The spectral radius of graphs on surfaces

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Abstract

This paper provides new upper bounds on the spectral radius ρ (largest eigenvalue of the adjacency matrix) of graphs embeddable on a given compact surface. Our method is to bound the maximum rowsum in a polynomial of the adjacency matrix, using simple consequences of Euler's formula. Let γ denote the Euler genus (the number of crosscaps plus twice the number of handles) of a fixed surface Σ. Then (i) for $n \geq 3$, every n-vertex graph embeddable on Σ has $\rho \leq 2 + \sqrt{2n + 8\gamma - 6}$, and (ii) a 4-connected graph with a spherical or 4-representative embedding on Σ has $\rho \leq 1 + \sqrt{2n + 2\gamma - 3}$. Result (i) is not sharp, as Guiduli and Hayes have recently proved that the maximum value of ρ is $3/2 + \sqrt{2n} + o(1)$ as $n \to \infty$ for graphs embeddable on a fixed surface. However, (i) is the only known bound that is computable, valid for all $n \geq 3$, and asymptotic to $\sqrt{2n}$ like the actual maximum value of ρ. Result (ii) is sharp for the sphere or plane ($\gamma = 0$), with equality holding if and only if the graph is a 'double wheel' $2K_1 + C_{n-2}$ (+ denotes join). For other surfaces we show that (ii) is within $O(1/n^{1/2})$ of sharpness. We also show that a recent bound on ρ by Hong can be deduced by our method.

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1. Introduction

Let G be an n-vertex graph with adjacency matrix A. In this paper *graph* means a simple graph, with no loops or multiple edges. The *spectral radius* ρ of G is the largest eigenvalue of A.

Schwenk and Wilson [**11**] suggested the study of the eigenvalues of planar graphs. At about the same time, the spectral radius of planar graphs became of interest to geographers as a measure of overall connectivity of a planar network (see [**1**] or [**3**, Subsection 6.2] for references). The first significant result was by Hong [**7**], who used a result about the spectral radius of graphs in general to show that for n-vertex planar graphs $\rho \leq \sqrt{5n-11}$. This bound was improved to $4+\sqrt{3n-9}$ by Cao and Vince [**2**], who also conjectured that the planar graph of a given order with largest spectral radius was $K_2 + P_{n-2}$ (+ denotes join). This conjecture had been proposed independently by Boots and Royle [**1**] based on computer studies of planar graphs on up to 11 vertices; they noted that the conjecture is not true for $n = 7$ and 8, but suggested it was true for all $n \geq 9$.

In further work, Hong [**8**] improved the bound for planar graphs to $\rho \leq 2\sqrt{2} + \sqrt{3n - \frac{15}{2}}$. In that paper Hong also gave the first bound on the spectral radius of graphs on an arbitrary surface. In terms of the Euler genus γ (the number of crosscaps plus twice the number of handles), Hong's bound had the form $\rho \leq \sqrt{6(n+\gamma-2)f(\gamma)}$ where $f(\gamma)$ is $24\gamma + O(\sqrt{\gamma})$. In his Ph.D. thesis Guiduli [**5**] showed that $\rho \leq 1 + \sqrt{6\gamma} + \sqrt{3n}$, and Hong [**9**] showed that $\rho \leq 1 + \sqrt{3n + 6\gamma - 8}$. Recently Guiduli and Hayes [**6**] showed that on any fixed surface the maximum value of ρ is $\frac{3}{2} + \sqrt{2n} + o(1)$ as $n \to \infty$. Moreover, they showed that on the plane the Boots-Royle-Cao-Vince conjecture is true for sufficiently large n.

Other work on the spectral radius of planar graphs has included work on outerplanar graphs by Rowlinson [**10**], Cao and Vince [**2**], and Guiduli and Hayes [**6**]. Guiduli and Hayes also disproved a second conjecture of Cao and Vince on planar graphs with minimum degree at least 4.

