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The Spread of the Potential on a Homogeneous Tree (*).

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Summary. – *We compute explicitly the solution of the heat equation on a homogeneous tree Γ whose edges have suitable positive conductances and are identified with copies of segments* $[0, 1]$ *with the condition that the sum of the weighted normal exterior derivatives is* 0 *at every node (Kirchhoff type condition). Furthermore we find the expression of the semigroup of linear operators on* $L^2(\Gamma, c)$ *having* Δ *as infinitesimal generator. These results derive from the equation governing the spread of the potential along the dendrites of a neuron.*

1. – Introduction.

The aim of this paper is to study the equation governing the spread of the potential

(1) $$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} - V$$

on an infinite homogeneous tree whose edges have suitable positive conductances and are identified with copies of segments $[0, 1]$ with the condition that the sum of the weighted normal exterior derivatives is 0 at every node (Kirchhoff type condition).

Problems of this type arise studying the model introduced to determine the spread of the potential along neurons. Neurons are the main components of nervous tissue, and they have a very irregular shape. In every neuron we can distinguish a cellular body containing the nucleus and prolongations having different shape and structure called axon, dendrites and additional prolongations (sometimes absent). The axon is usually composed by only one prolongation whose diameter remains constant up to the terminal ramification that can also happen at a big distance from the cellular body. The equations governing the spread of the potential along the axon are not linear. On the other hand, the dendrites are the lines of transmission of the potential where the in-

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duction is negligible (see e.g. J. RINZEL-W. RALL, ref. [15], [16]). In abstract, we can compare the dendrites to cylinders having variable diameter and joined to form a structure that we can represent as a tree (i.e. a connected graph without circuits). This tree can be considered as a particular case of a ramified space, according to the definition introduced by S. NICAISE (see ref. [9]).

The equations governing the spread of the potential on every edge of the dendritic tree are (see e.g. J. RINZEL-W. RALL, ref. [15], [16]):

$$ri = -\frac{\partial V}{\partial x}, \qquad \frac{\partial i}{\partial x} = -c\frac{\partial V}{\partial t} - gV, \tag{2}$$

where

i is the axial current,

V is the interior potential,

r is the resistance per unit length,

c is the capacity per unit length,

g is the conductibility per unit length.

Moreover the potential must be continuous and the sum of the currents arriving at every node must be 0. By a suitable change of variables in equations (2) one obtains equation (1) and the weighted Kirchhoff type conditions which appear in equation (12) below.

The study of equation (1) on finite trees or, more generally, on finite graphs has been dealt by S. NICAISE and J. P. ROTH in the framework of the abstract theory introduced by G. LUMER (see ref. [5]). In particular, S. NICAISE determined the spectrum of the operator $\Delta u = \partial^2 u/\partial x^2$ with Kirchhoff type conditions, Dirichlet type conditions, and mixed type conditions on finite graphs or on Z_+, the infinite graph represented by the one way path (see ref. [8], [10]). J. P. ROTH constructed the fundamental solution of the heat equation on finite graphs and applied it to the study of the asymptotic behaviour of the functions associated with the spectrum of the operator Δ (see ref. [17]). The operator Δ is a particular case of the operator L defined on graphs by $Lu = -(\partial/\partial x)(p(x)(\partial u/\partial x)) + q(x)\,u$, for every edge. The operator L arises in a natural way in various physical problems (oscillations of elastic nets, oscillations in hydraulic and electrical networks, electron oscillations of complex molecules, etc.). The operator L on finite graphs has been studied by several authors. For example see I. G. KARELINA, O. M. PENKIN, YU. V. POKORNYI (see ref. [13], [14]).

In this paper we will determine the solution of the spread equation (1) with Kirchhoff type condition at every node, when Γ is an infinite homogeneous tree of degree $q \geqslant 3$ (i.e. every vertex has exactly q edges branching out from it) and every edge of Γ has a positive conductance. Biologically, this represents a very simplified model with no Dirichlet type conditions (in particular there are no terminal nodes). We observe that the method adopted in this paper can be extended to more general trees, for example, to trees whose vertices have uniformly bounded degrees.

