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THE SPECTRAL GEOMETRY OF A RIEMANNIAN MANIFOLD

PETER B. GILKEY

Introduction

Let M be a compact smooth d-dimensional Riemannian manifold without boundary. Let $X = (X_1, \cdots, X_d)$ be a system of local coordinates centred at x_0. The metric tensor is given by

$$ds^2 = g_{ij} dX_i \otimes dX_j \qquad (\text{summed over } i, j = 1, \cdots, d) .$$

We adopt the convention of summing over repeated indices except where otherwise indicated. Let (g^{ij}) denote the inverse of the matrix (g_{ij}).

Let V be a smooth vector bundle over M and let D be a second order differential operator on V. Let $e = (e_1, \cdots, e_r)$ be a local frame for V defined near x_0. The coordinate system and frame e comprise a local system which identifies a neighborhood of M with R^d and a portion of V with $R^d \times R^r$. Using this local system, we express the operator D:

$$D = -\left(h^{ij} \frac{d^2}{dx_i dx_j} + a_i \frac{d}{dx_i} + b\right) ,$$

where h^{ij}, a_i, and b are square $r \times r$ matrices. Let $\xi \in T^*M$ and define

$$a^2(x, \xi) = h^{ij} \xi_i \xi_j , \quad a^1(x, \xi) = -i a_i \xi_i , \quad a^0(x, \xi) = -b .$$

The leading order symbol of D is a^2, which is defined invariantly. The lower order terms depend upon the local system chosen.

For the rest of this paper, we assume that the leading symbol is given by the metric tensor, i.e., that $h^{ij} = g^{ij} I = g^{ij}$, which implies $a^2(x, \xi) = |\xi|^2$. We omit multiplication by the identity matrix on V, and apply the functional calculus to define the operator $\exp(-tD)$ for $t > 0$. $\text{Exp}(-tD)$ is an infinitely smoothing operator from $L^2(V) \to C^\infty(V)$. It is defined by a kernel function $K(t, x, y, D)$ such that:

$$\exp(-tD) u(x) = \int K(t, x, y, D) u(y) d\,\text{vol}(y) ,$$

$K(t, x, y, D)$ maps V_y to V_x, $d\,\text{vol}(y)$ is the Riemannian measure. Seeley [8] proved that $K(t, x, x, D)$ has an asymptotic expansion as $t \to 0^+$ of the form:

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$$K(t, x, x, D) \sim \sum_{n=0}^{n} E_n(x, D)t^{(n-d)/2}, \qquad t \to 0^+ .$$

The E_n's are certain endomorphisms defined on the fibre. They vanish for odd n since D is a differential operator. Although the defining relation is global, we can compute them in terms of the derivatives of the symbol of the operator. They are local invariants of the differential operator D. In the first section we review the work of Seeley [8] to obtain explicit combinatorial formulas for E_n.

In the second section we apply invariance theory to investigate the form which E_n has. We will express E_n in terms of noncommutative polynomials in the covariant derivatives of certain tensors. By using H. Weyl's theorem [9], this will express E_n as a sum of various contractions of these tensors with unknown coefficients. In the third section we will evaluate these coefficients to determine E_0, E_2, E_4, E_6. In the final section we apply these results to the Laplace operator acting on functions.

Let V have an inner product (,) and suppose that D is self-adjoint. Take a spectral resolution of D into eigenvalues λ_i and corresponding eigenfunctions ϕ_i. Let

$$K(t, x, y, D) = \sum \exp(-t\lambda_i)\phi_i(x) \otimes \phi_i(y) .$$

Let $B_n(x, D) = \text{Trace}\,(E_n(x, D)$, $B_n(D) = \int B_n(x, D)d\,\text{vol}\,(x)$. Then

$$\text{Tr}\,(K(t, x, x, D)) = \sum \exp(-t\lambda_i)(\phi_i, \phi_i)(x) \sim \sum_{n=0}^{\infty} B_n(x, D)t^{(n-d)/2} .$$

We integrate both sides of this expansion. The ϕ_i were an orthonormal basis so they integrate to 1. Consequently

$$\exp(-t\lambda_i) \sim \sum_{n=0}^{\infty} \left(\int_M B_n(x, D)d\,\text{vol}\,(x)\right)t^{(n-d)/2} \sim \sum_{n=0}^{\infty} B_n(D)t^{(n-d)/2} .$$

The numbers $B_n(D)$ are invariants of the differential operator, which depend only on its spectrum.

Let D be some Laplacian of differential geometry. The invariants $B_n(x, D)$ will be certain expressions in the derivatives of the metric. We suppose that D_p is the Laplace-Belltrami operator acting on p-forms. Sakai [7] has computed a formula for $B_6(D_0)$. Using this formula, he proved

Theorem (*Sakai*). *Let M, M' be compact, connected orientable Einstein manifolds of dimension 6 which have the same Euler characteristic. Suppose that the spectrum of D_0 is the same for both manifolds. Then M is symmetric if and only if M' is symmetric.*

Donnely [2] has been able to improve this result as follows: his major contribution has been to remove the restriction that $d = 6$.

Theorem (*Donnely*). *Let M be an Einstein manifold which has the same spectrum for all the operators D_p, $p = 0, \cdots, d$, as a symmetric space N. Then N is Einstein and M is symmetric.*

Donnely's proof goes as follows: he applied a theorem of Patodi's [6] to show under these assumptions that N is Einsteinian. This result uses the computation of Patodi of the invariants $B_4(D_p)$. Let P denote the Pfaffian in dimension 6. P can be defined for all values of d. Then it was shown in (3, 5) that if $d = 6$,

$$P = \Sigma(-1)^p B_6(x, D_p) .$$

By applying the functorial properties of these invariants, this implies that P must be a combination of the invariants $B_6(x, D_p)$ for any $d \geq 6$, and therefore that the number $\int P$ is a spectral invariant for any d. Donnely used this observation together with the computation of Sakai to complete the proof.

In this paper, we derive a general formula for the endomorphism E_6. In the last section, we use this to derive Sakai's formula for $B_6(x, D_0)$. In a later paper, we will apply this formula to determine $B_6(x, D_p)$ as well as to determine B_6 for other second order operators which occur in geometry. We hope that these additional computations will enable us to remove the hypothesis that M is Einsteinian and therby show that the property of being a symmetric space is determined by the spectral geometry of the manifold.

Some of the computations in the determination of E_6 are long and combinatorial in nature. In an earlier paper [4], we computed the endomorphisms E_0, E_2 and E_4. We would like to express our appreciation to B. Galvannoni and M. Freidman at the IBT-CO for making computer time available us for use in the computation of E_6.

1. In this section, we derive a combinatorial formula for the endomorphisms E_n in terms of the derivatives of the symbol. We assume that the reader is familar with the calculus of pseudo-differential operators depending upon a complex parameter which was developed by Seeley [8]. Our arguments will be purely formal since the questions of convergence have already been dealt with by Seeley.

