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The Spectral Density Function for the Laplacian on High Tensor Powers of a Line Bundle

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Abstract. For a symplectic manifold with quantizing line bundle, a choice of almost complex structure determines a Laplacian acting on tensor powers of the bundle. For high tensor powers Guillemin–Uribe showed that there is a well-defined cluster of low-lying eigenvalues, whose distribution is described by a spectral density function. We give an explicit computation of the spectral density function, by constructing certain quasimodes on the associated principle bundle.

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1. Introduction

Let X be a compact $2n$-dimensional almost Kähler manifold, with symplectic form ω and almost complex structure J. *Almost Kähler* means that ω and J are compatible in the sense that

$$\omega(Ju, Jv) = \omega(u, v) \quad \text{and} \quad \omega(\cdot,\ J\cdot) \gg 0.$$

The combination thus defines an associated Riemannian metric $\beta(\cdot,\cdot) = \omega(\cdot,\ J\cdot)$. Any symplectic manifold possesses such a structure. We will assume further that ω is 'integral' in the cohomological sense. This means we can find a complex Hermitian line bundle $L \to X$ with Hermitian connection ∇ whose curvature is $-i\omega$.

Recently, beginning with Donaldson's seminal paper [5], the notion of 'nearly holomorphic' or 'asymptotically holomorphic' sections of $L^{\otimes k}$ has attracted a fair amount of attention. Let us recall that one natural way to define spaces of such sections is by means of an analogue of the $\overline{\partial}$-Laplacian [2, 3].

The Hermitian structure and connection on L induce corresponding structures on $L^{\otimes k}$. In combination with β this defines a Laplace operator Δ_k acting on

$C^\infty(X; L^{\otimes k})$. (Our convention is that the Laplacian is positive.) Then the sequence of operators

$$\mathcal{D}_k = \Delta_k - nk$$

has the same principal and subprincipal symbols as the $\overline{\partial}$-Laplacian in the integrable case; in fact in the Kähler case $\mathcal{D}_k$ *is* the $\overline{\partial}$-Laplacian. (By Kähler case we mean not only that J is integrable but also that L is Hermitian *holomorphic* with ∇ the induced connection.) The large k behavior of the spectrum of Δ_k was studied (in somewhat greater generality) by Guillemin and Uribe [6]. For our purposes, the main results can be summarized as follows:

THEOREM 1.1 ([6]). *There exist constants $a > 0$ and M (independent of k), such that for large k the spectrum of $\mathcal{D}_k$ lies in (ak, ∞) except for a finite number of eigenvalues contained in $(-M, M)$. The number n_k of eigenvalues in $(-M, M)$ is a polynomial in k with asymptotic behavior $n_k \sim k^n \mathrm{vol}(X)$. This polynomial can be computed exactly by a symplectic Riemann–Roch formula.*

Furthermore, if the eigenvalues in $(-M, M)$ are labeled $\lambda_j^{(k)}$, then there exists a spectral density function $q \in C^\infty(X)$ such that for any $f \in C(\mathbb{R})$,

$$\frac{1}{n_k}\sum_{j=1}^{n_k} f(\lambda_j^{(k)}) \longrightarrow \frac{1}{\mathrm{vol}(X)}\int_X (f \circ q)\frac{\omega^n}{n!},$$

as $k \to \infty$.

The proof of Theorem 1.1 is based on the analysis of generalized Toeplitz structures developed in [4].

By the remarks above, in the Kähler case all $\lambda_j^{(k)} = 0$, corresponding to eigenfunctions which are holomorphic sections of $L^{\otimes k}$. Hence $q \equiv 0$ for a true Kähler structure. In general, it is therefore natural to consider sections of $L^{\otimes k}$ spanned by the eigenvalues of $\mathcal{D}_k$ in $(-M, M)$ as being analogous to holomorphic sections.

The goal of the present paper is to derive a simple geometric formula for the spectral density function q. Our main result is:

THEOREM 1.2. *The spectral density function is given by*

$$q = -\tfrac{5}{24}|\nabla J|^2.$$

COROLLARY 1.3. *The spectral density function is identically zero iff (X, J, ω) is Kähler.*

It is natural to ask if one can choose J so that q is very small, i.e. if the symplectic invariant

$$j(X, \omega) := \inf\left\{ \left\| |\nabla J|^2 \right\|_\infty ;\ J \text{ a compatible almost complex structure}\right\}$$

is always zero. We have learned from Abreu that for Thurston's manifold $j = 0$; it would be very interesting to find (X, ω) with $j > 0$.

The proof of Theorem 1.2 starts with the standard and very useful observation that sections of $L^{\otimes k}$ are equivalent to equivariant functions on an associated principle bundle $\pi\colon Z \to X$. We endow Z with a 'Kaluza–Klein' metric such that the fibers are geodesic. Then the main idea exploited in the proof is the construction of approximate eigenfunctions (quasimodes) of the Laplacian Δ_Z concentrated on these closed geodesics. Such quasimodes are equivariant and thus naturally associated to sections of $L^{\otimes k}$. Moreover, the value of the spectral density function $q(x)$ is encoded in the eigenvalue of the quasimode concentrated on the fiber $\pi^{-1}(x) \subset Z$.

2. Preliminaries

The associated principle bundle to L is easily obtained as the unit circle bundle $Z \subset L^*$. There is a 1-1 correspondence between sections of $L^{\otimes k}$ and functions on Z which are k-equivariant with respect to the S^1-action, i.e. $f(z.\mathrm{e}^{i\theta}) = \mathrm{e}^{ik\theta} f(z)$.

The connection ∇ on L induces a connection 1-form α on Z. The curvature condition on ∇ translates to $\mathrm{d}\alpha = \pi^*\omega$, where $\pi\colon Z \to X$. Together with the Riemannian metric on X and the standard metric on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, this defines a 'Kaluza–Klein' metric g on Z such that the projection $Z \to X$ is a Riemannian submersion with totally geodesic fibers. With these choices the correspondence between equivariant functions and sections extends to an isomorphism between

$$L^2(X, L^{\otimes k}) \simeq L^2(Z)_k, \tag{2.1}$$

where $L^2(Z)_k$ denotes the kth isotype of $L^2(Z)$ under the S^1 action.

Let Δ_Z be the (positive) Laplacian on Z. By construction it commutes with the generator ∂_θ of the circle action, and so it also commutes with the 'horizontal Laplacian':

$$\Delta_h = \Delta_Z + \partial_\theta^2. \tag{2.2}$$

The action of Δ_h on $L^2(Z)_k$ is equivalent under (2.1) to the action of Δ_k on $L^2(X, L^{\otimes k})$.

