The Spectrum of Heavy Tailed Random Matrices

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Abstract: Take X_N to be a symmetric matrix with real independent (modulo the symmetry constraint) equidistributed entries with law P and denote $(\lambda_1, \cdots, \lambda_N)$ its eigenvalues. Then, Wigner [14] has shown that, if $\int x^2 dP(x)$ is finite, $N^{-1} \sum_{i=1}^N \delta_{\lambda_i/\sqrt{N}}$ converges in expectation towards the semi-circle distribution. In this paper, we consider the case where P has a heavy tail and belong to the domain of attraction of an α-stable law for $\alpha \in]0, 2[$. We show the convergence of $N^{-1} \sum_{i=1}^N \delta_{\lambda_i/N^{\frac{1}{\alpha}}}$ towards a law μ_α. We characterize and study μ_α, showing in particular that it is a symmetric measure with heavy tail.

1 Introduction

We study the asymptotic behavior of the spectral measure of large random real symmetric matrices with independent identically distributed heavy tailed entries. Let $(x_{ij}, 1 \leq i \leq j < \infty)$ be an infinite array of i.i.d real variables in a probability space $(\Omega, \mathbb{P})$, with common marginal distribution P. Denote by X_N the $N \times N$ symmetric matrix given by:

$$X_N(i, j) = x_{ij} \text{ if } i \leq j, \; x_{ji} \text{ otherwise.}$$

If the entries have a finite second moment $\sigma^2 = E[x_{ij}^2] = \int x^2 dP(x)$, and if $(\lambda_1, \cdots, \lambda_N)$ are the eigenvalues of $\frac{X_N}{\sqrt{N}}$ then Wigner's theorem (see [14] and generalizations in [9, 1]) asserts that the empirical spectral measure $\frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$ of the matrix $\frac{X_N}{\sqrt{N}}$ converges weakly almost surely to the semi-circle distribution

$$\sigma(dx) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} dx.$$

We will consider here the case of heavy tailed entries, when the second moment σ^2 is infinite. We will assume that the common distribution of the absolute values of the x_{ij}'s is in the domain of attraction of an α-stable law, for $\alpha \in]0, 2[$, i.e that there exists a slowly varying function L such that

$$P(|x| \geq u) = \mathbb{P}(|x_{ij}| \geq u) = \frac{L(u)}{u^\alpha}. \tag{1}$$

We introduce the normalizing constant a_N by:

$$a_N = \inf(u, \mathbb{P}[|x_{ij}| \geq u] \leq \frac{1}{N}). \tag{2}$$

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It is clear that a_N is roughly of order $N^{\frac{1}{\alpha}}$, indeed there exists another slowly varying function L_0 such that

$$a_N = L_0(N)N^{\frac{1}{\alpha}}. \tag{3}$$

We then consider the matrix $A_N := a_N^{-1}X_N$, its eigenvalues $(\lambda_1, \cdots, \lambda_N)$, and its spectral measure $\hat{\mu}_{A_N} := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$. Our main result is

Theorem 1.1. *Let $\alpha \in]0,2[$ and assume (1).*

1. *There exists a probability measure μ_α on $\mathbb{R}$ such that the mean spectral measure $\mathbb{E}[\hat{\mu}_{A_N}]$ converges weakly to μ_α.*

2. *$\hat{\mu}_{A_N}$ converges weakly in probability to μ_α. More precisely, for any bounded continuous function f, $\int f(x)d\hat{\mu}_{A_N}(x)$ converges in probability to $\int f(x)d\mu_\alpha(x)$.*

3. *Let $(N_k)_{k\geq 1}$ be an increasing sequence of integers such that $\sum_{k=1}^{\infty}N_k^{-\varepsilon} < \infty$ for some $\varepsilon < 1$, then the subsequence $\hat{\mu}_{A_{N_k}}$ converges almost surely weakly to μ_α.*

Remark 1.2. *We note that the hypothesis (1) concerns only the tail behavior of the distribution of the absolute values of the entries. We make no assumption about the skewness of the distribution of the entries, i.e about their right or left tails.*

Remark 1.3. *It would be useful to control better the fluctuations in Theorem 1.1 and establish almost sure convergence for the whole sequence $\hat{\mu}_{A_N}$.*

Our approach is classical. It consists in proving the convergence of the resolvent, i.e of the mean of the Stieltjes transform of the spectral measure, by proving tightness and characterizing uniquely the possible limit points. We first prove, in section 2, that it is possible , for all later purposes, to truncate the large values of the entries at appropriate levels. We then proceed, in section 3, to show tightness for the spectral measures of the truncated and original matrices A_N . We then introduce, in section 4, the following important quantity: for $z \in \mathbb{C}\backslash\mathbb{R}$, we define the probability measure L_N^z on $\mathbb{C}$ by

$$L_N^z = \frac{1}{N}\sum_{k=1}^{N}\delta_{(z-A_N)_{kk}^{-1}}$$

i.e the empirical measure of the diagonal elements of the resolvent of A_N at $z \in \mathbb{C}\backslash\mathbb{R}$. The classical Schur complement formula is our basic linear algebraic tool to study L_N^z recursively on the dimension, as is usual when the resolvent method is used (see e.g [9] or [1]). In section 5, using an argument of concentration of measure and borrowing classical techniques from the theory of triangular arrays of i.i.d random variables, we show that the limit points μ^z of L_N^z satisfy a fixed point equation in the space of probability measures on $\mathbb{C}$. Even though we cannot prove uniqueness of the solution to this equation, we manage in section 6 to prove the uniqueness of the solution to the resulting equation for $\int x^{\frac{\alpha}{2}}d\mu^z(x)$, which in turn gives the uniqueness of $\int xd\mu^z(x)$. This is enough to characterize uniquely the limit points of $\mathbb{E}[\hat{\mu}_{A_N}]$ and thus the convergence of $\mathbb{E}[\hat{\mu}_{A_N}]$ to μ_α.

Once the question of convergence is settled by Theorem 1.1, the next question is to describe the limiting measure μ_α. We will discuss in this article three different characterizations of μ_α. Our approach leads directly to the following first characterization of μ_α through its Stieltjes transform, defined for $z \in \mathbb{C}\backslash\mathbb{R}$ by:

$$G_\alpha(z) = \int (z-x)^{-1} d\mu_\alpha(x). \tag{4}$$

Define the entire function g on $\mathbb{C}$ by

$$g_\alpha(y) = \frac{2}{\alpha} \int_0^\infty e^{-v^{\frac{2}{\alpha}}} e^{-vy} dv \tag{5}$$

We will also need the constants $C(\alpha) = \frac{2e^{i\frac{\pi\alpha}{2}}}{\alpha\Gamma(\frac{\alpha}{2})}$ and $c(\alpha) = \cos(\frac{\pi\alpha}{4})$.

Theorem 1.4.   1. There exists a unique function Y_z, analytic on the half plane $\mathbb{C}^+ = \{z \in \mathbb{C}, Imz > 0\}$, tending to zero at infinity, and such that

$$C(\alpha)g_\alpha(c(\alpha)Y_z) = Y_z(-z)^\alpha$$

2. The probability measure μ_α of Theorem 1.1 is uniquely described by its Stieltjes transform given, for $z \in \mathbb{C}^+$, by

$$G_\alpha(z) = -\frac{1}{z} \int_0^\infty e^{-t} e^{-c(\alpha)t^{\frac{\alpha}{2}}Y_z} dt \tag{6}$$

Using the characterization given in Theorem 1.4, we prove in section 7 the following properties of μ_α.

Theorem 1.5. *The probability measure μ_α of Theorem 1.1 satisfies*

1. *μ_α is symmetric.*

2. *μ_α has unbounded support.*

3. *There exists a (possibly empty) compact subset of the real line K_α of capacity zero, such that the measure μ_α has a smooth density ρ_α on the open complement $U_\alpha = \mathbb{R}\backslash K_\alpha$.*

4. *μ_α has heavy tails. There exists a constant $L_\alpha > 0$ such that, when $|x| \to \infty$*

$$\rho_\alpha(x) \sim \frac{L_\alpha}{|x|^{\alpha+1}}$$

A second and different characterization of μ_α is proposed in the physics literature by Cizeau-Bouchaud [3]. This description has been controversial (see [4] for a discussion and numerical simulations). The strategy used in [3] is also based on the convergence of the resolvent, but on the real axis as opposed to our proof of convergence away from the real axis. We unfortunately cannot make sense of the strategy used in [3]. We discuss in section 8 the link between our characterization given in Theorem 1.4 and the Bouchaud-Cizeau characterization (after correction of a small typographical error in [3] already noted by [4]).

Remark 1.6. *We are not able to prove in general that the exceptional set K_α of Theorem 1.4 is empty, or reduced to zero, even though we conjecture this is true. Recent work in progress with A. Dembo indicates that K_α is at most the origin for $\alpha \leq 1$. This would say that μ_α has a smooth density everywhere (except may be at zero) as suggested by numerical simulations and accepted by the physics literature. This question is discussed further in Section 7.*

We also describe below (in section 9) a third characterization of μ_α, more combinatorial in nature. It is based on an extension (due to I.Zakharevich, [15]) of the classical moment method rather than the resolvent approach used both by [3] and us. Obviously because of the heavy tails and thus of the absence of moments, one would have to do it first for truncated matrices and then try to lift the truncation. More precisely if one truncates the entries at the level Ba_N, for a fixed $B > 0$ and define $x_{ij}^B = x_{ij}1_{|x_{ij}| \le Ba_N}$ one can compute the moments of the empirical measures $\hat{\mu}_{A_N^B}$ of the truncated matrix $A_N^B(ij) = a_N^{-1}x_{ij}^B$

$$\int x^k d\hat{\mu}_{A_N^B}(x) = \frac{1}{N}\text{tr}\left((A_N^B)^k\right).$$

and study their convergence when N tends to infinity. We establish in section 9 that

Theorem 1.7. *With the above notations,and under the hypothesis of Theorem (1.1)and the additional hypothesis:*

$$\lim_{u \to \infty} \frac{\mathbb{P}(x(ij) > u)}{\mathbb{P}(|x(ij)| > u)} = \theta \in [0, 1] \tag{7}$$

1. $\mathbb{E}[\hat{\mu}_{A_N^B}]$ *converges weakly to a probability measure μ_α^B uniquely determined by its moments and independent of the parameter θ. This measure μ_α^B has unbounded support and is symmetric.*

2. μ_α^B *converges weakly to μ_α as B tends to infinity.*

The moments of μ_α^B are described combinatorially in Section 9. Thus Theorem 1.7 gives a third, independent, description of the limiting measure μ_α. As we will see in Section 9, the first part of Theorem (1.7) is a direct consequence of a general combinatorial result of I.Zakharevich and its proof is essentially given in [15]. The convergence of these Zakharevich measures to our μ_α establishes a link between this combinatorial description and the one we have given in terms of Stieltjes transforms in Theorem 1.4. This link is far from transparent.

Remark 1.8. *We note that the limiting measure μ_α^B is in fact independent of the skewness parameter θ. Thus it is insensitive to the hypothesis (7) about the upper and lower tails of the distribution of the entries. This is coherent with Remark 1.2.*

Remark 1.9. *The case $\alpha = 2$ is covered neither by the classical Wigner theorem (which asks for a second moment) nor by our results so far. In fact it is easy to see, using the combinatorial approach of Theorem 1.7 that the limit law is then the semi-circle, even though the normalization differs from the usual one.*

Finally, let us mention that the behavior of the edge of the spectrum of heavy tailed matrices (when $\alpha \in]0, 2[$) has been established by Soshnikov [13]. The largest eigenvalues are asymptotically, in the scale a_N^2, distributed as a Poisson point process with intensity $\alpha^{-1}x^{-\alpha-1}dx$. This is in sharp contrast with the Airy determinantal process description of top eigenvalues for the case of light tailed entries [12] but in perfect agreement with our result about the tail of μ_α given in Theorem 1.5.

2 Truncating the entries

Since the entries of our random matrices have very few moments, it will be of importance later to truncate them. We introduce the appropriate truncated matrices in this section and show how their spectral measure approximate the spectral measure of the original matrices.

Let us consider X_N^B (resp. X_N^κ) the Wigner matrix with entries $x_{ij}1_{|x_{ij}|\leq Ba_N}$ for $B > 0$, respectively $x_{ij}1_{|x_{ij}|\leq N^\kappa a_N}$ for $\kappa > 0$. Also define

$$A_N = a_N^{-1}X_N, \quad A_N^B = a_N^{-1}X_N^B, \quad A_N^\kappa = a_N^{-1}X_N^\kappa$$

Let us remark here that the threshold a_N is precisely the scale of the largest entry in a row (or a column) of the random matrix X_N, while the scale of the largest entry (or of the largest eigenvalue) of the whole matrix is a_N^2 i.e roughly $N^{\frac{2}{\alpha}}$.

We want to state that the spectral measures of the matrices A_N, A_N^B and A_N^κ are very close in a well chosen distance, compatible with the weak topology. The standard Dudley distance d is defined on $\mathcal{P}(\mathbb{R})$ by

$$d(\mu,\nu) = \sup_{\|f\|_{\mathcal{L}}\leq 1}\left|\int f d\nu - \int f d\mu\right|$$

where the supremum is taken over all Lipschitz functions f on $\mathbb{R}$ such that $\|f\|_{\mathcal{L}} \leq 1$, where the norm $\|f\|_{\mathcal{L}}$ is defined by

$$\|f\|_{\mathcal{L}} := \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|} + \sup_x |f(x)|.$$

We will use the following variant d_1 of the Dudley distance.

$$d_1(\mu,\nu) = \sup_{\|f\|_{\mathcal{L}}\leq 1, f\uparrow}\left|\int f d\nu - \int f d\mu\right|$$

where the supremum is taken over non-decreasing Lipschitz functions such that $\|f\|_{\mathcal{L}} \leq 1$. The Dudley distance d is well known to be a metric compatible with the weak topology and the following Lemma shows that so is the variant d_1.

Lemma 2.1. d_1 *is compatible with the weak topology on* $\mathcal{P}(\mathbb{R})$, *i.e if μ is a positive measure on $\mathbb{R}$ such that there exists $\mu^n \in \mathcal{P}(\mathbb{R})$ so that*

$$\lim_{n\to\infty} d_1(\mu^n,\mu) = 0,$$

then μ^n converges weakly to μ and $\mu \in \mathcal{P}(\mathbb{R})$. Reciprocally, if μ_n converges to μ weakly, $d_1(\mu_n,\mu)$ goes to zero. If a sequence $\mu_n \in \mathcal{P}(\mathbb{R})$ is Cauchy for d_1, it converges weakly.

