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The spectrum on prism graph using circulant matrix

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ABSTRACT

Spectral graph theory discusses about the algebraic properties of graphs based on the spectrum of a graph. This article investigated the spectrum of prism graph. The method used in this research is the circulant matrix. The results showed that prism graph $P_{2,s}$ is a regular graph of degree 3, for s odd and s ≥ 3, $P_{2,s}$ is a circulantt graph with regular spectrum.

Introduction

Spectral theory is one of the studies in the field of algebra related to the eigenvalues and eigenvectors of an operator in a certain space. In its development, this theory can be applied to the study of graph theory. Spectral graphs theory examines a graph through algebraic properties such as the presence of characteristic polynomials, eigenvalues and eigenvectors from graph representation into a matrix. Some matrices that can represent a graph are adjacent matrix, Laplace matrix, signless Laplace matrix, normalized Laplace matrix and seidel adjacency matrix. One of the algebraic properties of a graph can be seen from its spectrum.

According to Biggs (1993), the graph spectrum is the arrangement of the eigenvalues of the graph representation matrix and its multiplicity. The spectrum of a type of graph has a regular form, but there is a spectrum of a type of graph that is irregular. One of the methods for determining the spectrum of graph is the circulant matrix (Biggs, 1993). The circulant matrix is a technique to change the adjacent graph matrix into a matrix in which the i th row elements, i = 2, 3, ..., n is obtained from the first row by rotating the first-row elements i - 1 steps to the right.

One of the important applications of the spectrum of a graph in quantum chemistry is calculating the energy levels of electrons in hydrocarbons. Also, the stability of such molecules is studied with the graph spectrum and corresponding eigenvectors. Meanwhile, in theoretical physics and quantum mechanics, spectral graph theory is used to minimize the energy of Hamiltonian systems. In general, physicists and chemists are interested in calculating the spectra from a graph for certain purpose, whereas graph theorists and combinatorialists are interested about the graph structure of a given spectra.

Theoretically, spectral graphs have been studied by Hasmawati (2008), Kristiana, *et al.* (2010), Nurshiami, *et al.* (2011), and Side, *et al.* (2013) to determine the shape of the spectrum in several classes of graphs. The graph spectrum that has been studied includes complete graph spectrum, Mobius ladder graph, cycle graph and hyperoctahedral graph. Furthermore, Selvia, *et. Al.* (2015) determined the spectrum of simple graphs using adjacent matrices, Laplacian matrices, signless Laplace matrices, normalized Laplace matrices and Seidel adjacency matrices. This research investigates the algebraic properties of graphs through the spectrum of k-regular graphs using circulant matrix.

Definitions and Notations

All the definitions and notations in this article refers to Biggs (1993) and Wilson *et al.* (1990).

Definition 1. *Graph G is defined as a pair of sets (V, E) denoted by G = (V, E), where V is a non-empty set of objects called vertices and E is a set of possible empty edges that connect a pair of vertices on the graph. If e is the edge joining point v_i with point v_j, then e can be written as $e = (v_i, v_j)$ where $v_i, v_j \in V$ and $(v_i, v_j) \in E$.*

Definition 2. *Two vertices v_i and v_j on graph G are said to be adjacent if (v_i, v_j) is an edge on graph G. For any edge $e = (v_i, v_j)$, edge e is said to be incident to vertex v_i and vertex v_j.*

Definition 3. *A graph is said to be regular with degree r (r-regular) if every vertex has degree r.*

Definition 4. *The Cartesian product of graphs G_1 and G_2 is a new graph which is denoted by $G = G_1 \times G_2$ with the set of vertices $(G) = V(G_1) \times V(G_2)$, i.e. each vertex in graph G is a pair (u, v), with $u \in V(G_1)$ and $v \in V(G_2)$, and two vertices (x, y) and (s, r) are adjacent in graph G if and only if $x = s$ and $(y, r) \in E(G_2)$ or $y = r$ and $(x, s) \in E(G_1)$.*

Definition 5. *Prism graph is a graph can be obtained by $P_2 \times C_s$. The prism graph $P_2 \times C_s$ will be denoted by $P_{2,s}$.*

Definition 6. *An Antiprism graph A_s is a graph that composed of outer cycle $v_1 v_2 \dots v_s$ and inner cycle $u_1 u_2 \dots u_s$, then between the two cycles are connected by edges $u_i v_i$ dan $u_i v_{1+i(mod\ s)}$ with $i = 1,2,3, \dots, s$.* (See Figure 1 for the illustration of Antiprism graph A_s.

