The spectrum of the operator $D(r, 0, 0, s)$ over the sequence spaces ℓ_p and bv_p

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Abstract

In this paper we have examined the spectra of the operator $D(r, 0, 0, s)$
on sequence spaces ℓ_p and bv_p.

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1. Introduction

Spectral theory is an important branch of mathematics due to its application in other branches of science. It has been proved to be a standard tool of mathematical sciences because of its usefulness and application oriented scope in different fields. In numerical analysis, the spectral values may determine whether a discretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge. In aeronautics, the spectral values may determine whether the flow over a wing is laminar or turbulent. In electrical engineering, it may determine the frequency response of an amplifier or the reliability of a power system. In quantum mechanics, it may determine atomic energy levels and thus, the frequency of a laser or the spectral signature of a star. In structural mechanics, it may determine whether an automobile is too noisy or whether a building will collapse in an earthquake. In ecology, the spectral values may determine whether a food web will settle into a steady equilibrium. In probability theory, they may determine the rate of convergence of a Markov process.

In summability theory, different classes of matrices have been investigated. Characterizations of matrix classes are found in Tripathy and Sen [29], Tripathy [30], Rath and Tripathy [21] and many others. There are particular types of

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summability methods like Nörlund mean, Riesz mean, Euler mean, Abel transformation etc. Matrix methods have been studied from different aspects recently by Altin et.al [10], Tripathy and Baruah [31] and others.

Spectral theory is a thrust area of research in Functional analysis. The spectra of different operators have been studied. There are different types of matrix operators on sequence spaces. The spectra of only a few of the matrix operators have been studied so far that to on some particular type of sequence spaces. The works those exist are mainly on cesàro, Schur, Hausdorff and some difference matrix operators.

Wenger [33] examined the fine spectrum of the integer power of the Cesàro operator in c and Rhoades [26, 27] generalized this result to the weighted mean methods and proposed a conjecture for their fine spectra on $B(\ell_p)$ respectively. Reade [23] worked the spectrum of the Cesàro operator in the sequence space c_0 and Rhoades [24] extended it to the fine spectrum of the weighted mean operators. The fine spectrum of the Cesàro operator on the sequence space ℓ_p has been studied by Gonzalez [17], where $1 < p < \infty$. Okutyi [19, 20] computed the spectrum of the Cesàro operator on the sequence spaces bv and $bv_0 = bv \cap c_0$ and Rhoades [27] extended that result to weighted mean methods over the space bv_0. Akhmedov and Basar [4, 5] have recently determined, independently than that of Gonalez [17], the fine spectrum of the Cesàro operator in the sequence spaces c_0, ℓ_∞ and ℓ_p, by the different way respectively, where $1 < p < \infty$.

The spectrum and the fine spectrum of the Rhally operators on the sequence spaces c_0 and c, under assumption that $\lim_{n \to \infty} (n + 1)a_n = L \neq 0$, have been examined by Yildirim [32]. Furthermore, Coskun [12] has studied the spectrum and fine spectrum for p-Cesàro operator acting on the space c_0. More recently, Malafosse [18] and Altay and Basar [8] and Akhmedov and Basar [4] have respectively studied the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0, c and $\ell_p, p \geq 1$; where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by $||x||_{s_r} = \sup_{k \in N} \frac{|x_k|}{r^k}, (r > 0)$.

Also, Akhmedov and Basar [3], and Altay and Basar [8] have determined the fine spectrum with respect to Goldberg's classification [16] of the difference operator $\triangle$ and the generalized difference operator $B(r, s)$ over the sequence spaces ℓ_p, bv_p and c_0 and c; respectively where the sequence space bv_p is defined in [7] by $bv_p = \{x = (x_k) \in w : \sum |x_k - x_{k-1}|^p < \infty\}, (1 \leq p < \infty)$.

Furthermore, the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces ℓ_1 and bv has been studied by Furkan, Bilgic and Kayaduman [13]. Recently the fine spectrum of the operator $B(r, s)$ over ℓ_p and bv_p has been studied by Bilgiç and Furkan [11]. More recently, the fine spectrum of $B(r, s, t)$ over the sequence spaces c_0 and c and ℓ_p and bv_p have been studied by Furkan et al. [14, 15]. Srivastava and Kumar [28] have determined the spectrum and fine spectrum of the operator $\triangle_a$ over the sequence space c_0, where

$\triangle_a : c_0 \to c_0$ is defined by

$$\triangle_a x = \triangle_a(x_n) = (a_n x_n - a_{n-1} x_{n-1})_{n=0}^{\infty} \text{ with } x_{-1} = 0,$$

where (a_k) is either constant or strictly decreasing sequence of positive real numbers satisfying $\underset{k\to\infty}{lim} a_k = a > 0$ and $a_0 \le 2a$.