For sufficiently large k, we let $\mathcal{H}_k \subset L^2(Z)_k$ denote the span of the eigenvectors with eigenvalues in the bounded range $(-M, M)$. The corresponding orthogonal projection is denoted $\Pi_k\colon L^2(Z) \to \mathcal{H}_k$. The following fact appears in the course of the proof of Theorem 1.1:

LEMMA 2.1 ([6]). *There is a sequence of functions $q_j \in C^\infty(X)$ such that*

$$\left\| \Pi_k \left(\Delta_h - nk - \sum_{j=0}^{N} k^{-j} \pi^* q_j \right) \Pi_k \right\| = \mathrm{O}(k^{-(N+1)}).$$

Moreover, the spectral density function q in Theorem 1.1 *is equal to q_0.*

3. Quasimodes on the Circle Bundle

The key to the calculation of the spectral density function at $x_0 \in X$ is the observation that, with the Kaluza–Klein metric, the assumptions on X imply the stability of the geodesic fiber $\Gamma = \pi^{-1}(x_0)$. Thus one should be able to construct an approximate eigenfunction, or *quasimode*, for Δ_Z which is asymptotically localized on Γ. The lowest eigenvalue of the quasimode (or rather a particular coefficient in its asymptotic expansion) will yield the spectral density function.

The computation is largely a matter of interpolating between two natural coordinate systems. From the point of view of writing down the Kaluza-Klein metric explicitly, the obvious coordinate system to use is given by first trivializing Z to identify a neighborhood of Γ with $S^1 \times U_{x_0}$, where U_{x_0} is a neighborhood of x_0 in X. (The base point x_0 will be fixed throughout this section.) On U_{x_0} we can introduce geodesic normal coordinates centered at x_0. These coordinates will be denoted $(\theta, x^1, \dots, x^{2n})$. The corresponding base point $z_0 \in \pi^{-1}(x_0)$, specified by $\theta = 0$, is arbitrary. In such coordinates the connection α takes the form $\alpha = \mathrm{d}\theta + \alpha_j \,\mathrm{d}x^j$.

We will follow the quasimode construction outlined in [1], which is essentially based in the normal bundle $N\Gamma \subset TZ$. Let $\psi\colon N\Gamma \to Z$ be the map defined on each fiber $N_z\Gamma$ by the restriction of the exponential map $\exp_z\colon T_zZ \to Z$. Of course, ψ is only a diffeomorphism near Γ. The *Fermi coordinate system* along Γ is defined by the combination of ψ and the choice of a parallel frame for $N\Gamma$. Let $\gamma(s)$ be a parametrization of Γ by arclength, with $\gamma(0) = z_0$, $\gamma'(0) = \partial_\theta$. Let $e_j(s)$ be the frame for $N_{\gamma(s)}\Gamma$ defined by parallel transport from the initial value $e_j(0) = \partial_j$, where ∂_j denotes $\partial/\partial x^j$. Then the Fermi coordinates are defined by the map

$$(s, y^j) \mapsto \psi(y^j e_j(s)).$$

Note that $s = \theta$ only on Γ.

3.1. THE ANSATZ

Now we can formulate the construction of an approximate solution of $(\Delta_Z - \lambda)f = 0$ as a set of parabolic equations on $N\Gamma$. Let κ be an asymptotic parameter (eventually to be related to k). Setting $f(s, y) = \mathrm{e}^{i\kappa s} U(s, y)$ we consider the equation

$$(\Delta_Z - \lambda)\, \mathrm{e}^{i\kappa s} U(s, y) = 0. \tag{3.1}$$

Since we are hoping to localize near $y = 0$ for large κ, the ansatz is to substitute $u^j = \sqrt{\kappa}\, y^j$ and do a formal expansion

$$\mathrm{e}^{-i\kappa s} \Delta_Z\, \mathrm{e}^{i\kappa s} = \kappa^2 + \kappa \mathcal{L}_0 + \sqrt{\kappa}\mathcal{L}_1 + \mathcal{L}_2 + \cdots. \tag{3.2}$$

This defines differential operators $\mathcal{L}_j$ on a neighborhood of the zero-section in $N\Gamma$, but since the coefficients are polynomial in the y^j variables, they extend naturally to all of $N\Gamma$. We also make an ansatz of formal expansions for λ and U:

$$\lambda = \kappa^2 + \sigma + \cdots, \quad U = U_0 + \kappa^{-1} U_1 + \cdots.$$

Substituting these expansions into (3.1) and reading off the orders gives the equations

$$\mathcal{L}_0 U_0 = 0, \quad \mathcal{L}_1 U_0 = 0, \quad \mathcal{L}_0 U_1 = -(\mathcal{L}_2 - \sigma) U_0. \tag{3.3}$$

Since $\mathcal{L}_j$ is well defined on $N\Gamma$, we can seek global solutions $U_j(s, y)$, subject to the boundary condition $\lim_{|y|\to\infty} U_j = 0$. In the right coordinates, we will see that $\mathcal{L}_0 U_0 = 0$ is simply a harmonic oscillator Schrödinger equation. Furthermore, the second equation will be satisfied if and only if U_0 is taken to be the ground-state solution this Schrödinger equation. Hence these two equations will determine U_0 up to normalization. Solutions of the third equation exist only for a certain value of σ, and the main goal of this section is to compute this quantity.

By pulling back with ψ, we can use (θ, x) as an alternate coordinate system on $N\Gamma$ (near the zero section). We'll use $\bar\beta_{ij}, \bar\alpha_i, \bar\omega_{ij}, \bar J^i_j$ to denote the various tensors lifted from X and written in these coordinates (so all are independent of θ). Also $\bar\Gamma^\sigma_{\mu\nu}$ will denote the Christoffel symbols of the Kaluza–Klein metric g in the (θ, x) coordinates. The index convention is that Greek indices range over $0, \dots, 2n$ and Roman over $1, \dots, 2n$. To reduce notational complexity insofar as possible, we will adopt the convention that unbarred expressions involving $\beta_{ij}, \alpha_i, \omega_{ij}, J^i_j$ and their derivatives are to be evaluated at the base point $x_0 \in X$, e.g.

$$\beta_{ij} = \bar\beta_{ij}|_{x=0}, \quad \partial_k \beta_{ij} = \frac{\partial}{\partial x^k} \bar\beta_{ij}|_{x=0}.$$

The Christoffel symbols of β_{ij} (evaluated at x_0) will be denoted by F^l_{jk}, with the same convention for evaluation of derivatives as above. (Thus $F^l_{jk} = 0$ because the coordinates are geodesic normal at x_0, but $\partial_m F^l_{jk}$ is nonzero.) The freedom in the trivialization of Z may be exploited to assume that

$$\alpha_j = 0, \quad \partial_j \alpha_k = \tfrac{1}{2} \omega_{jk},$$

where throughout the computation ∂_j denotes the vector field $\partial/\partial x^j$ on (or lifted from) X.