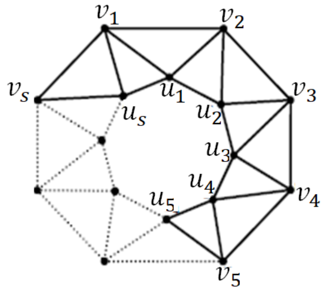

Figure 1. Antiprism graph A_s

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Definition 7. *Let G be a simple graph with n vertices. The adjacency matrix of graph G, denoted by* $A(G) = [a_{ij}]$ *is a square matrix with n vertices whose elements are 0 and 1 where* $a_{ij} = 1$ *if* $v_i s$ *adjacent to* v_j *and* $a_{ij} = 0$ *if the* v_i *is not adjacent to* v_j*.*

Definition 8. *Circulant matrix is an* $n \times n$ *matrix with i-th row elements,* $i = 2,3, \ldots, n$ *obtained from the first row by rotating the first-row elements* $i - 1$ *steps to the right.*

The general form of the circulant matrix C is as follows.

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ \vdots & c_1 & c_0 & \cdots & \vdots \\ c_{n-2} & \cdots & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

The circulant matrix has the input elements in the first row. Furthermore, the elements of the i-th row with $i = 2,3, \ldots, n$ are obtained from the elements of the row $(i - 1)$ with the nth column element as the first element of the i-th row, the i-th row element of the next column successively obtained from the elements of the row to $(i - 1)$ from column 1 to column to $(n - 1)$. A graph whose adjacency matrix can be converted into a circulant matrix is called a circulant graph (Biggs, 1993).

Definition 9. *Given the graphs G = (V,E) and H = (V,E). An isomorphic graph G with graph H, denoted by* $G \cong H$*, if there is a bijective mapping* $\boldsymbol{f}: \boldsymbol{V(G)} \rightarrow \boldsymbol{V(H)}$ *such that* $\forall(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{E(G)} \Leftrightarrow (\boldsymbol{f(u)}, \boldsymbol{f(v)}) \in \boldsymbol{E(H)}$*.*

Definition 10. *Spectrum of a graph G, denoted by* $Spec(G)$ *is the arrangement of the different eigenvalues of the adjacent matrix* $A(G)$ *and their multiplicity. For example, there are s different eigenvalues from A(G) namely* $\lambda_0, \lambda_1, \ldots, \lambda_{s-1}$ *with* $\lambda_0 > \lambda_1 > \cdots > \lambda_{s-1}$ *and* $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$ *is the multiplicity of each eigenvalue, then the spectrum of graph G is*

$$Spec(G) = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_{s-1}) \end{pmatrix}.$$

Proposition 1. *(Biggs, 1993). Let G be a r-regular graph with n vertices, then:*

1. *r is the eigenvalue of G;*
2. *if G is a connected regular graph, then the multiplicity of r is 1;*
3. *for each eigenvalue of G, then* $|\lambda| \leq r$*.*

Proposition 2. *(Biggs, 1993). Suppose S is a circulant matrix of size n×n whose first-row element is* $[s_1, s_2, s_3, \ldots, s_n]$*, then the eigenvalues are*

$$\lambda_r = \sum_{j=1}^{n} s_j \omega^{(j-1)r}, \text{ for every } r = 0,1, \ldots, n-1 \text{ with } \omega = e^{\left(\frac{2\pi}{n} i\right)}.$$

