the noise temperature of the appropriate circuit elements, the question of the width of the measuring system bandwidth is important.

Too narrow a bandwidth will remove noise contributions from the oscillator and give optimistic data of its frequency stability; conversely, too large a bandwidth will introduce excessive noise in the first stages of the measuring apparatus which may corrupt the oscillator signal. The latter case is important when low power signals, such as those from maser oscillations, are considered. Since the measurement of the bandwidth of the output noise is an important aspect of the measurement, the measuring equipment should have some of the properties of a spectrum analyzer.

In general, frequency stability data is usually obtained in two ways: by measuring the time periods of zero crossings using a reference clock [2]–[6], or by observing the instantaneous phase difference between an oscillator and a reference source. Both methods have limitations imposed by apparatus bandwidth that compromise the rigor of the description of oscillator performance.

It is worth noting that in the zero crossing method, one has really a measurement of elapsed time for a given phase and not the measurement of elapsed phase for a given time interval. This reversal of the role of the independent and dependent variable is contrary to the definition of frequency stability. In practice, the problem is not severe for stable oscillators since the elapsed time (or period) measurements are made to have a small spread compared to the elapsed time between the measurements.

The bandwidth limitations become apparent if one considers the rigorous definition of frequency stability over a given time interval τ given by Cutler, Searle, Baghdady, and others [7]–[9].

$$\left.\frac{\Delta f}{f}\right|_\tau = \frac{1}{2\pi}\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} S\dot{\phi}(\omega)\left(\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right)^2 d\omega} \qquad (1)$$

where $(\Delta f/f)|_\tau$ is the fractional frequency stability over a time interval τ obtained by making a large number of independent measurements of frequency, each over a time interval τ, and taking the root mean square deviation of the frequencies obtained and dividing by the average frequency. $S\dot{\phi}$ is the two-sided power spectral density of the frequency difference between the oscillator being measured and the reference, and is related to $S\phi(\omega)$ by $S\dot{\phi}(\omega)=\omega^2 S\phi(\omega)$. If the oscillator and its ref-

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erence are identical, then the performance for each oscillator will be described by the foregoing expression divided by $\sqrt{2}$.

The quantity $\sigma^2(\tau)$ is defined as

$$\sigma^2(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S\phi(\omega)\left\{\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right\}^2 d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} S\phi(\omega)\left\{\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right\}^2 d\omega.$$

The integral ranges from zero frequency to infinite frequency, and $\sigma^2(\tau)$ does not depend on bandwidth cutoff at either the high- or the low-frequency limits.

If the bandwidth of the measuring systems is limited at the upper or lower frequencies, the quantity σ^2 depends on three parameters τ, ω_c, and ω_0, where ω_c and ω_0 are the high- and low-frequency cutoff. The problems of low-frequency cutoff have been described elsewhere by Barnes and Allen [10] in the case of nonstationary effects described as flicker frequency noise given by $S\phi(\omega) = A/|\omega|$.

In the case where the bandwidth is limited at high frequency, the quantity σ^2 depends on two parameters and is given by

$$\sigma^2(\tau, \omega_c) = \frac{1}{\pi}\int_{0}^{\omega_c} S\phi(\omega)\left\{\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right\}^2 d\omega.$$

The behavior of an oscillator depends on the function $S\phi(\omega)$ and it is convenient to describe $S\phi(\omega)$ in some region of interest in the frequency domain. Both an upper limit encompassing the important high-frequency power contributions and a lower limit related in some way to the available observing time are needed.

In the following illustration, the effects of a sharply rectangular high-frequency cutoff are shown, calculated for $S\phi(\omega)$ described as a series of terms related to noise processes that, to date, have some significance.

$$S\phi(\omega) = \frac{A}{|\omega|} + B + C\omega + D\omega^2$$

$$S\phi(\omega) = \frac{A}{|\omega|^3} + \frac{B}{|\omega|^2} + \frac{C}{|\omega|} + D.$$

The terms can separately be related to the following processes:

$S\phi_A(\omega) = \frac{A}{|\omega|}$ "flicker" frequency noise

$S\phi_B(\omega) = B$ "white" frequency noise

$S\phi_C(\omega) = C\omega$ "flicker" phase noise

$S\phi_D(\omega) = D\omega^2$ "white" phase noise.

There is no reason to ignore higher and lower powers of ω; however, for the purposes of the illustration only these will be considered. The output from an oscillator can be described by such a series provided care is taken to avoid divergences near zero frequency. This problem has been discussed by Barnes elsewhere in this issue [11].

If the terms of $S\phi(\omega)$ are applied separately to the defining equation for $(\Delta f/f)|_\tau$, it is seen that only in the case of white frequency noise does convergence result without some consideration of high- and low-frequency limits. The question of what the limits are due to must now be faced.

If the limits are due to a filter-like behavior of the output circuit of the oscillator, then the integral of $S\phi(\omega)$ must be modified by a filter function $H_{LP}(j\omega)$ due to the oscillator. This will result in convergence of the integral to a value characterizing the oscillator,

$$\sigma^2(\tau) = \frac{1}{\pi}\int_{0}^{\infty} S\phi(\omega)\,|H_{LP}(j\omega)|^2\left\{\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right\}^2 d\omega,$$

and this is the desired result. In describing $S\phi(\omega)$ for an oscillator, one could equally well write

$$S_0\phi(\omega) = |H_{LP}(j\omega)|^2 S\phi(\omega)$$

where $S_0\phi(\omega)$ is the bounded spectral output of the oscillator.

