The Spectral Analysis of Impulse Processes*

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An expression for the spectral density of the impulse process $s(t) = \sum_{-\infty}^{\infty} \alpha_n \delta(t - t_n)$ is derived under the assumption that $\{\alpha_n\}$ is a stationary process, and that $\{t_n\}$ is a stationary point process independent of $\{\alpha_n\}$. The spectral density appears as an infinite series in terms of the correlation of $\{\alpha_n\}$ and the interval statistics of $\{t_n\}$. The same result was obtained by Leneman by a different argument under considerably more restrictive conditions of validity.

Various models of impulse processes are discussed relative to random sampling of random processes. Random and systematic loss of samples, separate read-in and read-out jitters, and correlated random scaling errors can all be represented by appropriate assumptions on $\{\alpha_n\}$ and $\{t_n\}$.

Finally, closed form expressions are calculated for the spectral density of $s(t)$ and the sampled process under combinations of the sampling errors mentioned in the preceding paragraph.

I. INTRODUCTION

The impulse process

$$s(t) = \sum_{-\infty}^{\infty} \alpha_n \delta(t - t_n) \qquad (1.1)$$

is often encountered in the analysis of communication and control systems. Although (1.1) is an obvious model for noise appearing as sharp pulses (Stratonovich, 1963; Mazzetti, 1962 and 1964; Khurgin, 1957; Lee, 1960), $s(t)$ plays an even more important role as an intermediary or modulating process. For instance, $s(t)$ may be multiplied by another random process $x(t)$ to represent impulse sampling; the t_n are (possibly randomly irregular) sampling times, and the α_n can be chosen to introduce random multiplicative errors, plus-minus sampling, etc. In other applications, such as to shot noise, $s(t)$ is passed through a linear time-invariant filter. Further examples dealing with modulation processes (Lee, 1960 and Nelsen, 1964) and pulse-modulated control systems (see Gupta and Jury, 1962 for a bibliography) are scattered throughout the literature.

Whenever an impulse process (1.1) appears in a system, successful analysis or synthesis demands that at least some of the statistics of $s(t)$ be known. Although the studies cited above attempt to obtain such statistics, they do so only under severely restrictive assumptions on $\{t_n\}$, and their ad hoc approach fails to suggest systematic techniques applicable to large classes of impulse processes. Typically, it is supposed that sampling is periodic except for small jitter perturbations (Balakrishnan, 1962 and Brown, 1963), or that intervals between noise pulses are independent, perhaps even with an interval distribution related to the exponential (Banta, 1964; Mazzetti, 1962). Even so, the resulting calculations are usually tedious and difficult.

A recent paper by Leneman (1966a) suggests a new and more powerful approach to the statistical analysis of pulse processes. It is based on general properties of stationary point processes (Beutler and Leneman, 1966a, 1966b) (hereafter abbreviated s.p.p.), a class of $\{t_n\}$ that seems to embrace all those point processes of interest to communication theorists. Using these properties, Leneman (1966a) was able to both simplify and generalize earlier results on pulse processes.

Subsequent to the appearance of Leneman's paper (Leneman, 1966a), the authors have made further progress in the analysis of impulse processes. The general expression for second-order statistics is now derived by a different method that permits relaxed assumption, and is also of greater intuitive appeal. Furthermore, the authors have gained experience with the application of the principal formula, and are now able to apply it to a larger variety of impulse processes of more general type.

In Part II, first- and second-order statistics of impulse processes

(1.1) are derived. For this purpose $\{t_n\}$ may be any s.p.p. with finite second moment. A remarkably simple formula is then obtained for the correlation of $s(t)$; this formula corresponds to equation (69) of Leneman (1966a) but the intervals between successive t_n need be neither identically distributed nor mutually independent[1] as in Leneman (1966a).

Part III introduces models of impulse processes, and discusses their physical interpretation and applicability. Combinations of randomly-skipped and variously jittered sampling sequences are considered, correlated amplitude errors being taken into account. Special cases include plus-minus sampling, systematic skipping (as in time multiplexing), combined read-in and read-out jitters, etc. Computation of spectra (or correlations) for each of these models is carried out in Part IV.

II. FIRST- AND SECOND-ORDER STATISTICS

Of the statistics of a random process, the first and second moments are perhaps the most useful. Correlations and spectra, in particular, are required for the analysis of bandwidth occupancy, signal detectability, and message reconstruction (Leneman, 1966b; Leneman and Lewis, 1966). Indeed, first- and second-order moments provide necessary and sufficient information for those problems involving linear systems and/or Gaussian random processes. It is therefore natural that efforts to obtain statistical knowledge of impulse processes be centered on means and second moments.

Consider then the impulse process

$$s(t) = \sum_{-\infty}^{\infty} \alpha_n \delta(t - t_n), \qquad (2.1)$$

consisting of an infinite train of delta functions occurring at random times t_n with random intensities α_n. It is assumed throughout that $\{\alpha_n\}$ is a stationary discrete parameter random process, with $\{t_n\}$ an s.p.p. (cf. Beutler and Leneman, 1966a, 1966b) independent of $\{\alpha_n\}$. In order that the second moment of $s(t)$ be finite, the finiteness of the

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[1] S.p.p. for which the t_n are dependent and/or differently distributed from one another include many models of physical interest. Typical examples of such s.p.p. are jitter processes, processes with dependent skips (burst erasures), and non-uniformly spaced periodic sampling times. See Beutler and Leneman (1966a, 1966b) for precise definitions, discussion of properties, and examples.

expectations $E[(\alpha_n)^2]$ and $E[\{N(t, x)\}^2]$ are supposed.[2] Further hypotheses are not required for the validity of the formulas for the second moments of $s(t)$. Although these formulas hold generally, they appear as infinite sums, and summation to closed form becomes convenient only when additional conditions are imposed.

In order to avoid expectations of delta function products, and to utilize the knowledge of moments of $N(t, x)$, it is convenient to define moments of the impulse process indirectly. To this end, let $N(t)$ be a stationary increment stochastic process which is continuous from the right with

$$N(0) = 0, \tag{2.2}$$

and such that for $u \leqq v$

$$N(v) - N(u) = \sum_{n=-\infty}^{\infty} \alpha_n I_{(u,v]}(t_n). \tag{2.3}$$

In (2.3), $I_{(u,v]}(t_n)$ is an indicator function that is one or zero according as $t_n \in (u, v]$ or not. This means that

$$N(t) = \sum_{n=k}^{k+m-1} \alpha_n \tag{2.4}$$

if $t_k \leqq t$ is the first point to the right of the origin, and there are m points in the interval $(0, t]$. Since $\{\alpha_n\}$ is stationary, the moments of $N(t)$ depend only on m, and not on k.