Proposition 3. *(Biggs, 1993). If G is a circulant graph with n vertices and the first-row element of the adjacency matrix is* $[0, a_2, a_3, \dots, a_n]$, *then the eigenvalues of graph G are*

$$\lambda_r = \sum_{j=2}^{n} a_j \omega^{(j-1)r} \quad , for\ every\ r = 0,1, \dots, n-1\ with\ \omega = e^{\left(\frac{2\pi}{n} i\right)}..$$

Method

The research method used is the study of literature and journals. The steps in this study are:

1. Constructing circulant matrix of prism graph $P_{2,s}$, $s \geq 3$;
2. Determining eigen value λ of circulant matrix;
3. Determining multiplicity of eigen value λ;
4. Investigating spectrum of prism graph $P_{2,s}$, $s \geq 3$.

Results and Discussion

Proposition 4. *If* s_{odd}, $s \geq 3$ *then the prism graph* $P_{2,s}$ *is a circulant graph.*

Proof of Proposition 4. Given a prism graph $P_{(2,s)}$ withn $s_{odd} \geq 3$ (See the illustration in Figure 2). The vertex and edge set $P_{(2,s)}$ can be defined by $V(P_{(2,s)}) = \{ v_i \mid 1 \leq i \leq 2s \}$ and $E(P_{(2,s)}) = \{ (v_i, v_{i+1}) \mid 1 \leq i \leq s-1\} \cup \{(v_1, v_s)\} \cup \{ (v_i, v_{i+1}) \mid s+1 \leq i \leq 2s-1\} \cup \{ (v_{s+1}, v_{2s}) \} \cup \{ (v_i, v_{i+s}) \mid 1 \leq i \leq s \}$.

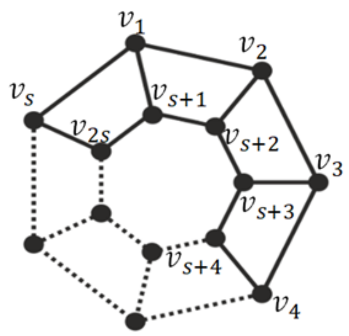

Figure 2. Prism graph $P_{(2.s)}$

Based on this definition, the form of the adjacency matrix $P_{(2,s)}$ can be expressed as

$$A(P_{(2,s)}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{s-1} \\ v_s \\ v_{s+1} \\ v_{s+2} \\ \vdots \\ v_{2s-2} \\ v_{2s-1} \\ v_{2s} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} v_1 & v_2 & v_3 & \dots & v_{s-1} & v_s & v_{s+1} & v_{s+2} & & v_{2s-2} & v_{2s-1} & v_{2s} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \ddots & 0 & 0 & 0 & 1 & \ddots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 1 & 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 & \ddots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 & \dots & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

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The adjacency matrix of a prism graph $A(P_{2,s})$ is not a circulant matrix. Therefore, the prism graph $P_{(2,s)}$ is reconstructed so that the adjacency matrix is a circulant matrix. Prism graph reconstruction $P_{(2,s)}$ is done by rotating the inner cycle graph by 180° and **enlarge it** to the outer cycle graph. See graph $Q_{(2,s)}$ in Figure 3.

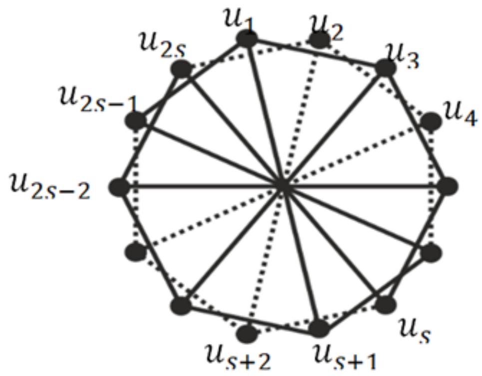

Figure 3. Graph $Q_{2,s} \cong P_{2,s}$

The general form of the adjacency matrix of the graph $Q_{2,s}$ is

$$A(Q_{2,s}) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_s \\ u_{s+1} \\ u_{s+2} \\ \vdots \\ u_{2s-2} \\ u_{2s-1} \\ u_{2s} \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} u_1 & u_2 & u_3 & u_4 & \cdots & u_s & u_{s+1} & u_{s+2} & & & u_{2s-1} & & \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \ddots & 0 & 0 & 1 & \ddots & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \ddots & 1 & 0 & 0 & \ddots & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \ddots & 0 & 0 & 1 & \ddots & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \ddots & 1 & 0 & 0 & \ddots & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \ddots & 0 & 0 & 1 & \ddots & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

