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THE SPECTRUM OF THE LAPLACIAN ON A MANIFOLD OF NEGATIVE CURVATURE. I

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1. Introduction

Let M be a simply connected complete two-dimensional Riemannian manifold. On M we have the Laplacian, a self-adjoint negative semidefinite operator on the Hilbert space $L^2(M)$. In case the curvature is everywhere $\leq -k^2 < 0$, it has been shown [1] that $\lambda_1 \geq k^2/4$, where λ_1 is the lower bound of the spectrum of the negative Laplacian.

The purpose of this note is to determine more accurate bounds for λ_1. We assume the following conditions on M:

(A) M possesses a global system of geodesic polar coordinates with respect to some point $0 \in M$.

Thus we are considering R^2 with a Riemannian metric of the form $ds^2 = dr^2 + G(r,\theta)^2 d\theta^2$ where $G(0^+,\theta) = 0$, $G_r(0^+,\theta) = 1$.

(B) $G_r(r,\theta) \geq 0$; $G(r,\theta) \geq g(r)$ where $g(r)$ is nondecreasing with $\lim_{r\to\infty} g(r) = \infty$.

Both of these conditions are satisfied in case the curvature is everywhere nonpositive. Finally we need the technical condition

(C) $|(G_{rr}/G_r)_\theta| \leq \text{const}$, when $r \geq r_0$, $0 \leq \theta \leq 2\pi$.

Our main result states that

$$\inf_M (G_r/G) \leq \sqrt{4\lambda_1} \leq \inf_{0\leq\theta\leq 2\pi} \overline{\lim_{r\to\infty}} (G_{rr}/G_r) . \tag{1.1}$$

This result shows, for instance, that when the curvature is constant and equal to $-k^2$, then $\lambda_1 = \frac{1}{4}k^2$; no explicit calculations with special functions are needed in our approach.

To prove the lower half of (1.1) we modify the methods used in [1]. To obtain the upper bound we first obtain a comparison function and apply the variational characterization of λ_1. It is shown that if G_{rr}/G_r satisfies an upper bound on a sufficiently thick sector, then a corresponding upper bound can be obtained.

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2. The lower bound

To prove the lower bound, we recall the variational characterization of λ_1:

$$\lambda_1 = \inf_{f \neq 0} \frac{\int_0^{2\pi} \int_0^\infty (f_r^2 + f_\theta^2/G^2) G dr d\theta}{\int_0^{2\pi} \int_0^\infty f^2 G dr d\theta}, \tag{2.1}$$

where the infimum is taken over continuous, piecewise C^1 functions f with compact support. We assume that $G_r/G \geq \delta > 0$ (if $\delta = 0$ there is nothing to prove). Following [1] we have

$$\int_0^\infty f^2 G dr \leq \frac{1}{\delta} \int_0^\infty f^2 G_r dr = -\frac{2}{\delta} \int_0^\infty f f_r G dr . \tag{2.2}$$

Therefore by Schwarz's inequality,

$$\left(\int_0^\infty f^2 G dr \right)^2 \leq \frac{4}{\delta^2} \left(\int_0^\infty f^2 G dr \right) \left(\int_0^\infty f_r^2 G dr \right), \tag{2.3}$$

with the conclusion

$$\int_0^\infty f_r^2 G dr \geq \frac{\delta^2}{4} \int_0^\infty f^2 G dr . \tag{2.4}$$

When we add in the angular term, do the angular integration, and divide by the denominator of (2.1), we see that for any $f \neq 0$ this quotient is bounded below by $\delta^2/4$, which was to be proved.

3. The upper bound

The main result of the section is

Lemma 3.1. *Assume that $G_{rr}/G_r \leq m$ for $R_0 \leq r \leq R_1$, $\alpha \leq \theta \leq \beta$. Then*

$$\lambda_1 \leq \frac{m^2}{4} + \frac{\pi^2}{(R_1 - R_0)^2} + \frac{\pi^2}{(\beta - \alpha)^2 g(R_0)^2} .$$

Proof. Let

$$f(r, \theta) = \exp\left(-\tfrac{1}{2} m r\right) \sin \frac{\pi(r - R_0)}{R_1 - R_0} \sin \frac{\pi(\theta - \alpha)}{\beta - \alpha}$$

in the indicated region, and let $f = 0$ elsewhere. By direct computation f is a solution of the differential equation

$$f_{rr} + m f_r + [m^2/4 + \pi^2/(R_1 - R_0)^2] f = 0 , \tag{3.1}$$

with the end condition $f(R_0, \theta) = 0$, $f(R_1, \theta) = 0$. Thus

$$f_{rr} + (G_r/G)f_r + [m^2/4 + \pi^2/(R_1 - R_0)^2]f = (G_r/G - m)f_r \,. \tag{3.2}$$