The desired impulse process (2.1) is now obtained from $N(t)$ by differentiation, i.e.,

$$s(t) = \frac{dN(t)}{dt}. \tag{2.5}$$

Thus, the discontinuities of $N(t)$ become impulses of $s(t)$ with the intensity of each delta function determined by the corresponding α_n. This relationship, with typical $N(t)$ and $s(t)$ processes, is depicted by Fig. 2.1.

The computation of the expectation of a linear functional of a random process is often facilitated by interchanging the functional operation

[2] Here $N(t, x)$ refers to the number of points of $\{t_n\}$ occurring in the interval $(t, t + x]$. This notation, as well as that used elsewhere in the paper in referring to s.p.p., is consistent with that of Beutler and Leneman (1966a, 1966b). The reader is advised to familiarize himself with these two references, as properties of s.p.p. are basic to the present work.
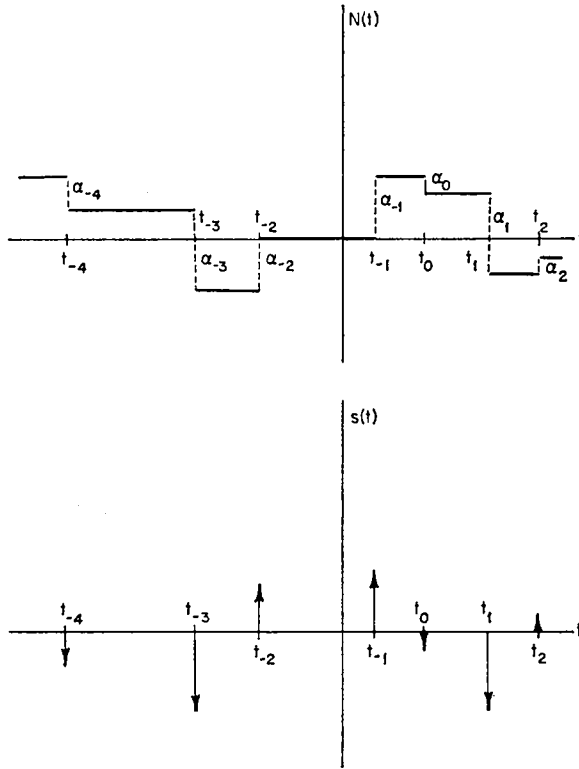
Fig. 2.1. Stationary Increment Process, and Corresponding Impulse Process.

with the expectation. Without inquiring into its validity here, we employ this interchange to determine moments of $s(t)$. Since $s(t)$ is the derivative of $N(t)$, its joint kth order moment becomes

$$E[s(t_1)s(t_2) \cdots s(t_k)] = \frac{\partial^k E[N(t_1)N(t_2) \cdots N(t_k)]}{\partial t_1 \partial t_2 \cdots \partial t_k}. \qquad (2.6)$$

In applying (2.6), it may be supposed without loss of generality that all the t_j are nonnegative and ordered according to magnitude. The ordering is irrelevant if it can be assumed that the order of differentiation is immaterial, and the time origin is unimportant because $N(t)$ is a stationary increment process.

Use of (2.6) makes it easy to show that the mean of $s(t)$ is

$$E[s(t)] = \alpha\beta \qquad (2.7)$$

where α is defined by

$$E[\alpha_n] = \alpha \tag{2.8}$$

and β is the average number of points per unit time interval. To verify (2.7), first apply the expectation to $N(t)$ as given by (2.3). Taking the expectation first with respect to $\{\alpha_n\}$ produces α, and $E[\sum I_{(0,t]}(t_n)] = E[N(0, t)] = \beta t$ by (5.4) of Beutler and Leneman (1966a). Thus,

$$E[N(t)] = \alpha\beta t \tag{2.9}$$

for $t \geqq 0$. Differentiating both sides on t and applying (2.6) to the left side then yields (2.7).

Calculation of the correlation of $s(t)$ is again based on (2.6); hence, $E[N(u)N(v)]$ must be obtained first. The expression for the latter is simplified by the expansion

$$E[N(u)N(v)]$$
$$= \tfrac{1}{2}E[N^2(u)] + \tfrac{1}{2}E[N^2(v)] - \tfrac{1}{2}E[\{N(v) - N(u)\}^2] \tag{2.10}$$

from which it follows that only $E[N^2(t)]$ need be determined. Indeed, $N(t)$ is a stationary increment process with $N(0) = 0$, so that the last term of (2.10) can be replaced by $E[N^2(v - u)]$ whenever $0 \leqq u \leqq v$. With this substitution (2.10) becomes

$$E[N(u)N(v)] = \tfrac{1}{2}\{E[N^2(u)] + E[N^2(v)] - E[N^2(v - u)]\}. \tag{2.11}$$

In view of the representation (2.4) for $N(t)$, each of the expectation terms on the right of (2.11) above is

$$E[N^2(t)] = \sum_{n=1}^{\infty} E\left[\left\{\sum_{k=1}^{n} \alpha_k\right\}^2\right] p(n, t), \tag{2.12}$$

in which $p(n, t)$ represents the probability that there are precisely n points in an interval of length t. Now $p(n, t) = G_n(t) - G_{n+1}(t)$, where G_n is the distribution of the waiting time from the origin up to the nth point occurring thereafter (compare (2.12) of Beutler and Leneman, 1966b). This means that (2.12) can be rewritten as

$$E[N^2(t)] = \sum_{n=1}^{\infty} E\left[\left\{\sum_{k=1}^{n} \alpha_k\right\}^2\right]\{G_n(t) - G_{n+1}(t)\}. \tag{2.13}$$

But

$$E\left\{\left[\sum_{k=1}^{n} \alpha_k\right]^2\right\} \leqq n^2 E[\alpha_j^2],$$

and $n^2 G_{n+1} \leqq n^2 G_n \to 0$, as shown in the proof of Theorem 3.2.2 in Beutler and Leneman (1966b); hence, a change in the index of summation on the G_{n+1} term in (2.13) is legitimate, and the same argument as in Theorem 3.2.2 of Beutler and Leneman (1966b) produces

$$E[N^2(t)] = E[\alpha_1{}^2]G_1(t) + \sum_{n=1}^{\infty} E\left[{}^{1+u}\!\rho\left(\alpha_{n+1} + 2\sum_{k=1}^{n}\alpha_k\right)\right]G_{n+1}(t). \quad (2.14)$$