The adjacency matrix of the graph $Q_{2,s}$ is a circulant matrix, so the prism graph $P_{2,s}$ for $s_{odd} \geq 3$ is a circulant graph. ■

Proposition 5. *For $s_{odd} \geq 3$,, the circulant prism graph $P_{2,s}$ has eigenvalues*

$$\lambda_r = 2\cos\left(\frac{2r\pi}{s}\right) + (-1)^r \text{ for every } r = 0, 1, 2, \ldots, 2s-1.$$

Proof of Proposition 5. Based on adjacency matrix of the graph $Q_{2,s}$, the first-row element of this matrix is $[a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_{s-1} \quad a_s \quad a_{s+1} \quad a_{s+2} \quad \ldots \quad a_{2s-2} \quad a_{2s-1} \quad a_{2s}]$, with $s_{odd} \geq 3$ and

$$a_{1j} = \begin{cases} 1 & \text{, for } j = 3, j = s+1, \text{ and } j = 2s-1 \\ 0 & \text{, for other } j \end{cases}$$

Based on Proposition 3, eigen value λ_r of this matrix for every $r = 0,1,\dots,n-1$ successively can be determined as follows

$$\begin{aligned}\lambda_0 &= \sum_{j=2}^{2s} a_j\omega^{(j-1)0}\\
&= a_2\omega^{(2-1)0} + a_3\omega^{(3-1)0} + a_4\omega^{(4-1)0} + \cdots + a_{s-2}\omega^{(s-3)0} + a_{s-1}\omega^{(s-2)0} + a_s\omega^{(s-1)0} +\\
&\quad a_{s+1}\omega^{(s)0} + a_{s+2}\omega^{(s+1)0} + \cdots + a_{2s-2}\omega^{(2s-3)0} + a_{2s-1}\omega^{(2s-2)0} + a_{2s}\omega^{(2s-1)0}\\
&= a_2\omega^0 + a_3\omega^0 + a_4\omega^0 + \cdots + a_{s-2}\omega^0 + a_{s-1}\omega^0 + a_s\omega^0 + a_{s+1}\omega^0 + a_{s+2}\omega^0 + \cdots + a_{2s-2}\omega^0 +\\
&\quad a_{2s-1}\omega^0 + a_{2s}\omega^0\\
&= 0.\omega^0 + 1.\omega^0 + 0.\omega^0 + \cdots + 0.\omega^0 + 0.\omega^0 + 0.\omega^0 + 1.\omega^0 + 0.\omega^0 + \cdots + 0.\omega^0 + 1.\omega^0 + 0.\omega^0\\
&= 0 + 1 + 0 + \cdots + 0 + 0 + 0 + 1 + 0 + \cdots + 0 + 1 + 0\\
&= 3.\end{aligned}$$