Multiply (3.2) by fG and integrate on (R_0, R_1); thus

$$-\int_{R_0}^{R_1} f_r^2 G dr + \left[\frac{m^2}{4} + \frac{\pi^2}{(R_1 - R_0)^2}\right]\int_{R_0}^{R_1} f^2 G dr = \int_{R_0}^{R_1} (G_r - mG)ff_r dr \,. \tag{3.3}$$

We now integrate the right-hand member of (3.3) by parts. The boundary term is zero, and the new integrand has the same sign as $mG_r - G_{rr}$ which is non-negative by assumption. Therefore

$$\int_{R_0}^{R_1} f_r^2 G dr \le \left[\frac{m^2}{4} + \frac{\pi^2}{(R_1 - R_0)^2}\right]\int_{R_0}^{R_1} f^2 G dr \,. \tag{3.4}$$

To treat the θ-terms in (2.1) we note that f also satisfies $f_{\theta\theta} + (\pi^2/(\beta - \alpha)^2)f = 0$ with $f(r, \alpha) = 0 = f(r, \beta)$. Multiplying this equation by f and integrating on $(0, 2\pi)$ we have

$$\int_0^{2\pi} f_\theta^2 d\theta = \frac{\pi^2}{(\beta - \alpha)^2}\int_0^{2\pi} f^2 d\theta \,.$$

By hypothesis (B) we can make the following estimations:

$$\begin{aligned}\int_{R_0}^{R_1}\int_\alpha^\beta (f_\theta^2/G) d\theta dr &\le g(R_0)^{-1}\int_{R_0}^{R_1}\int_\alpha^\beta f_\theta^2 d\theta dr \\ &= \pi^2(\beta - \alpha)^{-2} g(R_0)^{-1}\int_{R_0}^{R_1}\int_\alpha^\beta f^2 d\theta dr \\ &\le \pi^2(\beta - \alpha)^{-2} g(R_0)^{-2}\int_{R_0}^{R_1}\int_\alpha^\beta f^2 G d\theta dr \,.\end{aligned}$$

Combining this with (3.4) gives

$$\begin{aligned}&\int_\alpha^\beta\int_{R_0}^{R_1} (f_r^2 + f_\theta^2/G^2) G dr d\theta \\ &\quad \le \left[\frac{m^2}{4} + \frac{\pi^2}{(R_1 - R_0)^2} + \frac{\pi^2}{(\beta - \alpha)^2 g(R_0)^2}\right]\int_\alpha^\beta\int_{R_0}^{R_1} f^2 G dr d\theta \,.\end{aligned}$$

Inserting the above f into the variational characterization (2.1), we have proved the lemma.

We can now turn to the proof of the upper half of (1.1). For this purpose, let $\bar{m} = \inf_\theta \overline{\lim_{r\to\infty}} (G_{rr}/G_r)$. Given $\varepsilon > 0$ we can find $R_0' > 0$ and (α, β) such that $G_{rr}/G_r \le \bar{m} + \varepsilon$ when $r \ge R_0'$ and $\alpha \le \theta \le \beta$. Let R_0 be such that $g(R_0) > \pi/[\varepsilon(\beta - \alpha)]$ and $R_0 \ge R_0'$. Therefore for any $R_1 > R_0$ we have, by Lemma 3.1,

$$\lambda_1 \le \tfrac{1}{4}(\bar{m} + \varepsilon)^2 + \pi^2/(R_1 - R_0)^2 + \varepsilon^2 \,. \tag{3.4}$$

In this inequality we let $R_1 \to \infty$. Since the resulting inequality is valid for every $\varepsilon > 0$, we have proved that $\lambda_1 \leq \frac{1}{4}\bar{m}^2$.

4. Discussion of the result-applications and examples

In previous works upper and lower bounds for λ_1 were obtained in terms of the curvature. Recall that in a system of geodesic polar coordinates we have

$$- K = G_{rr}/G , \tag{4.1}$$

where K is the Gaussian curvature. Thus we obtain the two-dimensional case of [1]:

Corollary 4.1. *Suppose that $K \leq -k^2 < 0$ on M. Then $\lambda_1 \geq \frac{1}{4}k^2$.*

Proof. G satisfies the inequality $G_{rr} \geq k^2 G$ with $G(0^+, \theta) = 0$, $G_r(0^+, \theta) = 1$. Thus $h = G_r/G$ satisfies the inequality $h_r + h^2 \geq k^2$, $\lim_{r\to 0} rh(r, \theta) = 1$. Therefore $h(r, \theta) \geq k \coth kr$ with the conclusion that $(G_r/G) \geq k$ everywhere. Applying the lower bound of (1.1), we obtain the stated result.