In measuring the output from an oscillator, one must be careful to account for this bandwidth in the measuring apparatus. However, if the measuring apparatus has an excessively large bandwidth, then the additive noise from sources outside the oscillator can corrupt the signal and the term $S\phi_D(\omega)$ will predominate.

In Fig. 1, the function

$$\sigma_B{}^2(\tau, \omega_c) = \frac{2}{\pi}\frac{B}{\tau}\int_{0}^{\omega_c\tau/2}\left\{\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}\right\}^2 d\frac{\omega\tau}{2}$$

is shown plotted and, as expected, is seen to converge to a value $\sigma^2(\tau) = B/\tau$ when ω_c becomes infinitely large.

Fig. 2 describes the effect of flicker phase noise and it is seen that no convergence occurs when $\omega_c \rightarrow \infty$, and that the dependence on the upper frequency limit is serious. Here, the function

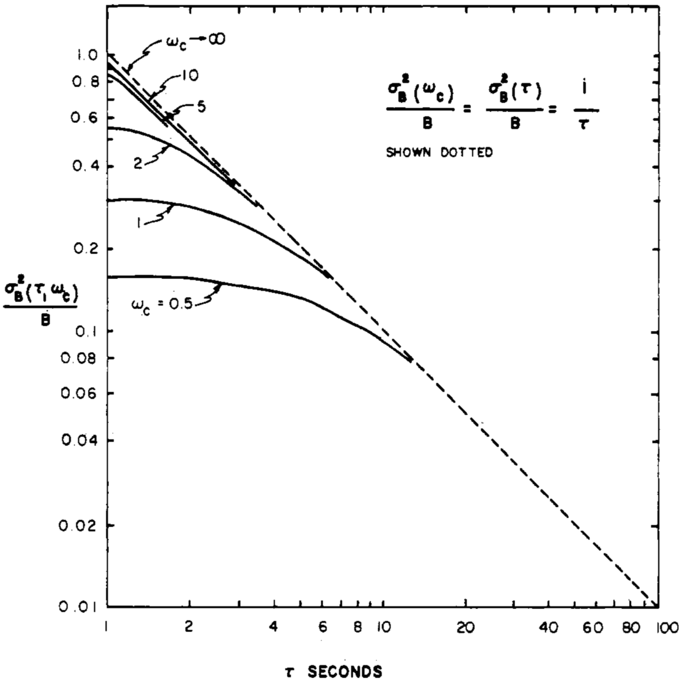

Fig. 1.

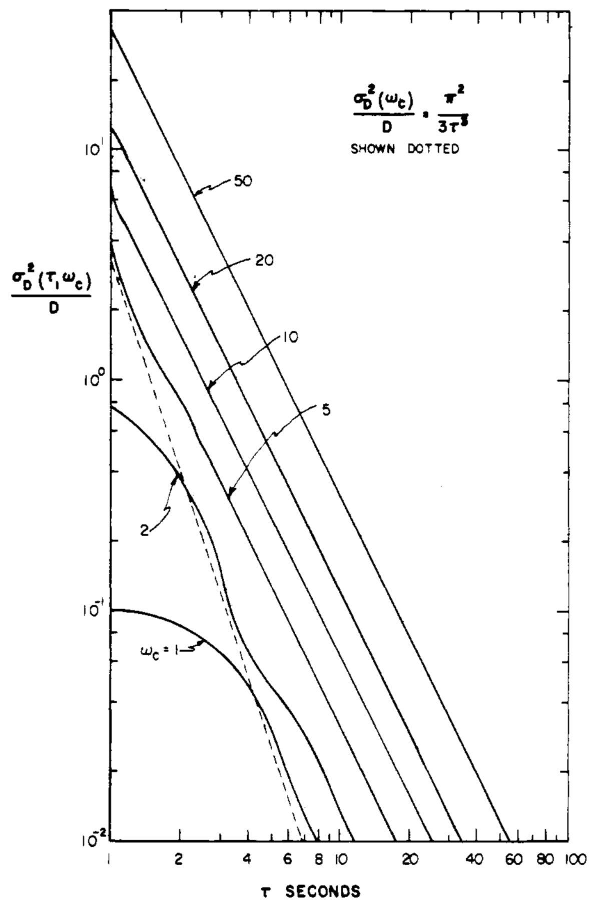

Fig. 3.