$$\begin{aligned}\lambda_1 &= \sum_{j=2}^{2s} a_j\omega^{(j-1)1}\\
&= a_2\omega^{(2-1)1} + a_3\omega^{(3-1)1} + a_4\omega^{(4-1)1} + \cdots + a_{s-2}\omega^{(s-3)1} + a_{s-1}\omega^{(s-2)1} + a_s\omega^{(s-1)1} +\\
&\quad a_{s+1}\omega^{(s)1} + a_{s+2}\omega^{(s+1)1} + \cdots + a_{2s-2}\omega^{(2s-3)1} + a_{2s-1}\omega^{(2s-2)1} + a_{2s}\omega^{(2s-1)1}\\
&= a_2\omega^1 + a_3\omega^2 + a_4\omega^3 + \cdots + a_{s-2}\omega^{(s-3)} + a_{s-1}\omega^{(s-2)} + a_s\omega^{(s-1)} + a_{s+1}\omega^{(s)} + a_{s+2}\omega^{(s+1)} +\\
&\quad \cdots + a_{2s-2}\omega^{(2s-3)} + a_{2s-1}\omega^{(2s-2)} + a_{2s}\omega^{(2s-1)}\\
&= 0.\omega^1 + 1.\omega^2 + 0.\omega^3 + \cdots + 0.\omega^{(s-3)} + 0.\omega^{(s-2)} + 0.\omega^{(s-1)} + 1.\omega^{(s)} + 0.\omega^{(s+1)} + \cdots +\\
&\quad 0.\omega^{(2s-3)} + 1.\omega^{(2s-2)} + 0.\omega^{(2s-1)}\\
&= 0 + \omega^2 + 0 + \cdots + 0 + 0 + 0 + \omega^s + 0 + \cdots + 0 + \omega^{(2s-2)} + 0\\
&= \omega^2 + \omega^s + \omega^{2s}\omega^{-2}\end{aligned}$$

By substituting $= e^{\left(\frac{2\pi i}{2s}\right)} = \cos\frac{2\pi}{2s} + i\sin\frac{2\pi}{2s}$,

to $\omega^{2s} = \left(e^{\left(\frac{2\pi i}{2s}\right)}\right)^{2s} = \cos 2\pi + i\sin 2\pi = 1 + 0 = 1$, then it is obtained that

$$\begin{aligned}\lambda_1 &= \left(e^{\frac{2\pi i}{2s}}\right)^2 + \left(e^{\frac{2\pi i}{2s}}\right)^s + \left(e^{\frac{2\pi i}{2s}}\right)^{-2}\\
&= \cos\frac{2.2\pi}{2s} + i\sin\frac{2.2\pi}{2s} + \cos\frac{2s\pi}{2s} + i\sin\frac{2s\pi}{2s} + \cos\frac{2.2\pi}{2s} - i\sin\frac{2.2\pi}{2s}\\
&= 2\cos\frac{2.2\pi}{2s} + \cos\frac{2s\pi}{2s} + i\sin\frac{2s\pi}{2s}\\
&= 2\cos\frac{2\pi}{s} + \cos\pi + i\sin\pi\\
&= 2\cos\frac{2\pi}{s} + (-1) + 0\\
&= 2\cos\frac{2\pi}{s} + (-1).\end{aligned}$$

$$\begin{aligned}\lambda_2 &= \sum_{j=2}^{2s} a_j\omega^{(j-1)2}\\
&= a_2\omega^{(2-1)2} + a_3\omega^{(3-1)2} + a_4\omega^{(4-1)2} + \cdots + a_{s-2}\omega^{(s-3)2} + a_{s-1}\omega^{(s-2)2} + a_s\omega^{(s-1)2} +\\
&\quad a_{s+1}\omega^{(s)2} + a_{s+2}\omega^{(s+1)2} + \cdots + a_{2s-2}\omega^{(2s-3)2} + a_{2s-1}\omega^{(2s-2)2} + a_{2s}\omega^{(2s-1)2}\\
&= a_2\omega^2 + a_3\omega^4 + a_4\omega^6 + \cdots + a_{s-2}\omega^{(2s-6)} + a_{s-1}\omega^{(2s-4)} + a_s\omega^{(2s-2)} + a_{s+1}\omega^{(2s)} +\\
&\quad a_{s+2}\omega^{(2s+2)} + \cdots + a_{2s-2}\omega^{(4s-6)} + a_{n-1}\omega^{(4s-4)} + a_n\omega^{(4s-2)}\\
&= 0.\omega^2 + 1.\omega^4 + 0.\omega^6 + \cdots + 0.\omega^{(2s-6)} + 0.\omega^{(2s-4)} + 0.\omega^{(2s-2)} + 1.\omega^{(2s)} + 0.\omega^{(2s+2)} + \cdots +\\
&\quad 0.\omega^{(4s-6)} + 1.\omega^{(4s-4)} + 0.\omega^{(4s-2)}\\
&= 0 + \omega^4 + 0 + \cdots + 0 + 0 + 0 + \omega^{2s} + 0 + \cdots + 0 + \omega^{(4s-4)} + 0\\
&= \omega^4 + \omega^{2s} + \omega^{4s}\omega^{-4}\end{aligned}$$