In section 2 we introduce our notation and the basic facts. Namely, we describe the tree Γ as a CW-complex, so that we can introduce a natural topology on Γ. Furthermore we compute the number of all the paths joining two points of Γ, and we illustrate the conditions on the conductances assigned to the edges of Γ (these conditions are more general than in the biological model of J. RINZEL and W. RALL (see ref. [15], [16])).

In section 3 we define the spaces $L^2(\Gamma, c)$ and $H^m(\Gamma, c)$. We also illustrate the variational formulation of our problem. In particular we show how we can define the Laplacian Δ on a tree using the setting of J. L. LIONS (see ref. [4]), and we describe the domain and the basic properties of Δ.

In section 4 we introduce the fundamental solution of the heat equation on a homogeneous tree. First we define such a solution abstractly and we study its regularity properties. Then, in section 5, we verify that this function is actually a fundamental solution. Moreover, we observe that if there exists a function with the properties characterizing the fundamental solution of the heat equation, then this function is just the function defined in section 4. In order to do this, we generalize the method adopted by J. P. ROTH (see ref. [17]) for finite graphs and so we also obtain that the fundamental solution of the heat equation on Γ is the sum of two terms: the source solution of the heat equation on R (solution suitably weighted) and a series obtained by imposing Kirchhoff type conditions at every node of Γ.

We notice that a construction of the heat kernel for homogeneous trees appears in a paper of B. GAVEAU - M. OKADA - T. OKADA (see ref. [2]) under the more restrictive assumption that all the conductances are equal to 1. In this paper of B. GAVEAU-M. OKADA - T. OKADA, the heat kernel is expressed as a contour integral and some asymptotic formulas are deduced (see also T. OKADA, ref. [11]).

In final section 6, we obtain the solution in $H^2(\Gamma, c)$ of the Cauchy problem associated to the spread equation of the potential i.e., the solution of the following Cauchy problem on the Hilbert space $L^2(\Gamma, c)$

$$\text{(3)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - u\,, \quad t > 0\,, \\ u(0) = f\,, \end{cases}$$

In order to do this, first we determine the solution in $H^2(\Gamma, c)$ of the following Cauchy problem on the Hilbert space $L^2(\Gamma, c)$

$$\text{(3')} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u\,, \quad t > 0\,, \\ u(0) = f\,. \end{cases}$$

We prove that, for every f in $L^2(\Gamma, c)$, the solution of (3′), denoted by $P_t f$, is expressed by integrating on Γ the initial value f against the fundamental solution, and moreover it turns out that P_t is the semigroup of linear operators on $L^2(\Gamma, c)$ having Δ as infinitesimal generator. Then, by the general theory of semigroups (see e.g. A. PAZY, ref. [12]), we know that the solution of (3) is $(\exp(-t))P_t f$.

2. - Notation.

Let $\Gamma = (V, E)$ be a homogeneous tree of degree $q \geq 3$ where V is the vertex set and E the edge set. Let us fix a reference vertex o in V which we will also call root of Γ.

We call geodesic distance between two vertices v and v' the smallest number of edges joining v to v'.

We say that two edges e and e' are neighbours, we write $e \sim e'$, if they have a common endpoint (i.e. a common vertex). We denote by E_e' the set of all the edges which are neighbours of e (note that E_e' has $2(q-1)$ elements).

We identify every edge e of Γ with the real interval $[0, 1]$, for istance in such a way that the endpoint having the smaller geodesic distance from o is identified with 0 (the other endpoint of the edge e is then identified with 1). In this way we associate with Γ a CW-complex having only cells of dimensions 0 and 1 (see e.g. J. R. MUNKRES, ref. [7]). The cells of dimension 1 are the intervals $[0, 1]$ which we have identified with the edges e of E, and the cells of dimension 0 correspond to the vertices v of V. The same letter Γ will be used from now on to denote the associated CW-complex . Note that Γ is a metric space in a natural way.