The same problem, in the case when the sequence (a_k) is assumed to be constant except for finitely many elements was investigated by Akhmedov [2]. Ahmadov and Shabrawy [1] have studied the spectrum of the operator $\Delta_{a,b}$ over the sequence space c. Spectra of some particular type of matrix operator have been investigated from different aspects by Rath and Tripathy [22].

2. Preliminaries and Definition

Let X be a linear space. By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, where X is a Banach space then the adjoint operator T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$.

Let $T : D(T) \to X$ be a linear operator, defined on $D(T) \subset X$, where $D(T)$ denote the domain of T and X is a complex normed linear space. For $T \in B(X)$ we associate a complex number α with the operator $(T - \alpha I)$ denoted by T_α defined on the same domain $D(T)$, where I is the identity operator. The inverse $(T - \alpha I)^{-1}$, denoted by T_α^{-1} is known as the resolvent operator of T.

A regular value is a complex number α of T such that
(R_1) T_α^{-1} exists,
(R_2) T_α^{-1} is bounded and
(R_3) T_α^{-1} is defined on a set which is dense in X.

The resolvent set of T is the set of all such regular values α of T, denoted by $\rho(T)$. Its complement is given by $C \setminus \rho(T)$ in the complex plane C is called the spectrum of T, denoted by $\sigma(T)$. Thus the spectrum $\sigma(T)$ consist of those values of $\alpha \in C$, for which T_α is not invertible.

Classification of spectrum:

The spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows:

(i) The point(discrete) spectrum $\sigma_{pt}(T)$ is the set such that T_α^{-1} does not exist. Further $\alpha \in \sigma_{pt}(T)$ is called the eigen value of T.

(ii) The continuous spectrum $\sigma_c(T)$ is the set such that T_α^{-1} exists and satisfies (R_3) but not (R_2) that is T_α^{-1} is unbounded.

(*iii*) The residual spectrum $\sigma_r(T)$ is the set such that T_α^{-1} exists (may be bounded or not) but not satisfy (R_3), that is, the domain of T_α^{-1} is not dense in X .

This is to note that in finite dimensional case, continuous spectrum coincides with the residual spectrum and equal to the empty set and the spectrum consists of only the point spectrum.

Let E and F be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in N = \{0, 1, 2, ...\}$. Then, we say that A defines a matrix mapping from E into F, denote by $A : E \to F$, if for every sequence $x = (x_n) \in E$ the sequence $Ax = \{(Ax)_n\}$ is in F where $(Ax)_n = \sum\limits_{k=0}^{\infty} a_{nk}x_k$, provided the right hand side converges for every $n \in N$ and $x \in E$.

Our main focus in this paper is on the operator $D(r, 0, 0, s)$, where

$$
D(r,0,0,s) = \begin{pmatrix}
r & 0 & 0 & 0 & 0 & ... \\
0 & r & 0 & 0 & 0 & ... \\
0 & 0 & r & 0 & 0 & ... \\
s & 0 & 0 & r & 0 & ... \\
0 & s & 0 & 0 & r & ... \\
. & . & . & . & & ...
\end{pmatrix},
$$

Here we assume that r and s are complex parameters and $s \neq 0$.
Remark: In particular if we consider $r = -1$ and $s = 1$ then $D(-1, 0, 0, 1) = \triangle_3$

2.1. Lemma. *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

2.2. Lemma. *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_1 norms of the rows of A is bounded.*

2.3. Lemma. *T has a dense range if and only if T^* is one to one, where T^* denote the adjoint operator of T.*

3. The spectrum of the operator $D(r, 0, 0, s)$ on the sequence space ℓ_p, $(1 < p < \infty)$.

3.1. Theorem. *$D(r, 0, 0, s) : \ell_p \to \ell_p$ is a bounded linear operator satisfying the inequalities $(|r|^p + |s|^p)^{\frac{1}{p}} \leq ||D(r, 0, 0, s)||_{\ell_p} \leq |r| + |s|$.*

