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THE SPECTRUM OF THE SUB-LAPLACIAN ON THE HEISENBERG GROUP

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Abstract. We give a complete analysis of the spectrum of the unique self-adjoint extension of the sub-Laplacian on the one-dimensional Heisenberg group.

1. The sub-Laplacian on the Heisenberg group.

If we identify $\boldsymbol{R}^2$ with the complex plane $\boldsymbol{C}$ via the obvious identification

$$\boldsymbol{R}^2 \ni (x, y) \leftrightarrow z = x + iy \in \boldsymbol{C}\,,$$

and we let

$$\boldsymbol{H} = \boldsymbol{C} \times \boldsymbol{R}\,,$$

then $\boldsymbol{H}$ becomes a noncommutative group when it is equipped with the multiplication $\cdot$ given by

$$(z, t) \cdot (w, s) = \left(z + w, t + s + \frac{1}{4}[z, w]\right), \quad (z, t), (w, s) \in \boldsymbol{H}\,,$$

where $[z, w]$ is the symplectic form of z and w defined by

$$[z, w] = 2\,\mathrm{Im}(z\overline{w})\,.$$

In fact, $\boldsymbol{H}$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $dz\,dt$.

Let $\mathfrak{h}$ be the Lie algebra of all left-invariant vector fields on $\boldsymbol{H}$. Then a basis for $\mathfrak{h}$ is given by X, Y and T, where

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t}\,,$$

$$Y = \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial t}\,,$$

and

$$T = \frac{\partial}{\partial t}\,.$$

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The sub-Laplacian $\mathcal{L}$ on $\boldsymbol{H}$ is then defined by

$$\mathcal{L} = -(X^2 + Y^2)\,.$$

Let $\partial/\partial z$ and $\partial/\partial\overline{z}$ be partial differential operators on $\boldsymbol{C}$ defined by

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\,.$$

Then we look at the vector fields Z and $\overline{Z}$ on $\boldsymbol{H}$ given by

$$Z = X - iY = \frac{\partial}{\partial z} + \frac{1}{2} i\overline{z}\frac{\partial}{\partial t}$$

and

$$\overline{Z} = X + iY = \frac{\partial}{\partial \overline{z}} - \frac{1}{2} iz\frac{\partial}{\partial t}\,.$$

$\overline{Z}$ is the celebrated Hans Lewy operator in [9] that defies local solvability on $\boldsymbol{R}^3$, and

$$\mathcal{L} = -\frac{1}{2}(Z\overline{Z} + \overline{Z}Z).$$

A simple computation gives

$$\mathcal{L} = -\Delta - \frac{1}{4}(x^2 + y^2)\frac{\partial^2}{\partial t^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}\,,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,.$$

The symbol $\sigma(\mathcal{L})$ of $\mathcal{L}$ is then given by

$$\sigma(\mathcal{L})(x, y, t\,; \xi, \eta, \tau) = \left(\xi + \frac{1}{2}y\tau\right)^2 + \left(\eta - \frac{1}{2}x\tau\right)^2$$

for all (x, y, t) and (ξ, η, τ) in $\boldsymbol{H}$. It is then easy to see that $\mathcal{L}$ is nowhere elliptic on $\boldsymbol{R}^3$. Since

$$[X, Y] = T\,,$$

it follows from a theorem of Hörmander [8, Theorem 1.1] that $\mathcal{L}$ is hypoelliptic.

The aim of this paper is to compute the spectrum of the unique positive and self-adjoint extension $\mathcal{L}_0$ of the sub-Laplacian $\mathcal{L}$ as an unbounded linear operator from $L^2(\boldsymbol{H})$ into $L^2(\boldsymbol{H})$ with dense domain given by the Schwartz space $\mathcal{S}(\boldsymbol{H})$. This is carried out by using the spectral analysis of the twisted Laplacians obtained by taking the inverse Fourier transform of the sub-Laplacian with respect to the center, i.e., the time t of the Heisenberg group.

We recall in Section 2 the twisted Laplacians and their spectral analysis. The essential self-adjointness of $\mathcal{L}$ as an unbounded linear operator from $L^2(\boldsymbol{H})$ int $L^2(\boldsymbol{H})$ with dense domain $\mathcal{S}(\boldsymbol{H})$ is recalled in Section 3. The spectrum of the unique self-adjoint extension $\mathcal{L}_0$ of $\mathcal{L}$ is then computed in Section 4.

The results in this paper are valid for the sub-Laplacian on the n-dimensional Heisenberg group $\boldsymbol{H}_n$, $n > 1$, in which the underlying space is $\boldsymbol{C}^n \times \boldsymbol{R}$, but we have chosen to present the results for the one-dimensional Heisenberg group $\boldsymbol{H}$ for the sake of simplicity and transparency.