Proof. A compactly supported Lipschitz function f can be written as

$$f(x) = f(0) + \int_0^x g(y)dy$$

where g is a borelian function bounded by the Lipschitz norm of f. Writing

$$f(x) - f(0) = \int_0^x 1_{g(y)\geq 0}g(y)dy - \int_0^x |g(y)|1_{g(y)<0}dy$$

we see that f can be written as the difference of two non-decreasing Lipschitz functions. Hence, if $d_1(\mu^n, \mu)$ goes to zero as n goes to infinity, $\int f d\mu_n$ converges to $\int f d\mu$ for all Lipschitz compactly supported functions. Hence, μ_n converges to μ for the vague topology. On the other hand, if μ^n converges to μ for d_1, we must have, taking $f = 1$,

$$\mu(1) = \lim_{n \to \infty} \mu^n(1) = 1$$

which is enough to guarantee also the weak convergence. Indeed, if we now take $f \in \mathcal{C}_b(\mathbb{R})$, and g compactly supported with values in $[0, 1]$,

$$|\mu_n(f) - \mu(f)| \leq \|f\|_\infty \left(\mu(1) + \mu_n(1) - \mu(g) - \mu_n(g) \right) + |\mu_n(fg) - \mu(fg)|$$

Letting first n going to infinity and then taking g approximating the unit, we obtain the result. The second statement is clear since $d_1 \leq d$ with d the standard Dudley distance (obtained by taking the supremum over all Lipschitz functions with norm bounded by one) and the result is well known to hold for d. Finally, if a sequence μ_n is Cauchy for d_1, it converges for the vague topology (as it is tight for the vague topology, and the property of being Cauchy uniquely prescribes the limit) and then for the weak topology by the mass property. $\qquad\square$

We next show that truncation does not affect much the spectral measures in the d_1 distance.

Theorem 2.2. *1. For every $\epsilon > 0$ there exists $B(\epsilon) < \infty$ and $\delta(\epsilon, B) > 0$ when $B > B(\epsilon)$ such that, for N large enough*

$$P\left(d_1(\hat{\mu}_{A_N}, \hat{\mu}_{A_N^B}) > \epsilon \right) \leq e^{-\delta(\epsilon, B)N}.$$

2. For $\kappa > 0$, and $a \in]1 - \alpha\kappa, 1[$, there exists a finite constant $C(\alpha, \kappa, a)$ such that for all $N \in \mathbb{N}$,

$$P\left(d_1(\hat{\mu}_{A_N}, \hat{\mu}_{A_N^\kappa}) > N^{a-1} \right) \leq e^{-CN^a \log N}.$$

Remark 2.3. *This result depends crucially on the proper choice of the truncation level. Had we truncated the entries at a lower level, say $N^\kappa a_N$ with $\kappa < 0$, then the limit law would be the semi-circle. Thus the effect of the heavy tails would have been completely canceled by the truncation.*

Proof. Let X and Y be two $N \times N$ Hermitian matrices, and $\hat{\mu}_X$ and $\hat{\mu}_Y$ be their spectral measures. Then Lidskii's theorem implies (see e.g [8] p. 500) that, if d is the rank of $X - Y$, then

$$d_1(\hat{\mu}_X, \hat{\mu}_Y) \leq \frac{2d}{N} \tag{8}$$

Consequently, the following Lemma implies Theorem (2.2). $\qquad\square$

Lemma 2.4. *1. For every $\epsilon > 0$, there exists $B(\epsilon) > 0$ and $\delta(\epsilon, B) > 0$ when $B > B(\epsilon)$ such that*

$$\mathbb{P}(rank(X_N - X_N^B) \geq \epsilon N) \leq e^{-\delta(\epsilon, B)N}$$

2. For $\kappa > 0$, and $a \in]1 - \alpha\kappa, 1[$ there exists a finite constant $C(\alpha, \kappa, a)$ such that for all $N \in \mathbb{N}$,

$$\mathbb{P}(rank(X_N - X_N^\kappa) \geq N^a) \leq e^{-CN^a \log N} \tag{9}$$

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Proof. (of Lemma 2.4.)

Let $M_i^- = 1$ (resp. $M_i^+ = 1$) if there exists a $j \leq i$ (resp. $j > i$ such that $|x_{ij}| > Ba_N$, and $M_i^- = 0$ (resp. $M_i^+ = 0$) otherwise. Define

$$M^- = \sum_{i=1}^N M_i^- \ \text{ and } \ M^+ = \sum_{i=1}^N M_i^+.$$

Now let M be the number of non zero rows of the matrix $X_N - X_N^B$, obviously

$$\text{rank}(X_N - X_N^B) \leq M \leq M^- + M^+, \tag{10}$$

so that

$$\mathbb{P}(\text{rank}(X_N - X_N^B) \geq \epsilon N) \leq \mathbb{P}(M^- \geq \frac{\epsilon N}{2}) + \mathbb{P}(M^+ \geq \frac{\epsilon N}{2}).$$

But if we denote by $p_i = \mathbb{P}(M_i^- = 1)$, we have

$$p_i = \mathbb{P}(\exists j \leq i, |x_{ij}| > Ba_N) = 1 - (1 - \frac{L(Ba_N)}{(Ba_N)^\alpha})^i \leq 1 - (1 - \frac{c}{NB^\alpha})^i$$

where the later inequality holds for $c > 1$ when N is large enough since

$$\lim_{N \to \infty} \frac{NL(Ba_N)}{a_N^\alpha} = 1. \tag{11}$$

As a consequence we can estimate the sum

$$\sum_{i=1}^N p_i \leq N - \frac{1 - (1 - \frac{c}{NB^\alpha})^{N+1}}{1 - (1 - \frac{c}{NB^\alpha})} \sim NC(B) \tag{12}$$

where we denoted $A_N \sim B_N$ if A_N/B_N goes to one as N goes to infinity and

$$C(B) = 1 - \frac{B^\alpha}{c}(1 - e^{-\frac{c}{B^\alpha}}). \tag{13}$$

For any $\lambda > 0$, the independence of the M_i^-'s gives

$$\mathbb{E}(\exp \lambda M^-) = \prod_{i=1}^N (1 + p_i(e^\lambda - 1)) \leq \exp[(e^\lambda - 1)(\sum_{i=1}^N p_i)]$$

So that we get the exponential upper bound, for N large enough

$$\mathbb{P}(M^- \geq \frac{\epsilon N}{2}) \leq e^{-\lambda \frac{\epsilon N}{2}} \mathbb{E}(\exp \lambda M^-) \leq \exp[-N\phi_-(\lambda, \epsilon, B)],$$

with

$$\phi_-(\lambda, \epsilon, B) = \frac{\lambda \epsilon}{2} - (e^\lambda - 1)C(B).$$

Obviously, since $\lim_{B \to \infty} C(B) = 0$, for any $\epsilon > 0$, there exists a $B(\epsilon) > 0$ (of order $\epsilon^{-\frac{1}{\alpha}}$) such that when $B > B(\epsilon)$,

$$\delta_-(\epsilon, B) := \sup_{\lambda > 0} \phi_-(\lambda, \epsilon, B) > 0$$

and

$$\mathbb{P}(M^- \geq \frac{\epsilon N}{2}) \leq \exp[-N\delta_-(\epsilon, B)].$$

The analogous result for M^+

$$\mathbb{P}(M^+ \geq \frac{\epsilon N}{2}) \leq \exp[-N\delta_+(\epsilon, B)]$$

is obtained similarly. Using the crude rank estimate (10) proves the first claim of Lemma (2.4).

In order to prove the second claim of Lemma (2.4), we simply replace B by $B(N) = N^\kappa$ and ϵ by $\epsilon(N) = N^{a-1}$ in the proof above. We get then that

$$\delta_-(\epsilon(N), B(N)) \sim \frac{1}{2}(a - 1 + \alpha\kappa)(N^a \log N)$$

and similarly for $\delta_+(\epsilon(N), B(N))$, which proves our second claim.

$\square$

Remark 2.5. *We now let $A_N^\kappa = a_N^{-1} X_N^\kappa$. We note that centering the entries of the matrix A_N^κ defines a perturbation of rank one. Hence, Lidskii's theorem (see (8)) shows that*

$$d_1(\hat{\mu}_{X_N^\kappa}, \hat{\mu}_{A_N^\kappa - \mathbb{E}[A_N^\kappa]}) \leq \frac{2}{N}.$$

Thus we may assume that A_N^κ is centered without changing its limiting spectral distribution.

3 Tightness

We prove in this section that the mean of the spectral measures of the random matrices A_N and of their truncated versions A_N^B or A_N^κ are tight.

Lemma 3.1. *1. The sequence $(\mathbb{E}[\hat{\mu}_{A_N}]; N \in \mathbb{N})$ is tight for the weak topology on $\mathcal{P}(\mathbb{R})$.*

2. For every $B < \infty$, and $\kappa > 0$, the sequences $(\mathbb{E}[\hat{\mu}_{A_N^B}]; N \in \mathbb{N})$ and $(\mathbb{E}[\hat{\mu}_{A_N^\kappa}]); N \in \mathbb{N})$ are tight for the weak topology on $\mathcal{P}(\mathbb{R})$.

Proof. We will use the following classical result about truncated moments (Theorem VIII.9.2 of [6]): For any $\zeta \geq \alpha$

$$\lim_{t \to \infty} \frac{E[|x_{ij}|^\zeta 1_{|x_{ij}| < t}]}{t^{\zeta - \alpha} L(t)} = \frac{\alpha}{\zeta - \alpha}. \tag{14}$$

Therefore, using (11), we have

$$E[|x_{ij}|^\zeta 1_{|x_{ij}| < Ba_N}] \sim \frac{\alpha}{\zeta - \alpha} B^{\zeta - \alpha} \frac{a_N^\zeta}{N} \tag{15}$$

or equivalently:

$$E[|A_N^B(ij)|^\zeta] \sim \frac{\alpha}{\zeta - \alpha} B^{\zeta - \alpha} \frac{1}{N}. \tag{16}$$

The version for the truncated matrix A_N^κ will also be useful:

$$E[|A_N^\kappa(ij)|^\zeta] \sim \frac{\alpha}{\zeta - \alpha} N^{\kappa(\zeta-\alpha)-1}. \tag{17}$$

Using these estimates with $\zeta = 2$, one sees that

$$\sup_{N \in \mathbb{N}} \mathbb{E}[\frac{1}{N} \text{tr}((A_N^B)^2)] \sim \frac{\alpha}{2 - \alpha} B^{2-\alpha} \tag{18}$$

and that

$$\sup_{N \in \mathbb{N}} \mathbb{E}[\frac{1}{N} \text{tr}((A_N^\kappa)^2)] \sim \frac{\alpha}{2 - \alpha} N^{\kappa(2-\alpha)}. \tag{19}$$

(18) shows that $\mathbb{E}[\hat\mu_{A_N^B}]$ belongs to the compact set $K_C := \{\mu \in \mathcal{P}(\mathbb{R}); \mu(x^2) \le C\}$ for any $C > \frac{\alpha}{2-\alpha} B^{2-\alpha}$ and N large enough. Hence, the sequence $(\mathbb{E}[\hat\mu_{A_N^B}]); N \in \mathbb{N})$ is tight, and thus any subsequence of $\mathbb{E}[\hat\mu_{A_N^B}]$ has converging subsequences. We denote by μ_B a limit point, i.e the limit of a converging subsequence. By a diagonal procedure, we can insure that this subsequence is the same for all $B \in \mathbb{N}$, and in particular, since d_1 is compatible with the weak topology, we can find an increasing function ϕ so that for any $\delta > 0$, $B_0 < \infty$, there exists $N_0 < \infty$ so that for $N \ge N_0$, and all $B \le B_0$,

$$d_1(\mathbb{E}[\mu_{A_{\phi(N)}^B}], \mu_B) \le \delta.$$

By Lemma 2.4, and Lidskii's estimate (8), we have for all $\epsilon > 0$,

$$d_1(\mathbb{E}[\mu_{A_{\phi(N)}}], \mathbb{E}[\hat\mu_{A_{\phi(N)}^B}]) \le \mathbb{E}[d_1(\hat\mu_{A_{\phi(N)}}, \hat\mu_{A_{\phi(N)}^B})] \le 2\epsilon + e^{-\delta(\epsilon,B)\phi(N)} \tag{20}$$

with $\delta(\epsilon, B) > 0$ if $B > B(\epsilon)$.