Let $g_{\mu\nu}$ to denote the Kaluza–Klein metric expressed in the (θ, x) coordinates. The horizontal lift of ∂_j to Z is

$$E_j = \partial_j - \bar\alpha_j \partial_\theta. \tag{3.4}$$

The Kaluza–Klein metric is specified by the conditions:

$$g(E_j, \partial_\theta) = 0, \quad g(\partial_\theta, \partial_\theta) = 1, \quad g(E_j, E_k) = \bar\beta_{jk}.$$

Substituting in with (3.4) we quickly see that

$$g_{00} = 1, \quad g_{j0} = \bar\alpha_j, \quad g_{jk} = \bar\beta_{jk} + \bar\alpha_j \bar\alpha_k.$$

In block matrix form we can write

$$g = \begin{pmatrix} 1 & \bar{\alpha} \\ \bar{\alpha} & \bar{\beta} + \bar{\alpha}\bar{\alpha} \end{pmatrix}, \tag{3.5}$$

from which

$$g^{-1} = \begin{pmatrix} 1 + \bar{\alpha}\bar{\beta}^{-1}\bar{\alpha} & -\bar{\beta}^{-1}\bar{\alpha} \\ -\bar{\beta}^{-1}\bar{\alpha} & \bar{\beta}^{-1} \end{pmatrix}. \tag{3.6}$$

We will use $G_{\mu\nu}$ to denote the Kaluza–Klein metric written in the Fermi coordinates (s, y), i.e.

$$G_{00} = g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right), \quad G_{0j} = g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial y^j}\right), \quad G_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right).$$

$G_{\mu\nu}$ is well defined in a neighborhood of $y = 0$, and with the ansatz above we only need to know its Taylor series to determine $\mathcal{L}_j$. As noted above, the heart of the calculation will be the change of coordinates from (θ, x) to (s, y).

By assumption $G_{\mu\nu} = \delta_{\mu\nu}$ to second order in y. After the substitution $u_j = \sqrt{\kappa}\, y_j$, we can write the Taylor expansions of various components as

$$\begin{aligned} G_{00} &= 1 + \kappa^{-1}a^{(2)} + \kappa^{-3/2}a^{(3)} + \kappa^{-2}a^{(4)} + \cdots, \\ G_{0j} &= \kappa^{-1}b_j^{(2)} + \kappa^{-3/2}b_j^{(3)} + \cdots, \\ G_{jk} &= \delta_{jk} + \kappa^{-1}c_{jk}^{(2)} + \cdots, \end{aligned} \tag{3.7}$$

where superscript (l) denotes the term which is a degree l polynomial in u. Then using the definition

$$\Delta_Z = -\frac{1}{\sqrt{G}}\partial_\mu\left[\sqrt{G}G^{\mu\nu}\partial_\nu\right],$$

we can substitute the expansions (3.7) into (3.2) and read off the first few orders in κ:

$$\begin{aligned} \mathcal{L}_0 &= -2i\partial_s - a^{(2)} - \partial_u^2, \\ \mathcal{L}_1 &= -a^{(3)} + 2ib^{j\,(2)}\frac{\partial}{\partial u^j} + i\left(\frac{\partial}{\partial u^j}b^{j\,(3)}\right), \\ \mathcal{L}_2 &= -\partial_s^2 + 2ia^{(2)}\partial_s - a^{(4)} + (a^{(2)})^2 + (b^{(2)})^2 + \\ &\quad + i\left[-\tfrac{1}{2}\partial_s \mathrm{Tr} c^{(2)} + 2b^{j\,(3)}\frac{\partial}{\partial u^j} + \left(\frac{\partial}{\partial u^j}b^{j\,(3)}\right)\right] + \\ &\quad + c^{jk\,(2)}\frac{\partial}{\partial u^j}\frac{\partial}{\partial u^k} + \left(\frac{\partial}{\partial u^j}c^{jk\,(2)}\right)\frac{\partial}{\partial u^k} - \tfrac{1}{2}\frac{\partial}{\partial u^j}[a^{(2)} + \mathrm{Tr} c^{(2)}]\frac{\partial}{\partial u^j}. \end{aligned} \tag{3.8}$$

3.2. THE METRIC IN FERMI COORDINATES

For use in the calculation, let us first work out some simple implications of $J^2 = -1$. Using conventions as above, this means $\bar{J}^k_j \bar{J}^m_k = -\delta^m_j$. Differentiating at the base point x_0 gives us

$$(\partial_l J^k_j) J^m_k = -J^k_j (\partial_l J^m_k), \quad J^k_j (\partial_l J^j_k) = 0.$$

The other basic fact is $\mathrm{d}\omega = 0$, which translates to

$$\partial_l \omega_{jk} + \partial_j \omega_{kl} + \partial_k \omega_{lj} = 0.$$

LEMMA 3.1. $\partial_l J^l_j = 0$.

Proof. Using the fact that $J^l_j = \omega_{jk}\beta^{kl}$ we have

$$\begin{aligned} J^k_j(\partial_l J^l_k) &= -(\partial_l J^k_j) J^l_k \\ &= -(\partial_l \omega_{jk})\omega^{kl} \\ &= -\tfrac{1}{2}(\partial_l \omega_{jk} - \partial_k \omega_{jl})\omega^{kl} \\ &= \tfrac{1}{2}(\partial_j \omega_{kl})\omega^{kl} \\ &= -\tfrac{1}{2}(\partial_j J^l_k) J^k_l \\ &= 0. \end{aligned}$$

□

A similar fact, which will also be needed, is

LEMMA 3.2. *For any vector v^j we have*

$$(\partial_l J^m_j) v^j (\omega v)_m = 0.$$

Proof.

$$\begin{aligned} (\partial_l J^m_j) v^j (\omega v)_m &= (\partial_l J^m_j) v^j J^s_m v_s \\ &= -(\partial_l J^s_m) v^j J^m_j v_s \\ &= -(\partial_l \omega_{ms})(Jv)^m v^s \\ &= (\partial_l \omega_{sm})(Jv)^m v^s \\ &= -(\partial_l J^m_s)(\omega v)_m v^s. \end{aligned}$$

□

To proceed, we must determine the terms in the Taylor expansion of $G_{\mu\nu}$ in terms of the geometric data β, ω, J, α. Let us expand the parallel frame $e_j(s)$ in the basis $\{\partial_k\}$ as $T^k_j \partial_k$. The parallel condition on $e_j(s)$ is then

$$\frac{\partial}{\partial s} T^k_j = -\Gamma^k_{0l} T^l_j,$$

where

$$\Gamma^k_{0l} = \tfrac{1}{2}\beta^{km}(\partial_l\alpha_m - \partial_m\alpha_l) = \tfrac{1}{2}\beta^{km}\omega_{lm} = \tfrac{1}{2}J^k_l.$$

The solution is

$$T^k_j = (\mathrm{e}^{-s/2J})^k_j.$$

Since this is the matrix relating the x-frame to the y-frame at $x = 0$, we have $\frac{\partial x^k}{\partial y^j}|_{x=0} = T^k_j$. This makes it convenient to introduce an auxiliary coordinate $z^k = T^k_j y^j$.