2. The twisted Laplacians. For $\tau \in \boldsymbol{R} \setminus \{0\}$, let Z_τ and $\overline{Z}_\tau$ be partial differential operators given by

$$Z_\tau = \frac{\partial}{\partial z} + \frac{1}{2}\tau\overline{z},$$

and

$$\overline{Z}_\tau = \frac{\partial}{\partial \overline{z}} - \frac{1}{2}\tau z\,.$$

Then we are interested in the twisted Laplacian L_τ defined by

$$L_\tau = -\frac{1}{2}(Z_\tau\overline{Z}_\tau + \overline{Z}_\tau Z_\tau).$$

More explicitly,

$$L_\tau = -\Delta + \frac{1}{4}(x^2 + y^2)\tau^2 - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\tau\,.$$

The fundamental connection between the sub-Laplacian and the twisted Laplacians is given by the following theorem [1, 3, 4].

THEOREM 2.1. *Let $u \in \mathcal{S}'(\boldsymbol{H}) \cap C^\infty(\boldsymbol{H})$ be such that $\check{u}(z,\tau)$ is a tempered distribution of τ on $\boldsymbol{R}$ for each z in $\boldsymbol{C}$, where $\check{u}$ is the inverse Fourier transform of u with respect to time t. Then for almost all τ in $\boldsymbol{R} \setminus \{0\}$,*

$$(\mathcal{L}u)^\tau = L_\tau u^\tau\,,$$

where

$$(\mathcal{L}u)^\tau(z) = (\mathcal{L}u)^\vee(z,\tau)\,, \quad z \in \boldsymbol{C}\,,$$

and

$$u^\tau(z) = \check{u}(z,\tau)\,, \quad z \in \boldsymbol{C}\,.$$

REMARK 2.2. We note that the Fourier transform $\hat{f}$ of a function f in $L^1(\boldsymbol{R})$ is taken to be the one defined by

$$\hat{f}(\xi) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx\,, \quad \xi \in \boldsymbol{R}.$$

In order to study the spectral theory of L_τ, we first introduce the τ-Fourier-Wigner transform $V_\tau(f, g)$ of the functions f and g in the Schwartz space $\mathcal{S}(\boldsymbol{R})$ by

$$V_\tau(f, g)(q, p) = (2\pi)^{-1/2}|\tau|^{1/2} \int_{-\infty}^{\infty} e^{i\tau q y} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy$$

for all q and p in $\boldsymbol{R}$. If $\tau = 1$, then we get

$$V_1(f, g) = V(f, g)\,,$$

which is the classical Fourier-Wigner transform in, for instance, [6, 13, 16]. It can be shown easily that

$$V_\tau(f, g)(q, p) = |\tau|^{1/2} V(f, g)(\tau q, p)\,.$$

For $\tau \in \boldsymbol{R} \setminus \{0\}$ and $k = 0, 1, 2, \ldots$, we define the function $e_{k,\tau}$ on $\boldsymbol{R}$ by

$$e_{k,\tau}(x) = |\tau|^{1/4} e_k(\sqrt{|\tau|}x), \quad x \in \boldsymbol{R},$$

where e_k is the Hermite function given by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x)\,, \quad x \in \boldsymbol{R}\,,$$

and

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k (e^{-x^2})\,, \quad x \in \boldsymbol{R}\,.$$

For $j, k = 0, 1, 2, \ldots$, we define the function $e_{j,k,\tau}$ on $\boldsymbol{C}$ by

$$e_{j,k,\tau} = V_\tau(e_{j,\tau}, e_{k,\tau})\,,$$

and an easy computation gives

$$e_{j,k,1} = V_1(e_{j,1}, e_{k,1}) = V(e_j, e_k)\,,$$

where $V(e_j, e_k)$ is the classical Hermite function on $\boldsymbol{C}$ studied in [16].

The following result is an analog of [16, Proposition 21.1].

PROPOSITION 2.3. *The set* $\{e_{j,k,\tau}\,;\, j, k = 0, 1, 2, \ldots\}$ *is an orthonormal basis for* $L^2(\boldsymbol{C})$.

The following theorem gives a complete spectral analysis of L_τ, $\tau \in \boldsymbol{R} \setminus \{0\}$.

THEOREM 2.4. *For* $j, k = 0, 1, 2, \ldots$,

$$L_\tau e_{j,k,\tau} = (2k + 1)|\tau| e_{j,k,\tau}\,.$$

A proof of Theorem 2.4 can be modeled on the proof of [16, Theorem 22.2].