Proof. The linearity of $D(r, 0, 0, s)$ is trivial and so is omitted. Let us consider $e = (1, 0, 0, ...) \in \ell_p$. Then $D(r, 0, 0, s)e = (r, 0, 0, s, 0, ...)$ and $\frac{||D(r,0,0,s)e||_{\ell_p}}{||e||_{\ell_p}} = (|r|^p + |s|^p)^{\frac{1}{p}}$ which give us $(|r|^p + |s|^p)^{\frac{1}{p}} \leq ||D(r, 0, 0, s)||_{\ell_p}$, for any $p > 1$.
Next let $x = (x_k) \in \ell_p$ then by using Minkowski's inequality and taking $x_{-3} = x_{-2} = x_{-1} = 0$, we have,

$$\|D(r,0,0,s)x\|_{\ell_p} = (\sum_{k=0}^{\infty} |sx_{k-3} + rx_k|^p)^{\frac{1}{p}}$$

$$\leq (\sum_{k=0}^{\infty} |sx_{k-3}|^p)^{\frac{1}{p}} + (\sum_{k=0}^{\infty} |rx_k|^p)^{\frac{1}{p}} = (|r|+|s|)\|x\|_{\ell_p}$$

This implies $\|D(r,0,0,s)\|_{\ell_p} \leq |r| + |s|$. This completes the proof. $\qquad\square$

3.2. Lemma. *Let* $1 < p < \infty$ *and let* $A \in (\ell_\infty, \ell_\infty) \cap (\ell_1, \ell_1)$ *then* $A \in (\ell_p, \ell_p)$.

3.3. Theorem. $\sigma(D(r,0,0,s), \ell_p) = \{\lambda \in C : |r - \lambda| \leq |s|\}$.

Proof. First, we prove that $(D(r,0,0,s) - \alpha I)^{-1}$ exists and is in (ℓ_p, ℓ_p) for $|r-\alpha| > |s|$ and then we have to show that the operator $(D(r,0,0,s) - \alpha I)$ is not invertible for $|r - \alpha| \leq |s|$.
Let $\alpha \notin \{\lambda \in C : |r - \lambda| \leq |s|\}$. Since $s \neq 0$ we have $\alpha \neq r$ and so $(D(r,0,0,s) - \alpha I)$ is triangle, hence $(D(r,0,0,s) - \alpha I)^{-1}$ exists.
Let,

$$\begin{pmatrix} r-\alpha & 0 & 0 & 0 & 0 & \cdots \\ 0 & r-\alpha & 0 & 0 & 0 & \cdots \\ 0 & 0 & r-\alpha & 0 & 0 & \cdots \\ s & 0 & 0 & r-\alpha & 0 & \cdots \\ 0 & s & 0 & 0 & r-\alpha & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \begin{pmatrix} p_0 & 0 & 0 & 0 & 0 & \cdots \\ p_1 & p_0 & 0 & 0 & 0 & \cdots \\ p_2 & p_1 & p_0 & 0 & 0 & \cdots \\ p_3 & p_2 & p_1 & p_0 & 0 & \cdots \\ p_4 & p_3 & p_2 & p_1 & p_0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

Then we have

$p_0 = \frac{1}{r-\alpha}$
$p_1 = 0$
$p_2 = 0$
$p_3 = -\frac{s}{(r-\alpha)^2}$
$p_4 = 0$
$p_5 = 0$
$p_6 = \frac{s^2}{(r-\alpha)^3}$
- - -
we obtain
$p_{3k} = \frac{(-s)^k}{(r-\alpha)^{k+1}}$, $(k \geq 0)$
and
$p_{3k+1} = 0$, $(k \geq 0)$
and
$p_{3k+2} = 0$, $(k \geq 0)$.