It is seen that the mean square value of $N(t)$ appears in terms of the correlation function for $\{\alpha_n\}$, which is hereafter denoted by

$$\rho(k) = E[\alpha_{j+k}\alpha_j] \quad (\text{any } j). \quad (2.15)$$

With this notation, (2.14) is written

$$E[N^2(t)] = \rho(0)G_1(t) + \sum_{n=1}^{\infty}\left[\rho(0) + 2\sum_{k=1}^{n}\rho(k)\right]G_{n+1}(t). \quad (2.16)$$

It is possible to express (2.16) in even more compact form, using the interval distribution functions[3] F_n in place of the forward occurrence (waiting) time distribution G_n. From $g_n = s_n - s_{n-1}$ (lower case symbols represent derivatives of the corresponding upper case letters) and (6.1) of Beutler and Leneman (1966a), one deduces that

$$g_n(t) = \beta[F_{n-1}(t) - F_n(t)], \quad (2.17)$$

with $F_0(t) = 1$ for $t > 0$. Thus, substituting (2.17) into (2.16) yields

$$E[N^2(t)] = \beta \int_0^t \Big\{\rho(0)[1 - F_1(u)]$$
$$+ \sum_{n=1}^{\infty}\left[\rho(0) + 2\sum_{k=1}^{n}\rho(k)\right][F_n(u) - F_{n+1}(u)]\Big\}\,du \quad (2.18)$$

in which the interchange of integration and summation is valid because the integrand summands are dominated by an integrable series of non-negative terms (compare proof of Theorem 3.3.3 in Beutler and Leneman, 1966b). Further, $nF_n \to 0$ as $n \to \infty$, so that a change in the index of summation on the right-most term in (2.18) leads to the simplified form

$$E[N^2(t)] = \beta\rho(0)t + 2\beta \int_0^t \sum_{n=1}^{\infty} \rho(n)F_n(u)\,du. \quad (2.19)$$

[3] As noted in Section 3.4 of Beutler and Leneman (1966b), F_n is actually the conditional probability distribution function for the nth point after a given starting reference (say t), given that there is a point at t. In other words, F_n may be viewed as the distribution function for n successive intervals.

It is convenient to call

$$K(x) = \sum_{n=1}^{\infty} \rho(n)F_n(x);$$ (2.20)

then substitution of (2.19) into (2.11) yields for the correlation of $N(t)$

$$E[N(u)N(v)] = \beta\rho(0)u + \beta\left\{\int_0^u K(x)\,dx\right.$$

$$\left. + \int_0^v K(x)\,dx - \int_0^{v-u} K(x)\,dx\right\},$$ (2.21)

whenever $0 \leqq u \leqq v$. Considerations of symmetry lead to the same expression for $v < u$, except that these two symbols are interchanged in (2.21). The cases $u \leqq v$ and $v < u$ can therefore be combined in a single relation that holds for both:

$$E[N(u)N(v)] = \beta\left\{\rho(0)\min(u,v) + \int_0^u K(x)\,dx\right.$$

$$\left. + \int_0^v K(x)\,dx - \int_0^{|v-u|} K(x)\,dx\right\}.$$ (2.22)

It remains to differentiate (2.22) in accordance with (2.6) to obtain the correlation function for the impulse process $s(t)$ from the correlation of $N(t)$. For the derivative on u, one has

$$\frac{\partial\{E[N(u)N(v)]\}}{\partial u}$$ (2.23)

$$= \beta\{\rho(0)U(v-u) + K(u) + K(|v-u|)\operatorname{sgn}(v-u)\},$$

where U is the unit step function. Upon differentiating (2.23) with respect to v, one finally obtains for the correlation $R_s(u,v)$

$$E[s(u)s(v)] = \beta\rho(0)\delta(v-u) + \beta\sum_{n=1}^{\infty}\rho(n)f_n(|v-u|).$$ (2.24)

Here f_n is the derivative of F_n; f_n may well contain delta functions corresponding to discontinuities of F_n. Because of the intuitive meaning imputed to F_n, f_n can be regarded as the probability density function associated with n consecutive intervals of the s.p.p. $\{t_n\}$.

Unless the sum in (2.24) can be evaluated, further explicit results cannot be readily obtained. Frequently, however, summation in closed form becomes possible when (2.24) is subjected to a Fourier transforma-

tion. This suggests that it may be easier to calculate directly the spectral density $\Phi_s(\omega)$ of $s(t)$, than to evaluate (2.24) to find its correlation. For this purpose, let

$$f_n^*(s) = \int_0^\infty f_n(t)e^{-st} \, dt \qquad (2.25)$$

which is related to the characteristic function for n successive interval lengths, since $f_n(t) = 0$ for negative argument. The definition (2.25) can be extended to negative indices by taking

$$f_{-n}^*(i\omega) = f_n^*(-i\omega) \qquad (2.26)$$

and adopting the convention $f_0(i\omega) = 1$. Taking the Fourier transform of the correlation $R_s(\tau)$ (with $\tau = v - u$) furnished by (2.24) yields

$$\Phi_s(\omega) = \beta \sum_{n=-\infty}^\infty \rho(n)f_n^*(i\omega). \qquad (2.27)$$

The doubly-infinite sum results because

$$\int_{-\infty}^\infty \rho(n)f_n(|\tau|) \, e^{-i\omega\tau} \, d\tau = \int_0^\infty \rho(n)f_n(\tau)e^{-i\omega\tau} \, d\tau$$
$$\qquad (2.28)$$
$$+ \int_0^\infty \rho(n)f_n(\tau)e^{+i\omega\tau} \, d\tau = \rho(n)f_n^*(i\omega) + \rho(n)f_n^*(-i\omega),$$

and $\rho(n) = \rho(-n)$. Equation (2.27) and its equivalent

$$\Phi_s(\omega) = \beta\{\rho(0) + \sum_{n=1}^\infty \rho(n)[f_n^*(i\omega) + f_n^*(-i\omega)]\} \qquad (2.29)$$

are relations central to this paper. They represent remarkably simple expressions for the spectrum of $s(t)$, and may be explicitly evaluated for a wide variety of pulse trains.