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$$= \left(e^{\frac{2\pi i}{2s}}\right)^{4} + \left(e^{\frac{2\pi i}{2s}}\right)^{2s} + \left(e^{\frac{2\pi i}{2s}}\right)^{-4}$$
$$= \cos\frac{2.4\pi}{2s} + i\sin\frac{2.4\pi}{2s} + \cos\frac{2.2s\pi}{2s} + i\sin\frac{2.2s\pi}{2s} + \cos\frac{2.4\pi}{2s} - i\sin\frac{2.4\pi}{2s}$$
$$= 2\cos\frac{2.4\pi}{2s} + \cos\frac{2.2s\pi}{2s} + i\sin\frac{2.2s\pi}{2s}$$
$$= 2\cos\frac{2.2\pi}{s} + \cos 2\pi + i\sin 2\pi$$
$$= 2\cos\frac{2.2\pi}{s} + 1 + 0$$
$$= 2\cos\frac{2.2\pi}{s} + 1.$$

$$\lambda_3 = \sum_{j=2}^{2s} a_j\omega^{(j-1)3}$$
$$= a_2\omega^{(2-1)3} + a_3\omega^{(3-1)3} + a_4\omega^{(4-1)3} + \cdots + a_{s-2}\omega^{(s-3)3} + a_{s-1}\omega^{(s-2)3} + a_s\omega^{(s-1)3} + a_{s+1}\omega^{(s)3} + a_{s+2}\omega^{(s+1)3} + \cdots + a_{2s-2}\omega^{(2s-3)3} + a_{2s-1}\omega^{(2s-2)3} + a_{2s}\omega^{(2s-1)3}$$
$$= a_2\omega^3 + a_3\omega^6 + a_4\omega^9 + \cdots + a_{s-2}\omega^{(3s-9)} + a_{s-1}\omega^{(3s-6)} + a_s\omega^{(3s-3)} + a_{s+1}\omega^{(3s)} + a_{s+2}\omega^{(3s+3)} + \cdots + a_{2s-2}\omega^{(6s-9)} + a_{2s-1}\omega^{(6s-6)} + a_{2s}\omega^{(6s-3)}$$
$$= 0.\omega^3 + 1.\omega^6 + 0.\omega^9 + \cdots + 0.\omega^{(3s-9)} + 0.\omega^{(3s-6)} + 0.\omega^{(3s-3)} + 1.\omega^{(3s)} + 0.\omega^{(3s+3)} + \cdots + 0.\omega^{(6s-9)} + 1.\omega^{(6s-6)} + 0.\omega^{(6s-3)}$$
$$= 0 + \omega^6 + 0 + \cdots + 0 + 0 + 0 + \omega^{3s} + 0 + \cdots + 0 + \omega^{(6s-6)} + 0$$
$$= \omega^6 + \omega^{3s} + \omega^{6s}\omega^{-6}$$
$$= \left(e^{\frac{2\pi i}{2s}}\right)^{6} + \left(e^{\frac{2\pi i}{2s}}\right)^{3s} + \left(e^{\frac{2\pi i}{2s}}\right)^{-6}$$
$$= \cos\frac{2.6\pi}{2s} + i\sin\frac{2.6\pi}{2s} + \cos\frac{2.3s\pi}{2s} + i\sin\frac{2.3s\pi}{2s} + \cos\frac{2.6\pi}{2s} - i\sin\frac{2.6\pi}{2s}$$
$$= 2\cos\frac{2.6\pi}{2s} + \cos\frac{2.3s\pi}{2s} + i\sin\frac{2.3s\pi}{2s}$$
$$= 2\cos\frac{2.3\pi}{s} + \cos 3\pi + i\sin 3\pi$$
$$= 2\cos\frac{2.3\pi}{s} + (-1) + 0$$
$$= 2\cos\frac{2.3\pi}{s} + (-1).$$