Let D be as described in the introduction. The symbol of D is given by:

$$\sigma(D)(x, \xi) = a^2(x, \xi) + a^1(x, \xi) + a^0(x) ,$$
$$a^2(x, \xi) = |\xi|^2 , \quad a^1(x, \xi) = -ia_j\xi_j , \quad a^0(x, \xi) = -b ,$$
$$D = -\left(g^{ij}\frac{d^2}{dx_i dx_j} + a_j\frac{d}{dx_j} + b\right) ,$$

where the a^j are homogeneous of order j in the dual variable ξ.

We introduce the following notational conventions:

$$\alpha = (\alpha_1, \cdots, \alpha_d) \text{ is a multi-index },$$
$$|\alpha| = \alpha_1 + \cdots + \alpha_d \text{ is the order of } \alpha ,$$
$$\alpha ! = \alpha_1 ! \cdots \alpha_d ! ,$$
$$d_\alpha^\xi = (d/d\xi_1)^{\alpha_1} \cdots (d/d\xi_d)^{\alpha_d} ,$$
$$D_x^\alpha = (-i)^{|\alpha|}(d/dx_1)^{\alpha_1} \cdots (d/dx_d)^{\alpha_d} .$$

Let c be a matrix or function. Let $c_{/\alpha} = d_x^\alpha(c)$. We will also use the notation $c_{/i_1 \cdots i_k} = d/dx_{i_1} \cdots d/dx_{i_k}(c)$. We introduce the following notation for the formal derivatives of the symbol of the operator:

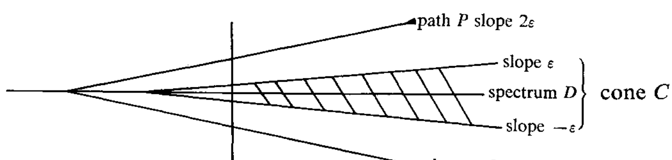
$$g_{ij/\alpha} = d_x^\alpha(g_{ij}) \text{ is defined to have order } |\alpha| ,$$
$$a_{i/\alpha} = d_x^\alpha(a_i) \text{ is defined to have order } 1 + |\alpha| ,$$
$$b_{/\alpha} = d_x^\alpha(b) \text{ is defined to have order } 2 + |\alpha| .$$

Let P denote the noncommutative algebra in these formal variables and let P_n be the linear subspace of all polynomials which are homogeneous of order n. For P in P_n, define $P(X, e, D)$ by evaluation in the local system (X, e) on the symbol of the operator D. If the endomorphism defined by P is independent of the particular local system chosen and depends only on the differential operator D, then P is said to be invariant. Let Q be the subalgebra of all invariant expressions in the derivatives of the symbol. Let Q_n denote the subspace of invariant polynomials of order n. We will study Q_6 in detail in the next section.

The leading symbol of D is self-adjoint, positive, nonzero. Let $\varepsilon > 0$ be given. The spectrum of D lies in a cone C of slope ε about the real axis. Let P be a path about the cone C with slope 2ε outside some compact set. For λ on P, the operator $(D - \lambda)^{-1}$ is a uniformly bounded compact operator from $L^2(V) \to L^2(V)$. The integral

$$-\frac{1}{2\pi i} \int_P \exp(-t\lambda)(D - \lambda)^{-1} d\lambda$$

converges absolutely for $t > 0$ and defines the operator $\exp(-tD)$.



We construct a pseodo-differential operator to approximate the resolvant

$(D-\lambda)^{-1}$ as follows: let $b(x,\xi,\lambda) \sim b_0(x,\xi,\lambda) + \cdots + b_n(x,\xi,\lambda) + \cdots$. Let the complex parameter λ have homogeneity 2. Let the b_i be homogeneous of order $-2-i$ in the variables (ξ,λ). This infinite sum defines b asymptotically. The symbol of the composition of the operator defined by b is given by

$$\sigma(B(D-\lambda)) \sim \sum_\alpha (d_\xi^\alpha b)\cdot(D_x^\alpha(\sigma(D-\lambda)))/\alpha! \ .$$

Define

$$\tilde{a}^2 = |\xi|^2 - \lambda \ , \quad \tilde{a}^1 = a^1 = -ia_j\xi_j \ , \quad \tilde{a}^0 = a^0 = -b \ .$$

Decompose this sum into orders of homogeneity:

$$\sigma(B(D-\lambda)) \sim \sum_{n=0}^{\infty} (\Sigma_{n=j+|\alpha|+2-k}(d_\xi^\alpha b_j)\cdot(D_x^\alpha a_k)/\alpha!) \ .$$

The sum is over terms which are homogeneous of order $-n$. We wish to define b so that

$$\sigma(B(D-\lambda)) \sim I \ .$$

This yields the equations

$$\begin{aligned} I &= \sum_{0=j+|\alpha|+2-k} (d_\xi^\alpha b_j)(D_x^\alpha \tilde{a}_k)/\alpha! = b_0(|\xi|^2-\lambda) \ , \\ 0 &= \sum_{n=j+|\alpha|+2-k} (d_\xi^\alpha b_j)(D_x^\alpha \tilde{a}_k)/\alpha! \\ &= b_n(|\xi|^2-\lambda) + \sum_{\substack{n=j+|\alpha|+2-k \\ j<n}} (d_\xi^\alpha b_j)(D_x^\alpha \tilde{a}_k)/\alpha! \ . \end{aligned}$$

These equations define the b_n inductively. In the sum in the second equation, if $k=2$ and $j<n$, then $|\alpha| \neq 0$. Consequently we replace $\tilde{a}_k$ by a_k. Define

$$b_0 = (|\xi|^2-\lambda)^{-1} \ , \quad b_n = -b_0\left(\sum_{j<n} (d_\xi^\alpha b_j)(D_x^\alpha a_k)/\alpha!\right)$$
$$\text{for } n = j + |\alpha| + 2 - k.$$

It is clear that b_0 is a scalar matrix.

Lemma 1.1.

(1) $b_n = \sum_\alpha b_{n,\alpha}(x)\xi^\alpha b_0^{k(n,\alpha)}$ *for* $k(n,\alpha) = \frac{1}{2}(|\alpha| + n + 2)$ *is an integer,*

(2) *the* $b_{n,\alpha}$ *belong to* P_n.

The proof of this lemma is by induction. It follows immediately from the inductive definition given of the b_n. The fact that a_2 is a scalar matrix is essential in order for us to express the dependence of b_n upon the complex parameter in this fashion. This assumption fails when we consider the ETA invariant defined by Atiyah-Patodi-Singer. It is this fact which makes the computation of the local pole of the ETA function at zero so dificult.