For a general discussion of eigenspaces of graphs we refer the reader to the book by Cvetković, Rowlinson and Simić [**4**], and for a discussion of the spectral radius in particular see the survey paper by Cvetković and Rowlinson [**3**].

In this paper we provide a new upper bound on the spectral radius of an n-vertex graph embedded in a surface of Euler genus γ. Our method is to bound the maximum rowsum in a polynomial of the adjacency matrix, using simple consequences of Euler's formula. For planar graphs, our bound is off by $\frac{1}{2}+o(1)$ from the Boots-Royle-Cao-Vince conjecture, but it is valid for all n, unlike Guiduli and Hayes' verification of the conjecture, which is valid only for large n. For graphs on arbitrary surfaces our bound is also off by $\frac{1}{2}+o(1)$ from the asymptotic expression obtained by Guiduli and Hayes, but our result yields computable numerical bounds, whereas their asymptotic expression does not. Moreover, the way in which the bound varies with the Euler genus is clear for our bound, but not for theirs.

We also obtain a bound on the spectral radius of a 4-connected embedded graph where the embedding is either spherical or 4-representative. In this case, our bound is sharp for the plane, and we characterize exactly when equality occurs. For arbitrary surfaces our bound is proved to be within $O(1/n^{1/2})$ of sharpness.

Finally, we show that the most recent bound of Hong [**9**] can be obtained by our methods, and we compare Hong's bound to our new one.

2. Lemmas

In this section we provide the necessary definitions and the lemmas on which our main results rely. We begin with some simple matrix results. Let B be an $m \times n$ matrix. Then $s_i(B)$ will denote the *i-th rowsum of B*, i.e. $s_i(B) = \sum_{j=1}^{n} B_{ij}$, where $1 \leq i \leq m$.

Lemma 2.1. *Let B be a real symmetric $n \times n$ matrix, and let λ be an eigenvalue of B with an eigenvector x all of whose entries are nonnegative. Then*

$$\min_{1\leq i\leq n} s_i(B) \ \leq\ \lambda\ \leq\ \max_{1\leq i\leq n} s_i(B).$$

Moreover, if the rowsums of B are not all equal and if all entries of x are positive, then both inequalities above are strict.

Proof. Since B is symmetric, we may consider x to be a row vector such that $xB = \lambda x$. Without loss of generality we may assume that $\sum_{j=1}^{n} x_j = 1$. Then

$$\begin{aligned}\lambda = \lambda(\sum_{j=1}^{n} x_j) &= \sum_{j=1}^{n}(\lambda x)_j = \sum_{j=1}^{n}(xB)_j \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} x_i B_{ij} = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} B_{ij} = \sum_{i=1}^{n} x_i s_i(B).\end{aligned}$$

In other words, since the entries of x are nonnegative and sum to 1, λ is a convex combination of the rowsums of B, and the result follows. ∎

Lemma 2.2. *Let G be a connected n-vertex graph and A its adjacency matrix, with spectral radius ρ. Let p be any polynomial. Then*

$$\min_{v \in V(G)} s_v(p(A)) \;\leq\; p(\rho) \;\leq\; \max_{v \in V(G)} s_v(p(A)).$$

Moreover, if the rowsums of $p(A)$ are not all equal then both inequalities are strict.

Proof. By the well known Perron-Frobenius Theorem (see for example [**11**, Theorem 4.1]), A has an eigenvector x with all entries positive for ρ. Now $p(A)$ has x as an eigenvector for the eigenvalue $p(\rho)$, and we may apply Lemma 2.1. ∎

Now we begin to look at graphs embedded in surfaces; we assume that the reader is familiar with this notion. A *surface* here is a compact connected 2-manifold. The *Euler genus* $\gamma = \gamma(\Sigma)$ of a surface Σ is defined to be $2 - \chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic, so that a sphere with h handles has Euler genus $2h$ and a sphere with k crosscaps has Euler genus k. The *faces* of an embedding are the connected components that remain when the embedded graph is deleted from the surface. An embedding is *k-representative* if no noncontractible closed curve in the surface intersects the embedded graph at fewer than k points. An embedding is *cellular* if every face is homeomorphic to an open disk; an embedding is cellular if and only if the graph is connected and either the surface is the sphere or the embedding is 1-representative. In a cellular embedding each face is bounded by a closed walk, whose length is the *degree* of the face. An embedding of a graph in a surface induces a circular order on the neighbours of any given vertex v, which we call the *embedded order around v*.