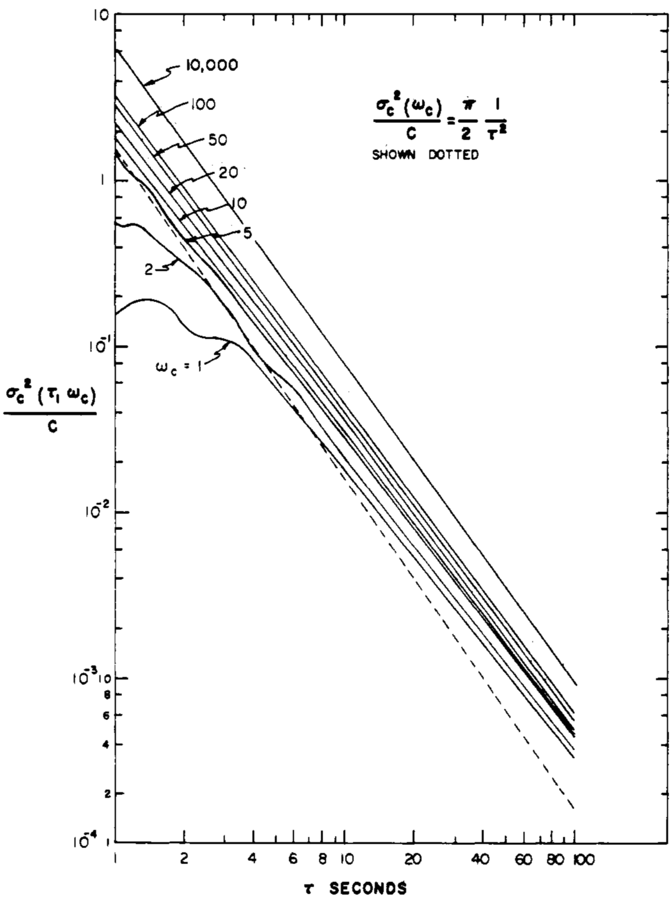

Fig. 2.

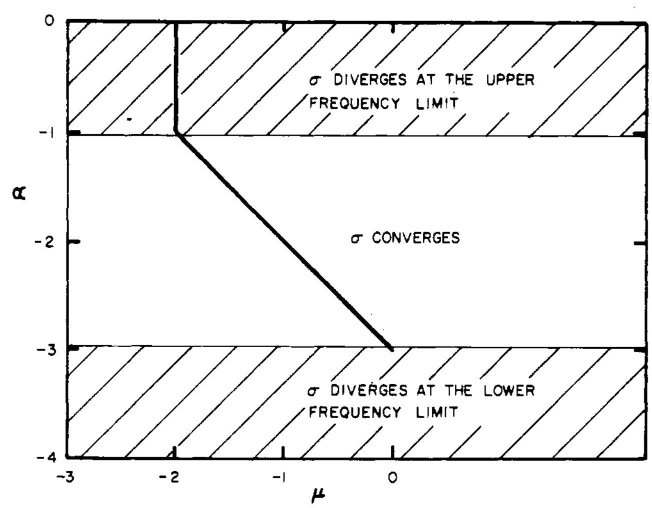

Fig. 4. Mapping of μ on α for $\omega_c \to \infty$.

$$\sigma_C^2(\tau, \omega_c) = \frac{4C}{\pi\tau^2}\int_0^{\omega_c\tau/2} \frac{\sin^2 \frac{\omega\tau}{2}}{\frac{\omega\tau}{2}} d\frac{\omega\tau}{2}$$

is plotted. This is seen to diverge as $\omega_c \rightarrow \infty$; however, the slope of σ_C^2 vs. τ on a log-log plot converges to -2.

Figure 3, where the effect of white phase noise is considered, shows a more strongly divergent result than the previous one as ω_c is increased.

$$\sigma_D^2(\tau, \omega_c) = \frac{8D}{\pi\tau^3}\int_0^{\omega_c\tau/2} \sin^2 \frac{\omega\tau}{2} d\frac{\omega\tau}{2}.$$

This is seen to converge to a slope of -2 as $\omega_c \rightarrow \infty$.

The behavior of the asymptotic values of $\sigma(\tau, \omega_c)$ for $\omega_c \rightarrow \infty$ given by

$$\sigma = \alpha(\omega) \mid \tau \mid^\mu$$
$$\lim \omega_c \rightarrow \infty$$

may be plotted against the value of α, where

$$S\phi(\omega) = h \mid \omega \mid^\alpha.$$

For $\alpha \geq -1$ the value of σ diverges at the upper frequency limit; however, its slope remains finite and constant at the value $\mu = -2$. For $\alpha \leq -3$, the value of σ diverges at the lower frequency limit. In the range $-3 < \alpha < 1$, σ converges to a finite value for $\omega_c \rightarrow \infty$.

A mapping of μ on α is presented in Fig. 4 and shows essentially the same characteristics as described by Allen [2].

II. The Application of the Cross-Correlation Technique for Measuring the Short-Term Properties of Atomic Hydrogen Masers

In describing the frequency stability of oscillators, several authors have made use of a model where the rate of phase change of a carrier signal amplitude phasor is calculated by considering the effect of adding random noise signals in quadrature with it. These expressions have the general form [1], [11]–[15].

$$\left.\frac{\Delta f}{f}\right|_\tau = \frac{1}{Q_l}\sqrt{\frac{kT}{2P\tau}} \qquad (1)$$

where Δf is the standard deviation of a set of frequency measurements, each made over an interval of time τ. Q_l is the quality factor of the resonant circuit determining the frequency, and in the case of masers is related to the linewidth of the transition involved in generating the energy. The above expression describes the frequency stability of a signal that is considered to exist in the absence of added noise; the only influence of noise that has been included in the expression is that due to the response of the signal vector due to noise energy that lies within the bandwidth determined by the oscillator Q.