Our final formula will express E_n in terms of the matrices $b_{n,\alpha}$. This will imply that $E_n(x, D)$ lies in P_n. Since $E_n(x, D)$ is independent of the particular local system chosen, it is invariant. We will exploit this invariance in the next section.

We use this approximation to the resolvant to define an approximation of $\exp(-tD)$:

$$e_n(x, \xi, t) = -\frac{1}{2\pi i}\int_P \exp(-t\lambda) b_n(x, \xi, \lambda) d\lambda \, .$$

Let $E(t)$ have symbol $e_0 + \cdots + e_n + \cdots$. $E(t)$ is a pseudo-differential approximation of $\exp(-tD)$. Let H_s denote the Sobelev space defined as the completion of the smooth functions in the s-norm to measure L^2 derivatives. Let A be any pseudo-differential operator. Define $|A|_{s,s'}$ to be the operator norm (possibly infinite) of A as a map from H_s to $H_{s'}$. The following estimates were proved by Seeley:

$$|E(t) - \exp(-tD)|_{s,s'} \leq C(s, s', k)t^k \qquad \text{as } t \to 0.$$

The constants $C(s, s', k)$ are finite for all s, s', k. Consequently, the difference of these two operators is an infinitely smoothing operator. The diference has a kernel function which dies to infinite order as $t \to 0$. Consequently, the asymptotic behavior of the kernel function of the operator $\exp(-tD)$ is the same as that of the kernel function of the operator $E(t)$.

The kernel of a pseudo-differential A is given by the equation:

$$K(x, y) = \int \exp(\xi\cdot(x - y))\sigma(A)(x, \xi)d\xi\cdot(2\pi)^{-d}$$

provided that this integral converges absolutely. The normalizing constant $(2\pi)^{-d}$ arises from the reverse Fourrier transform. We compute the kernel of $E(t)$:

$$\begin{aligned} e_n(x, \xi, t) &= -\frac{1}{2\pi i}\int_P b_n(x, \xi, \lambda)\exp(-t\lambda)d\lambda \\ &= -\frac{1}{2\pi i}\sum_\alpha \int_P b_{n,\alpha}(x)\xi^\alpha b_0^{k(n,\alpha)}\exp(-t\lambda)d\lambda \\ &= -\sum_\alpha b_{n,\alpha}(x)\xi^\alpha \int_P \frac{1}{(2\pi)^d}(|\xi|^2 - \lambda)^{-k(n,\alpha)}\exp(-t\lambda)d\lambda \, . \end{aligned}$$

We evaluate this countour integral using Cauchy's formula. This yields:

$$e_n(x, \xi, t) = \Sigma b_{n,\alpha}(x)\xi^\alpha t^{k-1} \exp(-t|\xi|^2)/(k-1)!\ ,$$

where $k = k(n, \alpha)$. This function dies exponentially as the dual variable tends to infinity. It therefore defines a smooth kernel function. On the diagonal, we compute

$$K_n(t, x, x) = \sum_\alpha b_{n,\alpha}(x)t^{k-1} \int \frac{1}{(2\pi)^d(k-1)!} \exp(-t|\xi|^2)\xi^\alpha d\xi\ .$$

We change variables in the integral. This gives rise to the formula

$$K_n(t, x, x) = \sum_\alpha b_{n,\alpha}(x)t^{k-1-|\alpha|/2-d/2}c_{d,\alpha}/(k-1)!\ ,$$

$$c_{d,\alpha} = \int \frac{1}{(2\pi)^d}\xi^\alpha \exp(-|\xi|^2)d\xi\ .$$

Since $k(n, \alpha) - 1 - \frac{1}{2}|\alpha| = \frac{1}{2}n$, this proves that

$$K_n(t, x, x) = t^{(n-d)/2} \sum_\alpha c_{d,\alpha}b_{n,\alpha}(x)/(k-1)!\ .$$

Since the kernel function for $E(t)$ is given by $K_0 + \cdots + K_n + \cdots$, and the kernel function for $E(t)$ asymptotically approximates the kernel function for $\exp(-tD)$, this proves the convergence of the series given in the introduction and shows that

$$E_n(x, D) = \sum_\alpha b_{n,\alpha}(x)c_{d,\alpha}/(k-1)!\ .$$

To complete the formula for E_n, we must evaluate the harmonic integral defining $c_{d,\alpha}$. Let $\alpha = (\alpha_1, \cdots, \alpha_d)$. If any of the α_i is an odd integer, this integral vanishes. Consequently, we may suppose that $\alpha = 2\beta$. Since $n = 2k - |\alpha| - 2$, E_n is zero unless n is even.

Lemma 1.2. $c_{d,\alpha} = (4\pi)^{-d/2}(2\beta)!/(\beta!\,4^{|\beta|})$.

Note that this formula agrees with the formula given in the author's thesis, which was, however, expressed differently.

Proof. The identity

$$\sqrt{\pi} = \int_0^\infty \exp(-r)(r)^{.5} = \int_{-\infty}^\infty \exp(-r^2)dr$$

implies that

$$\begin{aligned}\int_{-\infty}^\infty r^{2k}\exp(-r^2)dr &= \int_0^\infty r^{(k-.5)}\exp(-r)dr = (k-.5)(k-1.5)\cdots(.5)\sqrt{\pi}\\ &= \sqrt{\pi}(2k-1)(2k-3)\cdots(1)/2^k\\ &= \sqrt{\pi}(2k)(2k-1)\cdots(1)/((2k)(2k-2)\cdots(2)2^k)\\ &= \sqrt{\pi}(2k)!/(k!\,4^k)\ .\end{aligned}$$

Consequently

$$c_{d,\alpha} = \int \frac{1}{(2\pi)^d} \xi^{2\beta} \exp(-|\xi|^2) d\xi = \prod_{i=1}^{d} \int \frac{1}{(2\pi)^d} (\xi_i^{2\beta_i} \exp(-\xi_i^2) d\xi_i) .$$
$$= (\pi)^{d/2}(2\beta)!/(\beta!\, 4^{|\beta|}(2\pi)^d) = (4\pi)^{-d/2}(2\beta)!/(\beta!\, 4^{|\beta|}) .$$

We summarize our conclusions in

Theorem 1.3.

(1) *Define* $b_0 = (|\xi| - \lambda)$ *and* $b_n = -b_0 \Sigma (d_\xi^\alpha b_j)(D_x^\alpha a_k)/\alpha!$ *for* $n = j + |\alpha| + 2 - k,\ j < n$.

(2) $b_n = \sum_\alpha b_{n,\alpha} \xi^\alpha b_0^{k(n,\alpha)}$ *for* $k(n, \alpha) = \frac{1}{2}(2 + n + |\alpha|)$.

(3) $E_n(x, D) = (4\pi)^{-d/2} \Sigma b_{n,2\beta}(x)(2\beta)!/(\beta!\, 4^{|\beta|}(k-1)!)$.