By substituting $r = 2s - 1$ then the obtained eigen value of λ_{2s-1} as follows

$$\lambda_{2s-1} = \sum_{j=2}^{2s} a_j\omega^{(j-1)(2s-1)}$$
$$= a_2\omega^{(2-1)(2s-1)} + a_3\omega^{(3-1)(2s-1)} + a_4\omega^{(4-1)(2s-1)} + \cdots + a_{s-2}\omega^{(s-3)(2s-1)} + a_{s-1}\omega^{(s-2)(2s-1)} + a_s\omega^{(s-1)(2s-1)} + a_{s+1}\omega^{(s)(2s-1)} + a_{s+2}\omega^{(s+1)(2s-1)} + \cdots + a_{2s-2}\omega^{(2s-3)(2s-1)} + a_{2s-1}\omega^{(2s-2)(2s-1)} + a_{2s}\omega^{(2s-1)(2s-1)}$$
$$= a_2\omega^{(2s-1)} + a_3\omega^{2(2s-1)} + a_4\omega^{3(2s-1)} + \cdots + a_{s-2}\omega^{(2s^2-s-6s+3)} + a_{s-1}\omega^{(2s^2-s-4s+2)} + a_s\omega^{(2s^2-s-2s+1)} + a_{s+1}\omega^{(2s^2-s)} + a_{s+2}\omega^{(2s^2-s+2s-1)} + \cdots + a_{2s-2}\omega^{2s(2s-1)-3(2s-1)} + a_{2s-1}\omega^{2s(2s-1)-2(2s-1)} + a_{2s}\omega^{2s(2s-1)-(2s-1)}$$
$$= 0.\omega^{(2s-1)} + 1.\omega^{2(2s-1)} + 0.\omega^{3(2s-1)} + \cdots + 0.\omega^{(2s^2-s-6s+3)} + 0.\omega^{(2s^2-s-4s+2)} + 0.\omega^{(2s^2-s-2s+1)} + 1.\omega^{s(2s-1)} + 0.\omega^{(2s^2-s+2s-1)} + \cdots + 0.\omega^{2s(2s-1)-3(2s-1)} + 1.\omega^{2s(2s-1)-2(2s-1)} + 0.\omega^{2s(2s-1)-(2s-1)}$$
$$= 0 + \omega^{2(2s-1)} + 0 + \cdots + 0 + 0 + 0 + \omega^{s(2s-1)} + 0 + \cdots + 0 + \omega^{2s(2s-1)-2(2s-1)} + 0$$
$$= \omega^{2(2s-1)} + \omega^{s(2s-1)} + \omega^{2s(2s-1)}\omega^{-2(2s-1)}$$
$$= \left(e^{\frac{2\pi i}{2s}}\right)^{2(2s-1)} + \left(e^{\frac{2\pi i}{2s}}\right)^{s(2s-1)} + \left(e^{\frac{2\pi i}{2s}}\right)^{-2(2s-1)}$$

$$= \cos\frac{2.2(2s-1)\pi}{2s} + i\sin\frac{2.2(2s-1)\pi}{2s} + \cos\frac{2s(2s-1)\pi}{2s} + i\sin\frac{2s(2s-1)\pi}{2s} + \cos\frac{2.2(2s-1)\pi}{2s} - i\sin\frac{2.2(2s-1)\pi}{2s}$$
$$= 2\cos\frac{2.2(2s-1)\pi}{2s} + \cos\frac{2s(2s-1)\pi}{2s} + i\sin\frac{2s(2s-1)\pi}{2s}$$
$$= 2\cos\frac{2(2s-1)\pi}{s} + \cos(2s-1)\pi + i\sin(2s-1)\pi$$
$$= 2\cos\frac{2(2s-1)\pi}{s} + (-1)^{(2s-1)} + 0$$
$$= 2\cos\frac{2(2s-1)\pi}{s} + (-1)^{(2s-1)}.$$