Lemma 2.3. *Let G be a pseudograph (a graph where loops and multiple edges are permitted) cellularly embedded in a surface of Euler genus γ in such a way that each face has degree at least g, $g \geq 3$. Then*

$$|E(G)| \leq \left(\frac{g}{g-2}\right)(|V(G)| + \gamma - 2).$$

Moreover, equality holds precisely when every face has degree g.

Proof. This is a standard result derived from Euler's formula. ∎

Given a graph G, a vertex v, and $i \ge 0$, let $N_i(v, G)$ denote the set of vertices at distance i from v, and let $n_i(v, G) = |N_i(v, G)|$. Note that $n_1(v, G) = \deg_G(v)$ is just the degree of v in G. Let $M_{ij}(v, G)$ denote the set of edges with one end in $N_i(v, G)$ and the other in $N_j(v, G)$, and let $m_{ij}(v, G) = |M_{ij}(v, G)|$.

The following lemma contains most of the structural analysis used in proving our main results.

Lemma 2.4. *Let G be a graph on at least two vertices, with adjacency matrix A and with a cellular embedding Ψ in a surface of Euler genus γ. Let v be any vertex of G, and let $c(v, \Psi)$ be the number of edges that join two vertices of $N_1(v, G)$ that are not consecutive in the embedded order around v. Then if $n_1(v, G) \ge 3$ we have*

(i) $s_v(A^2) \le 4n_1(v, G) + 2n_2(v, G) + 2c(v, \Psi) + 2\gamma - 2$,

(ii) $c(v, \Psi) \le n_1(v, G) + 3\gamma - 3$, *and*

(iii) $s_v(A^2) \le 6n_1(v, G) + 2n_2(v, G) + 8\gamma - 8$.

Proof. We abbreviate $N_i(v, G)$ and $n_i(v, G)$ to N_i and n_i respectively, and $c(v, \Psi)$ to c. Designate a local clockwise direction at v. Let the neighbours of v be $u_0, u_1, \ldots, u_{n_1-1}$ in clockwise order around v, with subscripts interpreted modulo n_1. For each i, $0 \le i \le n_1 - 1$, let F_i denote the face extending clockwise from the edge vu_i to the edge vu_{i+1}.

Since G is simple, every face of G has degree at least 3. From G form a new embedded graph H as follows. For each i, $0 \le i \le n_1 - 1$, we (a) do nothing if u_iu_{i+1} is a boundary edge of F_i, (b) move the edge u_iu_{i+1} to cross F_i if it occurs in G but is not a boundary edge of F_i, or (c) add the edge u_iu_{i+1} crossing F_i if it does not already occur in G. Since $n_1 \ge 3$, no multiple edges are created. Due to changes of type (b), the embedding of H may no longer be cellular, but if so we may make it cellular again by replacing each non-disk face with one or more disks. Thus, we get a cellular embedding of H on a surface of Euler genus $\gamma' \le \gamma$. Then $N_i(v, H) = N_i = N_i(v, G)$ for every i, and the edges of $M_{11}(v, H)$, which is a superset of $M_{11}(v, G)$, may be divided into two classes: there are n_1 edges that are boundary edges of faces of the form $vu_iu_{i+1}v$, and there are c other edges.