The transformation to Fermi coordinates may now be written as

$$\theta = s + A(s,z), \quad x^j = z^j + B^j(s,z).$$

The functions A and B are determined by the condition that the ray $t \mapsto (s, ty)$ be a geodesic. Of course, we are really just interested in the Taylor expansions:

$$\begin{aligned} A &= \kappa^{-1}A^{(2)} + \kappa^{-3/2}A^{(3)} + \kappa^{-2}A^{(4)} + \cdots, \\ B^j &= \kappa^{-1}B^{j\,(2)} + \kappa^{-3/2}B^{j\,(3)} + \cdots, \end{aligned}$$

where degrees are labeled as above.

Denoting the t derivative by a dot, the geodesic equations are

$$\begin{aligned} \ddot\theta &= -\bar\Gamma^0_{00}\dot\theta^2 - 2\bar\Gamma^0_{0l}\dot\theta\dot x_l - \bar\Gamma^0_{jl}\dot x_j\dot x_l, \\ \ddot x_k &= -\bar\Gamma^k_{00}\dot\theta^2 - 2\bar\Gamma^k_{0l}\dot\theta\dot x_l - \bar\Gamma^k_{jl}\dot x_j\dot x_l. \end{aligned} \tag{3.9}$$

The Christoffel symbols of g_{ij} are

$$\begin{aligned} \bar\Gamma^0_{00} &= \bar\Gamma^j_{00} = 0, \\ \bar\Gamma^0_{0j} &= \tfrac{1}{2}(\bar J\bar\alpha)_j, \\ \bar\Gamma^0_{jk} &= \tfrac{1}{2}[\partial_j\bar\alpha_k + \partial_k\bar\alpha_j + \bar\alpha_j(\bar J\bar\alpha)_k + \bar\alpha_k(\bar J\bar\alpha)_j] - \bar F^l_{jk}\bar\alpha_l, \\ \bar\Gamma^j_{0k} &= -\tfrac{1}{2}\bar J^j_k, \\ \bar\Gamma^j_{lk} &= -\tfrac{1}{2}\bar J^j_l\bar\alpha_k - \tfrac{1}{2}\bar J^j_k\bar\alpha_l + \bar F^j_{lk}. \end{aligned}$$

Substituting the Taylor expansion of the Christoffel symbols at x_0 into (3.9) and equating coefficients, we find $A^{(2)} = 0$, $B^{(2)} = 0$,

$$\begin{aligned} A^{(3)} &= -(\partial_m\partial_j\alpha_l)z^m z^j z^l, \\ A^{(4)} &= -\tfrac{1}{24}(\partial_k\partial_m\partial_j\alpha_l)z^k z^m z^j z^l - \tfrac{1}{24}(\partial_k F^i_{jl})z^k z^j z^l(\omega z)_i, \\ B^{k\,(3)} &= -\tfrac{1}{6}(\partial_m F^k_{jl})z^m z^j z^l. \end{aligned} \tag{3.10}$$

Using $x = z + \kappa^{-3/2}B^{(3)} + \cdots$, we can then determine the coefficients of the expansion of $\bar{\alpha}_k$:

$$\begin{aligned}
\bar{\alpha}_k^{(1)} &= -\tfrac{1}{2}(\omega z)_k, \\
\bar{\alpha}_k^{(2)} &= \tfrac{1}{2}(\partial_l \partial_m \alpha_k) z^l z^m n, \\
\bar{\alpha}_k^{(3)} &= \tfrac{1}{6}(\partial_j \partial_l \partial_m \alpha_k) z^j z^l z^m + \tfrac{1}{12}\omega_{ki}(\partial_m F^i_{jl}) z^m z^j z^l.
\end{aligned} \tag{3.11}$$

The Fermi coordinate vector fields are

$$\begin{aligned}
&\partial_s = (1 + \partial_s A)\partial_0 + (z'^l + B'^l)\partial_l, \\
&\frac{\partial}{\partial y^j} = \left(\frac{\partial}{\partial y^j} A\right)\partial_0 + \left(T^l_j + \frac{\partial}{\partial y^j} B^l\right)\partial_l.
\end{aligned}$$

Note that $z^j = T^j_k(s) y^k$, so $z'^j = -1/2(Jz)^j$. To compute $a^{(l)}$, we use (3.10) and (3.11) to expand $G_{00} = g(\partial_s, \partial_s)$. The second order term is

$$a^{(2)} = 2\alpha_l^{(1)} z'^l + z'^l z'_l = -\frac{z^2}{4}. \tag{3.12}$$

At third order we have

$$\begin{aligned}
a^{(3)} &= 2(A^{(3)})' + 2\alpha_m^{(2)} z'^m \\
&= -\tfrac{1}{3}(\partial_j \partial_l \alpha_m)[2z'^j z^l z^m + z^j z^l z'^m] + (\partial_j \partial_l \alpha_m) z^j z^l z'^m \\
&= \tfrac{1}{3}(\partial_j \partial_l \alpha_m)(Jz)^j z^l z^m - \tfrac{1}{3}(\partial_j \partial_l \alpha_m) z^j z^l (Jz)^m \\
&= -\tfrac{1}{3}(\partial_l \omega_{jm}) z^j z^l (Jz)^m.
\end{aligned}$$

Thus, by Lemma 3.2 we have

$$a^{(3)} = 0. \tag{3.13}$$

The fourth-order term is somewhat more complicated:

$$\begin{aligned}
a^{(4)} = {} & 2A'^{(4)} + 2\alpha_m^{(3)} z'^m + 2\alpha_m^{(1)} (B'^m)^{(3)} + \\
& + z'^l(\beta_{lm}^{(2)} + \alpha_l^{(1)}\alpha_m^{(1)}) z'^m + 2z'_m (B'^m)^{(3)}.
\end{aligned}$$

We will expand the first term,

$$\begin{aligned}
2A'^{(4)} = {} & \tfrac{1}{24}(\partial_k \partial_m \partial_j \alpha_l)[3z^k z^m (Jz)^j z^l + z^k z^m z^j (Jz)^l] + \\
& + \tfrac{1}{24}(\partial_k F^i_{jl})[(Jz)^k z^j z^l (\omega z)_i + 2z^k z^j (Jz)^l (\omega z)_i + z^k z^j z^l z_i]
\end{aligned}$$

and the second,

$$2\alpha_k^{(3)} z'^k = -\tfrac{1}{6}(\partial_j \partial_l \partial_m \alpha_k) z^j z^l z^m (Jz)^k - \tfrac{1}{12}\omega_{ki}(\partial_m F^i_{jl}) z^m z^j z^l (Jz)^k.$$