The effect of the inevitable added noise power per unit frequency bandwidth kT on the signal, prior to its use or analysis by a particular system, is not included. Additive noise is a source of contamination of signals in other oscillators such as crystal oscillators; however, in the case of quantum mechanical oscillators the effect is very serious, due to the low power output levels of the desired signals. In the case of hydrogen masers the effect is the dominant source of the observed instabilities for observation times up to about one minute, depending on the measuring apparatus noise figure and the system bandwidth. For observation time τ greater than about one minute, the maser stability thus far has been limited by systematic processes, and the value of $(\Delta f/f)|_\tau$ levels off to about 8×10^{-14} as τ increases. For time intervals greater than one minute, the effect of the thermal noise as given by (1) leading to a $\tau^{-1/2}$ law is obscured by the cavity pulling effect [16], [17] given by

$$f - f_0 = \frac{Q_c}{Q_l}(f - f_t) \qquad (2)$$

where $f_0 \doteq 1.42 \times 10^9$ is the resonance hyperfine frequency of the hydrogen atom, f is the output frequency of the maser, and f_t is the resonance frequency of the cavity. Since the ratio Q_c/Q_l is of the order of 10^{-5}, a few cycles change in f_t will cause the systematic variation previously mentioned. Improvements in the stability of masers will result from increasing Q_l by extending the interaction time of the atoms with the RF magnetic field [18]. The result will be that for a given output power level, the effects due to (1) and (2) will diminish in the same ratio and, while the stability of the masers will be improved, the $\tau^{-1/2}$ behavior will probably continue to be obscured.

It must be remembered, however, that the noise power bandwidth of systems connected to an oscillator will be finite. This limitation occurs, in the case of a maser, due to the bandwidth of the RF cavity, typically some tens of kilocycles. The total signal plus noise power spectrum of the maser can be described as thermal noise over the frequency response of the cavity plus a signal spectrum at its center. If the receiver bandwidth is opened further than that of the cavity, the cavity will be the overall limiting factor and no further significant degradation of the frequency stability will result to additive noise in the oscillator. The receiver, however, will continue to degrade the signal as its bandwidth is extended. It is possible, as pointed out by Edson [12], to improve the apparent stability of an oscillator by using an external filter before going to the receiver or measuring system; however, in defining the boundary between the oscillator and the measuring system, one should be careful to state whether or not this filter is included.

The spectrum of each of the masers, nominally at 1.4 kMc/s, is transformed to a low frequency by a pair of double heterodyne receivers employing common local oscillators as shown in Fig. 5. The signals, each rep-

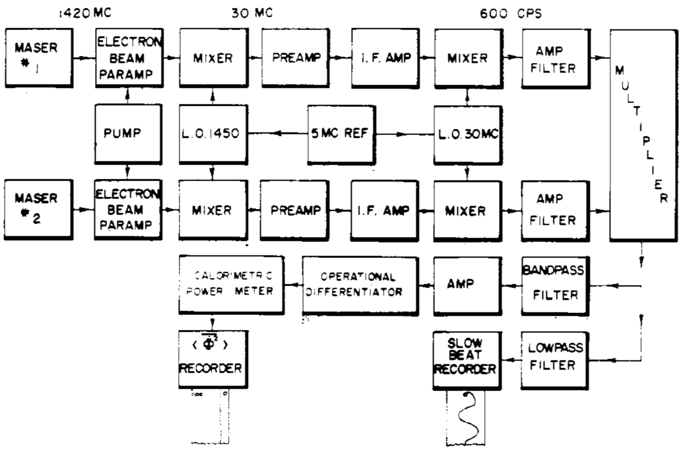

Fig. 5. Short-term noise measurement.

resenting a maser plus thermal noise and receiver noise, may be described in the following manner:

$$V_i(t) = [x_i(t) + A] \cos \omega_i t - y_i(t) \sin \omega_i t$$

$$i = 1, 2$$

$$\omega_1 - \omega_2 = \Omega \quad \text{usually 1 radian/minute or less.}$$

$y_i(t)$ and $x_i(t)$ are instantaneous variations of noise amplitude and are defined as consisting of two components,

$$x_i(t) = w_i(t) + z_i(t)$$

$$y_i(t) = u_i(t) + v_i(t)$$

where

$w_i(t)$ is the in-phase component of oscillator noise
$z_i(t)$ is the in-phase component of thermal and receiver noise
$u_i(t)$ is the quadrature component of oscillator noise, and
$v_i(t)$ is the quadrature component of thermal and receiver noise.

$\langle w_i v_i \rangle_{avg} = 0 \quad i = 1, 2$
$\langle u_i z_i \rangle_{avg} = 0 \quad i = 1, 2.$
$\langle w_1 u_2 \rangle_{avg} = 0$
$\langle w_2 u_1 \rangle_{avg} = 0$
$\langle z_1 v_2 \rangle_{avg} = 0$
$\langle z_2 v_1 \rangle_{avg} = 0$