To prove (i), form an embedded graph B_0 from H by deleting all vertices in $N_3 \cup N_4 \cup \ldots$ and all edges in $M_{22}(v, H) = M_{22}(v, G)$. Then form B from B_0 by subdividing every edge in

$M_{11}(v, H)$. B is connected and bipartite, with bipartition $(\{v\} \cup S \cup N_2, N_1)$, S being the set of new vertices created by subdivision. If the embedding of B is not cellular, we again make it cellular by appropriate face replacements. Thus, we have B cellularly embedded on a surface of Euler genus $\gamma'' \leq \gamma' \leq \gamma$.

Note that $s_v(A^2)$ is exactly the number of walks of length 2 in G that begin at v. By counting these walks according to their middle vertex we see that (1) $s_v(A^2) = \sum_{u \in N_1} \deg_G(u)$. Now, (2) the degrees of the vertices of N_1 are the same in H and B, (3) every edge of B is incident with exactly one vertex of N_1, and (4) since B is bipartite we can apply Lemma 2.3 to B with $g = 4$. Thus,

$$\begin{aligned}
s_v(A^2) &= \sum_{u \in N_1} \deg_G(u) \qquad \text{by (1)} \\
&\leq \sum_{u \in N_1} \deg_H(u) \\
&= \sum_{u \in N_1} \deg_B(u) \qquad \text{by (2)} \\
&= |E(B)| \qquad \text{by (3)} \\
&\leq 2(|V(B)| + \gamma'' - 2) \qquad \text{by (4)} \\
&\leq 2(|V(B)| + \gamma - 2) = 2(1 + n_1 + n_2 + |S| + \gamma - 2) \\
&= 2(1 + n_1 + n_2 + (n_1 + c) + \gamma - 2) = 4n_1 + 2n_2 + 2c + 2\gamma - 2,
\end{aligned}$$

proving (i).

To prove (ii), let J be the subgraph of H induced by $\{v\} \cup N_1$. If J is not cellularly embedded we modify the embedding as usual, obtaining a cellular embedding of J on a surface of Euler genus $\gamma''' \leq \gamma$. Now we may apply Lemma 2.3 to J with $g = 3$, giving

$$\begin{aligned}
2n_1 + c = |E(J)| &\leq 3(|V(J)| + \gamma''' - 2) \\
&\leq 3(|V(J)| + \gamma - 2) \\
&= 3((1 + n_1) + \gamma - 2) = 3n_1 + 3\gamma - 3
\end{aligned}$$

from which $c \leq n_1 + 3\gamma - 3$, proving (ii).

Finally, (iii) follows by substituting (ii) into (i). ∎

In the above proof, we need to be careful in how we define H. The essential property of H that we need to make our counting arguments work is that the faces around v are triangles whose edges opposite v form a cycle. One might think that we could obtain such an H in a way independent of v by adding edges to make G into a triangulation. This works on the plane, but for arbitrary surfaces this can result in the creation of multiple edges, and then the faces around some vertices do not have the correct structure.

All parts of Lemma 2.4 are sharp. To see this, take a simple (no loops or multiple edges) triangulation J of Σ of Euler genus γ having a vertex v adjacent to all other vertices (such triangulations exist, for example, when there is a complete graph with a triangular embedding in Σ). Into every face not incident with v insert a vertex of degree 3 to obtain G. Then in the proof of Lemma 2.4 we have $H = G$ since G is a triangulation. The B in part (i) of the proof has all faces of degree 4, and $n = 1 + n_1 + n_2$, so that (i) is sharp. The J in part (ii) of the proof is the J we started with, which has all faces of degree 3, so that (ii) is sharp. It follows that (iii) is also sharp.

3. Main results

Now we state our main results. The general bounds are given in Theorem 3.1, with specialization to the plane in Corollary 3.2. After stating and proving the bounds we discuss their sharpness.