(4) $E_n(x, D)$ *belongs to* Q_n.

2. In this section, we will exhibit a basis for the vector space Q_6. We have previously constructed bases for Q_0, Q_1, Q_4 in [4]. Let D be as in section one and let ∇ be any connection on V. In a local system, we express $\nabla_i(e) = \nabla_{d/dx_i}(e) = w_i(e)$ where w_i is an $r \times r$ matrix called the connection form.

Since M is a Riemannian manifold, let ∇^r be the Levi-Civita connection on TM. The Christoffel symbols $\Gamma_{ij}{}^k$ are defined by the equation

$$\nabla_i^r(d/dx_j) = \Sigma \Gamma_{ij}{}^k d/dx_k .$$

We extend the connection to T^*M in the natural way. Then

$$\nabla_i^r(dx_j) = -\sum_k \Gamma_{ik}{}^j (dX_k) .$$

The connection on TM and V induce connections on the complete tensor algebra. The metric tensor is a map from $T^*M \otimes T^*M$ to R. We define the operator D_∇ by

$$D_\nabla : C^\infty(V) \xrightarrow{\nabla} C^\infty(V \otimes T^*M) \xrightarrow{\nabla} C^\infty(V \otimes T^*M \otimes T^*M) \xrightarrow{\text{metric}} C^\infty(V) .$$

In polar geodesic coordinates, D is given by the formula

$$D_\nabla(s) = \sum_{i,j} - g^{ij} \nabla_i \nabla_j(s) ,$$

where s is any smooth section to V.

We determine a unique connection from the differential operator D as follows. The operator $(D - D_\nabla)$ is a first order operator for any connection. The leading symbol of this operator is

$$\sigma(D - D_\nabla)(x, \xi) = \Sigma \xi_i (a_i - 2g^{ij} w_j + g^{jk} \Gamma_{jk}{}^i) + \text{zero order terms.}$$

The first order part is invariantly defined. We define the connection uniquely by requiring that $(D - D_\nabla)$ is a 0-th order operator. This defines the w_i by the equations

$$a_i - 2g^{ij}w_j + g^{jk}\Gamma_{jk}{}^i = 0 \qquad \text{for } i = 1, \cdots, d .$$

We fix this invariantly defined connection henceforth.

The connection was defined so that $E = (D - D_\nabla)$ is an invariantly defined 0-th order operator. This implies that E is an endomorphism.

Theorem 2.1. *Let D be given. There are a unique connection ∇ on V and a unique endomorphism E of V such that $D = D_\nabla + E$.*

The derivatives of the symbol of D can be computed in terms of the derivatives of the metric, the derivatives of the connection form w_i, and the endomorphism E. Let X be geodesic polar coordinates at x_0, and $e(x_0)$ an arbitrary frame for the fibre at x_0. Extend e to a smooth frame near x_0 by parallel transport along the geodesic rays from x_0. If we require that $g_{ij}(x_0) = \delta_{i,j}$, then this choice of coordinates is unique up to the action of $0(d)$, and the choice of frame is unique up to the action of $GL\,(\dim(V))$.

Let

$$R_{ijkm} = G((\nabla_i\nabla_j - \nabla_j\nabla_i)d/dx_k, d/dx_m) ,$$
$$W_{ij} = w_{j/i} - w_{i/j} + w_iw_j - w_jw_i ,$$

where R_{ijkm} is the curvature tensor of the Levi-Civita connection on TM, and W_{ij} is an $r \times r$ matrix giving the curvature tensor of the connection on V. We covariantly differentiate these tensors and the endomorphism E to form the tensors

$$R_{ijkm;i_1\cdots i_s} , \quad W_{ij;i_1\cdots i_s} , \quad E_{;i_1\cdots i_s} .$$

These tensors are of order $2 + s$ in the derivatives of the symbol of the operator. The notation ";" denotes covariant differentiation.

Since X is a system of geodesic polar coordinates, we can express the ordinary derivatives of the metric tensor in terms of the $R_{ijkm;\dots}$ tensors at x_0. Furthermore, we can express the ordinary derivatives of the connection form w_i at x_0 in terms of the values of the $R_{ijkm;\dots}$ and $W_{ij;\dots}$ tensors at x_0. Finally, it is clear that we can compute the ordinary derivatives of the endomorphism E in terms of the $E_{;\dots}$ tensors and the derivatives of the metric and connection forms. (A proof of these facts is to be found in the appendix to [1]). Consequently, we can express any element of Q_n in terms of the tensors listed above.

Since we are considering endomorphism valued invariants, the action of $GL(\dim(V))$ on the choice of frame can be ignored. We apply H. Weyl's theorem [9] on the invariants of the orthogonal group to deduce that every element of Q_n can be constructed in terms of contractions of indices. Since the algebra of invariant polynomials is noncommutative, we must consider contractions of all possible noncommutative expressions. This proves

Theorem 2.2. *A basis for Q_n can be constructed, which consists of contractions of various noncommutative expressions in the tensors listed above which are of order n. We contract these tensors by summing over repeated indices.*

We first consider those invariants which depend on the metric tensor alone. Donnely [2] has computed all the invariants of the metric tensor, which are of order 6. These are listed in the first column of table I of the appendix. Next, we consider those invariant expressions depending only on the metric and connection curvature tensors. After reducing by the Bianchi identities, there are a total of 11 such expressions which are listed in the third column of table I. Finally, we consider those invariants which involve the endomorphism E. There are 18 such expressions which are listed in fifth column of table I. The computations showing that these 46 invariants are linearly independent and span Q_6 are routine in nature and are therefore omitted. In the next section, we express E_6 in terms of these 46 invariants.

3. In an earlier paper [4], we computed that

$$\begin{aligned} E_0 &= (4\pi)^{-d/2} I \,, \\ E_2 &= (4\pi)^{-d/2}(E - \tfrac{1}{6}R_{ijij}) \,, \\ E_4 &= (4\pi)^{-d/2}((-\tfrac{1}{30}R_{ijij;kk} + \tfrac{1}{72}R_{ijij}R_{kmkm} - \tfrac{1}{180}R_{ijik}R_{njnk} \\ &\quad + \tfrac{1}{180}R_{ijkn}R_{ijkn}) - \tfrac{1}{6}R_{ijij}E + \tfrac{1}{2}E^2 + \tfrac{1}{12}W_{ij}W_{ij} + \tfrac{1}{6}E_{;kk}) \,. \end{aligned}$$

We sum over repeated indices in any orthonormal frame for TM. The $R_{ijkm;\ldots}$ tensor acts on V by scalar multiplication.