The terms involving $\partial_m \alpha_k$ combine to form factors of ω_{mk}:

$$2A'^{(4)} + 2\alpha_k^{(3)} z'^k = -\tfrac{1}{8}(\partial_j \partial_l \omega_{mk}) z^j z^l z^m (Jz)^k + \tfrac{1}{24}(\partial_k F^i_{jl})(Jz)^k z^j z^l (\omega z)_i + \\ + \tfrac{1}{12}(\partial_k F^i_{jl}) z^k z^j (Jz)^l (\omega z)_i + \tfrac{1}{8}(\partial_k F^i_{jl}) z^k z^j z^l z_i.$$

After noting that $2\alpha_m^{(1)}(B'^m)^{(3)} + 2z'_m (B'^m)^{(3)} = 0$, we are left with the term

$$z'^l(\beta_{lm}^{(2)} + \alpha_l^{(1)}\alpha_m^{(1)}) z'^m = \tfrac{1}{8}(\partial_j \partial_k \beta_{lm})(Jz)^l z^j z^k (Jz)^m + \frac{z^4}{16}$$

So in conclusion,

$$\begin{aligned} a^{(4)} &= -\tfrac{1}{8}(\partial_j \partial_l \omega_{mk}) z^j z^l z^m (Jz)^k + \tfrac{1}{24}(\partial_k F^i_{jl})(Jz)^k z^j z^l (\omega z)_i \\ &\quad + \tfrac{1}{12}(\partial_k F^i_{jl}) z^k z^j (Jz)^l (\omega z)_i + \tfrac{1}{8}(\partial_k F^i_{jl}) z^k z^j z^l z_i \\ &\quad + \tfrac{1}{8}(\partial_j \partial_k \beta_{lm})(Jz)^l z^j z^k (Jz)^m + \frac{z^4}{16}. \end{aligned} \tag{3.14}$$

For $b_j = g(\partial_s, \partial_{y^j})$ the third-order term will prove irrelevant, so we compute only

$$\begin{aligned} b_j^{(2)} &= \partial_{y^j} A^{(3)} + \alpha_m^{(2)} T_j^m \\ &= -\tfrac{1}{6}(\partial_k \partial_l \alpha_m)[2T_j^k z^l z^m + z^k z^l T_j^m] + \tfrac{1}{2}(\partial_k \partial_l \alpha_m) z^k z^l T_j^m \\ &= -\tfrac{1}{3}(\partial_k \partial_l \alpha_m) T_j^k z^l z^m + \tfrac{1}{3}(\partial_k \partial_l \alpha_m) z^k z^l T_j^m \\ &= \tfrac{1}{3}(\partial_l \omega_{km}) z^k z^l T_j^m. \end{aligned} \tag{3.15}$$

Finally, we have $c_{lm} = g(\partial_{y^l}, \partial_{y^m})$. It is convenient to insert factors of T:

$$\begin{aligned} T_j^l c_{lm}^{(2)} T_k^m &= \beta_{jk}^{(2)} + \alpha_j^{(1)}\alpha_k^{(1)} + (\partial_{z^j} B_k^{(3)}) + (\partial_{z^k} B_j^{(3)}) \\ &= \tfrac{1}{2}(\partial_l \partial_m \beta_{jk}) z^l z^m + \tfrac{1}{4}(\omega z)_j (\omega z)_k - \tfrac{1}{6}(\partial_j F_{ilk}) z^i z^l - \\ &\quad - \tfrac{1}{3}(\partial_m F_{jlk}) z^m z^l - \tfrac{1}{6}(\partial_k F_{ilj}) z^i z^l - \tfrac{1}{3}(\partial_m F_{klj}) z^m z^l. \end{aligned} \tag{3.16}$$

3.3. PARABOLIC EQUATIONS

With the computation of $a^{(2)}$ in (3.12), we now have that

$$\mathcal{L}_0 = -2i\partial_s + \frac{u^2}{4} - \partial_u^2.$$

The equation $\mathcal{L}_0 U_0 = 0$ is then the harmonic oscillator as promised. The 'ground state' solution is

$$U_0 = \mathrm{e}^{-ins/2}\, \mathrm{e}^{-u^2/4}. \tag{3.17}$$

Now $e^{i\kappa s}U$ is required to be periodic in s, which means that

$$\kappa - \frac{n}{2} \in \mathbb{Z}.$$

A function on z which is $e^{iks} \times$(periodic) comes from a section of L^k, so the relation between the two asymptotic parameters is $k = \kappa - n/2$. Recall that the leading term in the eigenvalue λ was

$$\kappa^2 = k^2 + nk + \frac{n^2}{4}. \tag{3.18}$$

The nk correction at first order exhibits the spectral drift accounted for by subtracting nk from Δ_k.

By the well-known analysis of the quantum harmonic oscillator, a complete set of solutions to $\mathcal{L}_0 U = 0$ can be generated by application of the 'creation operator'

$$\Lambda_j^* = -i\, e^{-is/2}\left(\partial_{u^j} - \frac{u_j}{2}\right).$$

We will need

$$U_{ij} = \Lambda_i^* \Lambda_j^* U_0, \quad U_{ijkl} = \Lambda_i^* \Lambda_j^* \Lambda_k^* \Lambda_l^* U_0,$$

which are easily computed explicitly:

$$\begin{aligned} U_{ij} &= (-u_j u_k + \delta_{ij})\, e^{-is} U_0, \\ U_{ijkl} &= \big(u_i u_j u_k u_l - \delta_{ij} u_k u_l - \delta_{ik} u_j u_l - \delta_{il} u_j u_k - \delta_{jk} u_i u_l - \\ &\qquad - \delta_{kl} u_i u_j - \delta_{lj} u_i u_k + \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\big)\, e^{-2is} U_0. \end{aligned}$$

Since $a^{(3)} = 0$ and $\partial_{u^j} b^{j\,(2)} = 0$, the next operator is

$$\mathcal{L}_1 = 2ib^{j\,(2)} \frac{\partial}{\partial u^j}.$$

It then follows from $b_j^{(2)} u^j = 0$ that $\mathcal{L}_1 U_0 = 0$. Moreover, it is easy to check, using the creation operators, that U_0 is the unique solution of $\mathcal{L}_0 U = 0$ for which this is true.