Theorem 3.1. *Let G be an n-vertex graph, $n \geq 3$, with spectral radius ρ. Suppose G can be embedded on a surface of Euler genus γ (or Euler characteristic $\chi = 2 - \gamma$).*

(i) *Then $\rho \leq 2 + \sqrt{2n + 8\gamma - 6} = 2 + \sqrt{2n + 10 - 8\chi}$.*

(ii) *If G is 4-connected and either the surface is the sphere or the embedding is 4-representative then $\rho \leq 1 + \sqrt{2n + 2\gamma - 3} = 1 + \sqrt{2n + 1 - 2\chi}$.*

Proof. We may make the following assumptions. First, the embedding is cellular, as otherwise we can embed G on a surface of smaller Euler genus. Second, $n \geq 4$, as the theorem is clearly true for $n = 3$. Third, every vertex has degree at least 3, as otherwise we may add edges to obtain a graph with larger spectral radius embedded on the same surface, as follows. If v is a vertex of degree 1, with neighbour u, then since $n \geq 4$ there are at least two other vertices besides u on the boundary of the unique face with which v is incident, and we may join v to both of those without creating any multiple edges. If v has degree 2, then since $n \geq 4$ at least one of the faces with which v is incident is not a triangle (if both were triangles G would have a multiple edge), so there is a vertex to which v may be joined without creating a multiple edge.

Let A be the adjacency matrix of G. Note that $s_v(A) = \deg_G(v) = n_1(v, G)$ for every vertex v. Fix a vertex v. Since $\deg_G(v) = n_1(v, G) \geq 3$, we may apply Lemma 2.4(iii). Using the notation from the proof of that lemma, we have

$$\begin{aligned} s_v(A^2 - 4A) &= s_v(A^2) - 4s_v(A) = s_v(A^2) - 4n_1 \\ &\leq 2n_1 + 2n_2 + 8\gamma - 8 \end{aligned}$$

Since $n \geq 1 + n_1 + n_2$, we have $s_v(A^2 - 4A) \leq 2n + 8\gamma - 10$. As this holds for every vertex v, Lemma 2.2 implies that $\rho^2 - 4\rho \leq 2n + 8\gamma - 10$. Solving the quadratic gives the upper bound of (i).

Now if G is 4-connected, and if the embedding Ψ of G is 4-representative when it is not on the sphere, then for each vertex v we must have $n_1(v, G) = \deg_G(v) \geq 4$, and also $c(v, \Psi) = 0$. For, any edge joining two non-consecutive neighbours u_i, u_j of v produces a nonfacial triangle vu_iu_jv which is either separating (violating 4-connectivity) or nonseparating and hence noncontractible (violating 4-representativity).

Again fix a vertex v. Using Lemma 2.4(i), and using the notation from the proof of that lemma, we have

$$\begin{aligned} s_v(A^2 - 2A) &= s_v(A^2) - 2s_v(A) = s_v(A^2) - 2n_1 \\ &\leq (4n_1 + 2n_2 + 2c + 2\gamma - 2) - 2n_1 = 2n_1 + 2n_2 + 2c + 2\gamma - 2 \end{aligned}$$

But since $c = 0$ and $n \geq 1 + n_1 + n_2$, we have $s_v(A^2 - 2A) \leq 2n + 2\gamma - 4$. As this holds for every vertex v, Lemma 2.2 implies that $\rho^2 - 2\rho \leq 2n + 2\gamma - 4$. Solving the quadratic gives the upper bound of (ii). ∎

Corollary 3.2. *Let G be an n-vertex planar graph, $n \geq 3$, with spectral radius ρ.*

(i) *Then $\rho \leq 2 + \sqrt{2n - 6}$.*

(ii) *If G is 4-connected then $\rho \leq 1 + \sqrt{2n - 3}$.*

Unfortunately, Theorem 3.1(i), or even Corollary 3.2(i), is not sharp. We know from the work of Guiduli and Hayes [**6**] that for graphs on any fixed surface, the maximum value of ρ is