In [2] it was proved that $K \geq - k^2$ implies $\lambda_1 \leq \frac{1}{4}k^2$. Although our method does not yield this result, we do obtain a related localized result.

Corollary 4.2. *Suppose that $k^2(\theta) = \lim_{r\to\infty} (- K(r, \theta))$ exists for $\theta \in (\alpha, \beta)$ and satisfies $k^2(\theta) \leq k^2$. Then $\lambda_1 \leq \frac{1}{4}k^2$.*

Proof. We have $G_{rr}/G_r = (- K)/h$ where $h = G_r/G$ is a solution of the equation $h_r + h^2 = - K$. Using a comparison estimate when $r \to \infty$ [3] we have $\lim_{r\to\infty} h(r, \theta) = k(\theta)$. (This also follows by an appropriate use of the Rauch comparison theorem.) Therefore $\lim_{r\to\infty} (G_{rr}/G_r) = k(\theta)$. Applying the upper half of (1.1), we see that $\lambda_1 \leq \frac{1}{4}k^2$, which was to be proved.

Using the same idea, we can obtain another variation of Cheng's result.

Corollary 4.3. *Suppose that $K(r, \theta) \geq -k^2$ and $(G_r/G)(r, \theta) \geq k_1$ for $\theta \in (\alpha, \beta)$ and $r \geq R_1$. Then $\lambda_1 \leq \frac{1}{4}k^2/k_1$.*

Proof. In this case we have $G_{rr}/G_r \leq k^2/k_1$ for $\theta \in (\alpha, \beta)$ and $r \geq R_1$, whence the result follows.

From (1.1) we see that the upper bound depends only on the details of the metric in a neighborhood of infinity. It is natural to ask whether a lower bound can be obtained which only depends on the metric in a neighborhood of infinity. The following example shows, in particular, that in the lower bound part of (1.1), the infimum cannot be replaced by lim inf when $r \to \infty$.

Example 4.4 Let $K(r)$ be a C^∞ function such that

$$\begin{aligned} K(r) &= - 1 , & 0 \leq r \leq R_1 , \\ K(r) &= - 4 , & R_1 + 1 \leq r < \infty , \end{aligned} \tag{4.2}$$

where $R_1 > \pi\sqrt{4/3}$; let $G(r)$ be the solution of $G_{rr} + KG = 0$, $G(0^+) = 0$,

$G_r(O^+) = 1$. Thus $G(r) = \sinh r$ and $G_{rr}/G_r = \tanh r$ for $0 \le r \le R_1$. Following the proof of Lemma 3.1 we let $f(r) = \exp(-\frac{1}{2}mr) \sin(\pi r/R_1)$ for $0 \le r \le R_1$ and $f = 0$ for $r > R_1$, where $m = \tanh R_1$. Substituting in the variational characterization (1.1), we have

$$\lambda_1 \le \tfrac{1}{4}(\tanh R_1)^2 + \pi^2/R_1^2 < 1 .$$

On the other hand, it is clear from (4.2) that $\lim_{r\to\infty} (G_r/G)$ exists and is equal to 2. Hence $\liminf_{r\to\infty} (G_r/G) = 2 > \sqrt{4\lambda_1}$.

By modifying the constants in Example 4.4 and taking R_1 sufficiently large, we obtain the following proposition: *Given $0 < a < b$, there exists a rotationally invariant metric $G(r)$ with curvature function $K(r)$ such that $\lim_{r\to\infty} [-K(r)] = b^2$ and $\lambda_1 < \frac{1}{4}a^2$.*

It is natural to ask if, under suitable regularity conditions, λ_1 depends only on the details of the metric in a neighborhood of infinity. We have the following result.

Corollary 4.5. *Assume that G_r/G is nonincreasing along each ray and that $\lim_{r\to\infty} K(r,\theta) = -k^2(\theta)$ exists for each $\theta \in [0, 2\pi]$. Then $\sqrt{4\lambda_1} = \inf_{0\le\theta\le2\pi} k(\theta)$.*

Proof. From the proof of Corollary 4.2, we have $\lim_{r\to\infty} (G_{rr}/G_r) = k(\theta)$. On the other hand, the same argument shows that $\lim_{r\to\infty} (G_r/G) = k(\theta) = \inf_{r>0} (G_r/G)$. Hence the left- and right-hand members of (1.1) are equal in this case, and the proof is complete.

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