Correlations may exist between u_i and w_i for $i = 1$ or 2, as well as between z_i and v_i for $i = 1$ or 2. In the present work the latter are assumed to be zero. Also

$$\langle v_1^2 \rangle_{avg} = \langle z_1^2 \rangle_{avg}$$

and

$$\langle v_2^2 \rangle_{avg} = \langle z_2^2 \rangle_{avg}.$$

Putting

$$r = [(x + A) + y^2]^{1/2}$$

and

$$\phi = \tan^{-1} \frac{y}{A + x},$$

then

$$V_i(t) = r_i(t) \cos [\omega_i t + \phi(t)].$$

Let the two voltages V_1 and V_2 be led to a function multiplier and let $E(t)$ be the output from the multipler.

$$\begin{aligned} E(t) &= K V_1(t) V_2(t) \\ &= K r_1 r_2 \cos (\omega t + \phi_1) \cos (\omega t + \phi_2 + \Omega t) \\ &= K \frac{r_1 r_2}{2} [\cos \Omega t + (\phi_1 - \phi_2) \sin \Omega t] \end{aligned}$$

when $x \ll A$, $y \ll A$, and when the terms involving cos 2 ωt are averaged to zero.

The signal is thus observed to involve a slow beat component cos Ωt that is not necessarily constant in frequency. It is recorded on one channel of the recorder in order to determine the reative phase of the two oscillators. Short-term variations, due to the terms $x(t)$ or $y(t)$, can be determined by observing the short-term variations at times when $\Omega t = 0$ or $\pi/2$, respectively.

At $\Omega t = 0$,

$$\begin{aligned} E_0(t) &= K \frac{r_1 r_2}{2} \\ &= \frac{KA^2}{2} + \frac{KA}{2} (x_1 + x_2) \end{aligned}$$

where $A_1 = A_2$ and $x_1 \ll A$. The first term is the amplitude of the slow beat signal. The second term involves short-term amplitude variations.

At $\Omega t = \frac{\pi}{2}$,

$$\begin{aligned} E_{\pi/2}(t) &= K \frac{r_1 r_2}{2} (\phi_1 - \phi_2) \\ &= \frac{KA^2}{2} (\phi_1 - \phi_2) \quad \text{or} \quad \frac{KA}{2} (y_2 - y_1) \end{aligned}$$

since

$$\phi_1 \doteq \frac{y_1}{A}.$$

The filtered signal from the multiplier is given by

$$\tilde{E}_r = \frac{KA^2}{2}\left[1 + \frac{\tilde{x}_1}{A} + \frac{\tilde{x}_2}{A}\right] \cdot \left[\cos \Omega t + \left(\frac{\tilde{y}_2}{A} - \frac{\tilde{y}_1}{A}\right) \sin \Omega t\right]$$

where the signal within the bandwidth 0 to ω_c is represented by the tilde.

Differentiating this signal, squaring, and averaging over a time long compared to ω_c^{-1}, one obtains, since the beat frequency $\Omega/2\pi$ can be made arbitrarily small for the period of measurement,

$$\langle \dot{E}^2 \rangle_{\substack{\text{avg} \\ \Omega \to 0}} = \left(\frac{KA^2}{2}\right)^2 \left[\frac{\langle \dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{y}_1^2 \rangle_{\text{avg}}}{2A^2} + \frac{\langle \dot{x}_1^2 + \dot{x}_2^2 - \dot{y}_2^2 - \dot{y}_1^2 \rangle_{\text{avg}}}{2A^2} \cos 2\Omega t + \frac{\langle \dot{x}_1 \dot{y}_1 - \dot{x}_2 \dot{y}_2 \rangle_{\text{avg}}}{2A^2} \sin 2\Omega t\right].$$

The terms $\dot{x}_1$, $\dot{y}_1$ and $\dot{x}_2$, $\dot{y}_2$ are due to correlation of amplitude and phase. If these exist in oscillator number 1, and oscillator number 2 has a known or zero value, it is possible to estimate the magnitude of this effect. Normally, if the oscillators are statistically similar, these terms are equal and the expression becomes

$$\langle \dot{E}^2 \rangle_{\substack{\text{avg} \\ \Omega \to 0}} = \frac{K^2 A^2}{8} \left[\langle \dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{y}_1^2 \rangle_{\text{avg}} + \langle \dot{x}_1^2 + \dot{x}_2^2 - \dot{y}_2^2 - \dot{y}_1^2 \rangle_{\text{avg}} \cos 2\Omega t\right]$$

where $\dot{x}_1 \dot{y}_1 = \dot{x}_2 \dot{y}_2$.

By observing the output signal for variations that have a periodicity of $2\pi/2\Omega$, an estimate of the magnitude of the amplitude and the phase fluctuations can be found.

$$\langle \dot{E}^2 \rangle_{\substack{\text{avg} \\ \Omega t = \pi/2}} = \frac{K^2 A^4}{4} \langle \dot{\phi}_2^2 + \dot{\phi}_1^2 \rangle$$

where

$$\dot{\phi}_1^2 = \frac{\widetilde{\dot{(u_i(t) + v_i(t))^2}}}{A^2}.$$

Assuming that there are no correlations between the two quadrature components of noise, we then can write

$$\dot{\phi}^2 = \sum_{i=1,2} \left(\frac{\widetilde{\dot{u_i(t)}}^2}{A^2} + \frac{\widetilde{\dot{v_i(t)}}^2}{A^2}\right).$$