These two inequalities imply that $(\mu_B, B \in \mathbb{N})$ is a Cauchy sequence for the modified Dudley metric d_1 and thus converges when B tends to ∞. Indeed, if we choose $\epsilon, \epsilon', \delta > 0$ and $B_0 > B(\epsilon) \vee B(\epsilon')$, we find that for $B, B' \in [B(\epsilon) \vee B(\epsilon'), B_0]$ and $N > N_0$

$$d_1(\mathbb{E}[\hat\mu_{A_{\phi(N)}}], \mu_B) \le \delta + 2\epsilon + e^{-\delta(\epsilon,B)\phi(N)} \text{ and } d_1(\mathbb{E}[\hat\mu_{A_{\phi(N)}}], \mu_{B'}) \le \delta + 2\epsilon' + e^{-\delta(\epsilon',B')\phi(N)} \tag{21}$$

and therefore

$$d_1(\mu_B, \mu_{B'}) \le 2\delta + 2\epsilon + 2\epsilon' + e^{-\delta(\epsilon,B)\phi(N)} + e^{-\delta(\epsilon',B')\phi(N)}. \tag{22}$$

Letting N going to infinity, and then δ to zero and B_0 to infinity we finally deduce that

$$d_1(\mu_B, \mu_{B'}) \le 2\epsilon + 2\epsilon'$$

provided that B and B' are greater than $B(\epsilon) \vee B(\epsilon')$. Hence, μ_B is a Cauchy sequence for d_1 and thus converges weakly by Lemma 2.1 as B goes to infinity. As a consequence of (21) we also find that $\mathbb{E}[\hat\mu_{A_{\phi(N)}}]$ converges to this limit as N goes to infinity. The same holds for the truncated versions $\mathbb{E}[\hat\mu_{A_{\phi(N)}^\kappa}^{\phi(N)}]$. Thus, we have proved that $(\mathbb{E}[\hat\mu_{A_N}], E[\hat\mu_{A_N^\kappa}])_{N \in \mathbb{N}}$ are tight. $\square$

This lemma (3.1) can be strengthened into a partial almost-sure tightness result. Consider an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that $\sum_{N \ge 0} \frac{1}{\phi(N)} < \infty$, then

Lemma 3.2. *The sequences* $(\hat\mu_{A_{\phi(N)}^B})_{N \in \mathbb{N}}, (\hat\mu_{A_{\phi(N)}})_{N \in \mathbb{N}}, (\hat\mu_{A_{\phi(N)}^\kappa})_{N \in \mathbb{N}}$ *are almost surely tight.*

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Proof. We note that the truncated moments bound given in (16) can be strengthened into a bound in probability as follows. Let $M > 0$ and $C > \frac{\alpha}{2-\alpha}B^{2-\alpha}$, Chebychev's inequality reads

$$
\begin{aligned}
\mathbb{P}\left(\frac{1}{N}\mathrm{tr}((A_N^B)^2) \geq M + C\right) &\leq \frac{1}{M^2}\mathbb{E}\left[\left(\frac{1}{N}\mathrm{tr}((A_N^B)^2) - \mathbb{E}[\frac{1}{N}\mathrm{tr}((A_N^B)^2)]\right)^2\right] \\
&= \frac{1}{M^2}\mathbb{E}\left[\left(\frac{1}{N^2}\sum_{i,j=1}^N (A_N^B(i,j)^2 - \mathbb{E}[A_N^B(i,j)^2])\right)^2\right] \\
&\leq \frac{4}{M^2N^2}\sum_{i\leq j}\mathbb{E}\left[\left(A_N^B(i,j)^2 - \mathbb{E}[A_N^B(i,j)^2]\right)^2\right] \\
&\leq \frac{2}{M^2}\mathbb{E}\left[A_N^B(11)^4\right] \\
&\sim \frac{2\alpha B^{4-\alpha}}{4-\alpha}\frac{1}{M^2N}
\end{aligned}
$$

where we used the independence of the entries at the third step and the truncated moments estimate (16) for $\zeta = 4$ at the last step. Then Borel Cantelli's lemma implies that for any $C > \frac{\alpha}{2-\alpha}B^{2-\alpha}$

$$
\limsup_{N\to\infty} \frac{1}{\phi(N)}\mathrm{tr}((A_{\phi(N)}^B)^2) \leq C \quad a.s
$$

which insures the almost sure tightness of $(\hat{\mu}_{A_{\phi(N)}^B}^{\phi(N)})_{N\in\mathbb{N}}$. From this point, all the above arguments apply to show the almost sure tightness of $(\hat{\mu}_{A_{\phi(N)}}^{\phi(N)})_{N\in\mathbb{N}}$ and $(\hat{\mu}_{A_{\phi(N)}^\kappa}^{\phi(N)})_{N\in\mathbb{N}}$. $\qquad\square$

4   Induction over the dimension of the matrices

We borrow the following idea from [3]: in order to prove the vague convergence of $(\mathbb{E}[\hat{\mu}_{A_N}])_{N\in\mathbb{N}}$ we study the asymptotic behavior, for z a complex number, of the probability measure L_N^z on $\mathbb{C}$ given, for $f \in \mathcal{C}_b(\mathbb{C})$, by

$$
L_N^z(f) = \mathbb{E}\left[\frac{1}{N}\sum_{k=1}^N f\left(((z - A_N)^{-1})_{kk}\right)\right].
$$

L_N^z is thus the empirical measure of the diagonal entries of the resolvent of A_N. In contrast to [3], we will only consider these measures when $z \in \mathbb{C}\backslash\mathbb{R}$, where everything is well defined since $z - A_N$ is invertible.

Note that for $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$, and for $k \in \{1,\cdots,N\}$, the diagonal term $((z - A_N)^{-1})_{kk}$ belongs to the set $D := \mathbb{C}^- \cap \{x \in \mathbb{C} : |x| \leq |\Im(z)|^{-1}\}$. L_N^z is thus a probability measure on the compact subset D of $\mathbb{C}$.

If we choose the function $f(x) = x$ then

$$
L_N^z(f) = \mathbb{E}[\frac{1}{N}\mathrm{tr}((z - A)^{-1})]
$$

is the Stieltjes transform of $\mathbb{E}[\hat{\mu}_{A_N}]$.

Thus, the weak convergence of L_N^z for all $z \in \mathbb{C}^+$ (or even for all z in a set with accumulation points) would be enough to prove the vague convergence of $\mathbb{E}[\hat{\mu}_{A_N}]$. Indeed the latter is a consequence of the convergence of its Stieltjes transform, which, as an analytic function on $\mathbb{C}^+$, is uniquely determined by its values on a set with accumulation points.

In the following, given a $z \in \mathbb{C}^+$, we will prove an equation on the limit points of L_N^z (more precisely of its analogue where A_N is replaced by its truncation A_N^κ for some well chosen $\kappa > 0$). Our main tool will be a recursion on the dimension N, and the Schur complement formula. We first investigate how these measures depend on the dimension.

We let $\bar{A}_{N+1}$ be the $(N+1) \times (N+1)$ matrix obtained by adding to A_N a first row and a first column $A_N(0,k) = A_N(k,0) = N^{-\frac{1}{\alpha}} x_{0k}$. Hence, $\bar{A}_{N+1}$ has the same law as $(\frac{N+1}{N})^{\frac{1}{\alpha}} A_{N+1}$.

We then let $\hat{A}_N$ be the $(N+1) \times (N+1)$ matrix obtained by adding as first row and column the zero vector.

We also define for $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\bar{G}_{N+1}(z) := (z - \bar{A}_{N+1})^{-1} \quad G_N(z) = (z - A_N)^{-1} \quad \hat{G}_N(z) = (z - \hat{A}_N)^{-1}$$

We finally denote by $.^\kappa$ all quantities where A_N has been replaced by its truncated version A_N^κ. Thus for $z \in \mathbb{C}\backslash\mathbb{R}$ we define

$$L_N^{z,\kappa} = \frac{1}{N} \sum_{k=1}^N \delta_{G_N^\kappa(z)_{kk}}, \quad \hat{L}_N^{z,\kappa} = \frac{1}{N+1} \sum_{k=0}^N \delta_{\hat{G}_N^\kappa(z)_{kk}}, \quad \bar{L}_{N+1}^{z,\kappa} = \frac{1}{N} \sum_{k=1}^N \delta_{\bar{G}_{N+1}(z)_{kk}}$$

.

Lemma 4.1.   *1. $\hat{G}_N^\kappa(z)_{kk}$ is equal to $G_N^\kappa(z)_{kk}$ for $k \geq 1$ and to z^{-1} for $k = 0$.*

2.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N \mathbb{E}[|\bar{G}_{N+1}^\kappa(z)_{kk} - \hat{G}_N^\kappa(z)_{kk}|] = 0.$$

3. For $\kappa \in]0, \frac{1}{2-\alpha}[$ and $0 < \eta < \frac{1}{2}(1 - \kappa(2 - \alpha))$,

$$\lim_{N\to\infty} P(d(L_N^{z,\kappa}, \bar{L}_{N+1}^{z,\kappa}) > N^{-\eta}) = 0$$

Here, as above, d is the Dudley distance on $\mathcal{P}(\mathbb{C})$.

Proof. We note that

$$(z - \hat{A}_N^\kappa) = \begin{pmatrix} z & 0 \\ 0 & z - A_N^\kappa \end{pmatrix} \Rightarrow \hat{G}_N^\kappa(z) = \begin{pmatrix} z^{-1} & 0 \\ 0 & (z - A_N^\kappa)^{-1} \end{pmatrix} \tag{23}$$

which immediately yields the first point. For the second, let us write

$$\begin{aligned} \bar{G}_{N+1}^\kappa(z)_{kk} - \hat{G}_N^\kappa(z)_{kk} &= \left(\bar{G}_{N+1}^\kappa(z)(\bar{A}_{N+1}^\kappa - \hat{A}_N^\kappa)\hat{G}_N^\kappa(z) \right)_{kk} \\ &= \sum_l \bar{G}_{N+1}^\kappa(z)_{kl}(\bar{A}_{N+1}^\kappa - \hat{A}_N^\kappa)_{l0}\hat{G}_N^\kappa(z)_{0k} \\ &\quad + \sum_l \bar{G}_{N+1}^\kappa(z)_{k0}(\bar{A}_{N+1}^\kappa - \hat{A}_N^\kappa)_{0l}\hat{G}_N^\kappa(z)_{lk} \\ &= \bar{G}_{N+1}^\kappa(z)_{k0} \sum_l A_N^\kappa(0l)\hat{G}_N^\kappa(z)_{lk} \end{aligned}$$

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where we noticed above that $\hat{G}_N^\kappa(z)_{0k}$ is null for $k \neq 0$ by (23). Therefore, we find that

$$\mathbb{E}[|\bar{G}_{N+1}^\kappa(z)_{kk} - \hat{G}_N^\kappa(z)_{kk}|]^2 \leq \mathbb{E}[|\bar{G}_{N+1}^\kappa(z)_{k0}|^2]\mathbb{E}[|\sum_l A_N^\kappa(0l)\hat{G}_N^\kappa(z)_{lk}|^2]$$

by Cauchy-Schwartz's inequality. We recall that we have seen in remark 2.5 that we can assume that the entries of the matrix A_N^κ are centered. Using then the independence of A_{0l}^κ and $\hat{G}_N(z)$, summing over $k \in \{1, \cdots, N\}$ and with a further use of Cauchy-Schwartz's inequality, we find that,

$$\frac{1}{N}\sum_{k=1}^N \mathbb{E}[|\bar{G}_{N+1}^\kappa(z)_{kk} - \hat{G}_N^\kappa(z)_{kk}|] \leq \mathbb{E}[(A_N^\kappa(00))^2]^{\frac{1}{2}}\left(\frac{1}{N}\sum_{k=1}^N \mathbb{E}[|\bar{G}_{N+1}^\kappa(z)_{k0}|^2]\right)^{\frac{1}{2}}\mathbb{E}\left[\frac{1}{N}\sum_{l,k=1}^N |\hat{G}_N^\kappa(z)_{lk}|^2\right]^{\frac{1}{2}}$$

We now note that the entries of the resolvent $\hat{G}_N(z)$ are uniformly bounded in modulus. Indeed observe that, if U is a basis of eigenvectors of $\hat{A}_N^\kappa$, with associated eigenvalues $(\lambda_i, 1 \leq i \leq N) \in \mathbb{R}^N$, for any $k, l \in \{0, \cdots, N\}^2$,

$$\begin{aligned}
|\hat{G}_N^\kappa(z)_{kl}| &= |\sum_r u_{kr}(z - \lambda_r)^{-1}u_{rl}| \\
&\leq \frac{1}{|\Im(z)|}(\sum_r |u_{kr}|^2)^{\frac{1}{2}}(\sum_r |u_{rl}|^2)^{\frac{1}{2}} \leq \frac{1}{|\Im(z)|} \quad (24)
\end{aligned}$$

and the same holds for $\bar{G}_{N+1}^\kappa(z)$. Moreover, since the spectral radius of $\hat{G}_N(z)$ is bounded above by $1/|\Im(z)|$, we also have

$$\frac{1}{N}\sum_{l,k=0}^N |\hat{G}_N^\kappa(z)_{lk}|^2 = \frac{1}{N}\mathrm{tr}(\hat{G}_N^\kappa(z)\hat{G}_N^\kappa(z)^*) \leq \frac{N+1}{N|\Im(z)|^2}.$$

Hence, we deduce

$$\frac{1}{N}\sum_{k=1}^N \mathbb{E}[|\bar{G}_{N+1}^\kappa(z)_{kk} - \hat{G}_N^\kappa(z)_{kk}|] \leq \sqrt{\frac{N+1}{N}}\frac{1}{|\Im(z)|^2}\mathbb{E}[(A_N^\kappa(0l))^2]^{\frac{1}{2}}.$$

But we know how to control the truncated moments $\mathbb{E}[(A_N^\kappa(0l))^2]$. Indeed by the estimate (17) we see that

$$E[|A_N^\kappa(ij)|^2] \sim \frac{\alpha}{2-\alpha}N^{-\epsilon}$$

with $\epsilon = 1 - \kappa(2 - \alpha) > 0$. The proof of the second point is complete.

We finally deduce the last result simply by

$$\begin{aligned}
\mathbb{E}[d(\bar{L}_{N+1}^{z,\kappa}, \hat{L}_N^{z,\kappa})] &\leq \frac{1}{N+1}\sum_{k=0}^N \mathbb{E}[|\bar{G}_{N+1}^\kappa(z)_{kk} - \hat{G}_N^\kappa(z)_{kk}| \wedge 1] \\
&\leq \frac{1}{N+1} + \sqrt{\frac{N+1}{N}}\frac{1}{|\Im(z)|^2}N^{-\frac{\epsilon}{2}} \quad (25)
\end{aligned}$$

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and since $G_N(z)$ and $\hat{G}_N(z)$ differ at most by a rank one perturbation,

$$\hat{L}_N^{z,\kappa} = \frac{N}{N+1} L_N^{z,\kappa} + \frac{1}{N+1} \delta_{z^{-1}}$$

implies that

$$d(L_N^{z,\kappa}, \hat{L}_N^{z,\kappa}) \le \frac{2}{N+1}.$$

This shows by Chebychev's inequality that for all $\eta < \frac{\epsilon}{2}$

$$\lim_{N \to \infty} P(d(L_N^{z,\kappa}, \bar{L}_{N+1}^{z,\kappa}) > N^{-\eta}) = 0.$$

$\square$

To derive an equation for $L_N^{z,\kappa}$, our tool will be the Schur complement formula, which we now recall. Let $\bar{A}_{N+1}$ and A_N be as above.