$\frac{3}{2} + \sqrt{2n} + o(1)$ as $n \to \infty$. For the sphere or plane they show that for large n the maximum value of ρ is attained uniquely by the graph $G_n = K_2 + P_{n-2}$, for which

$$\tfrac{3}{2} + \sqrt{2n - \tfrac{15}{4}} - \frac{8}{8n - 15 - \sqrt{8n-15}} \; < \; \rho \; < \; \tfrac{3}{2} + \sqrt{2n - \tfrac{15}{4}}.$$

On the other hand, Guiduli and Hayes' result for planar graphs applies only for large n, whereas Corollary 3.2(i) applies for all n. Moreover, in the case of surfaces other than the sphere Guiduli and Hayes give an asymptotic expression, but no actual numbers can be derived from this, whereas our Theorem 3.1(i) gives a number for each n. For our result, it is clear how the bound varies with γ, but the asymptotic expression does not explicitly mention γ at all. Therefore, Theorem 3.1(i) and Corollary 3.2(i) provide information that Guiduli and Hayes' results do not.

With Corollary 3.2(ii) we are more fortunate. It is sharp, as is shown by the 'double wheel' $D_n = 2K_1 + C_{n-2}$. The spectral radius of D_n is easily calculated to be $1 + \sqrt{2n-3}$, using the fact that the vertices fall into only two orbits under the action of the automorphism group, so that the entries of the eigenvector for ρ have only two distinct values. In Theorem 3.3 below we show that D_n is in fact the unique extremal graph for every $n \geq 6$.

It is interesting to note that Cao and Vince [**2**] conjectured the double wheel to be the graph with maximum spectral radius over planar graphs with minimum degree at least 4 of a given order. This conjecture is in fact false, as Guiduli and Hayes [**6**] have shown that minor modifications to $G_n = K_2 + P_{n-2}$ give a graph G'_n with minimum degree 4 and spectral radius larger than D_n when $n \geq 113$ (and possibly for smaller values of n too). Moreover, they can show [personal communication] that G'_n is in fact the extremal graph for large values of n.

We do not know whether Theorem 3.1(ii) is sharp for surfaces other than the sphere. However, on every given surface Σ other than the sphere we can construct 4-connected graphs with 4-representative embeddings having a spectral radius of $1 + \sqrt{2n} - O(1/n^{1/2})$, so that Theorem 3.1(ii) differs from the best possible result by at most $O(1/n^{1/2})$. To construct these examples take a fixed 4-connected 4-representative embedded graph G_0 on Σ, having a face F of degree 4. (Such a G_0 can easily be constructed by gluing together several copies of either $C_4 \times C_4$ embedded on the torus or a similar graph embedded on the projective plane.) If $|V(G_0)| = n_0$, then for every $n \geq n_0 + 1$ we may add $n - n_0$ vertices inside F in such a way that the $n - n_0$ new vertices and the four boundary vertices of F induce a planar graph $G_1 = 2K_1 + P_{n-n_0+2}$. Now, $2K_1 + P_{k-2}$ can be shown by standard methods (thinking of it as obtained by deleting one edge from D_k) to have spectral radius $1 + \sqrt{2k-3} - O(1/(k-2))$. Therefore, G_1, and hence the whole new graph G, have spectral radius at least $1 + \sqrt{2n - 2n_0 + 5} - O(1/(n - n_0 + 2)) = 1 + \sqrt{2n} - O(1/n^{1/2})$. It is not difficult to see that G is 4-connected and its embedding is 4-representative.

Now we prove our result on the sharpness of Corollary 3.2(ii).

Theorem 3.3. *Equality holds in Corollary 3.2(ii) if and only if G is a double wheel $D_n = 2K_1 + C_{n-2}$ for some $n \geq 6$.*

Proof. As described above, Corollary 3.2(ii) is sharp for D_n.

Suppose now that G is a 4-connected plane graph for which Corollary 3.2(ii) is sharp. G must be a triangulation, as otherwise we can add edges and obtain a new plane graph with larger spectral radius, violating the bound. As is well known, a plane triangulation is 4-connected if and only if it has no separating triangles.