The terms $\dot{\phi}_1^2$ and $\dot{\phi}_2^2$ are statistically similar and uncorrelated so that

$$\langle \dot{\phi}_1^2 + \dot{\phi}_2^2 \rangle = 2\langle \dot{\phi}_1^2 \rangle = 2\langle \dot{\phi}_2^2 \rangle.$$

The value of $\dot{\phi}(t)^2$ consists of two components: one, due to signals from the maser oscillator; the other, due to signals added to the maser because of thermal noise in the cavity, the isolator, and the receiver system. The thermal noise within atomic resonance results in a *one-sided spectral density* of the frequency given by

$$S_m \dot{\phi}(\omega) \doteq 4 \frac{kT}{P} \gamma^2$$

where γ is the relaxation rate of the atoms in the bulb and is related by $Q_l = \omega_0/2\gamma$, where ω_0 is the angular frequency of the hydrogen hyperfine transition. If this is used, along with the proper spectral density of the frequency for the additive noise outside the atomic resonance, we have

$$\langle \dot{\phi}^2 \rangle = \int_0^{\omega_c} S\dot{\phi}(\omega) d\omega$$

$$\langle \dot{\phi}^2 \rangle = \left\langle \int_0^{\omega_c} \left[\frac{4kT\gamma^2}{P_{\text{beam}}} + \frac{FkT\omega^2}{P_0}\right] \right\rangle$$

where

$$S\dot{\phi}(\omega) = B + D\omega^2$$

$$B = \frac{4kT\gamma^2}{P_b}$$

$$D = \frac{FkT}{P_{\text{out}}}$$

and

$$P_0 = P_b \frac{Q_c}{Q_{\text{ext}}}$$

where Q_c is the loaded cavity quality factor, Q_{ext} is the external cavity Q.

$$\langle \dot{\phi}^2 \rangle = \frac{4kT\gamma^2\omega_c}{P_{\text{beam}}} + \frac{FkT\omega_c^3}{3P_0}.$$

The quantity to be determined is

$$\sigma^2(\omega_c) = \frac{1}{2\pi} \langle \dot{\phi}^2 \rangle$$

($1/2\pi$ instead of $1/\pi$ since we are dealing with one-sided spectra)

$$\sigma^2(\omega_c) = \frac{4}{2\pi} \frac{kT\gamma^2\omega_c}{P_b} + \frac{FkT}{6\pi P_0} \omega_c^3.$$

Putting

$$\tau = \frac{1}{2f_c} = \frac{\pi}{\omega_c}$$

$$\sigma^2(\omega_c) = \frac{2kT\gamma^2}{P_b \tau} + \frac{FkT}{6P_b} \frac{\pi^2}{\tau^3} \frac{Q_{\text{ext}}}{Q_c}.$$

The first term, describing the effect of the noise within the linewidth of the oscillator, gives the correct value of $\sigma^2(\tau)$, as given by Cutler and Searle [1]. This term is equal to the second term when

$$\omega_c = 2\gamma \sqrt{\frac{3Q_c}{FQ_{ext}}}.$$

In a typical case,

$$\frac{Q_c}{Q_{ext}} = 0.18 \qquad F = 10 \qquad \gamma = 0.7 \text{ second}^{-1}$$

$$f_c = 0.052 \text{ c/s}.$$

For bandwidths of the order of cycles per second it is obvious that the first term should be small compared to the second term and that σ^2 will vary as $1/\tau^3$.

III. Calibration and Measurements

An important parameter of the system is the noise factor F, and this was measured by substituting an argon discharge tube at one of the inputs as shown in Fig. 6. The argon discharge tube when operating produces a matched termination at 10^{5}°C; when turned off, the termination is at room temperature. The filter was set to a constant value $f_c = 20$ c/s and the square-law voltmeter deflection D was noted for the ON and OFF conditions of the discharge.

$$D_{ON} = Rk(T_0F_1 + T - T_0)B'$$

$$D_{OFF} = RkF_1T_0B$$

$$F_1 = \frac{\left(\frac{T}{T_0} - 1\right)B'}{\left(\frac{D_{ON}}{D_{OFF}} - 1\right)B},$$

and

$$R = \frac{D_{OFF}}{kF_1T_0B}.$$

The value of F for each channel was measured before each run of data. Normally, F depended somewhat on the adjustment of the system and had values ranging from 7.3 to 11.2.

A second method of calibration employs a known applied frequency modulation on one of the masers. The frequency of the maser depends on the magnetic field squared averaged over the volume of the bulb [16]. The relation is given by

$$f_1 = f_0 + 2750H_0^2$$

where f_1 is the output frequency, f_0 is the zero field frequency, and H_0 is the magnetic field in oersteds.

If a small sinusoidal perturbation is applied to H_0 in the form $H = H_0 + H_m \sin \omega t$ we have

$$f_1 = f_0 + 2750[H_0^2 + 2H_mH_0 \sin \omega t + H_m^2 \sin^2 \omega t].$$

The frequency difference between two masers in radians per second is given by

$$\dot\phi_2 - \dot\phi_1 = KH_0^2\left[2\frac{H_m}{H_0}\sin \omega t + \frac{H_m^2}{H_0^2}\sin^2 \omega t\right].$$

The output from the square-law voltmeter D where $k = 2\pi \times 2750$, is given by:

$$D = R\langle\widetilde{(\dot\phi_2 - \dot\phi_1)^2}\rangle$$

$$= RK^2H_0^4\left[2\left(\frac{H_m}{H_0}\right)^2 + \frac{3}{8}\left(\frac{H_m}{H_0}\right)^4\right]$$

and the value of R can be determined. The system is shown schematically in Fig. 7.