3. Essential self-adjointness. In this section we look at the sub-Laplacian $\mathcal{L}$ as an unbounded linear operator from $L^2(\boldsymbol{H})$ into $L^2(\boldsymbol{H})$ with dense domain given by $\mathcal{S}(\boldsymbol{H})$.

PROPOSITION 3.1. *$\mathcal{L}$ is a symmetric operator from $L^2(\boldsymbol{H})$ into $L^2(\boldsymbol{H})$ with dense domain $\mathcal{S}(\boldsymbol{H})$. In fact, it is positive.*

The proof follows from a simple integration by parts and is hence omitted. So, the sub-Laplacian $\mathcal{L}$ is closable and we denote its closure by $\mathcal{L}_0$. Thus, $\mathcal{L}_0$ is a closed, symmetric and positive operator from $L^2(\boldsymbol{H})$ into $L^2(\boldsymbol{H})$. In fact, from the work [10] of Masamune, $\mathcal{L}$ is essentially self-adjoint in the sense that it has a unique self-adjoint extension, which is the same as $\mathcal{L}_0$. Details on essential self-adjointness can be found in [11, Theorem X.23].

4. The spectrum. Let A be a closed linear operator from a complex Banach space X into X with dense domain $\mathcal{D}(A)$. Then the resolvent set $\rho(A)$ of A is defined to be the set of all complex numbers λ for which $A - \lambda I : \mathcal{D}(A) \to X$ is bijective, where I is the identity operator on X. The spectrum $\Sigma(A)$ is simply the complement of $\rho(A)$ in $\boldsymbol{C}$.

Following Yosida [17], the point spectrum $\Sigma_p(A)$ of A is the set of all complex numbers λ such that $A - \lambda I$ is not injective. The continuous spectrum $\Sigma_c(A)$ of A is the set of all complex numbers λ such that the range $R(A - \lambda I)$ of $A - \lambda I$ is dense in X, $(A - \lambda I)^{-1}$ exists, but is unbounded. The residual spectrum $\Sigma_r(A)$ of A is the set of all complex numbers λ such that $(A - \lambda I)^{-1}$ is bounded, but the range $R(A - \lambda I)$ is not dense in X. It is easy to see that $\Sigma_p(A)$, $\Sigma_c(A)$ and $\Sigma_r(A)$ are mutually disjoint and

$$\Sigma(A) = \Sigma_p(A) \cup \Sigma_c(A) \cup \Sigma_r(A)\,.$$

Moreover, it is well-known that if A is a self-adjoint operator on a complex and separable Hilbert space X, then

$$\Sigma_r(A) = \emptyset\,.$$

The precise description of the spectrum of the sub-Laplacian on the Heisenberg group is given by the following theorem.

THEOREM 4.1. $\Sigma(\mathcal{L}_0) = \Sigma_c(\mathcal{L}_0) = [0, \infty)$.

PROOF. We first prove that $\mathcal{L}_0$ has no eigenvalues in $[0, \infty)$. We know from the paper [4] that 0 is not an eigenvalue of $\mathcal{L}_0$. Now, let λ be a positive number such that there exists a function u in $L^2(\boldsymbol{H})$ for which

$$\mathcal{L}_0 u = \lambda u\,.$$

Then

$$L_\tau u^\tau = \lambda u^\tau\,.$$

But this implies that $u^\tau = 0$ for all τ in $\boldsymbol{R} \setminus \{0\}$ with

$$|\tau| \neq \lambda/(2k+1)\,, \quad k = 0, 1, 2, \dots\,.$$

This proves that $u = 0$ and hence we get a contradiction. Since $\mathcal{L}_0$ is self-adjoint, it follows that

$$\Sigma(\mathcal{L}_0) = \Sigma_c(\mathcal{L}_0)\,.$$

So, it remains to prove that $\mathcal{L}_0 - \lambda I$ is not surjective for all λ in $[0, \infty)$. Suppose that $\mathcal{L}_0 - \lambda_0 I$ is surjective for some λ_0 in $[0, \infty)$. Then λ_0 is in the resolvent set $\rho(\mathcal{L}_0)$ of $\mathcal{L}_0$. Hence there exists an open interval I_{λ_0} such that $\lambda_0 \in I_{\lambda_0}$ and $I_{\lambda_0} \subset \rho(\mathcal{L}_0)$. Let f be the function on $\boldsymbol{H}$ defined by

$$f(x, y, t) = h(x, y)e^{-t^2/2}\,, \quad x, y, t \in \boldsymbol{R}\,,$$

where h is an arbitrary function in $L^2(\boldsymbol{R}^2)$. Then for all λ in I_{λ_0}, we can find a function u_λ in $L^2(\boldsymbol{H})$ such that