Lemma 4.2. *For any $z \in \mathbb{C}$,*

$$\left((\bar{A}_{N+1} - zI)^{-1}\right)_{00} = \left(A_N(00) - z - \sum_{k,l=1}^{N} A_N(0k) A_N(l0) \left((A_N - zI)^{-1}\right)_{kl}\right)^{-1}.$$

Proof. The proof is a direct consequence of Cramer's inversion formula:

$$\left((\bar{A}_{N+1} - zI)^{-1}\right)_{00} = \frac{\det(A_N - zI_{N-1})}{\det(\bar{A}_{N+1} - zI)}.$$

To get a more explicit formula for this ratio, write

$$\bar{A}_{N+1} - zI = \begin{pmatrix} A_N(00) - z & a_0 \\ a_0^T & A_N - z \end{pmatrix}$$

with $a_0 = (A(01), \cdots, A(0N))$, and use the representation

$$\begin{bmatrix} I & -BD^{-1} \\ -0 & I \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & 0 \\ C & D \end{bmatrix}$$

with $A = A(00) - z$, $B = a_0$, $C = a_0^T$ and $D = A_N - z$. Therefore, as $\det(AB) = \det(A)\det(B)$, we conclude that

$$\det(\bar{A}_{N+1} - zI) = \det(A_N - zI) \det\left[A(00) - z - \langle a_0, (A_N - zI)^{-1} a_0 \rangle\right].$$

This proves the lemma. $\square$

We now show that, in the Schur complement formula above, the off-diagonal terms in the sum in the right hand side are negligible.

Lemma 4.3. *For z with $|\Im(z)| \geq \delta > 0$, $0 < \kappa < \frac{1}{2(2-\alpha)}$, and $R > 0$*

$$P(|\sum_{k \neq l} A_N^\kappa(0k) A_N^\kappa(0l) \left((A_N^\kappa - z)^{-1}\right)_{kl}| > R) \leq \frac{1}{R^2 N^{2\epsilon-1}\delta^2}.$$

with $\epsilon = 1 - \kappa(2-\alpha) > \frac{1}{2}$.

Proof. As above we can always assume that the entries of A_N^κ are centered. By independence of $A_N^\kappa(0k)$ and A_N, we find that the first moment of the off-diagonal term vanishes:

$$\mathbb{E}[\sum_{\substack{1 \leq k,l \leq N \\ k \neq l}} A_N^\kappa(0k) A_N^\kappa(l0) \left((A_N^\kappa - zI)^{-1}\right)_{kl}] = 0$$

and that the second moment is small:

$$
\begin{aligned}
\mathbb{E}[|\sum_{k \neq l} A_N^\kappa(0k) A_N^\kappa(l0) \left((A_N^\kappa - z)^{-1}\right)_{kl}|^2] &\leq C\mathbb{E}[(A_N^\kappa(01))^2]^2 \mathbb{E}[\sum_{k,l} | \left((A_N^\kappa - z)^{-1}\right)_{kl} |^2] \\
&\leq N^{-2\epsilon} \mathbb{E}[\mathrm{tr}((A_N^\kappa - z)^{-1}(A_N^\kappa - \bar{z})^{-1})] \leq N^{-2\epsilon+1} \frac{1}{|\Im(z)|^2}.
\end{aligned}
$$

Chebychev's inequality concludes the proof. $\qquad\square$

We finally derive from the previous considerations a first approximation result for $L_N^{z,\kappa}$. This will be our first step to obtain a closed equation for the limit points of the spectral measure (such an equation will be derived in the next section).

Lemma 4.4. *For $0 < \kappa < \frac{1}{2(2-\alpha)}$, let $\epsilon = 1 - \kappa(2-\alpha) > \frac{1}{2}$. There exists a constant c such that, for any Lipschitz function f,*

$$|\mathbb{E}[L_N^{z,\kappa}(f)] - \mathbb{E}\left[f\left(\left(z - \sum_{k=1}^N A_N^\kappa(0k)^2 G_N^\kappa(z)_{kk}\right)^{-1}\right)\right]| \leq \frac{c\|f\|_{\mathcal{L}}}{|\Im(z)|^{\frac{5}{3}} N^{\frac{2\epsilon-1}{3}}}$$

Proof. It is clear, by Lemma 4.1, that it is sufficient to prove that, for a constant c', and every Lipschitz function f

$$|\mathbb{E}[\bar{L}_{N+1}^{z,\kappa}(f)] - \mathbb{E}\left[f\left(\left(z - \sum_{k=1}^N A_N^\kappa(0k)^2 G_N^\kappa(z)_{kk}\right)^{-1}\right)\right]| \leq \frac{c\|f\|_{\mathcal{L}}}{|\Im(z)|^{\frac{5}{3}} N^{\frac{2\epsilon-1}{3}}} \qquad (26)$$

We have proved above that, for $z \in \mathbb{C}\backslash\mathbb{R}$, there exists a random variable $\varepsilon_N(z)$,

$$P(|\varepsilon_N(z)| \geq R) \leq \frac{1}{R^2 N^{2\epsilon-1}|\Im(z)|^2}$$

such that

$$\bar{G}_{N+1}^\kappa(z)_{00} = \left(z - \sum_{k=1}^N A_N^\kappa(0k)^2 G_N^\kappa(z)_{kk} + \varepsilon_N(z)\right)^{-1}$$

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In particular we have for any Lipschitz function f,

$$\mathbb{E}[f(\bar{G}^\kappa_{N+1}(z)_{00})] = \mathbb{E}\left[f\left(\left(z - \sum_{k=0}^{N} A^\kappa_N(0k)^2 G^\kappa_N(z)_{kk} + \varepsilon_N(z)\right)^{-1}\right)\right]. \tag{27}$$

Observe that with $A^\kappa_N = U\operatorname{diag}(\lambda)U^*$,

$$G^\kappa_N(z)_{kk} = \sum_{i=1}^{N} |u_{ki}|^2 (z - \lambda_i)^{-1}$$

is such that

$$\Im(z)\Im\left(G^N_{00}(z)_{kk}\right) \leq 0, \quad |G^\kappa_N(z)_{kk}| \leq |\Im(z)|^{-1}.$$

In particular, we always have

$$\frac{\Im\left(z - \sum_{k=0}^{N} A^\kappa_N(0k)^2 G^\kappa_N(z)_{kk}\right)}{\Im(z)} \geq 1.$$

Thus, on $|\varepsilon_N(z)| \leq |\Im(z)|/2$, we obtain the control

$$\left|\left(z - \sum_{k=0}^{N} A^\kappa_N(0k)^2 G^\kappa_N(z)_{kk} + \varepsilon_N(z)\right)^{-1} - \left(z - \sum_{k=0}^{N} A^\kappa_N(0k)^2 G^\kappa_N(z)_{kk}\right)^{-1}\right| \leq \frac{2|\varepsilon_N(z)|}{|\Im(z)|}.$$

Hence, if f is Lipschitz,

$$\mathbb{E}[f(\bar{G}^\kappa_{N+1}(z)_{00})] = \mathbb{E}[f\left(\left(z - \sum_{k=0}^{N} A^\kappa_N(0k)^2 G^\kappa_N(z)_{kk}\right)^{-1}\right)] + O(\|f\|_{\mathcal{L}})\mathbb{E}[\frac{|\varepsilon_N(z)|}{|\Im z|} \wedge 1])$$

Now, the right hand side does not depend on the choice of the indices and so we have the same estimate for all $\mathbb{E}[f(\bar{G}^\kappa_{N+1}(z)_{kk})]$, for $k \in \{0, 1, \cdots, N\}$. Summing the resulting equalities we find that

$$\mathbb{E}[\bar{L}^{z,\kappa}_{N+1}(f)] = \mathbb{E}\left[f\left(\left(z - \sum_{k=0}^{N} A^\kappa_N(0k)^2 G^\kappa_N(z)_{kk}\right)^{-1}\right)\right] + O(\|f\|_{\mathcal{L}}\mathbb{E}[\frac{|\varepsilon_N(z)|}{|\Im(z)|} \wedge 1])$$

This proves the estimate (26) and thus the lemma.

$\square$

5   The limiting equation

We prove in this section that the limit points of the sequence of measures $\mathbb{E}[L^{z,\kappa}_N]$ satisfy an implicit equation. This section will rely heavily on a result about the convergence of sums of triangular arrays to complex stable laws. We have deferred to Appendix 10 the statements and proofs of these convergence results. We also refer to the same Appendix for notations and references about complex stable laws.

Hereafter $z \in \mathbb{C}^+$ will be fixed. We have seen that $\mathbb{E}[L_N^{z;\gamma}]$ is a compactly supported probability measure on $\mathbb{C}$ (since its support lies in the open ball with radius $1/|\Im(z)|$). Therefore, $(\mathbb{E}[L_N^{z;\gamma}])_{N \in \mathbb{N}}$ is tight, and we denote by μ^z a limit point. Recall that for $z \in \mathbb{C}^+$, μ^z is a probability measure on $\mathbb{C}^- \cap \{|y| \leq 1/|\Im(z)|\}$.

In order to state the main result of this section we will need the following notations. For $t, z \in \mathbb{C}$, we denote by $\langle t, z \rangle$ the scalar product of t and z seen as vectors in $\mathbb{R}^2$, i.e $\langle t, z \rangle = \Re(t)\Re(z) + \Im(t)\Im(z)$. For a probability measure μ on $\mathbb{C}$, and $t \in \mathbb{C}$, we define the numbers $\sigma^{\mu,\alpha}(t)$ and $\beta^{\mu,\alpha}(t)$ by:

$$\sigma_{\mu,\alpha}(t) = [\frac{1}{C_\alpha} \int |\langle t, z \rangle|^\alpha d\mu(z)]^{\frac{1}{\alpha}} \tag{28}$$

and

$$\beta_{\mu,\alpha}(t) = \frac{\int |<t,z>|^\alpha sign <t,z> d\mu(z)}{\int |<t,z>|^\alpha d\mu(z)} \tag{29}$$

where

$$C_\alpha^{-1} = \int_0^\infty \frac{\sin x}{x^\alpha} dx = \frac{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})}{1-\alpha} \tag{30}$$

Definition 5.1. *For a probability measure μ on $\mathbb{C}$, we define the probability measure P^μ on $\mathbb{C}$ by its Fourier transform*

$$\int e^{i<t,x>} dP^\mu(x) = \exp[-\sigma_{\mu,\frac{\alpha}{2}}(t)^{\frac{\alpha}{2}}(1 - i\beta_{\mu,\frac{\alpha}{2}}(t)\tan(\frac{\pi\alpha}{4}))]$$

P^μ is well defined by this Fourier transform, indeed P^μ is a complex stable distribution. For this description of P^μ see the appendix 10.

We can now state the main result of this section.

Theorem 5.2. *For $0 < \kappa < \frac{1}{2(2-\alpha)}$, the limit points μ^z of $\mathbb{E}[L_N^{z,\kappa}]$ satisfy the equation*

$$\int f d\mu^z = \int f\left(\frac{1}{z-x}\right) dP^{\mu^z}(x)$$

for every bounded continuous function f.

Proof. We consider a subsequence of $(\mathbb{E}[L_N^{z,\kappa}])$ converging to μ^z, i.e an increasing function $\phi(N)$ such that $(\mathbb{E}[L_{\phi(N)}^{z,\kappa}])$ converges weakly to μ^z. We denote by P_N^z the law of $\sum_{k=1}^N (A_N^\kappa(0k))^2 G_N^\kappa(z)_{kk}$. For $z \in \mathbb{C}^+$, P_N^z is a probability measure on $\mathbb{C}^-$ since then $G_N(z)_{kk} \in \mathbb{C}^-$ for all k. If f is Lispchitz, Theorem 5.2 is a direct consequence of the main result of the preceding section, i.e Lemma 4.4, and of the next crucial Lemma 5.3.

Lemma 5.3. *If $\mathbb{E}[L_{\phi(N)}^{z,\kappa}]$ converges weakly to μ^z as N goes to infinity, then $P_{\phi(N)}^z$ converges weakly to P^{μ^z} as N goes to infinity.*

It is then easy to see that the statement of Theorem 5.2 extends to any bounded continuous function. $\qquad\square$

We now have to prove Lemma (5.3).

Proof. We apply first the following concentration result for $L_N^{z,\kappa}$.

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Lemma 5.4. *For* $\kappa \in (0, \frac{1}{2-\alpha})$, *let* $\epsilon = 1 - \kappa(2-\alpha) > 0$. *There exists a finite constant c so that for* $z \in \mathbb{C}\backslash\mathbb{R}$ *and any Lispchitz function f on* $\mathbb{C}$

$$\mathbb{P}\left(\left|L_N^{z,\kappa}(f) - \mathbb{E}[L_N^{z,\kappa}(f)]\right| \geq \delta\right) \leq \frac{c\|f\|_{\mathcal{L}}^2}{|\Im(z)|^4 \delta^2} N^{-\epsilon}$$

This Lemma shows that since $\mathbb{E}[L_{\phi(N)}^{z,\kappa}]$ converges weakly to μ^z, then $L_{\phi(N)}^{z,\kappa}$ also converges almost surely to the non random probability μ^z. From there, one can apply Theorem 10.3 of Appendix 10 or more precisely its extension Theorem 10.4 which has been built to fit exactly our needs here, when applied to the variables $X_k = A(0,k)^2$. One must simply notice that the exponent α in Theorem 10.4 must be replaced here by $\frac{\alpha}{2}$. This concludes the proof of Lemma 5.3.