Samples of data are shown in Fig. 8. Each is a four-minute section of data giving D in arbitrary units and the phase of the beat signal. In the section marked "calibration run," the "C" field of both masers was raised from its usual setting of $H_0 = 1$ millioersted to $H_0 = 19.12$ millioersteds and the perturbing field ratio H_m/H_0 was set to 2×10^{-2}. A modulation frequency of one c/s was used and the low-pass cutoff filter set at $\omega_c/2\pi = 5$ c/s or $\tau = 0.1$ second. The maxima of D are seen to coincide with the values of $\omega t = \pi/2$ and $3\pi/2$, etc. The gain setting has been reduced by a factor of 10 from the unperturbed run shown above it.

As the bandwidth $\omega_c/2\pi$ is increased and approaches 60 c/s or $\tau = 8.33 \times 10^{-3}$, it is observed that amplitude fluctuations exceed phase fluctuations and D has maxima at $\Omega t = 0$ and π. It was extremely difficult to eliminate all the 60-cycle amplitude modulation, and in the region at which the data was taken the $\Omega t = \pi/2$ points.

Several runs of data were taken at different power levels and noise figures. The data are shown plotted in Fig. 9, and the experiment is shown to be in good agreement with the relationship predicting the effect of additive white noise on the signal of an atomic hydrogen maser as shown by the lines with slope $\tau^{-3/2}$.

The use of a filter in the maser system obviously has a large effect on the short-term stability observed. In most applications, the filter will be located after conversion to a lower frequency and the noise due to the converter will be an important source of frequency instability.

The effect of the thermal noise within the atomic resonance given by the expression

$$\frac{\Delta f}{f} = \frac{1}{Q_l}\sqrt{\frac{1}{2}\frac{kT}{P_T}}$$

has not been observed in these runs. Other data have been taken by making period measurements of beats between masers over long periods of time and have shown that the stability seems to reach a value $\Delta f/f = 8 \times 10^{-14}$ for time intervals of 10 seconds or longer. Systematic variations due to cavity pulling effects limit the measurements to this value for longer averaging times.

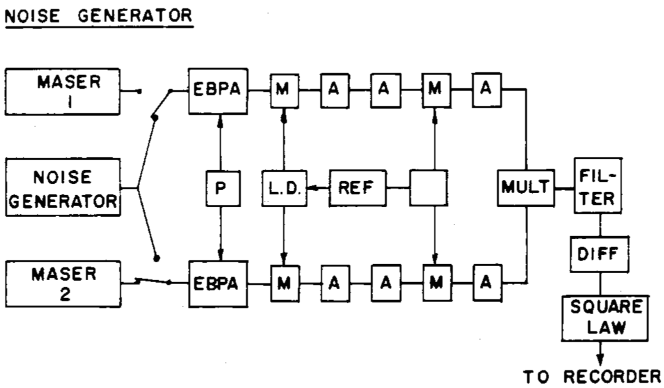

Fig. 6. Calibration method.

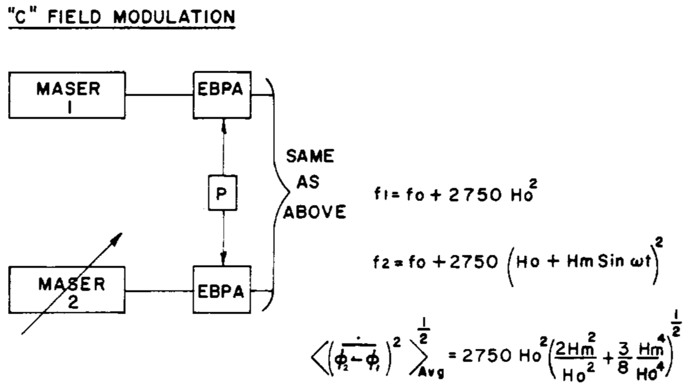

Fig. 7. Calibration method.