We can orient every edge e of Γ in two opposite ways. We call an arc an oriented edge and we denote by A the set of all the arcs of Γ. For every edge e, we denote by $+e$ the orientation (arc) of e such that the first endpoint has smaller geodesic distance from o than the second endpoint, and by $-e$ the opposite arc. We sometimes write $|e|$ to denote the unoriented edge e. If no confusion can arise, we denote by e both the oriented and the unoriented edge e. For every arc e we denote by $I(e)$ the initial vertex of e and by $T(e)$ the terminal vertex. So $I(+e) = T(-e)$ and $T(+e) = I(-e)$.

We define a path C to be a finite sequence of arcs $(e_0, \ldots, e_m)$ such that $T(e_j) = I(e_{j+1})$ for $0 \leq j \leq m-1$. We denote by $-C$ the path obtained by reversing all the orientations of the arcs of C i.e., if

$$C = (e_0, \ldots, e_m)$$

then

$$-C = (-e_m, \ldots, -e_0)$$

We call length of the path C, denoted by $l(C)$, the number of the arcs of C.

For every point x of Γ, we denote by E_x the set of all the edges containing x, and by E_x' the set of all the edges which are neighbours of the edges of E_x. So, if x is in V, then E_x has q elements, the q edges branching out from x, while if x is not in V, then E_x has only one element, the edge e containing x and moreover in this case, E_x' is equal to E_e'.

If x and y are points of the same edge, let us denote by $d(x, y)$ the (euclidean) distance between x and y.

Let x and y be points of Γ. We call path joining x to y a path whose first arc is one of the arcs obtained from the edges of E_x and whose last arc is one of the arcs obtained from the edges of E_y. We observe that if x and y belong to the same edge e, then there exist two paths joining x to y made by only one arc (i.e. $+e$, $-e$).

We call geodesic path joining x to y a path joining x to y having minimum length. We observe that if x and y do not belong to the same edge, then there exists only one geodesic path joining x to y. We will denote it by C^*. Note that $-C^*$ is the only geodesic path joining y to x. On the other hand, if x and y belong to the same edge e, then there exist two geodesic paths joining x to y (i.e. $+e$, $-e$), and moreover they are also the geodesic paths joining y to x.

Set

$$\varrho(x, y) = \begin{cases} 0 & \text{if } x \text{ and } y \text{ belong to the same edge}, \\ l(C^*) - 2 & \text{otherwise}. \end{cases} \tag{4}$$

Let x be a point of an edge e and let y be a point of an edge e'. We denote by $\boldsymbol{C}_m(e, e')$ the set of all the paths having length m, joining x to y and whose first arc is one of two arcs obtained by e and whose last arc is one of two arcs obtained by e' i.e.,

$$\boldsymbol{C}_m(e, e') = \{C \in \boldsymbol{C}\colon l(C) = m \quad \text{and} \quad C = (\pm e, e_1, \ldots, e_{m-2}, \pm e')\} \tag{5}$$

where $\boldsymbol{C}$ will denote throughout the paper the set of all the paths of Γ.

LEMMA 1. – *For all* m

$$\operatorname{card}(\boldsymbol{C}_m(e, e')) \leq 2q^{m-1}.$$

PROOF. – We denote by $\boldsymbol{C}_m(e)$ the set of all the paths having length m and whose first arc is one of two arcs obtained by the edge e i.e.,

$$\boldsymbol{C}_m(e) = \{C \in \boldsymbol{C}\colon l(C) = m \quad \text{and} \quad C = (\pm e, e_1, \ldots, e_{m-1})\}.$$

The paths of $\boldsymbol{C}_m(e, e')$ are particular paths of $\boldsymbol{C}_m(e)$, so it is enough to evaluate the cardinality of $\boldsymbol{C}_m(e)$. We observe that $\boldsymbol{C}_1(e)$ has exactly 2 elements. We fix a path $(\pm e, e_1, \ldots, e_{m-1})$ of $\boldsymbol{C}_m(e)$ and we consider the q paths $\{(\pm e, e_1, \ldots, e_{m-1}, \hat{e}_j), 1 \leq j \leq q\}$ of $\boldsymbol{C}_{m+1}(e)$ obtained by adding to the arcs of this path $(\pm e, e_1, \ldots, e_{m-1})$ any one of the arcs $\hat{e}_j$ obtained by the q edges of $E_{T(e_{m-1})}$ and having $I(\hat{e}_j) = T(e_{m-1})$. We get immediately that

$$\operatorname{card}(\boldsymbol{C}_m(e, e')) \leq \operatorname{card}((\boldsymbol{C}_m(e)) = 2q^{m-1}. \qquad \blacksquare$$