Proof of Lemma (5.4) . We prove this concentration lemma using standard martingales decomposition. We assume that f is continuously differentiable, the generalization to any Lipschitz function being deduced by density. We put

$$F_N(A_{kl}^\kappa, k \leq l) := L_N^{z,\kappa}(f) = \frac{1}{N}\sum_{k=1}^N f(G_N(z)_{kk})$$

Let $n = N(N-1)/2 + N$ and index the A_{kl}^κ by A_i^κ, $1 \leq i \leq n$ for some lexicographic order. Then, if we let $\mathcal{F}_i = \sigma(A_j^\kappa, 1 \leq j \leq i)$, the independence and identical distribution of the A_i^κ's shows that, if P_N denotes the law of A_i^κ (i.e the properly truncated and normalized version of P),

$$\mathbb{E}[(F_N - \mathbb{E}[F_N])^2]$$

$$= \sum_{i=0}^{n-1} \mathbb{E}[(\mathbb{E}[F_N|\mathcal{F}_{i+1}] - \mathbb{E}[F_N|\mathcal{F}_i])^2]$$

$$= \sum_{i=0}^{n-1} \int \left(\int F_N(x_1, \cdot, x_{i+1}, y_{i+2}, \cdot, y_n) dP_N^{\otimes n}(y) - \int F_N(x_1, \cdot, x_i, y_{i+1}, \cdot, y_n) dP_N^{\otimes n}(y)\right)^2 dP_N^{\otimes i+1}(x)$$

$$\leq \sum_{i=0}^{n-1} \int (F_N(x_1, \cdots, x_{i+1}, \cdots, x_n) - \int F_N(x_1, \cdots, x_i, y, x_{i+2} \cdots x_n) dP_N(y))^2 dP_N^{\otimes n}(x)$$

$$\leq \sum_{i=0}^{n-1} \|\partial_{x_{i+1}} F_N\|_\infty^2 \int (x-y)^2 dP_N^{\otimes 2}(x,y) \tag{31}$$

In our case, for all $k \in \{1, \cdots, N\}$, all $m, l \in \{1, \cdots, N\}$,

$$\partial_{A_{ml}} f(G_N(z)_{kk}) = f'(G_N(z)_{kk})(G_N(z)_{kl} G_N(z)_{mk} + G_N(z)_{km} G_N(z)_{lk})$$

which yields

$$\begin{aligned}\partial_{A_{ml}} F_N(A) &= \frac{1}{N}\sum_{k=1}^N f'(G_N(z)_{kk})(G_N(z)_{kl} G_N(z)_{mk} + G_N(z)_{km} G_N(z)_{lk}) \\ &= \frac{1}{N}\left([G_N(z)D(f')G_N(z)]_{ml} + [G_N(z)D(f')G_N(z)]_{lm}\right)\end{aligned}$$

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with $D(f')$ the diagonal matrix with entries $(f'(G_N(z)_{kk}))_{1\le k\le N}$. Note that the spectral radius of $G_N(z)D(f')G_N(z)$ is bounded by $\|f'\|_\infty/|\Im(z)|^2$ and so since for all $l,m\in\{1,\cdots,N\}^2$

$$\left|[G_N(z)D(f')G_N(z)]_{lm}\right| \le \|G_N(z)D(f')G_N(z)\|_\infty \le \|f'\|_\infty/|\Im(z)|^2$$

we conclude that for all $l,m\in\{1,\cdots,N\}^2$,

$$|\partial_{A_{ml}}F(A)| \le \frac{2\|f'\|_\infty}{N|\Im(z)|^2}.$$

Thus, (31) shows that

$$
\begin{aligned}
\mathbb{E}[(F_N-\mathbb{E}[F_N])^2] &\le \frac{4\|f'\|_\infty^2}{N^2|\Im(z)|^4}\frac{N^2}{2}\mathbb{E}[(A_{11}^\kappa-\mathbb{E}[A_{11}^\kappa])^2]\\
&\le \frac{2\|f'\|_\infty^2}{|\Im(z)|^4}N^{-\epsilon}
\end{aligned}
$$

where we used the truncated moment estimate(17). Chebychev's inequality then provides the announced bound. $\qquad\square$

$\square$

We now apply Theorem 5.2 for a particular choice of the function f. To this end, we need to define, for any $\alpha>0$, the proper branch of the power function $x\to x^\alpha$ defined as the analytic function on $\mathbb{C}\backslash\mathbb{R}^+$ such that $(i)^\alpha=e^{i\frac{\pi\alpha}{2}}$. This amounts to choosing, if $x=re^{i\theta}$ with $\theta\in]0,2\pi[$,

$$x^\alpha=r^\alpha e^{i\alpha\theta}.$$

This function is analytic on $\mathbb{C}\backslash\mathbb{R}^+$ and extends by continuity to $x=re^{i\theta}$ with θ decreasing to zero;

$$\lim_{\theta\downarrow0}(re^{i\theta})^\alpha=r^\alpha$$

the usual power function on $\mathbb{R}^+$ (for $x\in\mathbb{R}^+$, x^α will denote this classical real valued power function in the sequel). When $x=re^{i\theta}$ is on the other side of the cut $\mathbb{R}^+$, i.e when θ slightly exceeds 2π, the function jumps by a multiplicative factor $e^{2i\alpha\pi}$. We want to choose in (5.2) the analytic function $f(x)=x^{\frac{\alpha}{2}}$.

Theorem 5.5. *For $0<\kappa<\frac{1}{2(2-\alpha)}$, let μ^z be a limit point of $\mathbb{E}[L_N^{z,\kappa}]$ and define $X_{\mu^z}:=\int x^{\frac{\alpha}{2}}d\mu^z(x)$. Then*

1. X_{μ^z} is analytic in $\mathbb{C}^+$ and $|X_z|\le\frac{1}{|\Im(z)|^{\frac{\alpha}{2}}}$

2. X_{μ^z} is a solution of the following equation:

$$X_{\mu^z}=iC(\alpha)\int_0^\infty(it)^{\frac{\alpha}{2}-1}e^{itz}\exp\{-c(\alpha)(it)^{\frac{\alpha}{2}}X_{\mu^z}\}dt. \tag{32}$$

with $C(\alpha)=\frac{e^{i\frac{\pi\alpha}{2}}}{\Gamma(\frac{\alpha}{2})}$ and $c(\alpha)=\Gamma(1-\frac{\alpha}{2})$.

Proof. The first point is obvious. Indeed, for some increasing function ϕ,

$$X_{\mu^z} = \lim_{N \to \infty} X_z^{\phi(N)}, \quad X_z^N := \mathbb{E}[\frac{1}{N}\sum_{k=1}^{N} \left((z - A_N^\kappa)_{kk}^{-1}\right)^{\frac{\alpha}{2}}].$$

For each N, X_z^N is an analytic function on $\mathbb{C}^+$. Moreover, $|X_z^N| \leq \frac{1}{|\Im(z)|^{\frac{\alpha}{2}}}$ for all N. This entails that any limit point X_{μ^z} must also be analytic in $\mathbb{C}^+$.

In order to prove the second point and obtain the closed equation (32) we will need the following classical identity:

Lemma 5.6. *For all $z \in \mathbb{C}^+$,*

$$\left(\frac{1}{z}\right)^{\frac{\alpha}{2}} = iC(\alpha) \int_0^\infty (it)^{\frac{\alpha}{2}-1} e^{itz} dt$$

with $C(\alpha) = \frac{e^{i\frac{\pi\alpha}{2}}}{\Gamma(\frac{\alpha}{2})}$

This Lemma is proven by a simple contour integration, it is also a consequence of Lemma 6.2 , proven in the next section (plug y=0 in the statement of Lemma 6.2).

By Theorem (5.2), and since μ^z and P^{μ^z} are supported in $\mathbb{C}^-$, we can write

$$X_{\mu^z} = \int \left(\frac{1}{z-x}\right)^{\frac{\alpha}{2}} dP^{\mu^z}(x).$$

Applying Lemma 5.6 to $z \to z - x \in \mathbb{C}^+$ for P^{μ^z} almost all x, and integrating over the x's we have, by Fubini's theorem,

$$X_{\mu^z} = iC(\alpha) \int_0^\infty (it)^{\frac{\alpha}{2}-1} e^{itz} \int e^{-itx} dP^{\mu^z}(x) dt. \tag{33}$$

We now use Theorem 10.5 in the appendix, with $\nu = \mu^z$ here, and replacing α in Theorem 10.5 by $\frac{\alpha}{2}$ here. We see that:

$$\int e^{-itx} dP^{\mu^z}(x) = \exp\{-c(\alpha)(it)^{\frac{\alpha}{2}} \int x^{\frac{\alpha}{2}} d\mu^z(x)\}.$$

Plugging this equality into (33) yields

$$X_{\mu^z} = iC(\alpha) \int_0^\infty (it)^{\frac{\alpha}{2}-1} e^{itz} \exp\{-c(\alpha)(it)^{\frac{\alpha}{2}} \int x^{\frac{\alpha}{2}} d\mu^z(x)\} dt. \tag{34}$$

We have obtained the announced closed equation

$$X_{\mu^z} = iC(\alpha) \int_0^\infty (it)^{\frac{\alpha}{2}-1} e^{itz} \exp\{-c(\alpha)(it)^{\frac{\alpha}{2}} X_{\mu^z}\} dt. \tag{35}$$

$\square$

6 Proofs of Theorem 1.1 and of Theorem 1.4

In this section we gather the preceding arguments and prove Theorem 1.1 and Theorem 1.4. This proof will be based on the following uniqueness result for the closed equation (32). We recall the notation

$$g_\alpha(y) := \frac{2}{\alpha} \int_0^\infty e^{-v^{\frac{2}{\alpha}}} e^{-vy} dv = \int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \exp\{-t^{\frac{\alpha}{2}}y\} dt$$

Theorem 6.1. *1. There exists a unique analytic function X_z of $z \in \mathbb{C}^+$, such that $|X_z| = O(|Im(z)|^{-\frac{\alpha}{2}})$ at infinity, satisfying the equation*

$$X_z = iC(\alpha) \int_0^\infty (it)^{\frac{\alpha}{2}-1} e^{itz} \exp\{-c(\alpha)(it)^{\frac{\alpha}{2}} X_z\} dt. \tag{36}$$

2. This solution in fact also satisfies: $|X_z| = O(|z|^{-\frac{\alpha}{2}})$.

3. If one defines $Y_z := (-\frac{1}{z})^{\frac{\alpha}{2}} X_z$, then Y_z is the unique solution of the equation

$$(-z)^\alpha Y_z = C(\alpha) g_\alpha(c(\alpha) Y_z).$$

analytic on $\mathbb{C}^+$ and tending to zero at infinity. In fact $|Y_z| = O(|z|^{-\alpha})$

Proof. We already know that there exists such an analytic solution X_z. Indeed we have seen in the preceding section that, if μ^z is a limit point, then X_{μ^z} is such a solution. In order to prove uniqueness, we will use that:

Lemma 6.2. *For all $z \in \mathbb{C}^+$, and any $y \in \mathbb{C}$*

$$\left(-\frac{1}{z}\right)^{\frac{\alpha}{2}} g_\alpha(y) = i \int_0^\infty (it)^{\frac{\alpha}{2}-1} e^{itz} \exp[-(-z)^{\frac{\alpha}{2}} (it)^{\frac{\alpha}{2}} y] dt$$

Proof. We first remark that with our choice of the branch for the power function, we always have, for t real positive and $x \in \mathbb{C}\backslash\mathbb{R}^+$,

$$(tx)^{\frac{\alpha}{2}} = t^{\frac{\alpha}{2}} x^{\frac{\alpha}{2}}$$

Note that the equality also holds for all $t \in \mathbb{C}\backslash\mathbb{R}^+$ such that $tx \in \mathbb{C}\backslash\mathbb{R}^+$ (these identities can be checked by analytic continuation from $\mathbb{R}^-$).

We then write $z = re^{i\theta}$ with some $\theta \in]0, \pi[$. Since $f(u) = (u)^{\frac{\alpha}{2}-1} e^{uz} e^{-u^{\frac{\alpha}{2}}[(-z)^{\frac{\alpha}{2}}y]}$ is analytic in $\mathbb{C}\backslash\mathbb{R}^+$, for all $R > 0$ finite, its integral over the contour

$$\Gamma = \{it, \epsilon \leq t \leq R\} \cup \{e^{i\eta}R, \eta \in [\frac{\pi}{2}, \pi - \theta]\} \cup \{e^{i\pi-i\theta}t, R \leq t \leq \epsilon\} \cup \{e^{i\eta}\epsilon, \eta \in [\pi - \theta, \frac{\pi}{2}]\}$$

vanishes. Note that $\eta + \theta \in [\frac{\pi}{2} + \theta, \pi]$ so that $\Re(Re^{i\eta}z) = Rr\cos(\eta + \theta) < 0$ for all $\eta \in [\frac{\pi}{2}, \pi - \theta]$ and $\theta \in]0, \pi[$. This shows that

$$\lim_{R\to\infty} Rf(e^{i\eta}R) = 0 \quad \forall \eta \in [\frac{\pi}{2}, \pi - \theta] \Rightarrow \lim_{R\to\infty} R \int_{\eta\in[\frac{\pi}{2},\pi-\theta]} f(e^{i\eta}R)d\eta = 0.$$

Similarly,

$$\limsup_{\epsilon\to 0} |\int_{\eta\in[\frac{\pi}{2},\pi-\theta]} f(e^{i\eta}\epsilon)d\eta| < \infty \Rightarrow \lim_{\epsilon\to 0} \epsilon \int_{\eta\in[\frac{\pi}{2},\pi-\theta]} f(e^{i\eta}\epsilon)d\eta = 0$$

Hence, letting $R \to \infty$ and $\epsilon \to 0$, we find

$$i \int_0^\infty f(it)dt + \int_{+\infty}^0 f(e^{i(\pi-\theta)}t)e^{i(\pi-\theta)}dt = 0.$$

In other words,

$$i \int_0^\infty (it)^{\frac{\alpha}{2}-1}e^{itz}e^{-(it)^{\frac{\alpha}{2}}[(-z)^{\frac{\alpha}{2}}y]}dt = -\int_0^\infty (-e^{-i\theta}t)^{\frac{\alpha}{2}-1}e^{-t|z|}e^{-(-e^{-i\theta}t)^{\frac{\alpha}{2}}[(-z)^{\frac{\alpha}{2}}y]}e^{-i\theta}dt$$

$$= -z^{-1}\int_0^\infty (-z^{-1}t)^{\frac{\alpha}{2}-1}e^{-t}e^{-(-z^{-1}t)^{\frac{\alpha}{2}}[(-z)^{\frac{\alpha}{2}}y]}dt$$

where we finally did the change of variable $t' = |z|t$. Now, we note that, since z^{-1} and $-t$ belong to $\mathbb{C}\backslash\mathbb{R}^+$

$$(-z^{-1}t)^{\frac{\alpha}{2}-1} = (z^{-1})^{\frac{\alpha}{2}-1}(-t)^{\frac{\alpha}{2}-1} = (z^{-1})^{\frac{\alpha}{2}-1}e^{i\pi(\frac{\alpha}{2}-1)}(t)^{\frac{\alpha}{2}-1}.$$

We thus have proved that

$$i \int_0^\infty (it)^{\frac{\alpha}{2}-1}e^{itz}\exp\{-(-z)^{\frac{\alpha}{2}}(it)^{\frac{\alpha}{2}}y\}dt = (-z^{-1})^{\frac{\alpha}{2}}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}e^{-yt^{\frac{\alpha}{2}}}dt$$

which proves the claim. $\square$

By Lemma 6.2 we remark that, if X_z is a solution of the equation (36) and if $z = |z|e^{i\theta}$,

$$X_z = -e^{-i\theta}C(\alpha)\int_0^\infty (-e^{-i\theta}t)^{\frac{\alpha}{2}-1}e^{-t|z|}\exp\{-c(\alpha)(e^{-i\theta}t)^{\frac{\alpha}{2}}X_z\}dt \tag{37}$$

$$= -\frac{1}{z}C(\alpha)\int_0^\infty (-\frac{t}{z})^{\frac{\alpha}{2}-1}e^{-t}\exp\{-c(\alpha)(-\frac{t}{z})^{\frac{\alpha}{2}}X_z\}dt$$

$$= (-\frac{1}{z})^{\frac{\alpha}{2}}C(\alpha)\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}\exp\{-c(\alpha)t^{\frac{\alpha}{2}}(-\frac{1}{z})^{\frac{\alpha}{2}}X_z\}dt. \tag{38}$$

Hence, if $Y_z := (-\frac{1}{z})^{\frac{\alpha}{2}}X_z$, we obtain

$$(-z)^\alpha Y_z = C(\alpha)\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}\exp\{-c(\alpha)t^{\frac{\alpha}{2}}Y_z\}dt. \tag{39}$$

This equation for Y_z can be written simply as

$$(-z)^\alpha Y_z = C(\alpha)g(c(\alpha)Y_z).$$

We recall that we have assumed that there exists a constant C_1 such that $|X_z| \le C_1\Im(z)^{-\frac{\alpha}{2}}$.