Even for special cases of (2.1), the second-order properties are often not well known. Yet, a simple application of (2.24) may yield useful second-order properties. For example,

$$E\left[\sum_{-\infty}^\infty \delta(t - t_n)\delta(t + \tau - t_n) \right] = \beta\delta(\tau) \qquad (2.30)$$

follows immediately from (2.24) with $\rho(n) = \delta_{0n}$. Similarly, $\rho(0) = 2$, $\rho(\pm k) = 1$, and other $\rho(n) = 0$ leads to a result from which (2.30) is

subtracted to give

$$E\left[\sum_{n=-\infty}^{\infty} \delta(t + \tau - t_n)\delta(t - t_{n-k})\right.$$
$$\left. + \sum_{n=-\infty}^{\infty} \delta(t + \tau - t_n)\delta(t - t_{n+k})\right] = \beta f_k(|\tau|). \tag{2.31}$$

It is emphasized that (2.24), (2.27), and (2.29) require neither statistical independence nor identically-distributed interval lengths, as was required for the derivation in Leneman (1966a). In the latter (but not here), it was also supposed that the interval lengths are bounded away from zero [Leneman, 1966a, Eq. (27)]. Thus, the validity of many of the examples of spectral computation in the above paper was limited; the present paper permits results of greater generality.

III. MODELS OF IMPULSE PROCESSES

Although the spectrum for the $s(t)$ of (2.1) may be discussed without reference to specific physical applications, it appears desirable to emphasize those $\{\alpha_n\}$ and $\{t_n\}$ statistics that correspond to technologically interesting $s(t)$. A large class of $s(t)$ is associated with impulse sampling of another (ordinary wide-sense stationary) random process $x(t)$; most of the examples of the next section are motivated by this application.

The desirability of a spectrum (or correlation) in closed form restricts the $\{t_n\}$ that can be usefully considered. In particular, the infinite series in (2.24) or (2.27) ought to be summable to a closed form expression. One such $\{t_n\}$ is that of a Poisson s.p.p., in which the probability of n points in an interval of length x is $(\beta x)^n e^{-\beta x}/n!$. This type of sampling might be encountered as pseudo-randomly timed PAM, or as an approximation to outputs from a random access memory. For the Poisson s.p.p.,

$$f_n^*(i\omega) = \left(\frac{\beta}{i\omega + \beta}\right)^n. \tag{3.1}$$

It may be that certain of the samples are lost or expunged. If these deletions are random and mutually independent, the new point process is called a skip Poisson process. Then f_n^* has the same form (3.1) as before, except that the parameter β is replaced by $(1 - q)\beta$ where q is the probability that a given sample is skipped (compare Parzen (1962) and Beutler and Leneman (1966a)); thus, the skip Poisson process does not require separate analysis. Another variation of the Poisson point process is obtained through systematic skipping. For instance, if k separate

signals are sampled in turn (e.g., time multiplexed), the f_n^* for each signal becomes

$$f_n^*(i\omega) = \left(\frac{\beta}{i\omega + \beta}\right)^{kn},\tag{3.2}$$

in which β is the average number of points per unit time for all k signals combined.

The second major class of sampling schemes considered here represents uniformly-spaced sampling modified by the introduction of interference or errors. Random mutually-independent erasures (skips), each occurring with probability q, may be applied to uniform sampling of period T to yield

$$f_n^*(i\omega) = \left[\frac{(1 - q)e^{-i\omega T}}{1 - qe^{-i\omega T}}\right]^n.\tag{3.3}$$

The sample times t_n may also be jittered, that is, displaced from their nominal position (which may be called t_n') by a small[4] amount u_n, so that the actual sampling time is

$$t_n = t_n' + u_n.\tag{3.4}$$

Now let the jitters u_n applied to the t_n' [of a skip process as in (3.4)] be identically distributed and pairwise independent. If each u_n has characteristic function γ, the f_n^* for the jittered skip process is

$$f_n^*(i\omega) = |\gamma(i\omega)|^2 \left[\frac{(1 - q)e^{-i\omega T}}{1 - qe^{-i\omega T}}\right]^n,\tag{3.5}$$

as shown in equation (7.20) of Beutler and Leneman (1966a). Since (3.5) generalizes (3.3), which in turn generalizes uniformly spaced sampling, only the jittered skip process of (3.5) need be considered as an example in Section IV.

There are at least two ways of relating the impulse train

$$s(t) = \sum_{-\infty}^{\infty} \alpha_n \delta(t - t_n)\tag{3.6}$$

to the sampling of a (wide-sense stationary) random process $x(t)$. At first glance, it would seem that the computation of the spectral density

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[4] A "small" displacement is any that does not alter the original ordering of the t_n. It suffices, for instance, if the u_n are restricted in their range to $[-T/2, T/2)$ or $[0, T)$.

of the sample impulses is most easily achieved by taking $\alpha_n = x(t_n)$, so that $s(t)$ itself becomes the sampled sequence. But the $\rho(n)$ defined by (2.15) is then the correlation $R_x(t_{k+n} - t_k)$ of $x(t)$, which in general depends on $t_{k+n} - t_k$ rather than just on n. An awkward condition on $x(t)$ would then be required to assure that $R_x(t_{k+n} - t_k)$ is a function only of n for almost all realizations of $\{t_n\}$. For this reason it is generally preferable to proceed otherwise, regarding $s(t)$ as a linear modulation of $x(t)$. The sampled sequence now becomes $y(t) = s(t)x(t)$. If [as is usually true] $x(t)$ and $s(t)$ are independent random processes [or even only orthogonal], the correlation of $y(t)$ is obtained from $R_y(\tau) = R_s(\tau)R_x(\tau)$. In the frequency domain, the equivalent expression in terms of spectra becomes a convolution, viz.

$$\Phi_y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega - u)\Phi_s(u) \, du = \Phi_x * \Phi_s. \qquad (3.7)$$

Thus, the evaluation of Φ_y remains contingent on the computation of Φ_s, which is the purpose of this paper.