The formula for E_6 involves 46 terms and is much more complex. A basis for the invariants of order 6 is given in table I. Suppose that these invariants are denoted by $P_1, \cdots, P_{46}$. Since E_6 is an invariant of order 6, we can express $E = c_1 P_1 + \cdots + c_{46} E_{46}$ where the c_i's are certain universal constants. These constants are listed in table I. Thus our formula reads

$$E_6 = (4\pi)^{-d/2}(-\tfrac{18}{7}!\, R_{ijij;kknn} + \cdots + \tfrac{1}{180} E R_{ijkn} R_{ijkn}) \,.$$

The remainder of this section is devoted to the determination of the constants c_i. They are determined by considering the following special example: let M be the d-dimensional torus for some $d \geq 6$. Choose a metric of the form

$$ds^2 = g_i dx_i^2 \,.$$

Suppose that $g_{i/i}$ vanishes identically. Let $h_i = g_i^{-1}$ be the inverse function.

The Christoffel sumbols $\Gamma_{ij}{}^k$ vanish identically unless exactly two of the indices are equal. We compute that

$$\Gamma_{ii}{}^j = \tfrac{1}{2}h_j h_i^{-2} h_{j/i} , \qquad \Gamma_{ij}{}^i = \Gamma_{ji}{}^i = -\tfrac{1}{2}h_i^{-1}h_{i/j} .$$

The curvature tensor tensor R_{ijkm} vanishes unless at least two of the indices agree. We compute that

$$\begin{aligned} R_{ijij} &= \tfrac{3}{4}(h_i^{-3}(h_{i/j})^2 + h_j^{-3}(h_{j/i})^2 - \tfrac{1}{2}(h_i^{-2}h_{i/jj} + h_j^{-2}h_{j/ii}) \\ &\quad + \tfrac{1}{4} \sum_{k \neq i,j} h_i^{-2}h_j^{-2}h_{i/k}h_{j/k} , \\ R_{ijik} &= \tfrac{3}{4}(h_i^{-3}h_{i/j}h_{i/k}) - \tfrac{1}{2}(h_i^{-2}h_{i/jk}) \\ &\quad - \tfrac{1}{4}(h_j^{-1}h_{j/k}h_{i/j} + h_k^{-1}h_{k/j}h_{i/k}) \qquad \text{for } j \neq k . \end{aligned}$$

In these two formulas, we do not sum over repeated indices. The other nonzero curvatures can be obtained from these two by using the symmetries.

Let $V = M \times R^r$, and let $a_1, \cdots, a_d$ and b be $r \times r$ matrix valued functions. We suppose that a_i is not a function of x_i—i.e., $a_{i/i} = 0$. Let D be the differential operator

$$D = -(h_i d^2/dx_i^2 + a_i d/dx_i + b) \qquad \text{summed over repeated indices.}$$

This differential operator defines a connection on V. The connection form is

$$w_i = \tfrac{1}{2}h_i^{-1}(a_i + \sum_j h_j \Gamma_{jj}{}^i) = \tfrac{1}{2}h_i^{-1}a_i + \tfrac{1}{4}\sum_k h_k^{-1}h_{k/i} .$$

The sum over k ranges over $k \neq i$ since $h_{i/i} = 0$. The curvature form is

$$\begin{aligned} W_{ij} &= \tfrac{1}{4}h_i^{-1}h_j^{-1}(a_i a_j - a_j a_i) + \tfrac{1}{2}h_j^{-1}a_{j/i} - \tfrac{1}{2}h_i^{-1}a_{i/j} \\ &\quad - \tfrac{1}{2}h_j^{-2}h_{j/i}a_j + \tfrac{1}{2}h_i^{-2}h_{i/j}a_i . \end{aligned}$$

The endomorphism E defined by the operator D is given by

$$\begin{aligned} E &= b - \tfrac{1}{4}\sum_i h_i^{-1}a_i^2 + \sum_{i,k} (-\tfrac{1}{4}h_k^{-1}h_i h_{k/ii}) + \tfrac{5}{16}h_k^{-2}h_i(h_{k/i})^2 \\ &\quad + \sum_i \sum_{j<k} \tfrac{1}{8}h_j^{-1}h_k^{-1}h_{j/i}h_{k/i} . \end{aligned}$$

These formulas together with the formulas for differentiating tensors enables us to express all the invariants listed in table I in terms of the ordinary derivatives of the functions h_i and matrices a_i and b.

By using the combinatorial formulas obtained in the first section, we can express the endomorphism E_6 for this operator in terms of the ordinary derivatives of the functions h_i and matrices a_i and b. We have the identity $E = c_1P_1 + \cdots + c_{46}P_{46}$. This gives rise to a certain system of equations in the derivatives of these functions and matrices. This system is given in tables I-A and following. It is invertible and enables us to determine the c_i's.

We illustrate this method as follows: we apply the formula of the first section to compute that the coefficient of the monomial $b_{/1111}$ in E_6 is $\frac{1}{60}$. The only invariant of table I which contains the term $b_{/1111}$ is $E_{;iijj}$. Furthermore, the coefficient of $b_{/1111}$ in $E_{;iijj}$ is 1. This implies that the coefficient of $E_{;iijj}$ in the expansion of E_6 must be $\frac{1}{60}$ which is indicated in the sixth column of table I. The determination becomes more complicated for the other invariants. We are solving an upper-triangular system of equations which is very sparce. In tables I-A through I-H, we carry out the computations to determine the coefficients which are given in table I.

We consider a very special example in which the computations are particularly simple. This example gives us enough information to determine the coefficients c_i's and hence to determine E_6 for a general operator.

4. In this section, we apply the formula of table I to obtain Sakai's formula. Let D_0 be the Laplace-Beltrami operator acting on functions. For this operator, the connection ∇ on the vector bundle $M \times R$ is flat. The endomorphism E is zero. Consequently, $E_6(x, D_0)$ is given by summing over the first column of table I with the indicated coefficients. Since the vector bundle is 1-dimensional, $B_6(x, D_0) = E_6(x, D_0)$. In order to obtain Sakai's formula for the integral of $B_6(x, D_0)$, we must integrate by parts. We use the relations:

$$\int R_{ijij;kkmm} = 0\,,$$

$$\int R_{ijij;n}R_{kmkm;n} + R_{ijij}R_{kmkm;nn} = 0\,,$$

$$\int R_{ijik;n}R_{mjmk;n} + R_{ijik}R_{mjmk;nn} = 0\,,$$

$$\int R_{ijik;n}R_{mjmn;k} + R_{ijik}R_{mjmn;kn} = 0\,,$$

$$\int R_{ijik;n}R_{mjmn;k} = \int \tfrac{1}{4}R_{ijij;n}R_{kmkm;n} + R_{ijik}R_{mjmp}R_{qkqp} - R_{ijik}R_{mpmq}R_{jpkq}\,,$$

$$\begin{aligned}\int R_{ijkn}R_{imkp}R_{jmnp} = \int & -\tfrac{1}{4}R_{ijij;n}R_{kmkm;n} + R_{ijik;n}R_{mjmk;n} \\ & - \tfrac{1}{4}R_{ijkm;n}R_{ijkm;n} - R_{ijik}R_{jnmn}R_{kpmp} \\ & + R_{ijik}R_{npmp}R_{jnkm} + \tfrac{1}{2}R_{ijik}R_{jnmp}R_{knmp} \\ & - \tfrac{1}{4}R_{ijkn}R_{ijmp}R_{knmp}\,.\end{aligned}$$

We use these relations together with table I to prove

Theorem 4.1.