Now equality must hold in each of the inequalities which we used to derive Corollary 3.2(ii). In particular, by Lemma 2.2 every vertex v must have $s_v(A^2 - 2A) = 2n - 4$. Fix v, and use the notation of the proof of Lemma 2.4. From the proof of Theorem 3.1, for equality we must have $n = 1 + n_1 + n_2$, or in other words every vertex is at distance at most 2 from v. Next, consider the proof of Lemma 2.4(i). Since G is a triangulation, $H = G$. Since every vertex is at distance at most 2 from v, B is a subdivision of a spanning subgraph of G. By Lemma 2.3, for equality every face of B must have degree 4. We claim that therefore G has no induced $K_1 \cup K_3$ with v as the K_1. In other words, there is no triangle T all of whose vertices are nonadjacent to v. If there were, T would bound a face, since G has no separating triangles. Such a face would be contained in a face of B on whose boundary the vertices of T, elements of N_2, were pairwise separated by at least one vertex of N_1, giving rise to a face of degree 6 or more in B, a contradiction.

Since v was arbitrary, G is a 4-connected plane triangulation of diameter at most 2, and with no induced $K_1 \cup K_3$. We show that this implies that $G \cong D_n$.

Again fix an arbitrary vertex v of G. Let Z_i denote the subgraph of G induced by $N_i = N_i(v, G)$. Since G is a triangulation without separating triangles, Z_1 is a chordless cycle $u_0 u_1 \ldots$

$u_{n_1-1}u_0$. Since G is 4-connected, $\deg_G(u) \geq 4$ for every vertex u, and in particular $|V(Z_1)| = n_1 = \deg_G(v) \geq 4$. Think of G as embedded in the plane with v outside Z_1. Then Z_2 is induced by all vertices inside Z_1, since G has diameter 2. Since $|V(Z_1)| \geq 4$, Z_1 is not a face boundary, and so Z_2 is nonempty. If Z_2 is a single vertex then $G \cong D_n$, so we may suppose that $n_2 = |V(Z_2)| \geq 2$.

Since G is a triangulation, any cycle in Z_2 contains the vertices of a triangle, leading to an induced $K_1 \cup K_3$, and if Z_2 has two or more components then Z_1 must have a chord. Thus, Z_2 is a tree. Let w and x be two leaves of Z_2. Since $\deg_G(w) \geq 4$, the neighbours of w in Z_1 induce a subpath of Z_1 with 3 or more vertices, which we may without loss of generality suppose to be the subpath $u_0u_1 \ldots u_p$, where $p = \deg_G(w) - 2 \geq 2$. By planarity, x is not adjacent to v or any of $u_1, u_2, \ldots, u_{p-1}$. Thus, if $p \geq 3$ then x and the triangle vu_1u_2v are an induced $K_1 \cup K_3$, so $p = 2$ and $\deg_G(w) = 4$. Similarly, $\deg_G(x) = 4$ and the neighbours of x on Z_1 form a subpath $u_qu_{q+1}u_{q+2}$. Neither vu_0u_1v nor vu_1u_2v can induce a $K_1 \cup K_3$ with x, so x must be adjacent to both u_0 and u_2. Thus, $u_qu_{q+1}u_{q+2} = u_2u_3u_0$, so that Z_1 is a 4-cycle $u_0u_1u_2u_3u_0$.

Now u_0 and u_2 are the only vertices of Z_1 to which a vertex y of Z_2 other than w or x can be adjacent. Therefore $\deg_{Z_2}(y) \geq 2$ for every such y, and Z_2 has no leaves other than w and x. Thus, Z_2 is a path, and it follows that $G \cong D_n$, with u_0 and u_2 forming the $2K_1$ and the other vertices forming the C_{n-2}. ∎

4. Hong's bound

We conclude by discussing the bound $\rho \leq 1 + \sqrt{3n + 6\gamma - 8}$ obtained by Hong [**9**]. For a given γ, this bound is better than Theorem 3.1(i) when n is small. Here we show that Hong's bound can be derived using our methods, and we determine the values of n for which each bound, Theorem 3.1(i) or Hong's, is to be preferred.