Hence, we get

$$(D(r,0,0,s) - \alpha I)^{-1} = \begin{pmatrix} \frac{1}{r-\alpha} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{r-\alpha} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{r-\alpha} & 0 & 0 & \cdots \\ -\frac{s}{(r-\alpha)^2} & 0 & 0 & \frac{1}{r-\alpha} & 0 & \cdots \\ 0 & -\frac{s}{(r-\alpha)^2} & 0 & 0 & \frac{1}{r-\alpha} & \cdots \\ & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

Now, $\|(D(r,0,0,s) - \alpha I)^{-1}\|_{(\ell_1,\ell_1)} = \sup_k \sum_{n=k}^{\infty} \left|\frac{s}{r-\alpha}\right|^{n-k} \left|\frac{1}{r-\alpha}\right| = \left|\frac{1}{r-\alpha}\right| \sum_{k=0}^{\infty} \left|\frac{1}{r-\alpha}\right|^n < \infty.$

Similarly it can be verified that $\|(D(r,0,0,s) - \alpha I)^{-1}\|_{(\ell_\infty,\ell_\infty)} < \infty.$

This shows that $(D(r,0,s) - \alpha I)^{-1} \in (\ell_\infty, \ell_\infty) \cap (\ell_1, \ell_1)$ and hence by Lemma 3.2 $(D(r,0,0,s) - \alpha I)^{-1} \in (\ell_p, \ell_p)$ i.e. $\alpha \notin \sigma(D(r,0,0,s), \ell_p)$. This shows that $\sigma(D(r,0,0,s), \ell_p) \subseteq \{\lambda \in C : |r - \lambda| \le |s|\}.$

Conversely, let $\alpha \in \{\lambda \in C : |r - \lambda| \le |s|\}$

Case 1: Let $\alpha \ne r$.

Then $(D(r,0,0,s) - \alpha I)$ is triangle, and hence $(D(r,0,0,s) - \alpha I)^{-1}$ exists but for $y = (1,0,0,...) \in \ell_p$, $(D(r,0,0,s) - \alpha I)^{-1}y = (x_k)$ gives $x_{3k} = \frac{(-s)^k}{(r-\alpha)^{k+1}}$, for $(k \ge 0)$ and $x_{3k+1} = 0$, $x_{3k+2} = 0$ for $(k \ge 0)$ therefore $(x_k) \notin \ell_p$ since $|s| \ge |r - \alpha|$ i.e. $(D(r,0,0,s) - \alpha I)^{-1} \notin B(\ell_p)$ which implies $\alpha \in \sigma(D(r,0,0,s), \ell_p)$. Therefore $\{\lambda \in C : |r - \lambda| \le |s|\} \subseteq \sigma(D(r,0,0,s), \ell_p)$.

Case 2: Let $\alpha = r$.

Then the operator $(D(r,0,0,s) - \alpha I) = D(0,0,0,s)$ is represented by the matrix

$$(D(r,0,0,s) - \alpha I) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ s & 0 & 0 & 0 & 0 & \cdots \\ 0 & s & 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & & \cdots \end{pmatrix} = D(0,0,0,s).$$

Since $D(0,0,0,s)x = \theta$ implies $x = \theta$, $D(0,0,0,s) : \ell_p \to \ell_p$ is injective but not onto. Hence $D(0,0,0,s)$ is not invertible and so $\alpha \in \sigma(D(r,0,0,s), \ell_p)$. Therefore in this case also $\{\lambda \in C : |r - \lambda| \le |s|\} \subseteq \sigma(D(r,0,0,s), \ell_p)$. This completes the proof. $\qquad \square$

3.4. Theorem. $\sigma_{pt}(D(r,0,0,s), \ell_p) = \emptyset.$

Proof. Suppose that $D(r,0,0,s)x = \alpha x$ for $x \ne \theta = (0,0,0,...)$ in ℓ_p. Then by solving the system of linear equations we have

$rx_0 = \alpha x_0$
$rx_1 = \alpha x_1$
$rx_2 = \alpha x_2$
$sx_0 + rx_3 = \alpha x_3$
$sx_1 + rx_4 = \alpha x_4$
- - -
$sx_k + rx_{k+3} = \alpha x_{k+3}$

If $x_{n_0} \neq 0$ is the first non-zero entry of the sequence $x = (x_n)$, then $\alpha = r$ and from the equation $sx_{n_0} + rx_{n_0+3} = \alpha x_{n_0+3}$ we get $sx_{n_0} = 0$. Since $s \neq 0$. we must have $x_{n_0} = 0$, contradicting the fact that $x_{n_0} \neq 0$. This complete the proof.