It follows from (2.24) or (2.27) that $\{\alpha_n\}$ influences the second-order properties of $s(t)$ only through $\rho(n)$. For most sampling modulations $s(t)$, it suffices to consider $\{\alpha_n\}$ with rational spectral density (in $e^{i\omega}$); hence, one assumes that $\rho(n)$ is of the form

$$\rho(n) = \sum_{k=1}^{m} \sum_{j=0}^{r_k} a_{jk} \mid n \mid^j \rho_k^{\mid n \mid}. \qquad (3.8)$$

in which $-1 \leq \rho_k \leq +1$. However, the correlation and spectrum of $s(t)$ is linear in $\rho(n)$, so that it is enough to calculate R_s or Φ_s for a general term on the right of (3.8). For the sake of simplicity, we shall be content with the somewhat more specialized result corresponding to distinct roots in the spectrum of $\{\alpha_n\}$. Then one need only take $\rho(n)$ to be of the form

$$\rho(n) = \rho^{\mid n \mid}, \qquad -1 \leq \rho \leq +1. \qquad (3.9)$$

The $\rho(n)$ of (3.9) and its variants cover many cases of interest. For sampling without amplitude error, $\alpha_n \equiv 1$, so that $\rho(n) = 1$ for all n. Plus-minus sampling corresponds to $\rho(n) = (-1)^n$, whence $\rho = -1$. It is also possible to introduce amplitude error in the sampling procedure, taking

$$\alpha_n = 1 + w_n, \qquad (3.10)$$

w_n being a scaling error in the magnitude of the sample taken at t_n. If w_n has zero mean and variance σ^2, and the w_n are exponentially correlated, one has

$$\rho(n) = 1 + \sigma^2 \rho^{|n|}. \tag{3.11}$$

It is easy to specialize (3.11) to uncorrelated amplitude errors by taking $\rho = 0$, with $\rho^0 = 1$.

It will be observed that the correlation between the pulse amplitudes α_n appearing in (3.6) depends not on the time intervals between pulses, but only on the count of pulses. For instance, $\{t_n\}$ might be a skip process, in which case the correlation between successive pulse intensities would fail to depend on the number of skips intervening. The latter situation suggests that a more appropriate model (for some applications) would render such correlation a function of the time between samples. To this end, consider

$$s(t) = \sum_{-\infty}^{\infty} \alpha_n {}'y_n \delta(t - t_n), \tag{3.12}$$

in which each y_n is zero or unity according as the corresponding t_n is to be skipped or not. If skipping is again to occur with probability q, and the erasures are mutually independent, $E[y_{m+n}y_m] = (1 - q)^2 + q(1 - q)\delta_{0n}$, where δ_{0n} is the Kronecker delta. Further, it may be assumed that the α_n' are correlated in accordance with (3.9). Then, if one writes $\alpha_n = \alpha_n' y_n$ to reduce (3.12) to (3.6), one obtains for the correlation $\rho(n)$ of $\{\alpha_n\}$

$$\rho(n) = (1 - q)^2 \rho^{|n|} + q(1 - q)\delta_{0n}. \tag{3.13}$$

The model just described is called the time correlated skip model, whereas the earlier one is designated as the number interval correlated skip model.

Spectra of jittered sampling have been investigated by others (Balakrishnan, 1962; Brown, 1963), but the work presented here generalizes these studies by also permitting skips as well as correlated amplitude errors. Moreover, Brown's distinction between locked jitter and separate read-in and read-out jitters (Brown, 1963) can also be carried through here. As in Brown's paper, read-in jitter denotes the displacement of the samples in time by a jitter perturbation, whereas read-out jitter refers to a sample at t_n', [i.e., $x(t_n')$] being read out at a different time t_n. The appropriate expression for separate read-in and read-out jittered

sampling with skips and time interval correlated amplitude errors is

$$y(t) = \sum_{-\infty}^{\infty} \alpha_n' y_n x(t_n') \delta(t - t_n), \qquad (3.14)$$

in which $\{t_n'\}$ is the original sampling process with jitters u_n, i.e.,

$$t_n' = t_{00} + nT + u_n. \qquad (3.15)$$

Here t_{00} is uniformly distributed on the interval $[0, T)$, and the u_n (jitter process) are pairwise independent, independent of t_{00}, and distributed in some manner over $0 \leqq u_n < T$. $\{t_n\}$ is also a jitter process; the t_n are specified by

$$t_n = t_{00} + nT + v_n. \qquad (3.16)$$

It is assumed that $\{v_n\}$ is independent of t_{00} and $\{u_n\}$, and that the v_n are pairwise-independent, identically-distributed random variables, each taking on values in the interval $[0, T)$. The other notations and assumptions of (3.14) are identically those introduced earlier in the section. For locked jitter (with time correlated amplitude errors under skipping), (3.16) is replaced by $t_n = t_n'$; this corresponds precisely to the previously discussed (3.12).

   Although there are many other impulse processes whose spectra can (and have) been calculated by the methods suggested in this paper, computations for the above processes will suffice to illustrate the power and nature of the methods proposed herein.

IV. MOMENTS OF IMPULSE PROCESSES: EXAMPLES

   This section is devoted to the computation of spectra and/or correlations of $s(t)$ [or $y(t)$, as applicable]. Although the section's title might imply that other moments (the mean of $s(t)$ and cross-correlation R_{xy}) are also to be found, these latter will receive little further attention. Indeed, the mean of the impulse process $s(t)$ is obtained almost by inspection from (2.7), while for the cross-correlation one has

$$R_{xy}(\tau) = E[x(t + \tau)y(t)] = E[x(t + \tau)x(t)s(t)] = \alpha\beta R_x(\tau)$$

by virtue of the assumed independence of $x(t)$ and $s(t)$. Thus, the significant task is the computation of the correlation and/or spectrum of $s(t)$ [or $y(t)$], and it is with this that the remainder of the paper is concerned.