Now, consider the function of two complex variables $F(u,y) = ug_\alpha(y) - y$. Obviously $F(0,0) = 0$ and $\partial_y F(0,0) = -1$. By the local implicit function theorem, there exists $\epsilon_1 > 0$ and $\epsilon_2 > 0$, such that for every $u \in \mathbb{C}$ with $|u| < \epsilon_1$ there exists a unique $y(u) \in \mathbb{C}$ with $|y(u)| < \epsilon_2$ satisfying the equation $F(u,y(u)) = 0$, i.e $ug_\alpha(y(u)) = y(u)$. Moreover

$$|y(u)| \le C|u|. \tag{40}$$

For any $z \in \mathbb{C}^+$, such that $\Im(z) > L$, with $L^\alpha > \frac{1}{C(\alpha)\epsilon_1} \vee \frac{c(\alpha)C_1}{\epsilon_2}$, then $|X_z| \le C_1 L^{-\frac{\alpha}{2}}$ so that $|Y_z| \le \frac{C_1}{L^\alpha} \le \frac{\epsilon_2}{c(\alpha)}$. Thus for $z \in \mathbb{C}^+$, such that $\Im(z) > L$ we have that

$$\left|\frac{1}{C(\alpha)(-z)^\alpha}\right| \le \epsilon_1, \quad |c(\alpha)Y_z| \le \epsilon_2$$

Thus the uniqueness in the local implicit function theorem shows that Y_z is given by $Y_z = \frac{1}{c(\alpha)}y(\frac{1}{C(\alpha)(-z)^\alpha})$ and thus that $X_z = \frac{1}{(-\frac{1}{z})^{\frac{\alpha}{2}}}Y_z$. Since X_z is analytic on $z \in \mathbb{C}^+$ and uniquely determined on the set of $z \in \mathbb{C}^+$ such that $\Im(z) > L$ it is uniquely determined. This proves the claim of uniqueness for X_z. Using the bound (40) now proves the improved bound at infinity, i.e $|X_z| = O(|z|^{-\frac{\alpha}{2}})$. These arguments prove the second and third statements of the theorem. □

We can now deduce from this last uniqueness result the convergence of the mean of the normalized trace of the resolvent.

Theorem 6.3. *For any $\kappa \in]0, \frac{1}{2(2-\alpha)}[$, any $z \in \mathbb{C}^+$, $\mathbb{E}[\frac{1}{N}\sum_{k=1}^N G_N^\kappa(z)_{kk}]$ converges as N goes to infinity to*

$$G_\alpha(z) := i \int_0^\infty e^{itz}e^{-c(\alpha)(it)^{\frac{\alpha}{2}}X_z}dt = -\frac{1}{z}\int_0^\infty e^{-t}e^{-c(\alpha)t^{\frac{\alpha}{2}}Y_z}dt \tag{41}$$

Proof. For any $z \in \mathbb{C}^+$ and any limit point μ^z ,

$$\begin{aligned}
\int x d\mu^z(x) &= \int \frac{1}{z-x}dP^{\mu^z}(x) \\
&= i\int_0^\infty \int e^{it(z-x)}dP^{\mu^z}(x)dt \\
&= i\int_0^\infty e^{itz}e^{-c(\alpha)(it)^{\frac{\alpha}{2}}X_z}dt
\end{aligned}$$

The uniqueness of X_z implies that the mean of the resolvent $\mathbb{E}[N^{-1}\text{tr}(z - A_N^\kappa)^{-1}]$ has a unique limit point which is given by

$$G_\alpha(z) = i\int_0^\infty e^{itz}e^{-c(\alpha)(it)^{\frac{\alpha}{2}}X_z}dt$$

This shows that $\mathbb{E}[N^{-1}\text{tr}(z - A_N^\kappa)^{-1}]$ converges to $G_\alpha(z)$. In order to finish the proof, observe that for $z \in \mathbb{C}^+$, we can use the same arguments than in the proof of Lemma 6.2 to see that

$$\begin{aligned}
G_\alpha(z) &= i\int_0^\infty e^{itz}e^{-c(it)^{\frac{\alpha}{2}}X_z}dt \\
&= -\frac{1}{z}\int_0^\infty e^{-t}e^{-c(-tz^{-1})^{\frac{\alpha}{2}}X_z}dt \\
&= -\frac{1}{z}\int_0^\infty e^{-t}e^{-c(t)^{\frac{\alpha}{2}}Y_z}dt \tag{42}
\end{aligned}$$

□

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This last result enables us to conclude the proof of Theorem 1.1 and Theorem 1.4.

Proof of Theorem 1.1 and Theorem 1.4

By Lemma 3.1, $\mathbb{E}[\hat{\mu}_{A_N^\gamma}]$ is tight for the weak topology. Taking any subsequence, we see that any limit point μ is such that its Stieltjes transform must be equal to $G_\alpha(z)$ for all $z \in \mathbb{C}^+$. This prescribes uniquely the limit point μ and thus insures the convergence of $\mathbb{E}[\hat{\mu}_{A_N^\gamma}]$ towards $\mu \in \mathcal{P}(\mathbb{R})$ so that

$$\int (z-x)^{-1} d\mu(x) = G_\alpha(z), z \in \mathbb{C}^+.$$

By Corollary 2.2, and the fact that

$$d_1(\mathbb{E}[\hat{\mu}_{A_N^\kappa}], \mathbb{E}[\hat{\mu}_{A_N}]) \leq \mathbb{E}[d_1(\hat{\mu}_{A_N^\kappa}, \hat{\mu}_{A_N})]$$

we also conclude that $\mathbb{E}[\hat{\mu}_{A_N}]$ converges weakly towards μ. By Lemma 5.4, for any $z \in \mathbb{C}\backslash\mathbb{R}$, $L_N^{z,\kappa}(x) = \int (z-x)^{-1} d\hat{\mu}_{A_N^\kappa}(x)$ converges in probability towards $G_\alpha(z)$. This convergence holds as well for finite dimensional vectors $(\int (z_i - x)^{-1} d\hat{\mu}_{A_N^\kappa}(x), 1 \leq i \leq n)$. Since $\{(z-x)^{-1}, z \in \mathbb{C}\backslash\mathbb{R}\}$ is dense in the set $\mathcal{C}_0(\mathbb{R})$ of functions on $\mathbb{R}$ going to zero at infinity, we conclude that $\int f(x) d\hat{\mu}_{A_N^\kappa}(x)$ converges in probability towards $\int f(x) d\mu(x)$ for all $f \in \mathcal{C}_0(\mathbb{R})$. But also $\hat{\mu}_{A_N^\kappa}(1) = \mu(1) = 1$ and so this vague convergence can be strengthened in a weak convergence (see the proof of Lemma 2.1). We finally can remove the truncation by κ by using Corollary 2.2. Again by Lemma 5.4, $L_N^{z,\kappa}(x) = \int (z-x)^{-1} d\hat{\mu}_{A_N}(x)$ converges almost surely along subsequences $\phi(N)$ so that $\sum \phi(N)^{-\epsilon} < \infty$ by Borel-Cantelli Lemma. As $\epsilon = \frac{2}{\alpha} - \frac{2-\alpha}{\kappa}$ is as close to one as wished, for any sequence $\phi(N)$ so that $\sum \phi(N)^{-\varepsilon} < \infty$ for some $\varepsilon < 1$, we can choose κ close enough to one so that $L_{\phi(N)}^{z,\kappa}(x)$ converges almost surely to $G(z)$. This entails the almost sure weak convergence of $\hat{\mu}_{A_{\phi(N)}}^{\phi(N)}$ by the same arguments as above. $\qquad\square$

Remark 6.4. *If we could prove that the equation given in Theorem 5.2 admits a unique solution μ^z, at least for z in a set large enough, the convergence of $\mathbb{E}[L_N^{z,\kappa}]$ to this solution would be assured. We cannot prove this uniqueness result. But as we have seen we do not really need such a strong uniqueness statement either. We rather have proved a weaker statement, i.e the uniqueness of $\int x d\mu^z(x)$, which already entails the uniqueness of the limit points for $\mathbb{E}[\int x dL_N^{z,\kappa}(x)]$, i.e the mean Stieltjes transform of the spectral measure of A_N^κ. This is sufficient for our needs but the question of the uniqueness of solutions to the equation given in Theorem (5.2) remains intriguing.*

7   Study of the limiting measure. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. First, the fact that the limit measure μ_α is symmetric is obvious. It suffices to consider the case where the entries have symmetric distributions. To prove the other statements, we need to consider the limit of $G_\alpha(z)$ as z tends to a positive real number x. We first remark that the analytic function Y_z defined on $\mathbb{C}^+$ is univalent (i.e one-to-one). Indeed this is an obvious consequence of the equation, valid for $z \in \mathbb{C}^+$:

$$(-z)^\alpha Y_z = C(\alpha) g_\alpha(c(\alpha) Y_z).$$

In order to study the boundary behavior of $G_\alpha(z)$, we thus have to study the boundary behavior of the univalent function Y_z. For $x \in \mathbb{R}$, the cluster set $Cl(x)$ is defined as the set of limit points of Y_z when z tends to x (see [5] or [10]). It is easy to see that for any non zero $x \in \mathbb{R}$ the cluster set

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$Cl(x)$ is reduced to one point in $\mathbb{C} \cup \{\infty\}$. Indeed, assuming w.l.o.g that $x > 0$ we have, for any finite $v \in Cl(x)$, the equality $g_\alpha(v) = (-x)^\alpha v$. If $Cl(x)$ contains two points it is a continuum, i.e a compact connected set with more that one point (see [5]). By analytic continuation we would then get the equality $g_\alpha(v) = (-x)^\alpha v$ for every $v \in \mathbb{C}$ which is false. The only remaining possibility for $Cl(x)$ is to be reduced to one finite point or to the point at infinity. We define

$$K'_\alpha = \{x \in \mathbb{R}, Cl(x) = \{\infty\}\}$$

We first prove that K'_α is bounded. The proof of Theorem 6.1, using the local implicit function theorem at infinity, shows that Y_z admits and analytic extension to the set$\{z \in \mathbb{C}, |z| > L\}$ for L large enough, and that this extension satisfies $|Y_z| = O(|z|^{-\alpha})$. This obviously proves that, when $|x| > L$, the cluster set $Cl(x)$ is reduced to one finite point and thus that K'_α is bounded.

We consider the complement U'_α of K'_α. Let $x \in U'_\alpha$ and Y_x the unique point in the cluster set $Cl(x)$. By continuity, for $x > 0$, Y_x satisfies the equation

$$e^{i\pi\alpha} x^\alpha Y_x = C(\alpha) g(c(\alpha) Y_x).$$

The local implicit function theorem can be applied to this equation at (x, Y_x), except for the subset say F of $\mathbb{R}$ where the derivative vanishes. The exceptional set F must be bounded, since the derivative does not vanish at infinity, and its points must all be isolated. Thus F is finite. For any $x \in U_\alpha \backslash F$, the implicit function theorem shows that Y_z can be extended analytically on a complex neighborhood of x. Hence $U_\alpha := U'_\alpha \backslash F$ is open and its complement $K_\alpha = K'_\alpha \cup F \cup \{0\}$ is closed. K_α is also bounded and thus compact.

Finally we use Beurling's Theorem which states that the set K'_α has capacity zero, and thus also the set K_α (see [5] or [10]).

For any point x in the open set U_α the function Y_z admits an analytic extension to a complex neighborhood of x, and thus the Stieltjes transform $G_\alpha(z)$ admits a smooth extension, which proves that μ_α has a smooth density ρ_α on the open set U_α. Indeed, for $x \in U_\alpha$

$$\lim_{z \to x} G_\alpha(z) = H\mu_\alpha(x) - i\pi\rho_\alpha(x) = -\frac{1}{x} \int_0^\infty e^{-t} e^{-c(\alpha)t^{\frac{\alpha}{2}} Y_x} dt$$

In particular the density of the measure μ_α is given, if $Y_x = r_x e^{i\phi_x}$, by

$$\rho_\alpha(x) = \frac{1}{\pi x} \int_0^\infty e^{-t} e^{-c(\alpha)t^{\frac{\alpha}{2}} [r_x \cos(\phi_x)]} \sin[c(\alpha)t^{\frac{\alpha}{2}} r_x \sin(\phi_x)] dt. \tag{43}$$

Note that we now know that Y_x is well defined and smooth for x large enough. We also have seen that $Y_x = O(|x|^{-\alpha})$ and thus that $Y_x \sim e^{-i\pi\alpha} C(\alpha) g_\alpha(0) x^{-\alpha}$. Hence, when $x \to \infty$, the following asymptotic behavior holds for $G_\alpha(x) = \lim_{z \to x} G_\alpha(z)$:

$$G_\alpha(x) \sim \frac{1}{x} \int_0^\infty e^{-t}(1 - ct^{\frac{\alpha}{2}} Y_x(1 + o(1))) dt \approx \frac{1}{x}(1 - c \int_0^\infty e^{-t} t^{\frac{\alpha}{2}} dt Y_x(1 + o(1)))$$

Identifying the imaginary parts of both sides we get:

$$\rho_\alpha(x) \sim \pi^{-1} c\Gamma(\alpha) \frac{\Im(Y_x)}{x}.$$

Which proves the last statement of Theorem 1.5.