$$(\mathcal{L}_0 - \lambda I)u_\lambda = f\,.$$

Taking the inverse Fourier transform with respect to t, we get

$$(L_\tau - \lambda I)u_\lambda^\tau = he^{-\tau^2/2}$$

for almost all τ in $\boldsymbol{R} \setminus \{0\}$. So, $L_\tau - \lambda I$ is surjective for all τ in a set S_λ for which the Lebesgue measure $m(\boldsymbol{R} \setminus S_\lambda)$ of $\boldsymbol{R} \setminus S_\lambda$ is zero. Now, let $\tau \in \bigcap_{r \in I_{\lambda_0} \cap \boldsymbol{Q}} S_r$, where $\boldsymbol{Q}$ is the set of all rational numbers. Then $L_\tau - \lambda I$ is surjective and hence injective for all λ in $I_{\lambda_0} \cap \boldsymbol{Q}$. Hence $L_\tau - \lambda I$ is bijective for all λ in I_{λ_0} by the fact that the resolvent set of L_τ is an open set. On the other hand, $L_\tau - \lambda I$ is one to one if and only if

$$\lambda \neq (2k + 1)|\tau|\,, \quad k = 0, 1, 2, \ldots\,.$$

This is a contradiction if we choose τ in $\bigcap_{r \in I_{\lambda_0}} S_r$ to be a sufficiently small number such that $(2k + 1)|\tau| \in I_{\lambda_0}$ for some nonnegative integer k. □

As an application of Theorem 4.1, we first give the various essential spectra that are useful to us. Let A be a closed linear operator densely defined on a complex Banach space X. The essential spectrum $\Sigma_{DS}(A)$ of A due to Dunford and Schwartz [5] is the set of all complex numbers λ such that $R(A - \lambda I)$ is not closed in X. Now, let $\Phi_W(A)$ be the set of all complex numbers λ such that $A - \lambda I$ is Fredholm and let $\Phi_S(A)$ be the set of all complex numbers λ such that $A - \lambda I$ is Fredholm with zero index. Then the essential spectrum $\Sigma_W(A)$ and the essential spectrum $\Sigma_S(A)$ of A due to, respectively, Wolf [14, 15] and Schechter [12] are defined by

$$\Sigma_W(A) = \boldsymbol{C} \setminus \Phi_W(A)$$

and

$$\Sigma_S(A) = \boldsymbol{C} \setminus \Phi_S(A)\,.$$

It is obvious that

$$\Sigma_{DS}(A) \subseteq \Sigma_W(A) \subseteq \Sigma_S(A)\,.$$

In the situation of the sub-Laplacian on the Heisenberg group, we have the following result.

THEOREM 4.2. *The equalities*

$$\Sigma_{DS}(\mathcal{L}_0) = \Sigma_W(\mathcal{L}_0) = \Sigma_S(\mathcal{L}_0) = [0, \infty)$$

hold.

PROOF. It is enough to prove that

$$[0, \infty) \subseteq \Sigma_{DS}(\mathcal{L}_0)\,.$$

Suppose that $\lambda \in [0, \infty)$ is not in $\Sigma_{DS}(\mathcal{L}_0)$. Then the range $R(\mathcal{L}_0 - \lambda I)$ of $\mathcal{L}_0 - \lambda I$ is closed in $L^2(\boldsymbol{H})$. By Theorem 4.2, $\lambda \in \Sigma_c(\mathcal{L}_0)$ and hence $R(\mathcal{L}_0 - \lambda I)$ is dense in $L^2(\boldsymbol{H})$. Therefore $\mathcal{L}_0 - \lambda I$ is bijective, i.e., $\lambda \in \rho(\mathcal{L}_0)$. This is a contradiction. □

REMARK 4.3. The technique in this paper can be used to compute the spectrum of the unique self-adjoint extension $\Delta_{\boldsymbol{H},0}$ of the Laplacian $\Delta_{\boldsymbol{H}}$ on the Heisenberg group $\boldsymbol{H}$ given by

$$\Delta_{\boldsymbol{H}} = -(X^2 + Y^2 + T^2)\,.$$

In fact,

$$\Sigma(\Delta_{\boldsymbol{H},0}) = \Sigma_c(\Delta_{\boldsymbol{H},0}) = [0, \infty)\,,$$

which is a result in [7], and furthermore,

$$\Sigma_{DS}(\Delta_{\boldsymbol{H},0}) = \Sigma_W(\Delta_{\boldsymbol{H},0}) = \Sigma_S(\Delta_{\boldsymbol{H},0}) = [0, \infty)\,.$$

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