If $T : \ell_p \to \ell_p$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : \ell_p^* \to \ell_p^*$ is defined by the transpose of the matrix A. It is well-known that the dual space ℓ_p^* of ℓ_p is isomorphic to ℓ_q with $p^{-1} + q^{-1} = 1$. □

3.5. Theorem. $\sigma_{pt}(D(r,0,0,s)^*, \ell_p^*) = \{\lambda \in C : |r - \lambda| < |s|\}$.

Proof. Suppose that $D(r,0,0,s)^* x = \alpha x$ for $x \neq \theta$ in $\ell_p^* \cong \ell_q$ with $p^{-1} + q^{-1} = 1$. Then by solving the system of linear equations we have

$rx_0 + sx_3 = \alpha x_0$

$rx_1 + sx_4 = \alpha x_1$

$rx_2 + sx_5 = \alpha x_2$

$- - -$

$rx_k + sx_{k+3} = \alpha x_k$

$- - -$

we obtain that

$x_{3n} = \left(\frac{\alpha-r}{s}\right)^n x_0$,$(n \geq 1)$

and

$x_{3n+1} = \left(\frac{\alpha-r}{s}\right)^n x_1$,$(n \geq 1)$

and

$x_{3n+2} = \left(\frac{\alpha-r}{s}\right)^n x_2$,$(n \geq 1)$

From the above system of equations we have,

$\sum_{n=1}^{\infty} |x_n|^q = (|x_0| + |x_1| + |x_2|) \sum_{n=0}^{\infty} |\frac{\alpha-r}{s}|^{qn}$ this shows that $(x_n) \in \ell_q$ if and only if $|\alpha - r| < |s|$. This completes the proof. □

3.6. Theorem. $\sigma_r(D(r,0,0,s), \ell_p) = \{\lambda \in C : |r - \lambda| < |s|\}$.

Proof. We show that the operator $D(r,0,0,s) - \alpha I)$ has an inverse and $\overline{R(D(r,0,0,s) - \alpha I))} \neq \ell_p$ for $\alpha \in \{\lambda \in C : |r - \lambda| < |s|\}$. For $\alpha \neq r$ the operator $(D(r,0,0,s) - \alpha I)$ is triangle and has an inverse. For $\alpha = r$, the operator $(D(r,0,0,s) - \alpha I)$ is one to one and hence has an inverse. But by Theorem 3.5 implies that $(D(r,0,0,s)^* - \alpha I)$ is not one to one for $|r - \alpha| < |s|$. Now using the Lemma 2.3 we can conclude that the range of $(D(r,0,0,s) - \alpha I)$ is not dense in ℓ_p, i.e. $\overline{R(D(r,0,0,s) - \alpha I))} \neq \ell_p$. This completes the proof. □

3.7. Theorem. $\sigma_c(D(r,0,0,s), \ell_p) = \{\lambda \in C : |r - \lambda| = |s|\}$.

Proof. The proof immediately follows from the fact that the set of spectrum is the disjoint union of the point spectrum, residual spectrum and continuous spectrum, that is

$\sigma(D(r,0,0,s), \ell_p) = \sigma_{pt}(D(r,0,0,s), \ell_p) \cup \sigma_r(D(r,0,0,s), \ell_p) \cup \sigma_c(D(r,0,0,s), \ell_p)$.

□

4. The Spectrum of the operator $D(r,0,0,s)$ on the sequence space bv_p

4.1. Theorem. $D(r,0,0,s) \in B(bv_p)$.

Proof. The linearity of $D(r,0,0,s)$ is trivial and so is omitted. Let us take $x = (x_k) \in bv_p$ then by using Minkowski's inequality and taking the negative indices $x_{-k} = 0$, we have

$$||D(r,0,0,s)x||_{bv_p} = \left(\sum_{k=0}^{\infty} |(rx_k + sx_{k-3}) - (rx_{k-1} + sx_{k-4})|^p \right)^{\frac{1}{p}}$$

$$\leq (|r|^p \sum_{k=0}^{\infty} |x_k - x_{k-1}|^p)^{\frac{1}{p}} + (|s|^p \sum_{k=0}^{\infty} |x_{k-3} - x_{k-4}|^p)^{\frac{1}{p}} = (|r| + |s|)||x||_{bv_p}$$

This gives $||D(r,0,0,s)||_{bv_p} \leq |r| + |s|$. □

4.2. Theorem. $\sigma(D(r,0,0,s), bv_p) = \{\lambda \in C : |r - \lambda| \leq |s|\}$.

Proof. First, we prove that $(D(r,0,0,s) - \alpha I)^{-1}$ exists and is in (bv_p, bv_p) for $|r - \alpha| > |s|$ and then we have to show that the operator $(D(r,0,0,s) - \alpha I)$ is not invertible for $|r - \alpha| \leq |s|$.