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8 Cizeau and Bouchaud's characterization

In [3], the authors propose the following argument; they look at $G_N(z)_{00}$ for z *on the real line.* By arguments similar to those we used (but with no a priori bounds on the $G_N(z)_{kk}$) they argue that $G_N(z)_{00}$ converges in law as N goes to infinity. The limit law, that we will denote P_G to follow their notations (but which is μ^z in ours) is then given by the implicit equation (11) in [3]

$$\int f(y)dP_G(y) = \int f(\frac{1}{z-y})dP_S(y) = \int \frac{1}{y^2}f(y)dP_S(z-\frac{1}{y}).$$

$P_S = L_\alpha^{C(z),\beta(z)}$ is now a *real-valued* stable law with parameters $C(z)$ and $\beta(z)$ given self-consistently (see (12a) and (12b) in [3]) by

$$C(z) = \int |y|^{\frac{\alpha}{2}}dP_G(y) = \int |y|^{\frac{\alpha}{2}-2}dP_S(x-\frac{1}{y})$$

$$\beta(z) = \int |y|^{\frac{\alpha}{2}}\operatorname{sign}(y)dP_G(y)$$

where there was a typographical error in the definition of β in [3]. 12b which was already noticed in [4]. We in fact have that for any real t,

$$\int e^{-ity}dP_S(y) = e^{-C_\alpha^{-1}t^{\frac{\alpha}{2}}(C(z)-i\tan(\frac{\pi\alpha}{4})\beta(z))}$$

$$= e^{-\Gamma(\alpha-1)(it)^{\frac{\alpha}{2}}\int (x)^{\frac{\alpha}{2}}dP_G(x)} \tag{44}$$

where we used that $K_z := \int (x)^{\frac{\alpha}{2}}dP_G(x) = e^{-\frac{i\pi\alpha}{4}}[\cos(\frac{\pi\alpha}{4})C(z)-i\sin(\frac{\pi\alpha}{4})\beta(z)]$. So, we see that the description of the limit law is very similar to ours, except that z is supposed to belong to $\mathbb{R}$. Let us assume (as seems to be the case in [3]) that $C(z)$ and $\beta(z)$ are finite. Then, also K_z is finite and we see that for non negative real z's

$$K_z = \int (z-y)^{-\frac{\alpha}{2}}dP_S(y)$$

$$= -C(\alpha)\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-tz}e^{-\Gamma(\alpha-1)(it)^{\frac{\alpha}{2}}K_z}dt. \tag{45}$$

Hence, K_z and the X_z introduced in section 7 satisfy formally the same equation, except that X_z satisfies it for $z \in \mathbb{C}^+$ and K_z for real z's. Moreover, we have seen that X_z can be extended continuously to z real in $(K_\alpha')^c$ and then this extension X_z satisfies the same equation that K_z. This indicates that we expect K_z and X_z to be equal, at least on $(K_\alpha')^c$. In fact, X_z is the unique solution of this equation with an analytic extension to $\mathbb{C}^+$ and going to zero at infinity. In [3], under (12a-12b), it is claimed that the equations defining $C(z), \beta(z)$ have a unique solution, and so that K_z is also determined uniquely by (45). We could not prove the uniqueness of the solutions to this equation on the real line. In any case, if we beleive either that K_z extends analytically on $\mathbb{C}^+$ and goes to zero at infinity or that the above equation has a unique solution for $z \in \mathbb{R}$, we must have $X_z = K_z$ at least for $z \in (K_\alpha')^c$.

The second claim of [3] is that the density of the limiting spectral measure $\rho(z)dz = d\mu(z)$ is given, see [3] (14), by

$$\rho(z) = \frac{dP_S}{dz}(z).$$

Note that by Fourier inversion, if $K_z = X_z$, for $z > 0$, since P_S is a probability measure on $\mathbb{R}$ with Fourier transform given by (44),

$$
\begin{aligned}
\frac{dP_S}{dz}(z) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz}e^{-\Gamma(\alpha-1)(it)^{\frac{\alpha}{2}}X_z}dt \\
&= \frac{1}{\pi}\Im\left(\int_0^\infty e^{-itz}e^{-\Gamma(\alpha-1)(it)^{\frac{\alpha}{2}}X_z}dt\right) \\
&= -\frac{1}{\pi}\Im\left(\frac{1}{z}\int_0^\infty e^{-t}e^{-\gamma(\alpha-1)(t)^{\frac{\alpha}{2}}Y_z}dt\right)
\end{aligned}
$$

and therefore we miraculously recover our result (43). Hence, at least for $z \in (K'_\alpha)^c$, the prediction of [3] coincides with our result if we beleive that (45) has a unique solution.

9 The moment method. Proof of Theorem 1.7

We prove here Theorem 1.7 using the moment method developed by I. Zakharevich [15]. For any $B > 0$, we consider the matrix X_N^B with truncated entries $x_{ij}^B = x_{ij}1_{|x_{ij}|\leq Ba_N}$ and the normalized matrix $A_N^B = a_N^{-1}X_N^B$. Recall that work here under the additional hypothesis (7):

$$\lim_{u\to\infty}\frac{P(x > u)}{P(|x| > u)} = \theta \in [0,1]$$

We begin by the following estimate on moments of the entries of A_N^B.

Lemma 9.1. *For any integer $m \geq 1$, the following limit exists*

$$C_m = \lim_{N\to\infty}\frac{\mathbb{E}[A_N^B(ij)^m]}{N^{\frac{m}{2}-1}\mathbb{E}[A_N^B(ij)^2]^{\frac{m}{2}}}$$

Moreover, if $m = 2k$ is even

$$C_m = \frac{2-\alpha}{m-\alpha}\left(\frac{2-\alpha}{\alpha}B^\alpha\right)^{\frac{m}{2}-1}$$

If $m = 2k-1$ is odd

$$C_m = (2\theta-1)\frac{2-\alpha}{m-\alpha}\left(\frac{2-\alpha}{\alpha}B^\alpha\right)^{\frac{m}{2}-1}$$

Proof. It is a simple application of the classical result about truncated moments (Theorem VIII.9.2 of [6]) already used in Section 3.1, (15) : For any $\zeta \geq \alpha$

$$E[|x(ij)|^\zeta 1_{|x(ij)|<Ba_N}] \sim \frac{\alpha}{\zeta-\alpha}B^{\zeta-\alpha}\frac{a_N^\zeta}{N}$$

The first item of the Lemma is a direct consequence of this estimate for $\zeta = 2$ and $\zeta = 2k$. The second is also a consequence of this estimate, used for $x(ij)^+$ and $x(ij)^-$, and of the additional skewness hypothesis (7).

$\square$

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This lemma enables us to get the main result of this section, i.e the convergence of the moments of the spectral measure of the matrix A_N^B. We will need some more notations that we take verbatim from Zakharevich. For any integer $k \geq 1$, we define V_k as the set of all $(e_1, ..., e_l)$ such that $\sum_{i=1}^{l} e_i = k$ and $e_1 \geq e_2 \leq ... \geq e_l > 0$. For any $(e_1, ..., e_l) \in V_k$ define $T(e_1, ..., e_l)$ as the number of colored rooted trees with $k + 1$ vertices and $l + 1$ distinct colors, say $(c_1, ...c_l)$ satisfying the following conditions:

1. There are exactly e_i nodes of color c_i. The root node is the only node colored c_0

2. If nodes a and b are the same color then the distance from a to the root is the same as the distance from b to the root

3. If nodes a and b have the same color then their parents also have the same color

With these notations we have the following convergence result, directly implied by Zakharevich's results.

Lemma 9.2.   *1. For every integer $k \geq 1$, the following limit exists*

$$\lim_{N \to \infty} \mathbb{E}[\int x^k d\hat{\mu}_{A_N^B}(x)] =: m_k^B \tag{46}$$

2. $m_k^B = 0$ if k is odd, and $m_{2k}^B = \sum_{(e_1, ..., e_l) \in V_k} T(e_1, ..., e_l) \prod_{i=1}^{l} C_{2e_i}$.

3. There exists a probability measure μ_α^B uniquely determined by its moments m_k^B. μ_α^B is independent of the skewness parameter θ.

4. μ_α^B has unbounded support and is symmetric.

5. The mean spectral measure $E[\hat{\mu}_{A_N^B}]$ converges weakly to μ_α^B.

Proof. In order to prove the first and second items, it is enough to use the preceding Lemma, Corollary 6 and Theorem 2 in [15], plus the fact that

$$\lim_{N \to \infty} N\mathbb{E}[A_N^B(ij)^2] = \frac{\alpha}{2 - \alpha} B^{2-\alpha}$$

. The third item is a consequence of the estimate

$$C(m) \leq C\rho^m$$

with $\rho = (\frac{2-\alpha}{\alpha} B^\alpha)^{\frac{1}{2}}$ and of Proposition 10 in [15]. The fact that μ_α^B is independent of the skewness parameter θ is obvious since its moments only depend on the C_m for even m's, which are insensitive to the parameter θ. The fourth item is a consequence of Proposition 9 and Proposition 12 of [15]. The fifth one is a consequence of Theorem 1 of [15]. $\square$

This lemma proves the first part of Theorem 1.7. In order to prove the second part we simply remark that we have already done so, since we have seen, in the proof of Lemma 3.1, that μ_α^B converges and that its limit is the weak limit of $\mathbb{E}[\hat{\mu}_{A_N}]$.

10 Appendix: Convergence to stable distributions for triangular arrays

We begin here by recalling the notations for stable distributions, see for instance [11]. A real random variable Y has a stable distribution with exponent $\alpha \in (0,2)$, $\alpha \neq 1$, scale parameter $\sigma > 0$, skewness parameter $\beta \in [-1,1]$, and shift parameter $\mu \in \mathbb{R}$ (in short $Stable_\alpha(\sigma, \beta, \mu)$) iff its characteristic function is given by:

$$E[\exp(itY)] = \exp\left[-\sigma^\alpha |t|^\alpha (1 - i\beta sign(t)\tan(\frac{\pi\alpha}{2})) + i\mu t\right]$$

We will consider here only the case where $\alpha < 1$.

A complex random variable Y has an α-stable distribution with spectral representation (Γ, μ) if Γ is a finite measure on the unit circle S^1, and μ is a complex number such that the characteristic function of Y is given by:

$$E[\exp(i\langle t, Y\rangle)] = \exp\left[-\int_{S^1} |\langle t,s\rangle|^\alpha (1 - isign(\langle t,s\rangle)\tan(\frac{\pi\alpha}{2}))\Gamma(ds) + i\langle\mu, t\rangle\right]$$

We will need the constant

$$C_\alpha^{-1} = \int_0^\infty \frac{\sin x}{x^\alpha}dx = \frac{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})}{1-\alpha}$$

Throughout this section, we consider a sequence of i.i.d non negative random variables $(X_k)_{k\geq 1}$ and assume that their common distribution is in the domain of attraction of an α-stable distribution, with $\alpha \in (0,1)$, i.e that the tail is regularly varying:

$$P[X \geq u] = \frac{L(u)}{u^\alpha}$$

We introduce the normalizing constant a_N by:

$$a_N = \inf(u, P[X \geq u] \leq \frac{1}{N}) \tag{47}$$

We consider a triangular array of real or complex numbers $(G_{N,k}, 1 \leq k \leq N)$ and give sufficient conditions for the normalized sum:

$$S_N = \frac{1}{a_N}\sum_{k=1}^N G_{N,k}X_k$$

to converge in distribution to a (real or complex) stable distribution. We will always assume that the triangular array is bounded, i.e that

$$M := \sup(|G_{N,k}|, N \geq 1, 1 \leq k \leq N) < \infty$$

We begin with the case where the numbers $G_{N,k}$ are real.

Theorem 10.1. *Assume that the triangular array of real numbers $(G_{N,k}, N \geq 1, 1 \leq k \leq N)$ is bounded. Furthermore assume that the empirical measure*

$$\nu_N = \frac{1}{N} \sum_{k=1}^{N} \delta_{G_{N,k}}$$

converges weakly to a probability measure ν on the real line. Then the distribution of the normalized sum $S_N = \frac{1}{a_N} \sum_{k=1}^{N} G_{N,k} X_k$ converges to a $Stable_\alpha(\sigma, \beta, 0)$ distribution, with

$$\sigma^\alpha = \frac{1}{C_\alpha} \int |x|^\alpha d\nu(x),$$

$$\beta = \frac{\int |x|^\alpha sign(x) \nu(dx)}{\int |x|^\alpha \nu(dx)}$$

If $\sigma^\alpha = 0$, i.e if $\nu = \delta_0$, the above statement should of course be understood as: $S_N = \frac{1}{a_N} \sum_{k=1}^{N} G_{N,k} X_k$ converges in distribution to zero.

Proof of theorem 10.1. We begin with the particular case where the numbers $G_{N,k}$ are positive and bonded below. We assume that there exists an $\delta > 0$ such that for any $N \geq 1$ and $1 \leq k \leq N$

$$\delta \leq G_{N,k} \leq M. \tag{48}$$

In this context we will be able to apply directly classical theorems to the array of non negative independent random variables

$$U_{N,k} = \frac{1}{a_N} G_{N,k} X_k$$

For instance, we could apply the theorem in section XVII.7 of [6]. We rather choose to apply Theorem 8, chapter 5 of [7]. According to this last result, Theorem 10.1 will be proved in this restricted case if we can check the following three conditions. First the Uniform Asymptotic Negligibility (UAN) condition, for every $\epsilon > 0$

$$\lim_{N \to \infty} \max_{1 \leq k \leq N} P(U_{N,k} > \epsilon) = 0. \tag{49}$$

Second we must check that:

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \sum_{1 \leq k \leq N} Var[U_{N,k} 1_{(U_{N,k} < \epsilon)}] = 0, \tag{50}$$

and finally we must check that, for $x > 0$

$$\lim_{N \to \infty} P(\max_{1 \leq k \leq N} U_{N,k} \leq x) = \exp(-\frac{C_\alpha \sigma^\alpha}{x^\alpha}), \tag{51}$$

and that

$$\lim_{N \to \infty} P(\min_{1 \leq k \leq N} U_{N,k} \leq x) = 1. \tag{52}$$

We first note that

$$P(U_{N,k} > \epsilon) = P(X_k > \epsilon \frac{a_N}{G_{N,k}}) \leq \frac{L(\frac{\epsilon a_N}{G_{N,k}})}{(\frac{\epsilon a_N}{G_{N,k}})^\alpha}$$

which shows that (49) is thus a direct consequence of our assumption (48) and of the following lemma.