(1) $$B_6(x, D_0) = \frac{(4\pi)^{-d/2}}{7!}(-18R_{ijij;kkmm} + 17R_{ijij;k}R_{mnmn;k}$$
$$- 2R_{ijik;n}R_{mjmk;n} - 4R_{ijik;n}R_{mjmn;k} + 9R_{ijkm;n}R_{ijkm;n}$$
$$+ 28R_{ijij}R_{kmkm;nn} - 8R_{ijik}R_{mjmk;nn} + 24R_{ijik}R_{mjmn;kn}$$
$$+ 12R_{ijkm}R_{ijkm;nn} - \tfrac{35}{9}R_{ijij}R_{mnmn}R_{pqpq}$$
$$+ \tfrac{14}{3}R_{ijij}R_{mnmp}R_{qnqp} - \tfrac{14}{3}R_{ijij}R_{mnpq}R_{mnpq}$$
$$+ \tfrac{208}{9}R_{ijik}R_{jnmn}R_{kpmp} - \tfrac{64}{3}R_{ijik}R_{npmp}R_{jnkm}$$
$$+ \tfrac{16}{3}R_{ijik}R_{jnmp}R_{knmp} - \tfrac{44}{9}R_{ijkn}R_{ijmp}R_{knmp}$$
$$- \tfrac{80}{9}R_{ijkn}R_{imkp}R_{jmnp})\ .$$

(2) $$B_6(D_0) = \int_M B_6(x, D_0)d\ \mathrm{vol}\,(x)$$
$$= \frac{(4\pi)^{-d/2}}{7!}\int_M (-\tfrac{142}{9}R_{ijij;k}R_{mnmn;k} - \tfrac{26}{9}R_{ijik;n}R_{mjmk;n}$$
$$- \tfrac{7}{9}R_{ijkm;n}R_{ijkm;n} - \tfrac{35}{9}R_{ijij}R_{mnmn}R_{pqpq}$$
$$+ \tfrac{14}{3}R_{ijij}R_{mnmp}R_{qnqp} - \tfrac{14}{3}R_{ijij}R_{mnpq}R_{mnpq}$$
$$+ 4R_{ijik}R_{jnmn}R_{kpmp} - \tfrac{20}{9}R_{ijik}R_{npmp}R_{jnkm}$$
$$+ \tfrac{8}{9}R_{ijik}R_{jnmp}R_{knmp} - \tfrac{8}{3}R_{ijkn}R_{ijmp}R_{knmp})d\ \mathrm{vol}\ .$$

In these formulas we sum over repeated indices. This answer agrees with the formula given by Sakai for $B_6(D_0)$. In a later paper, we will apply the formula in table I to compute $B_6(D_p)$ as well as for the reduced Laplacian $(-g^{ij}\nabla_i\nabla_j)$ acting on tensors of all types.

Table I

Polynomial	Coeff.	Polynomial	Coeff.	Polynomial	Coeff.
$R_{ijij;kkmm}$	$-18/7!$	$W_{ij;k}W_{ij;k}$	$1/45$	$E_{;iijj}$	$1/60$
$R_{ijij;k}R_{nmnm;k}$	$17/7!$	$W_{ij;j}W_{ik;k}$	$1/180$	$EE_{;ii}$	$1/12$
$R_{ijik;n}R_{mjmk;n}$	$-2/7!$	$W_{ij;kk}W_{ij}$	$1/60$	$E_{;ii}E$	$1/12$
$R_{ijik;n}R_{mjmn;k}$	$-4/7!$	$W_{ij}W_{ij;kk}$	$1/60$	$E_{;i}E_{;i}$	$1/12$
$R_{ijkm;n}R_{ijkm;n}$	$9/7!$	$W_{ij}W_{jk}W_{ki}$	$-1/30$	E^3	$1/6$
$R_{ijij}R_{kmkm;nn}$	$28/7!$	$R_{ijkn}W_{ij}W_{kn}$	$-1/60$	$EW_{ij}W_{ij}$	$1/30$
$R_{ijik}R_{mjmk;nn}$	$-8/7!$	$R_{ijik}W_{jn}W_{kn}$	$1/90$	$W_{ij}EW_{ij}$	$1/60$
$R_{ijik}R_{mjmn;kn}$	$24/7!$	$R_{ijij}W_{kn}W_{kn}$	$-1/72$	$W_{ij}W_{ij}E$	$1/30$
$R_{ijkm}R_{ijkm;nn}$	$12/7!$	$R_{ijik}W_{kn;nj}$	0	$R_{ijij}E_{;kk}$	$-1/36$
$R_{ijij}R_{mnmn}R_{pqpq}$	$-35/9.7!$	$R_{ijij;k}W_{kn;n}$	0	$R_{ijik}E_{;jk}$	$-1/90$

Table I (Continued)

Polynomial	Coeff.	Polynomial	Coeff.	Polynomial	Coeff.
$R_{ijij}R_{mnmp}R_{qnqp}$	$14/3.7!$	$R_{ijkn;n}W_{ij;k}$	0	$R_{ijij;k}E_{;k}$	$-1/30$
$R_{ijij}R_{mnpq}R_{mnpq}$	$-14/3.7!$			$E_{;j}W_{ij;i}$	$-1/60$
$R_{ijik}R_{jnmn}R_{kpmp}$	$208/9.7!$			$W_{ij;i}E_{;j}$	$1/60$
$R_{ijik}R_{npmp}R_{jnkm}$	$-64/3.7!$			EER_{ijij}	$-1/12$
$R_{ijik}R_{jnmp}R_{knmp}$	$16/3.7!$			$ER_{ijij;kk}$	$-1/30$
$R_{ijkn}R_{ijmp}R_{knmp}$	$-44/9.7!$			$ER_{ijij}R_{knkn}$	$1/72$
$R_{ijkn}R_{imkp}R_{jmnp}$	$-80/9.7!$			$ER_{ijik}R_{njnk}$	$-1/180$
				$ER_{ijkn}R_{ijkn}$	$1/180$

The 46 invariants in this table are a basis for $P_6{}^i$. The coefficients next to each invariant should be multiplied by $(4\pi)^{-d/2}$ and summed to give E^6. Each invariant is to be summed over repeated indices for any orthonormal frame for the tangent bundle. The notation $R_{ijkn;}\ldots$, $W_{ij;}\ldots$, and $E_{;}\ldots$ is explained in section two.