Consider finally the third equation

$$\mathcal{L}_0 U_1 = -(\mathcal{L}_2 - \sigma) U_0, \tag{3.19}$$

from which we will determine σ. Since $\mathcal{L}_2 U_0$ has coefficients polynomial in u_j of order no more than four, we can expand

$$\mathcal{L}_2 U_0 = [C^{ijkl} u_i u_j u_k u_l + C^{ij} u_i u_j + C] U_0. \tag{3.20}$$

PROPOSITION 3.3. *Equations* (3.3) *have a solution* $U_0, U_1 \in C^\infty(N\Gamma)$ *if and only if*

$$\sigma = C + C_l{}^l + 3C_{kk}{}^{ll}, \tag{3.21}$$

where the coefficients C^{ijkl} *are assumed symmetrized.*

Proof. We have already remarked that U_0 is fixed by the first two equations of (3.3). In terms of the harmonic oscillator basis we can rewrite (3.20) as

$$\mathcal{L}_2 U_0 = \mathrm{e}^{2is} D^{ijkl} U_{ijkl} + \mathrm{e}^{is} D^{ij} U_{ij} + DU_0.$$

Observe that

$$\mathcal{L}_0(\mathrm{e}^{2is} U_{ijkl}) = -4U_{ijkl}, \quad \mathcal{L}_0(\mathrm{e}^{is} U_{ij}) = -2U_{ij}.$$

Therefore the equation $\mathcal{L}_0 U_1 = -(\mathcal{L}_2 - \sigma)U_0$ has a solution only if $\sigma = D$, and in this case we write the solution explicitly as

$$U_1 = \tfrac{1}{4}\mathrm{e}^{2is} D^{ijkl} U_{ijkl} + \tfrac{1}{2}\mathrm{e}^{is} D^{ij} U_{ij}.$$

To compute D we note

$$C^{ij} u_i u_j U_0 = -C^{ij}\, \mathrm{e}^{is} U_{ij} + C_l{}^l U_0,$$

and (with the symmetry assumption),

$$\begin{aligned} C^{ijkl} u_i u_j u_k u_l U_0 &= C^{ijkl}\, \mathrm{e}^{2is} U_{ijkl} + [6C_j{}^{jkl} u_k u_l - 3C_{kk}{}^{ll}]U_0 \\ &= C^{ijkl}\, \mathrm{e}^{2is} U_{ijkl} + (\dots)\, \mathrm{e}^{is} U_{jk} + 3C_{kk}{}^{ll} U_0. \end{aligned}$$

This means that

$$D = C + C_l{}^l + 3C_{kk}{}^{ll}.$$

□

To conclude the computation, we will examine $\mathcal{L}_2 U_0$ piece by piece and form the contractions of coefficients according to (3.21). From (3.8) we break up $\mathcal{L}_2 U_0 = W_1 + \cdots + W_6$, where

$$\begin{aligned} W_1 &= [-\partial_s^2 + 2ia^{(2)}\partial_s]U_0, \\ W_2 &= [-a^{(4)} + (a^{(2)})^2]U_0, \\ W_3 &= (b^{(2)})^2 U_0, \\ W_4 &= i\left[-\tfrac{1}{2}\partial_s \mathrm{Tr} c^{(2)} + 2b^{j\,(3)}\frac{\partial}{\partial u^j} + \left(\frac{\partial}{\partial u^j} b^{j\,(3)}\right)\right] U_0, \\ W_5 &= \left[c^{jk\,(2)}\frac{\partial}{\partial u^j}\frac{\partial}{\partial u^k} + \left(\frac{\partial}{\partial u^j} c^{jk\,(2)}\right)\frac{\partial}{\partial u^k}\right] U_0, \\ W_6 &= -\tfrac{1}{2}\frac{\partial}{\partial u^j}[a^{(2)} + \mathrm{Tr} c^{(2)}]\frac{\partial}{\partial u^j} U_0. \end{aligned}$$

By (3.17) we compute

$$W_1 = [-\partial_s^2 + 2ia^{(2)}\partial_s]U_0 = \left[\frac{n^2}{4} - \frac{nz^2}{4}\right]U_0.$$

The contribution to σ from W_1 is thus:

$$-\frac{n^2}{4}. \tag{3.22}$$

For W_2, from the calculations of $a^{(2)}$ and $a^{(4)}$ we have

$$\begin{aligned} -a^{(4)} + (a^{(2)})^2 \; = \; & \tfrac{1}{8}(\partial_j\partial_l\omega_{mk})z^j z^l z^m (Jz)^k - \tfrac{1}{24}(\partial_k F^i_{jl})(Jz)^k z^j z^l (\omega z)_i - \\ & - \tfrac{1}{12}(\partial_k F^i_{jl})z^k z^j (Jz)^l (\omega z)_i - \tfrac{1}{8}(\partial_k F^i_{jl})z^k z^j z^l z_i - \\ & - \tfrac{1}{8}(\partial_j\partial_k\beta_{lm})(Jz)^l z^j z^k (Jz)^m. \end{aligned}$$

We symmetrize and take the contractions to find the contribution to σ:

$$\begin{aligned} & \tfrac{1}{8}(\partial^j\partial_j\omega_{mk})\omega^{mk} + \tfrac{1}{4}(\partial_j\partial^l\omega_{lk})\omega^{jk} - \tfrac{1}{12}(\beta^{lm}\partial_k F^k_{lm}) - \tfrac{1}{6}(\partial^k F^l_{kl}) - \\ & \quad - \tfrac{1}{4}(\partial_j\partial_k\beta_{lm})\omega^{jl}\omega^{km} - \tfrac{1}{8}(\beta^{lm}\partial^k\partial_k\beta_{lm}). \end{aligned}$$

Let us simplify this expression. By $\mathrm{d}\bar\omega = 0$ we have

$$(\partial_j\partial^l\omega_{lk})\omega^{jk} = \tfrac{1}{2}(\partial^j\partial_j\omega_{mk})\omega^{mk}.$$

From $\bar\omega_{mk} = -\bar\beta_{mr}\bar J^r_k$ we derive

$$(\partial^j\partial_j\omega_{mk})\omega^{mk} = \beta^{lm}\partial^j\partial_j\beta_{lm} - (\partial^j\partial_j J^m_k)J^k_m.$$

Finally from $\bar J^2 = -1$ we obtain

$$(\partial^j\partial_j J^m_k)J^k_m = -(\partial_j J^m_k)(\partial^j J^k_m) = |\nabla J|^2.$$

Combining these facts gives

$$\tfrac{1}{8}(\partial^j\partial_j\omega_{mk})\omega^{mk} + \tfrac{1}{4}(\partial_j\partial^l\omega_{lk})\omega^{jk} = \tfrac{1}{4}\beta^{lm}\partial^j\partial_j\beta_{lm} - \tfrac{1}{4}|\nabla J|^2.$$