   For the first example, consider *Poisson sampling with exponentially*

correlated amplitudes. Here $\rho(n)$ is given by (3.9), and $f_n{}^*$ by (3.1); these are substituted into (2.29), which may also be written as

$$\Phi_s(\omega) = \beta\left\{\rho(0) + 2\Re\left[\sum_{n=1}^{\infty} \rho(n)f_n{}^*(i\omega)\right]\right\}. \qquad (4.1)$$

In (4.1), $\Re$ denotes "real part of," the formula being valid because $f_n{}^*(i\omega)$ and $f_n{}^*(-i\omega)$ are complex conjugates. The principal task in the evaluation of (4.1) is the summation of the series

$$\sum_{n=1}^{\infty} \rho(n)f_n{}^*(i\omega).$$

In the case of the Poisson s.p.p., the latter is a power series, and the relation

$$\sum_{n=1}^{\infty} z^n = z(1 - z)^{-1}$$

is applicable. Upon performing the indicated operations on the series, and taking the real part of the result, one obtains for the spectral density

$$\Phi_s(\omega) = \beta\,\frac{\omega^2 + \beta^2(1 - \rho^2)}{\omega^2 + \beta^2(1 - \rho)^2}\,. \qquad (4.2)$$

It is easy to find the spectral density for *plus-minus Poisson sampling* from (4.2); one simply sets $\rho = -1$ in (4.2). Then

$$\Phi_s(\omega) = \frac{\beta\omega^2}{\omega^2 + 4\beta^2}\,. \qquad (4.3)$$

Similarly, one could set $\rho = +1$ to model *ordinary Poisson sampling.* However, the behavior of $\Phi_s(\omega)$ near $\omega = 0$ is then incompletely specified. The possible anomaly can be avoided by using (2.24) to determine the correlation R_s. Since $\rho(n) = 1$, and

$$f_n(\tau) = \beta(\beta\tau)^{n-1}e^{-\beta\tau}/(n - 1)!, \qquad \tau \geqq 0, \qquad (4.4)$$

$$R_s(\tau) = \beta[\delta(\tau) + \beta], \qquad (4.5)$$

whence (taking the Fourier transform of R_s)

$$\Phi_s(\omega) = \beta[1 + 2\pi\beta\delta(\omega)]. \qquad (4.6)$$

The *alternated Poisson sampling* process consists of Poisson skip sampling with every other point deleted. This means that $f_n{}^*$ is specified by (3.2) with $k = 2$. One may again substitute in (4.1) with $\rho(n) = 1$,

thereby obtaining a power series which is well behaved except at $\omega = 0$. To find the value of $\Phi_s(0)$, it suffices to note that $s(t)$ has mean $\beta/2$, so that Φ_s has a delta function of intensity $\beta^2\pi/2$ at the origin. The complete expression for Φ_s is then

$$\Phi_s(\omega) = \beta\left[\frac{\omega^2 + 2\beta^2}{\omega^2 + 4\beta^2} + \frac{\pi\beta\delta(\omega)}{2}\right]. \tag{4.7}$$

This formula was found via a different approach by Mazzetti, 1962, and (along with the others related to Poisson sampling) also appears in Leneman, 1966a.

The remainder of the results presented here are motivated by consideration of errors arising in (supposedly) uniformly-spaced sampling. The spectral density of a *skip jittered sampling sequence with exponentially number interval correlated amplitudes* is calculated first. This means that $s(t)$ is given by (3.6), with the $\rho(n)$ of (3.9) and $f_n{}^*$ of (3.5). Evaluation of

$$\sum_{n=1}^{\infty} \rho(n)f_n{}^*(i\omega)$$

once more amounts to the summation of a power series, the sum being

$$\sum_{n=1}^{\infty} \rho(n)f_n{}^*(i\omega) = \frac{\rho(1 - q)e^{-i\omega T}}{1 - \mu e^{-i\omega T}}, \tag{4.8}$$

where the parameter μ is given by

$$\mu = (1 - q)\rho + q = \rho + (1 - \rho)q. \tag{4.9}$$

Now (4.8) is substituted into (4.1), in which $\beta = (1 - q)/T$ and $\rho(0) = 1$. Some manipulation leads to an expression for Φ_s in terms of the Poisson kernel (Hoffman, 1962)

$$P(\mu, \omega) = \frac{1 - \mu^2}{1 - 2\mu\cos\omega T + \mu^2}, \tag{4.10}$$

that is,

$$\Phi_s(\omega) = \frac{1 - q}{T}\left\{1 - \frac{(1 - q)\rho\,|\,\gamma(i\omega)\,|^2}{\mu}\right.$$
$$\left. + \frac{(1 - q)\rho\,|\,\gamma(i\omega)\,|^2}{\mu}\,P(\mu, \omega)\right\}. \tag{4.11}$$

The general result (4.11) may be specialized in various directions. If, for instance, there is no jitter, γ is taken as unity, while no skipping makes $q = 0$. Other interesting cases are those of the *ordinary skip jittered sampling sequence* and the *plus-minus skip jittered sampling sequence*.[5] For the former, $\rho(n) = 1$ (since all $\alpha_n = 1$), so that $\rho = 1$, and hence $\mu = 1$. The latter is characterized by $\rho(n) = (-1)^n$, whence $\rho = -1$ and $\mu = 2q - 1$. An attempt to apply (4.11) directly to ordinary skip jittered sampling then fails because both numerator and denominator of (4.10) are zero. However, $P(\mu, \omega)$ is an approximate identity (Hoffman, 1962) so that it is proper to use the interpretation (Hoffman, 1962; Lighthill, 1958)

$$\lim_{\mu \to +1} P(\mu, \omega) = \frac{2\pi}{T} \sum_{-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right). \tag{4.12}$$

The spectral density for ordinary skip jittered sampling then becomes

$$\Phi_s(\omega) = \frac{1 - q}{T}\left\{ 1 - (1 - q) \mid \gamma(i\omega) \mid^2 \right.$$
$$\left. + \frac{2\pi(1 - q)}{T} \sum_{n=-\infty}^{\infty} \left| \gamma\left(\frac{2\pi ni}{T}\right) \right|^2 \delta\left(\omega - \frac{2\pi n}{T}\right)\right\}. \tag{4.13}$$

Interestingly enough, the format of the spectrum for plus-minus skip jittered sampling depends on the skip probability. For high skip probabilities, i.e., $q > \frac{1}{2}$, $\mu > 0$, (4.11) remains applicable; as before, $P(\mu, \omega)$ attains its maxima at $\omega = 2\pi n/T$. When $q = \frac{1}{2}$, $\mu = 0$, so that direct evaluation of (4.11) is impracticable. However, one may return to (4.8), noting that twice its real part is now merely $-\cos \omega T$. According to (4.1), the spectrum is then

$$\Phi_s(\omega) = \frac{1}{2T} \{1 - \mid \gamma(i\omega) \mid^2 \cos \omega T\}. \tag{4.14}$$

With passage to smaller skip probabilities, $0 < q < \frac{1}{2}$, μ becomes negative. Since

$$P(\mu, \omega) = P\left(-\mu, \omega + \frac{\pi}{T}\right), \tag{4.15}$$

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[5] It is characteristic of this model that successive samples alternate in sign, regardless of the multiple of T that (due to skipping) separates adjacent sample times t_n .