Let $\alpha \notin \{\lambda \in C : |r - \lambda| \leq |s|\}$. Since $s \neq 0$ we have $\alpha \neq r$ and so $(D(r,0,0,s) - \alpha I)$ is triangle, hence $(D(r,0,0,s) - \alpha I)^{-1}$ exists.

Let $y = (y_k) \in bv_p$. This implies that $(y_k - y_{k-1}) \in \ell_p$. Solving the system of equations $(D(r,0,0,s) - \alpha I)x = y$ we have as in the proof of Theorem 3.3 that

$$x_k - x_{k-1} = \sum_{j=0}^{k} p_{k-j+1}(y_j - y_{j-1}); (k \in N), \text{ where } x_{-1} = y_{-1} = 0$$

i.e. $(x_k - x_{k-1}) = (D(r,0,0,s) - \alpha I)^{-1}(y_k - y_{k-1})$. Since $(D(r,0,0,s) - \alpha I)^{-1} \in (bv_p, bv_p)$ by Theorem 3.3, $(x_k - x_{k-1}) \in \ell_p$. This implies that $x = (x_k) \in bv_p$ and hence $(D(r,0,0,s) - \alpha I)^{-1} \in (bv_p, bv_p)$ this shows that $\alpha \notin \sigma(D(r,0,0,s), bv_p)$. Hence $\sigma(D(r,0,0,s), bv_p) \subseteq \{\lambda \in C : |r - \lambda| \leq |s|\}$.

Conversely, let $\alpha \in \{\lambda \in C : |r - \lambda| \leq |s|\}$. If $r \neq \alpha$, then $(D(r,0,0,s) - \alpha I)$ is triangle, hence $(D(r,0,0,s) - \alpha I)^{-1}$ exists, but $y = (1,0,0,...) \in bv_p$ gives $x = (x_k)$ with $x_{3k} = \frac{(-s)^k}{(r-\alpha)^{k+1}}$, for $(k \geq 0)$ and $x_{3k} = x_{3k+1} = 0$, for $(k \geq 0)$. Clearly, $(x_k) \notin bv_p$ for $|s| \geq |r - \alpha|$. This shows that $\alpha \in (D(r,0,0,s), bv_p)$. Next let, $r = \alpha$, then similar arguments as in the proof of Theorem 3.3. shows that the operator $D(r,0,0,s) - \alpha I = D(0,0,0,s)$ is not invertible, therefore in this case also $\alpha \in \sigma(D(r,0,0,s), bv_p)$. Thus, $\{\lambda \in C : |r - \lambda| \leq |s|\} \subseteq \sigma(D(r,0,0,s), bv_p)$. This completes the proof. □

Since the spectrum and fine spectrum of the matrix $D(r,0,0,s)$ as an operator on the sequence space bv_p are similar to that of the space ℓ_p in Section 3, to avoid the repetition of the similar statements we give the results in the following theorem without proof.

4.3. Theorem. *(i)* $\sigma_{pt}(D(r,0,0,s), bv_p) = \emptyset$.
(ii) $\sigma_{pt}(D(r,0,0,s)^*, bv_p^*) = \{\lambda \in C : |r - \lambda| < |s|\}$.
(iii) $\sigma_r(D(r,0,0,s), bv_p) = \{\lambda \in C : |r - \lambda| < |s|\}$.
(iv) $\sigma_c(D(r,0,0,s), bv_p) = \{\lambda \in C : |r - \lambda| = |s|\}$.

5. Conclusion

We can generalize our operator

$$(D(r,0,0,..(n-1)\text{times},s) = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & r & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ s & 0 & . & . & r & 0 & . & . & . \\ 0 & s & 0 & . & . & r & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}$$

If we take $r = -1$ and $s = 1$, then the operator $(D(r,0,0,..(n-1)\text{times},s)$ will be the same as the generalized difference operator $\triangle_n$. Further on considering the operator $(D(r,0,0,..(n-1)\text{times},s)$ in place of $D(r,0,0,s)$, one can get parallel all our results obtained in this paper.

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