Evaluating the Christoffel symbols gives

$$\begin{aligned} \beta^{lm}\partial_k F^k_{lm} \; &= \; \tfrac{1}{2}\beta^{lm}\partial^k[\partial_l\beta_{mk} + \partial_m\beta_{lk} - \partial_k\beta_{lm}] \\ &= \; \partial^k\partial^l\beta_{lk} - \tfrac{1}{2}\beta^{lm}\partial^k\partial_k\beta_{lm} \end{aligned}$$

and

$$\partial^k F^l_{kl} = \tfrac{1}{2}\beta^{lm}\partial^k\partial_k\beta_{lm}.$$

Thus the final contribution from W_2 to σ is

$$-\tfrac{1}{4}|\nabla J|^2 - \tfrac{1}{4}(\partial_j\partial_k\beta_{lm})\omega^{jl}\omega^{km} + \tfrac{1}{12}\beta^{lm}\partial^k\partial_k\beta_{lm} - \tfrac{1}{12}\partial^j\partial^l\beta_{jl}. \tag{3.23}$$

By our calculations,

$$(b^{(2)})^2 = \tfrac{1}{9}(\partial_l \omega_{km}) z^k z^l (\partial_i J_j^m) z^i z^j,$$

which (recalling that $\partial^j J_j^m = 0$) gives a contribution from W_3 of

$$\tfrac{1}{9}(\partial_l \omega_{km})(\partial^k \omega^{lm}) + \tfrac{1}{9}|\nabla J|^2.$$

By $d\bar{\omega} = 0$, we have

$$(\partial_l \omega_{km})(\partial^k \omega^{lm}) = -\tfrac{1}{2}(\partial_k \omega_{ml})(\partial^k \omega^{lm}) = +\tfrac{1}{2}|\nabla J|^2.$$

So the contribution from W_3 simplifies to

$$\tfrac{1}{6}|\nabla J|^2. \tag{3.24}$$

The terms in W_4 are purely imaginary and therefore must contribute zero because σ is real. This can easily be confirmed explicitly.

To compute W_5 we need to consider

$$c^{jk(2)} \partial_{u^j} \partial_{u^k} U_0 + (\partial_{u^j} c^{jk(2)}) \partial_{u^k} U_0.$$

Noting that $\partial_{u^j} U_0 = -(u_j/2) U_0$, this becomes

$$\left[\tfrac{1}{4} c^{(2)}_{jk} u^j u^k - \tfrac{1}{2}\beta^{jk} c^{(2)}_{jk} - \tfrac{1}{2} u_k (\partial_{u^j} c^{jk(2)})\right] U_0.$$

If $c^{jk(2)}$ is written $E^{jk}_{lm} u^l u^m$, then under contraction the contribution is

$$\begin{aligned}&\tfrac{1}{4}(\beta^{lm}\beta_{jk} E^{jk}_{lm} + E^{jk}_{jk} + E^{jk}_{kj}) - \tfrac{1}{2}\beta^{lm}\beta_{jk} E^{jk}_{lm} - \tfrac{1}{2}(E^{jk}_{jk} + E^{jk}_{kj})\\ &\quad = -\tfrac{1}{4}(\beta^{lm}\beta_{jk} E^{jk}_{lm} + E^{jk}_{jk} + E^{jk}_{kj})\end{aligned}$$

This is the same as the contribution of

$$-\tfrac{1}{4} c^{(2)}_{jk} u^j u^k = -\tfrac{1}{8}(\partial_j \partial_k \beta_{lm}) z^j z^k z^l z^m + \tfrac{1}{4}(\partial_m F_{jlk}) z^m z^k z^j z^l,$$

yielding

$$-\tfrac{1}{8}\beta^{lm}(\partial^j \partial_j \beta_{lm}) - \tfrac{1}{4}(\partial^j \partial^k \beta_{jk}) + \tfrac{1}{4}\beta^{lm}\partial_k F^k_{lm} + \tfrac{1}{2}(\partial^m F^k_{mk}),$$

which vanishes upon substitution of the F. Hence the total contribution of W_5 to σ is zero.

Finally, we evaluate the expression appearing in W_6:

$$\begin{aligned}\tfrac{1}{4} u^j \partial_{u^j}[a^{(2)} + \mathrm{Tr} c^{(2)}] &= \tfrac{1}{2}[a^{(2)} + \mathrm{Tr} c^{(2)}]\\ &= \tfrac{1}{4}(\beta^{lm}\partial_j \partial_k \beta_{lm}) z^j z^k - \tfrac{1}{6}(\partial_l F^l_{ik}) z^i z^k - \tfrac{1}{3}(\partial_m F^l_{il}) z^m z^i\end{aligned}$$

The contribution is

$$\tfrac{1}{4}(\beta^{lm}\partial^k \partial_k \beta_{lm}) - \tfrac{1}{6}(\beta^{ik}\partial_l F^l_{ik}) - \tfrac{1}{3}(\partial^m F^l_{ml}).$$

Substituting in for F^l_{ik} gives us a final contribution from W_6 of

$$\tfrac{1}{6}(\beta^{lm}\partial^k\partial_k\beta_{lm}) - \tfrac{1}{6}(\partial^k\partial^l\beta_{kl}). \tag{3.25}$$

Adding together (3.22), (3.23), (3.24), and (3.25) gives

$$\sigma = -\frac{n^2}{4} - \tfrac{1}{12}|\nabla J|^2 - \tfrac{1}{4}(\partial_j\partial_k\beta_{lm})\omega^{jl}\omega^{km} + \tfrac{1}{4}\beta^{lm}\partial^k\partial_k\beta_{lm} - \tfrac{1}{4}\partial^j\partial^l\beta_{jl}.$$

The last three terms on the right-hand side could be written in terms of the curvature tensors:

$$-\tfrac{1}{4}(\partial_j\partial_k\beta_{lm})\omega^{jl}\omega^{km} + \tfrac{1}{4}\beta^{lm}\partial^k\partial_k\beta_{lm} - \tfrac{1}{4}\partial^j\partial^l\beta_{jl} = \tfrac{1}{4}\left(R + \tfrac{1}{2}R_{ljkm}\omega^{lj}\omega^{km}\right).$$

To complete the calculation we cite a lemma which can be found, for example, in [7].