the periodic minima of $P(\mu, \omega)$ occur at $\omega = 2\pi n/T$. Since μ is negative, $P(\mu, \omega)/\mu$ has its maxima at these ω; hence, there are definite maxima at the same ω for all $0 < q < 1$ except $q = \frac{1}{2}$. Only $q = 0$ (no skipping) remains to be considered. This q corresponds to $\mu = -1$, so that it is necessary to use

$$\lim_{\mu \to -1} P(\mu, \omega) = \frac{2\pi}{T} \sum_{-\infty}^{\infty} \delta \left(\omega - \frac{(2n + 1)\pi}{T} \right) \qquad (4.16)$$

in applying (4.11) to plus-minus jittered sampling (without skips). The spectral density for this type of sampling is now

$$\Phi_s(\omega) = \frac{1}{T} \Big\{ [1 - | \gamma(i\omega) |^2]$$
$$+ \frac{2\pi}{T} \sum_{-\infty}^{\infty} \left| \gamma \left(\frac{[2n + 1]\pi i}{T} \right) \right|^2 \delta \left(\omega - \frac{(2n + 1)\pi}{T} \right) \Big\}. \qquad (4.17)$$

The calculations made earlier suffice for finding the spectral density of *skip jittered sampling with exponentially number interval correlated scaling errors*. In fact, the desired result is a weighted linear combination of (4.11) and (4.13), because of (3.11) and the linearity of Φ_s with ρ. For such sampling, therefore, the sampling sequence spectral density is given by

$$\Phi_s(\omega) = \frac{1 - q}{T} \Big\{ \sigma^2 \Big[1 - \frac{(1 - q)\rho \, | \gamma(i\omega) |^2}{\mu} \Big]$$
$$+ [1 - (1 - q) | \gamma(i\omega) |^2] + (1 - q) | \gamma(i\omega) |^2 \qquad (4.18)$$
$$\cdot \Big[\frac{2\pi}{T} \sum_{-\infty}^{\infty} \delta \left(\omega - \frac{2\pi n}{T} \right) + \frac{\rho \sigma^2}{\mu} P(\mu, \omega) \Big] \Big\}.$$

Each of the above skip jitter sampling models has a counterpart in which the impulse strengths are time correlated rather than number correlated. The impulse train is described by (3.12), with the $\rho(n)$ furnished by (3.13). Upon substituting (3.13) and $| \gamma(i\omega) |^2 e^{-i\omega n T'}$ into (4.1), one need only sum the resulting power series. The spectral density for a *skip jittered sampling sequence with exponentially time correlated amplitudes* is therefore

$$\Phi_s(\omega) = \frac{1 - q}{T} \{ 1 - (1 - q) | \gamma(i\omega) |^2$$
$$+ (1 - q) | \gamma(i\omega) |^2 P(\rho, \omega) \}. \qquad (4.19)$$

Term-by-term comparison of (4.19) with (4.11) (the corresponding spectral density for number interval correlated amplitudes) shows the two to be identical iff $\rho = 1$ (all pulses unit intensity) or $q = 0$ (no skipping), just as would be expected. In fact, both models of ordinary skip jittered sampling sequences represent the same impulse process. However, the *plus-minus skip jittered sampling sequence* derived from (3.12) differs from that considered earlier. Now (compare footnote 5) successive samples have the same sign if they are separated by an odd number of skipped t_n, and opposite signs if the number of skipped t_n is odd. In contrast to the dependence of the format of the spectral density on skip probability q as in the earlier model, the plus-minus skip jittered sampling of (3.12) always has line spectrum components at $\omega = (2n + 1)\pi/T$ (all integer n). More precisely, its spectral density is

$$\Phi_s(\omega) = \frac{1-q}{T}\left\{ 1 - (1-q) \mid \gamma(i\omega) \mid^2 \right.$$
$$\left. + \frac{2\pi(1-q)}{T} \sum_{-\infty}^{\infty} \left| \gamma\left(\frac{[2n+1]\pi i}{T}\right) \right|^2 \delta\left(\omega - \frac{(2n+1)\pi}{T}\right) \right\}. \tag{4.20}$$

A linear combination of (4.13) and (4.19) yields the spectral density of *skip jittered sampling with exponentially time correlated scaling errors*. This is the model represented by (3.14), with read-in and read-out times furnished by (3.15) and (3.16), respectively. Now (3.14) may be reduced to (3.6) by redefining $\alpha_n = \alpha_n' y_n x(t_n')$, and taking $\{t_n\}$ to be a jitter process without skips, the characteristic function of the jitters v_n being designated μ. Then the spectral density of the sampled signal is

$$\Phi_y(\omega) = \rho(0) + \mid \gamma(i\omega) \mid^2 \left(\sum_{-\infty}^{\infty} \rho(n)e^{-i\omega nT} - \rho(0) \right). \tag{4.22}$$

In order to complete the calculation of Φ_y, it is necessary to evaluate $\rho(n)$, and to find the indicated sum in (4.22). Because of the assumed independence of the pertinent random processes, and in view of their wide-sense stationarity,

$$\rho(n) = \rho'(n)E[R_x(t_{m+n}' - t_m')]E[y_{m+n}y_m]. \tag{4.23}$$

In (4.23), $\rho'(n)$ refers to $E[\alpha_{m+n}'\alpha_m']$,

$$E[y_{m+n}y_m] = \begin{cases} (1-q) & n = 0 \\ (1-q)^2 & n \neq 0 \end{cases} \tag{4.24}$$

as in the discussion following (3.12), and $E[R_x(t'_{m+n} - t_m')]$ is meant to be the expectation on $\{u_n\}$. The latter can be conveniently expressed in terms of the spectral density Φ_x of $x(t)$. If $\lambda'(n)$ is used to denote $E[R_x(t'_{m+n} - t_m')]$, an interchange of expectation and integration yields

$$\lambda'(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) E[e^{i\omega(t'_{m+n} - t_m')}] \, d\omega. \tag{4.25}$$

Now $t'_{m+n} - t_m' = nT + (u_{m+n} - u_m)$, so that for $n \neq 0$

$$\lambda'(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) \mid \mu(i\omega) \mid^2 e^{i\omega nT} \, d\omega, \tag{4.26}$$