So that it can be concluded to find the r-th eigenvalue of the prism graph $P_{2,s}$ ($s_{odd} \geq 3$) we can use the general formula as follows

$$\lambda_r = 2\cos\left(\frac{2r\pi}{s}\right) + (-1)^r \text{ for all } r = 0,1,2,\dots,2s-1. \qquad \blacksquare$$

Proposition 6. *For $s_{odd} \geq 3$, the spectrum of the circulant prism graph $P_{2,s}$ is as follow*

$$Spec\ P_{2,s} = \begin{pmatrix} \lambda_0 & \lambda_{1,2s-1} & \lambda_{2,2s-2} & \cdots & \lambda_{s-1,s+1} & \lambda_s \\ 1 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

Proof of Proposition 6. Based on Proposition 5, the *r*-th eigen value of $P_{2,s}$ $s_{odd} \geq 3$ is $\lambda_r = 2\cos\left(\frac{2r\pi}{s}\right) + (-1)^r$, for all $r = 0, 1, 2, \dots, 2s-1$.

Therefore, the graph spectrum $P_{(2,s)}$ is sought successively for each value of s as follows

(1) Spectrum of $P_{2,3}$

Eigen values $P_{2,3)}$ with $s = 3$ and $r = 0,1,2,\dots,5$ it is obtained $\lambda_0 = 3$ with multiplicity 1, $\lambda_1 = \lambda_5$, $\lambda_2 = \lambda_4$ and $\lambda_3 = 1$ with multiplicity 1.

$$\text{Spec } P_{2,3} = \begin{pmatrix} \lambda_0 & \lambda_{1,5} & \lambda_{2,4} & \lambda_3 \\ 1 & 2 & 2 & 1 \end{pmatrix}$$

(2) Spectrum of $P_{2,5}$

Eigen values $Q_{2,5}$ with $s = 5$ and $r = 0,1,2,\dots,9$, it is obtained

$$\text{Spec } P_{2,5} = \begin{pmatrix} \lambda_0 & \lambda_{1,9} & \lambda_{2,8} & \lambda_{3,7} & \lambda_{4,6} & \lambda_5 \\ 1 & 2 & 2 & 2 & 2 & 1 \end{pmatrix}$$

(3) Graph spectrum $P_{(2,7)}$ with $s = 7$ and $r = 0,1,2,\dots,13$, it is obtained

$$\text{Spec } P_{2,7} = \begin{pmatrix} \lambda_0 & \lambda_{1,13} & \lambda_{2,12} & \lambda_{3,11} & \lambda_{4,10} & \lambda_{5,9} & \lambda_{6,8} & \lambda_7 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \end{pmatrix}$$

Based on these results, it is obtained that the circulant prism graph $P_{2,s}$ ($s_{odd} \geq 3$) has $(s+1)$ different eigen values. The eigenvalue pattern is different from each graph $P_{2,s}$ is $\lambda_0 = 3$ with multiplicity 1, $\lambda_s = 1$ with multiplicity 1, and other eigenvalues obtained by finding the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{s-1}$ with multiplicity 2. So it can be concluded that the general shape of the prism spectrum graph $P_{2,s}$, for $s_{odd} \geq 3$ obtained from the different eigenvalues and their multiplicity, namely: $\text{Spec } P_{2,s} = \begin{pmatrix} \lambda_0 & \lambda_{1,2s-1} & \lambda_{2,2s-2} & \cdots & \lambda_{s-1,s+1} & \lambda_s \\ 1 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix}$. $\blacksquare$

Conclusion

Based on the results of the discussion, the following conclusions are obtained:

(1) Prism graph $P_{(2,s)}$ is a circulant graph;

(2) Prism graph is a regular graph of maximum degree 3 with one of its eigenvalues is 3 with a multiplicity of 1;

(3) Prism graph $P_{(2,s)}$ with $s_{odd} \geq 3$ is a circulant graph with

$$Spec\ P_{(2,s)} = \begin{pmatrix} \lambda_0 & \lambda_{1,2s-1} & \lambda_{2,2s-2} & \cdots & \lambda_{s-1,s+1} & \lambda_s \\ 1 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

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