Theorem 4.1 (Hong [**9**])**.** *Let G be an n-vertex graph, $n \geq 3$, with spectral radius ρ. Suppose G can be embedded on a surface of Euler genus γ (or Euler characteristic $\chi = 2 - \gamma$). Then $\rho \leq 1 + \sqrt{3n + 6\gamma - 8} = 1 + \sqrt{3n + 4 - 6\chi}$.*

Proof. We use the notation of Lemma 2.4, as well as abbreviating $m_{ij}(v, G)$ to m_{ij}. It is clear that for every vertex v,

$$s_v(A^2) \leq \sum_{u \in N_1} \deg_H(u) = 3n_1 + m_{12} + 2c.$$

By examining the graph B_0 of the proof of Lemma 2.4 and using Lemma 2.3 with $g = 3$, we may derive the inequality

$$c \leq n_1 + 3n_2 - m_{12} + 3\gamma - 3.$$

(This inequality may be shown to be sharp using the same graphs as for Lemma 2.4.) Now, we use this inequality together with Lemma 2.4(ii) to obtain

$$\begin{aligned} s_v(A^2) &\leq 3n_1 + (m_{12} + c) + c \\ &\leq 3n_1 + (n_1 + 3n_2 + 3\gamma - 3) + (n_1 + 3\gamma - 3) \\ &= 5n_1 + 3n_2 + 6\gamma - 6. \end{aligned}$$

Hence,

$$s_v(A^2 - 2A) = s_v(A^2) - 2n_1 \leq 3n_1 + 3n_2 + 6\gamma - 6$$

and since $n \geq 1 + n_1 + n_2$ we get $s_v(A^2 - 2A) \leq 3n + 6\gamma - 9$ for every vertex v. Thus, by Lemma 2.2 we have $\rho^2 - 2\rho \leq 3n + 6\gamma - 9$, giving $\rho \leq 1 + \sqrt{3n + 6\gamma - 8}$. ∎

In fact, by taking convex combinations of the bound on $s_v(A^2 - 2A)$ above and the bound on $s_v(A^2 - 4A)$ in the proof of Theorem 3.1, we can obtain a whole family of results, namely $\rho \leq (1 + \alpha) + \sqrt{(3 - \alpha)n + (6 + 2\alpha)\gamma + (\alpha^2 + \alpha - 8)}$ for every $\alpha \in [0, 1]$. However, for a given γ and n, this is always monotone as a function of α, so one of the extreme values $\alpha = 0$ (Hong's bound) or $\alpha = 1$ (Theorem 3.1(i)) provides the smallest bound. A little algebra reveals that the best overall bound we can get here is the following.

Corollary 4.2. *Let G be an n-vertex graph, $n \geq 3$, with spectral radius ρ. Suppose G can be embedded on a surface of Euler genus γ. Then*

$$\rho \leq \begin{cases} 1 + \sqrt{3n + 6\gamma - 8} & \text{if } 3 \leq n \leq 7 + 2\gamma + 4\sqrt{1 + 3\gamma}, \\ 2 + \sqrt{2n + 8\gamma - 6} & \text{if } n \geq 7 + 2\gamma + 4\sqrt{1 + 3\gamma}, \text{ or } n = 3 \text{ and } \gamma = 0. \end{cases}$$

Thus, even this extension of our method does not result in a bound of the form $\frac{3}{2} + \sqrt{2n} + o(1)$. Given that Lemma 2.4 is sharp, as is the upper bound on c we derived in the proof of Theorem

4.1, we cannot hope to get better bounds from the technique of examining maximum rowsums in quadratic polynomials in A. A different method will be needed to verify the Boots-Royle-Cao-Vince conjecture for all n, or to provide computable bounds that agree with Guiduli and Hayes' asymptotic result for arbitrary surfaces.

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