μ being the characteristic function of each read-in jitter variable u_n (the u_n are assumed pairwise independent). The role of λ' is clarified by dividing up the real line [interval of integration in (4.26)] into intervals of length 2π, so that (4.26) becomes

$$\lambda'(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{-\infty}^{\infty} \Phi_x(\omega - 2\pi k) \mid \mu(i\omega - 2\pi ik) \mid^2 \right] e^{i\omega n} \, d\omega; \tag{4.27}$$

this exhibits $\lambda'(n)$, $n \neq 0$, as the nth Fourier coefficient of the bracketed expression in (4.27). In general, one should choose the intervals into which the integral (4.26) is divided to be of length $2\pi/T$ rather than just 2π; however, for convenience (only) we are assuming here and henceforth that $T = 1$. Since $\lambda'(n)$ represents the Fourier coefficients of

$$\sum_{-\infty}^{\infty} \Phi_x(\omega - 2\pi k) \mid \mu(i\omega - 2\pi ik) \mid^2$$

only for $n \neq 0$, one defines

$$\lambda(n) = \begin{cases} \lambda'(n) & n \neq 0 \\ \dfrac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) \mid \mu(i\omega) \mid^2 d\omega & n = 0, \end{cases} \tag{4.28}$$

which gives the Fourier coefficients over $(-\pi, \pi)$ of this sum for all n. In other words

$$\sum_{-\infty}^{\infty} \lambda(n) e^{-i\omega n} = \sum_{-\infty}^{\infty} \Phi_x(\omega - 2\pi k) \mid \mu(i\omega - 2\pi ik) \mid^2. \tag{4.29}$$

With the further simplifying assumption (made without loss of gen-

erality) that $E[x^2(t)] = 1$, $\lambda'(0) = 1$, and the coefficients $\rho(n)$ in (4.22) become

$$\rho(0) = (1 - q)(1 + \sigma^2) \tag{4.30}$$

and since λ' is even,

$$\rho(n) = (1 - q)^2(1 + \sigma^2\rho^{|n|})\lambda'(|n|), \qquad n \neq 0. \tag{4.31}$$

Substitution of this $\rho(n)$ into (4.22) yields

$$\Phi_y(\omega) = (1 - q)\Bigg\{(1 + \sigma^2)[1 - (1 - q)|\gamma(i\omega)|^2\lambda(0)]$$

$$+ (1 - q)|\gamma(i\omega)|^2\Bigg[\sum_{-\infty}^{\infty}\lambda(n)e^{-i\omega n} \tag{4.32}$$

$$+ \sigma^2\sum_{-\infty}^{\infty}\lambda(n)\rho^{|n|}e^{-i\omega n}\Bigg]\Bigg\}.$$

The first sum in (4.32) is evaluated from (4.29). The second sum is the Fourier expansion at $z = \rho e^{-i\omega}$ of the analytic function specified on the unit circle by the right side of (4.29). It is well known (Hoffman, 1962) that this function is furnished by a convolution with the Poisson kernel[6], viz.

$$\sum_{-\infty}^{\infty}\lambda(n)\rho^{|n|}e^{-i\omega n} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty}\Phi_x(u - 2\pi k)|\mu(iu - 2\pi ik)|^2$$

$$\cdot P(\rho, \omega - u)\,du \tag{4.33}$$

$$= \Bigg[\sum_{-\infty}^{\infty}\Phi_x(\omega - 2\pi k)|\mu(i\omega - 2\pi ik)|^2\Bigg] * P(\rho, \omega).$$

Using (4.29) and (4.33) in (4.32) leads to the final form

$$\Phi_y(\omega) = (1 - q)\Bigg\{(1 + \sigma^2)\Bigg[1 - (1 - q)|\gamma(i\omega)|^2$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_x(\omega)|\mu(i\omega)|^2\,d\omega\Bigg]$$

$$\tag{4.34}$$

$$+ (1 - q)|\gamma(i\omega)|^2\Bigg[\sum_{-\infty}^{\infty}\Phi_x(\omega - 2\pi k)|\mu(i\omega - 2\pi ik)|^2$$

$$+ \sigma^2\Bigg(\sum_{-\infty}^{\infty}\Phi_x(\omega - 2\pi k)|\mu(i\omega - 2\pi ik)|^2\Bigg) * P(\rho, \omega)\Bigg]\Bigg\}.$$

[6] The right-hand sum in (4.29) converges almost everywhere, and is integrable over $[-\pi, \pi]$; hence the asserted representation as a convolution with the Poisson kernel is valid for $0 < \rho < 1$. For $-1 < \rho < 0$, the result is still valid, but with ρ replaced by $|\rho|$, and ω by $\omega + \pi$.

It is interesting to compare the above result with the spectral density for locked (in place of independent) jitters. If the jitters are locked, $s(t)$ is furnished by (3.12), since

$$y(t) = x(t)s(t) = \sum_{-\infty}^{\infty} \alpha_n' x(t_n) y_n \delta(t - t_n).$$

For an $s(t)$ of this form, the spectral density is already available from (4.21). Specializing the latter with $T = 1$, and carrying out the convolution of spectra implied by $y(t) = x(t)s(t)$, yields for a *sampled signal with locked read-in and read-out jitters, skips, and time-correlated scaling errors*

$$\Phi_y(\omega) = (1 - q)\left\{ (1 + \sigma^2)\left[1 - (1 - q)(\mid \gamma \mid^2 \mathbin{\bar{\divideontimes}} \Phi_x)\right] \right.$$

$$+ (1 - q)\left[\sum_{-\infty}^{\infty} \Phi_x(\omega - 2\pi k)\mid \gamma(\omega - 2\pi k)\mid^2 \right. \qquad (4.35)$$

$$\left. + \sigma^2 (\mid \gamma \mid^2 P(\rho, \cdot) \mathbin{\underline{*}} \Phi_x)\right]\right\}.$$

The symbol $\bar{\divideontimes}$ has been introduced to denote convolution over the entire real axis, rather than over the interval $[-\pi, \pi]$. Since (except in the trivial case of no jitter) "small" jitter variables preclude that γ be periodic with period 2π, (4.35) cannot be modified to a convolution form similar to that of (4.34).

Note added in proof: The authors thank one of the referees for directing their attention to the work of Kryukov, 1967. His results unify some earlier calculations of spectra of impulse processes by deriving a formula like our (2.24); however, he treats only impulses of the same fixed intensity and sampling points specified by a renewal process.

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