Table I–A

Polynomial	Coeff.	$B_{/1111}$	$BB_{/11}$	$B_{/11}B$	$B_{/22}\cdot H_{33/11}$	$B_{/11}\cdot H_{33/11}$	$A_{/1233}\cdot A_{2/1}$	$A_{2/1}\cdot A_{1/233}$	$A_{2/211}\cdot H_{33/11}$
$E_{;iijj}$	1/60	1	0	0	0	−1	0	0	1/2
$EE_{;ii}$	1/12	0	1	0	−1/4	−1/4	0	0	1/8
$E_{;ii}E$	1/12	0	0	1	−1/4	−1/4	0	0	1/8
$R_{ijij}E_{;kk}$	−1/36	0	0	0	−1	−1	0	0	1/2
$R_{ijik}E_{;jk}$	−1/90	0	0	0	0	−1/2	0	0	1/4
$W_{ij;kk}W_{ij}$	1/60	0	0	0	0	0	−1/2	0	0
$W_{ij}W_{ij;kk}$	1/60	0	0	0	0	0	0	−1/2	0
$R_{ijik}W_{kn;nj}$	0	0	0	0	0	0	0	0	−1/4
E_6	1	1/60	1/12	1/12	−1/72	−1/40	−1/120	−1/120	1/80

Table I–B

Polynomial	Coeff.	$B_{/1}\cdot B_{/1}$	$B_{/1}\cdot H_{33/111}$	$B_{/1}\cdot A_{1/22}$	$A_{1/22}\cdot B_{/1}$	$A_{1/23}\cdot A_{1/23}$	$A_{1/22}\cdot A_{1/33}$	$A_{1/33}\cdot H_{33/111}$	$A_{1/22}\cdot H_{33/111}$
$E_{;iijj}$	1/60	0	−1/2	−1	1	−2	−1/2	0	0
$E_{;i}E_{;i}$	1/12	1	−1/2	0	0	0	0	0	0
$R_{ijij;k}E_{;k}$	−1/30	0	−1	0	0	0	0	0	0
$E_{;j}W_{ij;i}$	−1/60	0	0	1/2	0	0	0	0	0
$W_{ij;i}E_{;j}$	1/60	0	0	0	1/2	0	0	0	0
$W_{ij;k}W_{ij;k}$	1/45	0	0	0	0	1	0	0	0
$W_{ij;j}W_{ik;k}$	1/180	0	0	0	0	0	1/4	0	0
$R_{ijij;k}W_{kn;n}$	0	0	0	0	0	0	0	1/2	1/2
$R_{ijkn;n}W_{ij;k}$	0	0	0	0	0	0	0	−1	0
E_6	1	1/12	−1/60	−1/40	1/40	−1/90	−1/144	0	0

Table I-C

Polynomial	Coeff.	B^3	$BA_{1/2} A_{2/1}$	$A_{1/2} BA_{2/1}$	$A_{1/2} A_{2/1}B$	$A_{1/2}A_{2/3} A_{3/1}$	$A_{1/3}A_{3/1} H_{33/11}$	$A_{1/2}A_{2/1} H_{33/11}$	$A_{5/6}A_{6/5} H_{33/11}$
E^3	1/6	1	0	0	0	0	0	0	0
$EW_{ij}W_{ij}$	1/30	0	−1/2	0	0	0	1/8	1/8	1/8
$W_{ij}EW_{ij}$	1/60	0	0	−1/2	0	0	1/8	1/8	1/8
$W_{ij}W_{ij}E$	1/30	0	0	0	−1/2	0	1/8	1/8	1/8
$W_{ij}W_{jk}W_{ki}$	−1/30	0	0	0	0	−1/8	0	0	0
$W_{ij}W_{ij;kk}$	1/60*	0	0	0	0	0	3/2	0	0
$W_{ij;kk}W_{ij}$	1/60*	0	0	0	0	0	−1/2	0	0
$R_{ijkn}W_{ij}W_{kn}$	−1/60	0	0	0	0	0	1/2	0	0
$R_{ijik}W_{jn}W_{kn}$	1/90	0	0	0	0	0	1/4	1/8	0
$R_{ijij}W_{kn}W_{kn}$	−1/72	0	0	0	0	0	1/2	1/2	1/2
E_6	1	1/6	−1/60	−1/120	−1/60	1/240	7/480	7/1440	1/288

*-see preceding tables for determination of the coefficients of this polynomial in E_6.

Table I-D

Polynomial	Coeff.	$B^2H_{11/22}$	$BH_{11/2222}$	$BH_{11/22}H_{11/22}$	$BH_{11/33}H_{22/33}$	$BH_{11/22}H_{33/44}$
$EE_{;ii}$**	1/6*	0	−1/4	7/8	1/4	0
E^3	1/6*	−3/4	0	3/16	3/8	3/8
E^2R_{ijij}	−1/12	−1	0	1/2	1	1
$ER_{ijij;kk}$	−1/30	0	−1	4	1	0
$ER_{ijij}R_{knkn}$	1/72	0	0	1	2	2
$ER_{ijik}R_{njnk}$	−1/180	0	0	1/2	1/2	0
$ER_{ijkn}R_{ijkn}$	1/180	0	0	1	0	0
E_6	1	−1/24	−1/120	3/160	1/80	1/44

*-see preceding tables for determination of this coefficient.
**-we have combined the entries for $EE_{;ii}$ and $E_{;ii}E$.

Table I-E

Polynomial	Coeff.**	$H_{22/111111}$	$H_{44/133}H_{22/111}$	$H_{22/133}H_{22/111}$	$H_{11/345}H_{22/345}$	$H_{11/345}H_{11/345}$
$E_{;iijj}$	28*	−1/4	1/2	7/2	3	15/2
$E_{;i}E_{;i}$	140*	0	1/8	1/8	0	0
$R_{ijij;k}E_{;k}$	−56*	0	1/2	1/2	0	0
$R_{ijij;kknn}$	−6	−1	2	15	12	36
$R_{ijij;k}R_{nmnm;k}$	17/3	0	2	2	0	0
$R_{ijik;n}R_{mjmk;n}$	−2/3	0	0	1/2	3	3/2

Table I-E (Continued)

Polynomial	Coeff.**	$H_{22/111111}$	$H_{44/133}H_{22/111}$	$H_{22/133}H_{22/111}$	$H_{11/345}H_{22/345}$	$H_{11/345}H_{11/345}$
$R_{ijik;n}R_{mjmn;k}$	$-4/3$	0	0	0	3	3/2
$R_{ijkm;n}R_{ijkm;n}$	3	0	0	0	0	6
E_6	1680	-1	17/6	17/2	6	9

**-We have multiplied the coefficients by 1680 to reduce the fractions involved. Thus the actual coefficient of $E_{;i}E_{;i}$, for example, is 140/1680.
*-see preceding tables for determination of this coefficient.