Lemma 10.2. *Let L be a slowly varying function and define a_N as in (47):*

$$a_N = \inf(u, P[|X| \geq u] \leq \frac{1}{N}) \tag{53}$$

Then , for any $0 < a < b$ and any $a < y < b$

$$\frac{L(ya_N)}{(ya_N)^\alpha} = \frac{1}{N}\frac{1}{y^\alpha}(1 + \epsilon(x, N)) \tag{54}$$

with

$$\lim_{N \to \infty} \sup_{a < y < b} \epsilon(x, N) = 0 \tag{55}$$

Proof of Lemma 10.2. Writing

$$\frac{L(ya_N)}{(ya_N)^\alpha} = \frac{L(ya_N)}{L(a_N)}\frac{NL(a_N)}{a_N^\alpha}\frac{1}{Ny^\alpha} \tag{56}$$

this lemma is clearly a direct consequence of the classical fact:

$$\lim_{N \to \infty} \frac{NL(a_N)}{(a_N)^\alpha} = 1 \tag{57}$$

and of the uniform convergence theorem for slowly varying functions ([2], Theorem 1.2.1), which asserts that the convergence

$$\lim_{t \to \infty} \frac{L(tx)}{L(t)} = 1 \tag{58}$$

is uniform for x's in a compact subset of $(0, \infty)$. $\square$

Next, in order to control the variance $Var[U_{N,k}1_{(U_{N,k} < \epsilon)}]$ and prove the validity of (50), we must use Karamata's theorem, or more directly Theorem VIII.9.2 of [6]which shows that

$$\lim_{t \to \infty} \frac{t^{\zeta - \alpha}L(t)}{E[X^\zeta 1_{X < t}]} = \frac{\zeta - \alpha}{\alpha}. \tag{59}$$

Using this for $\zeta = 1, 2$, we see that

$$Var[U_{N,k}1_{(U_{N,k} < \epsilon)}] \backsim \frac{\alpha}{2 - \alpha}\epsilon^2[\frac{L(\frac{\epsilon a_N}{G_{N,k}})}{(\frac{\epsilon a_N}{G_{N,k}})^\alpha}]. \tag{60}$$

Lemma (10.2) then shows that $Var[U_{N,k}1_{(U_{N,k} < \epsilon)}]$ is of order $\frac{\epsilon^2}{N}$, and thus that

$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \sum_{1 \leq k \leq N} Var[U_{N,k}1_{(U_{N,k} < \epsilon)}] = 0 \tag{61}$$

In order to complete the proof of Theorem 10.1 in the particular case where the numbers $G_{N,k}$ are positive and bounded below, we now only have to check (51) since (52) is obvious, the variables $u_{N,k}$ being non negative. For $x > 0$

$$\log P(\max_{1 \leq k \leq N} U_{N,k} \leq x) = \sum_1^N \log[1 - \frac{L(\frac{xa_N}{G_{N,k}})}{(\frac{xa_N}{G_{N,k}})^\alpha}].$$

Using again Lemma(10.2) we see that

$$\lim_{N\to\infty} \log P(\max_{1\le k\le N} U_{N,k} \le x) = -\frac{C_\alpha \sigma^\alpha}{x^\alpha}$$

since

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} G_{N,k}^\alpha = \int |x|^\alpha \nu(dx) = C_\alpha \sigma^\alpha.$$

This checks the condition (51) and finishes the proof in the particular case where the numbers $G_{N,k}$ are positive and bounded below. Now it is easy to prove theorem 10.1 in full generality. It is enough to split the sum into the three independent summands

$$S_N = \frac{1}{a_N} \sum_{k=1}^{N} G_{N,k} X_k = \sum_{k=1}^{N} U_{N,k} = S_N^{+,\epsilon} - S_N^{-,\epsilon} + S_{N,\epsilon}$$

with

$$S_N^{+,\epsilon} = \sum_{k=1}^{N} U_{N,k} 1_{\epsilon < G_{N,k}}$$

$$S_N^{-,\epsilon} = -\sum_{k=1}^{N} U_{N,k} 1_{G_{N,k} < -\epsilon}$$

$$S_{N,\epsilon} = \sum_{k=1}^{N} U_{N,k} 1_{|G_{N,k}| \le \epsilon}$$

We now know that, if ϵ and $-\epsilon$ are not atoms of ν, then $S_N^{+,\epsilon}$ (resp $S_N^{-,\epsilon}$) converges in distribution to a $Stable_\alpha(\sigma_{\alpha,\epsilon}^+, 1, 0)$ (resp $Stable_\alpha(\sigma_{\alpha,\epsilon}^-, 1, 0)$) with

$$C_\alpha \sigma_{\alpha,\epsilon}^+ = \int_{\epsilon}^{\infty} |x|^\alpha \nu(dx)$$

$$C_\alpha \sigma_{\alpha,\epsilon}^- = \int_{-\infty}^{-\epsilon} |x|^\alpha \nu(dx)$$

So that the sum $S_N^{+,\epsilon} + S_N^{-,\epsilon}$ converges in distribution, when N tends to ∞ to a $Stable_\alpha(\sigma_{\alpha,\epsilon}, \beta_{\alpha,\epsilon}, 0)$ with

$$C_\alpha \sigma_{\alpha,\epsilon} = \int_{|x|>\epsilon} |x|^\alpha \nu(dx)$$

$$\beta_{\alpha,\epsilon} = \frac{\int_{|x|>\epsilon} |x|^\alpha sign(x) \nu(dx)}{\int_{|x|>\epsilon} |x|^\alpha \nu(dx)}$$

It is clear that, since $\lim_{\epsilon\to 0} \sigma_{\alpha,\epsilon} = \sigma_\alpha$ and that $\lim_{\epsilon\to 0} \beta_{\alpha,\epsilon} = \beta_\alpha$, the distribution $Stable_\alpha(\sigma_{\alpha,\epsilon}, \beta_{\alpha,\epsilon}, 0)$ converge to $Stable_\alpha(\sigma_\alpha, \beta_\alpha, 0)$, when ϵ tends to zero. Thus there exists a sequence ϵ_N tending to zero such that the sum $S_N^{+,\epsilon_N} + S_N^{-,\epsilon_N}$ converges in distribution to a $Stable_\alpha(\sigma_\alpha, \beta_\alpha, 0)$ variable.

But S_{N,ϵ_N} converges to zero in probability when $N \to \infty$. Indeed, for any $x > 0$,

$$P(|S_{N,\epsilon_N}| > x) \leq P(\frac{1}{a_N} \sum_{k=1}^{N} X_k > \frac{x}{\epsilon_N})$$

so that

$$\lim_{N \to \infty} P(|S_{N,\epsilon_N}| > x) = 0 \qquad (62)$$

These two facts show that $S_N = \frac{1}{a_N} \sum_{k=1}^{N} G_{N,k} X_k$ converge in distribution to a $Stable_\alpha(\sigma_\alpha, \beta_\alpha, 0)$ variable as announced in Theorem 10.1. □

This result implies easily the following analogous result in the complex case.

Theorem 10.3. *Assume that the triangular array of complex numbers $(G_{N,k}, N \geq 1, 1 \leq k \leq N)$ is bounded. Furthermore assume that the empirical measure*

$$\nu_N = \frac{1}{N} \sum_{k=1}^{N} \delta_{G_{N,k}}$$

converges weakly to a probability measure ν on the complex plane. Then $S_N = \frac{1}{a_N} \sum_{k=1}^{N} G_{N,k} X_k$ converges in distribution to a complex stable distribution with spectral representation $(\Gamma_\nu, 0)$ where Γ_ν is the measure on S^1 obtained as the image of the measure $\frac{1}{C_\alpha} |z|^\alpha \nu(dz)$ on the complex plane by the map $z \to \frac{z}{|z|}$. Again if $\nu = \delta_0$ the above statement should be understood as: S_N converges in distribution to zero.

Proof. For any fixed $t \in \mathbb{C}$, a direct application of Theorem 10.1 to the array of real numbers $(\langle t, G_{N,k} \rangle)$ shows that $\langle t, S_N \rangle$ converges in distribution to a $Stable_\alpha(\sigma(t), \beta(t), 0)$ variable, where

$$\sigma(t)^\alpha = \frac{1}{C_\alpha} \int |\langle t, z \rangle|^\alpha d\nu(z)$$

and

$$\beta(t) = \frac{\int |\langle t, z \rangle|^\alpha sign \langle t, z \rangle d\nu(z)}{\int |\langle t, z \rangle|^\alpha d\nu(z)}.$$

As a consequence, we obtain that

$$\lim_{N \to \infty} E[\exp(i \langle t, S_N \rangle)] = \exp\left[\sigma(t)^\alpha (1 - i\beta(t) \tan(\frac{\pi\alpha}{2}))\right]$$

Note that, by definition of Γ_ν:

$$\sigma(t)^\alpha (1 - i\beta(t) \tan(\frac{\pi\alpha}{2})) = \int_{S^1} |\langle t, s \rangle|^\alpha (1 - isign(\langle t, s \rangle) \tan(\frac{\pi\alpha}{2})) \Gamma_\nu(ds)$$

These two last facts prove that the distribution of S_N converges to a complex α-stable distribution with spectral representation $(\Gamma_\nu, 0)$.

□

In Section 5 we need a slight variation of Theorem 10.3. We want to extend it to the case where the random variables X_k are truncated at a high enough level. More precisely, keeping the notations and hypothesis of Theorem 10.3, we define, for any $\delta > 0$, the truncated variables

$$X_k^\delta = X_k 1_{X_k \leq N^\delta a_N}$$

We then consider the normalized sum

$$S_N^\delta = \frac{1}{a_N} \sum_{k=1}^N G_{N,k} X_k^\delta$$

Theorem 10.4. *Assume that the triangular array of complex numbers $(G_{N,k}, N \geq 1, 1 \leq k \leq N)$ is bounded. Furthermore assume that the empirical measure*

$$\nu_N = \frac{1}{N} \sum_{k=1}^N \delta_{G_{N,k}} \tag{63}$$

converges weakly to a probability measure ν on the complex plane . Then S_N^δ converges in distribution to a complex stable distribution with spectral representation $(\Gamma_\nu, 0)$ where Γ_ν is the measure on S^1 obtained as the image of the measure $\frac{1}{C_\alpha}|z|^\alpha \nu(dz)$ on the complex plane by the map $z \to \frac{z}{|z|}$. Again if $\nu = \delta_0$ the above statement should be understood as: S_N converges in distribution to zero.

The proof of this variant is identical verbatim to the proof of Theorem 10.3, we omit it.

Finally we also need in Section 5 an information about the Fourier-Laplace transform of certain complex stable distributions. Consider a probability measure ν on $\mathbb{C}$ and define as above the measure Γ_ν. Let us denote by P^ν the complex $Stable_\alpha(\Gamma_\nu, 0)$ distribution.

Theorem 10.5. *Assume that the measure ν is compactly supported in the closure of $\mathbb{C}^-$. Then, for any $t > 0$:*

$$\int e^{-itx} dP^\nu(x) = \exp(-\Gamma(1-\alpha)(it)^\alpha \int x^\alpha d\nu(x)) \tag{64}$$

Proof. This is a simple consequence of the analogous result for real $Stable_\alpha(\sigma, 1)$ distributions. If X is a random variable with $Stable_\alpha(\sigma, 1)$ distribution, and if $\gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$, then

$$E(e^{-\gamma X}) = e^{-\frac{\sigma^\alpha}{\cos(\frac{\pi\alpha}{2})}\gamma^\alpha} \tag{65}$$

This result is classical when γ is real positive (see Proposition 1.2.12 of [11] for instance). The statement (65) is obtained by an easy analytic extension from the real case.

Consider now a sequence of i.i.d.r.v $(X_k)_{k\geq 1}$, with common distribution $Stable_\alpha(\sigma, 1)$. Furthermore consider a bounded array of complex numbers $(G_{N,k}) \in \mathbb{C}^-$, such that the empirical measure $\frac{1}{N} \sum_{k=1}^N \delta_{G_{N,k}}$ converges to ν when $N \to \infty$. As above define the normalized sum

$$S_N = \frac{1}{a_N} \sum_{k=1}^N G_{N,k} X_k$$

Then, if $\gamma_{N,k} = it\frac{G_{N,k}}{a_N}$, one has obviously

$$E(e^{-itS_N}) = \prod_{k=1}^{N} E(\exp(-\gamma_{N,k}X_k))$$

Noting that $\Re(\gamma_{N,k}) > 0$, it is then possible to use (65):

$$E(e^{-itS_N}) = \exp(-\frac{\sigma^\alpha}{\cos(\frac{\pi\alpha}{2})} \sum_{k=1}^{N} \gamma_{k,N}^\alpha)$$

Using the classical tail estimate for real $Stable_\alpha(\sigma, 1)$ distributions, when u tends to ∞:

$$P(X \geq u) \sim \frac{C_\alpha \sigma^\alpha}{u^\alpha}$$

one sees that $a_N \sim C_\alpha^{\frac{1}{\alpha}} N^{\frac{1}{\alpha}}$.

Thus, we get the estimate

$$E(e^{-itS_N}) \sim \exp(-\frac{(it)^\alpha}{C_\alpha \cos(\frac{\pi\alpha}{2})} \frac{1}{N} \sum_{k=1}^{N} G_{N,k}^\alpha).$$

But $\frac{1}{N} \sum_{k=1}^{N} G_{N,k}^\alpha$ converges to $\int x^\alpha d\nu(x)$. Using now the convergence theorem 10.3 we see that,

$$\int e^{-itx} dP^\nu(x) = \lim_{N\to\infty} E(e^{-itS_N}) = \exp(-\frac{1}{C_\alpha \cos(\frac{\pi\alpha}{2})}(it)^\alpha \int x^\alpha d\nu(x)).$$

Noting that

$$C_\alpha \cos(\frac{\pi\alpha}{2}) = \frac{1-\alpha}{\Gamma(2-\alpha)} = \frac{1}{\Gamma(1-\alpha)}$$

proves Theorem 10.5. □

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