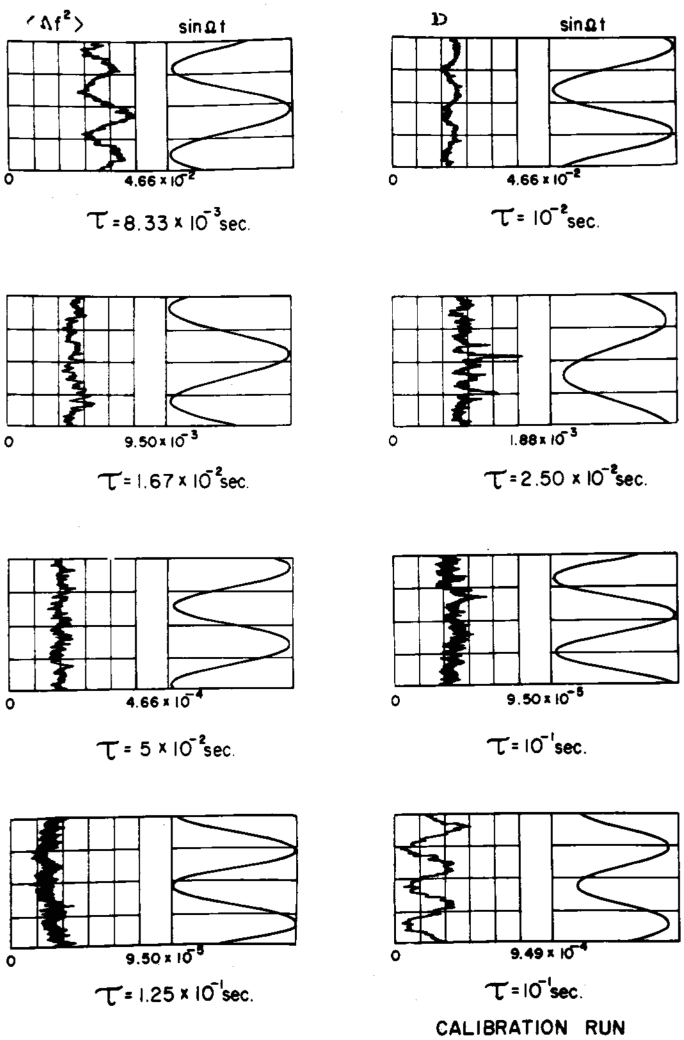

Fig. 8. Four-minute samples of data from cross correlation.

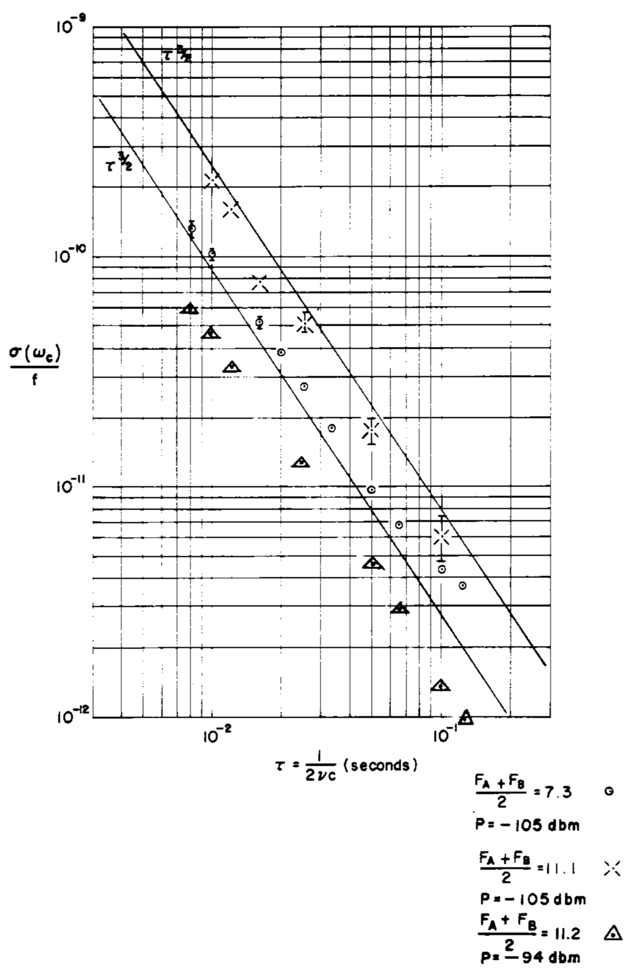

Fig. 9.

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Atomic Timekeeping and the Statistics of Precision Signal Generators

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***Abstract*—Since most systems that generate atomic time employ quartz crystal oscillators to improve reliability, it is essential to determine the effect on the precision of time measurements that these oscillators introduce. A detailed analysis of the calibration procedure shows that the third finite difference of the phase is closely related to the clock errors. It was also found, in agreement with others, that quartz crystal oscillators exhibit a "flicker" or $|\omega|^{-1}$ type of noise modulating the frequency of the oscillator.**

The method of finite differences of the phase is shown to be a powerful means of classifying the statistical fluctuations of the phase and frequency for signal generators in general. By employing finite differences it is possible to avoid divergences normally associated with flicker noise spectra. Analysis of several cesium beam frequency standards have shown a complete lack of the $|\omega|^{-1}$ type of noise modulation.

Introduction

An ORDINARY clock consists of two basic systems: a periodic phenomenon (pendulum), and a counter (gears, clock face, etc.) to count the periodic events. An atomic clock differs from this only in that the frequency of the periodic phenomenon is, in some sense, controlled by an atomic transition (atomic frequency standard). Since microwave spectroscopic techniques allow frequencies to be measured with a relative precision far better than any other quantity, the desirability of extending this precision to the domain of time measurement has long been recognized [1].

From the standpoint of precision, it would be desirable to run the clock (counter) directly from the atomic frequency standard. However, atomic frequency standards in general are sufficiently complex that reliable operation over very extended periods becomes somewhat doubtful (to say nothing of the cost involved). For this reason, a quartz crystal oscillator is often used as the source of the "periodic" events to run a synchronous clock (or its electronic equivalent). The frequency of this oscillator is then regularly checked by the atomic frequency standard and corrections are made.

These corrections can usually take on any of three forms: 1) correction of the oscillator frequency, 2) correction of the indicated time, or 3) an accumulating

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