Table I-F

Polynomial	Coeff.	$H_{11/3333}H_{22/44}$	$H_{11/2222}H_{11/33}$	$H_{11/3333}H_{22/33}$	$H_{11/2222}H_{11/22}$
$E_{;iijj}$	28*	0	3/4	3/4	9/2
$R_{ijij;kknn}$	-6*	0	3	3	20
$EE_{;ii}$	280**	1/16	1/16	1/16	1/16
$R_{ijij}E_{;kk}$	$-140/3$*	1/4	1/4	1/4	1/4
$R_{ijij;kk}E$	-56*	1/4	1/4	1/4	1/4
$R_{ijik}E_{;jk}$	$-56/3$*	0	0	1/8	1/8
$R_{ijij}R_{kmkm;nn}$	28/3	1	1	1	1
$R_{ijik}R_{mjmk;nn}$	$-8/3$	0	1/4	1/4	1/2
$R_{ijik}R_{mjmn;kn}$	8	0	0	1/4	1/4
$R_{ijkm}R_{ijkm;nn}$	4	0	0	0	1
E_6	1680	7/6	7/2	19/6	19/2

*-see preceding tables for computation of this coefficient. All coefficients have been multiplied by 1680 to reduce the number of fractions involved.
**-we have combined the terms in $EE_{;ii}$ and $E_{;ii}E$.

Table I-G

Polynomial	Coeff.	$H_{33/66}H_{11/44}H_{22/55}$	$H_{33/66}H_{11/45}H_{22/45}$	$H_{33/66}H_{11/45}H_{11/45}$
$EE_{;ii}$	280**	0	$-1/8$	$-5/16$
$R_{ijij}E_{;kk}$	$-140/3$*	0	$-1/2$	$-5/4$
E^3	280*	$-3/32$	0	0
E^2R_{ijij}	-140*	$-3/8$	0	0
$ER_{ijij;kk}$	-56*	0	$-1/2$	$-3/2$
$ER_{ijij}R_{kmkm}$	70/3*	$-3/2$	0	0
$ER_{ijik}R_{mjmk}$	$-28/3$*	0	$-1/4$	$-1/8$
$ER_{ijkm}R_{ijkm}$	28/3*	0	0	$-1/2$
$R_{ijij}R_{kmkm;nn}$	28/3*	0	-2	-6

Table I-G (Continued)

Polynomial	Coeff.	$H_{33/66}H_{11/44}H_{22/55}$	$H_{33/66}H_{11/45}H_{22/45}$	$H_{33/66}H_{11/45}H_{11/45}$
$R_{ijij}R_{mnmn}R_{opop}$	$-35/27$	-6	0	0
$R_{ijij}R_{mnmo}R_{pnpo}$	$14/9$	0	-1	$-1/2$
$R_{ijij}R_{mnop}R_{mnop}$	$-14/9$	0	0	-2
E_6	1680	$-35/36$	$-14/9$	$-7/3$

*-see preceding tables for computation of this coefficient. All coefficients have been multiplied by 1680 to reduce the number of fractions involved.
**-we have combined the terms in $EE_{;ii}$ and $E_{;ii}E$.

Table I-H

Polynomial	Coeff.	$H_{11/44}H_{11/55} \cdot H_{11/66}$	$H_{11/44}H_{11/55} \cdot H_{22/44}$	$H_{11/44}H_{11/44} \cdot H_{22/44}$	$H_{11/44}H_{11/44} \cdot H_{11/44}$	$H_{11/22}H_{22/33} \cdot H_{33/11}$
$E_{;iijj}$	28*	-3	$-3/4$	$-21/8$	$-79/8$	$-3/4$
$R_{abab;ccdd}$	-6*	-12	-3	-11	-46	-3
$EE_{;ii}$	280**	$-3/8$	$-3/16$	$-9/32$	$-7/32$	$3/16$
$R_{ijij}E_{;kk}$	$-140/3$*	$-3/2$	$-3/4$	$-9/8$	$-7/8$	$3/4$
$R_{ijik}E_{;jk}$	$-56/3$*	0	$-1/8$	$-9/16$	$-7/16$	$3/8$
$R_{ijij;kk}E$	-56*	$-3/2$	$-3/4$	$-5/4$	-1	$3/4$
$R_{abab}R_{ijij;kk}$	$28/3$*	-6	-3	-5	-4	3
$R_{abac}R_{ibic;jj}$	$-8/3$*	$-6/4$	$-1/2$	$-5/4$	-2	$3/4$
$R_{abac}R_{ibij;cj}$	8*	$6/8$	$-4/8$	$-9/8$	-1	0
$R_{abcd}R_{abcd;ee}$	4*	0	0	0	-4	0
E^3	280*	$-3/32$	$-3/32$	$-3/64$	$-1/64$	$-6/64$
E^2R_{ijij}	-140*	$-3/8$	$-3/8$	$-3/16$	$-1/16$	$-6/16$
$ER_{ijij}R_{abab}$	$70/3$*	$-3/2$	$-3/2$	$-3/4$	$-1/4$	$-3/2$
$ER_{abac}R_{dbdc}$	$-28/3$*	$-3/8$	$-1/4$	$-1/4$	$-1/8$	$-6/16$
$ER_{abcd}R_{abcd}$	$28/3$*	0	0	$-1/4$	$-1/4$	0
$R_{ijij}R_{abab}R_{cdcd}$	$-35/27$*	-6	-6	-3	-1	-6
$R_{ijij}R_{abac}R_{dbdc}$	$14/9$*	$-3/2$	-1	-1	$-1/2$	$-6/4$
$R_{ijij}R_{abcd}R_{abcd}$	$-14/9$*	0	0	-1	-1	0
$R_{ijik}R_{jnmn}R_{kpmp}$	$208/27$	$-6/8$	0	$-3/8$	$-2/8$	0
$R_{ijik}R_{npmp}R_{jnkm}$	$-64/9$	0	$-1/4$	$-1/4$	$-1/4$	$-6/8$
$R_{ijik}R_{jnmp}R_{knmp}$	$16/9$	0	0	$-1/4$	$-1/2$	0
$R_{ijkn}R_{ijmp}R_{knmp}$	$-44/27$	0	0	0	-1	0
$R_{ijkn}R_{imkp}R_{jmnp}$	$-80/27$	0	0	0	0	$-6/8$
E_6	1680	$-175/12$	$-21/4$	$-61/8$	$-305/24$	$-3/4$

*-see preceding tables for computation of this coefficient. All coefficients have been multiplied by 1680 to reduce the number of fractions involved.
**-we have combined the terms in $EE_{;ii}$ and $E_{;ii}E$.

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UNIVERSITY OF CALIFORNIA, BERKELEY