LEMMA 3.4. *For an almost Kähler manifold,*

$$R + \tfrac{1}{2}R_{ljkm}\omega^{lj}\omega^{km} = -\tfrac{1}{2}|\nabla J|^2.$$

This lemma leads us to the final result that

$$\sigma = -\frac{n^2}{4} - \tfrac{5}{24}|\nabla J|^2. \tag{3.26}$$

3.4. QUASIMODES

Let us introduce the function

$$h(x) = -\tfrac{5}{24}|\nabla J(x)|^2.$$

PROPOSITION 3.5. *Fix $x_0 \in X$ and let $\Gamma = \pi^{-1}(x_0)$. There exists a sequence $\psi_k \in L^2(Z)_k$ with $\|\psi_k\| = 1$ such that*

$$\|(\Delta_h - nk - h(x_0))\psi_k\| = \mathrm{O}(k^{-1/2}). \tag{3.27}$$

Moreover, ψ_k is asymptotically localized on Γ in the sense that if $\varphi \in C^\infty(Z)$ vanishes to order m on Γ, then

$$\langle\psi_k, \varphi\psi_k\rangle = \mathrm{O}(k^{-m/2}). \tag{3.28}$$

Proof. Let W be a neighborhood of Γ in which Fermi coordinates (s, y) are valid, and $\chi \in C^\infty(Z)$ a cutoff function with $\mathrm{supp}(\chi) \subset W$ and $\chi = 1$ in some neighborhood of Γ. Then we define the sequence $\psi_k \in C^\infty(Z)_k$ by

$$\psi_k(s, y) = \Lambda_k\chi\, \mathrm{e}^{i\kappa s}[U_0 + \kappa^{-1}U_1],$$

where $U_j(s, y)$ are the solutions obtained above, $\kappa = k + n/2$, and Λ_k normalizes $\|\psi_k\| = 1$. This could be written as

$$\psi_k(s, y) = \Lambda_k \chi\, e^{iks}[P_0 + P_2(y) + \kappa P_4(y)]\, e^{-\kappa y^2/4}, \tag{3.29}$$

where P_l is a polynomial of degree l (with coefficients independent of k). Since $P_0 = 1 + O(k^{-1})$, we have that

$$\Lambda_k \sim \left(\frac{k}{2\pi}\right)^{n/2} \quad \text{as } k \to \infty.$$

The concentration of ψ_k on Γ described in (3.28) then follows immediately from (3.29).

By virtue of the factor $e^{-\kappa y^2/4}$, we can turn the formal considerations used to obtain the operators $\mathcal{L}_j$ into estimates. With cutoff, $\chi \mathcal{L}_j$ could be considered an operator on Z with support in W. By construction we have

$$\chi[e^{-i\kappa s}\Delta_Z\, e^{i\kappa s} - \kappa^2 - \kappa\mathcal{L}_0 - \sqrt{\mathcal{L}}_1 - \mathcal{L}_2] = \sum_{l,m,|\beta|\leq 2} E_{l,m,\beta}(s, y)\kappa^l \partial_s^m \partial_y^\beta,$$

where $A_{l,m,\beta}$ is supported in W and vanishes to order $2l + |\beta| + 1$ at $y = 0$. We also have

$$(\kappa\mathcal{L}_0 + \sqrt{\kappa}\mathcal{L}_1 + \mathcal{L}_2 - \sigma)(U_0 + \kappa^{-1}U_1) = \kappa^{-1}(\sqrt{\kappa}\mathcal{L}_1 + \mathcal{L}_2 - \sigma)U_1.$$

Combining these facts with the definition of ψ_k we deduce that

$$(\Delta_Z - \kappa^2 - \sigma)\psi_k(s, y) = \Lambda_k \sum_{l\leq 4} k^l F_l(s, y)\, e^{-\kappa y^2/4},$$

where F_l is supported in W and vanishes to order $2l + 1$ at $y = 0$. Using this order of vanishing we estimate

$$\left\| \Lambda_k k^l F_l\, e^{-\kappa y^2/4} \right\|^2 = O(k^{-1}).$$

Noting that $\Delta_Z - \kappa^2 - \sigma = \Delta_h - nk - h(x_0)$ on $L^2(Z)_k$, we obtain the estimate (3.27). $\square$

4. Spectral Density Function

Let $\psi_k \in L^2(Z)_k$ be the sequence produced by Proposition 3.5. As in Section 2, we let Π_k denote the orthogonal projection onto the span of low-lying eigenvectors of $\Delta_h - nk$. Consider

$$\phi_k = \Pi_k\psi_k \qquad \eta_k = (I - \Pi_k)\psi_k.$$

By Theorem 1.1 (for k sufficiently large, which we will assume throughout),

$$\|(\Delta_h - nk)\phi_k\| < M, \quad \|(\Delta_h - nk)\eta_k\| > ak\,\|\eta_k\|\,.$$

By Proposition 3.5 we have a uniform bound

$$\|(\Delta_h - nk)\psi_k\| \leq C,$$

so these estimates imply in particular that

$$ak\,\|\eta_k\| < C + M.$$

Hence $\|\eta_k\| = \mathrm{O}(k^{-1})$.

From Lemma 2.1 we know that q satisfies

$$\langle \phi_k, (\Delta_h - nk - \pi^* q)\phi_k\rangle = \mathrm{O}(1/k).$$

Let $r_k = (\Delta_h - nk + h(x_0))\psi_k$, which by Proposition 3.5 satisfies $\|r_k\| = \mathrm{O}(k^{-1/2})$. So

$$\begin{aligned}
&\langle \phi_k, (\Delta_h - nk - \pi^* q)\phi_k\rangle \\
&\quad = \langle \phi_k, (h(x_0) - \pi^* q)\phi_k\rangle + \langle \phi_k, (\Delta_h - nk - h(x_0))\phi_k\rangle \\
&\quad = \langle \phi_k, (h(x_0) - \pi^* q)\phi_k\rangle + \langle \phi_k, r_k\rangle - \langle \phi_k, (\Delta_h - nk - h(x_0))\eta_k\rangle. \qquad (4.1)
\end{aligned}$$

The left-hand side is $\mathrm{O}(1/k)$, while the second term on the right is $\mathrm{O}(k^{-1/2})$, The third term term on the right-hand side is equal to

$$\langle (\Delta_h - nk)\phi_k, \eta_k\rangle < M\,\|\eta_k\| = \mathrm{O}(k^{-1}).$$

Therefore, the first term on the right-hand side of (4.1) can be estimated

$$\langle \phi_k, (h(x_0) - \pi^* q)\phi_k\rangle = \mathrm{O}(k^{-1/2}).$$

Because $\|\eta_k\| = \mathrm{O}(1/k)$ this implies also that

$$h(x_0) - \langle \psi_k, (\pi^* q)\psi_k\rangle = \mathrm{O}(k^{-1/2}).$$

Since q is smooth, the localization of ψ_k on Γ from Proposition 3.5 implies that

$$\langle \psi_k, (\pi^* q)\psi_k\rangle = q(x_0) + \mathrm{O}(k^{-1/2}).$$

Thus $q(x_0) = h(x_0)$. This proves Theorem 1.2.

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