We suppose assigned to every edge e of Γ a positive conductance $c(e)$ with the following condition

$$\frac{c(e)}{c(e')} \leq \kappa \quad \text{if the edges } e, e' \text{ are neighbours} \tag{6}$$

where κ is a positive constant such that $\kappa \geq 1$.

For every vertex v of Γ we denote by $c(v)$ the sum of the conductances of all the edges branching out from v i.e.

$$c(v) = \sum_{e \in E_v} c(e). \tag{7}$$

We call transfer coefficient from the arc e to the arc e', the following quantity

$$\varepsilon_{e,e'} = \begin{cases} 2c(|e|)/c(T(e)) & \text{if } T(e) = I(e'),\ e' \neq -e, \\ 2c(|e|)/c(T(e)) - 1 & \text{if } T(e) = I(e'),\ e' = -e, \\ 0 & \text{if } T(e) \neq I(e'). \end{cases} \tag{8}$$

We observe that

$$|\varepsilon_{e,e'}| \leq \varepsilon = \max\{1, 2\kappa/q\} \quad \text{for all arcs } e, e'. \tag{8'}$$

For every path $C = (e_0, \ldots, e_m)$ of Γ, we denote by ε_C the product of the transfer coefficients of all the pairs of consecutive arcs of C i.e.

$$\varepsilon_C = \prod_{j=0}^{m-1} \varepsilon_{e_j, e_{j+1}}. \tag{8''}$$

We denote by R_+ the set of the real numbers which are strictly positive.
Set

$$k(t,x) = \begin{cases} \dfrac{1}{2\sqrt{\pi t}} \exp(-x^2/4t) & \text{if } (t,x) \in R_+ \times R, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

$$h(t,x) = \frac{x}{t} k(t,x). \tag{10}$$

The function k is the source solution of the heat equation on R (see e.g. D. V. WIDDER, ref. [19]). We denote by the symbol $\star$ the convolution with respect to the time t. It is well known that h and k are C^∞ functions which satisfy the heat equation on $R^2 \setminus (0,0)$ and that

$$\begin{cases} k(t, x_1 + x_2) = k(t, x_1) \star h(t, x_2), \\ h(t, x_1 + x_2) = h(t, x_1) \star h(t, x_2), \end{cases} \tag{11}$$

(see e.g. G. DOETSCH, ref. [1]).

3. - The Laplacian on Γ.

We will identify any function u on Γ with a collection $\{u_e\}_{e \in E}$ of functions u_e defined on the edges e of Γ. Note that u_e can be considered a function on [0, 1]. In fact, we will use the same notation u_e to denote both the function on the edge e and the function on the real interval [0, 1] identified with e.

The integral on Γ of a positive function u is defined as follows

$$\int_{\Gamma} u(x)\,dx = \sum_{e\in E} c(e) \int_{e} u_e(x)\,dx = \sum_{e\in E} c(e) \int_0^1 u_e(x)\,dx\,.$$

We define the space $L^2(\Gamma, c)$ as the space of all the functions u on Γ such that $u_e \in L^2((0, 1))$ for every e in E, and $\sum_{e\in E} c(e)\|u_e\|^2_{L^2((0,1))} < \infty$.

Analogously, for every integer $m > 0$, we define the Sobolev space $H^m(\Gamma, c)$ as the space of all the functions u on Γ such that u is continuous on Γ, $u_e \in H^m((0, 1))$ for every e in E, and $\sum_{e\in E} c(e)\|u_e\|^2_{H^m((0,1))} < \infty$.

The above spaces are Hilbert spaces with inner products

$$(u, w)_{L^2(\Gamma, c)} = \sum_{e\in E} c(e)(u_e, w_e)_{L^2((0,1))}\,,$$

$$(u, w)_{H^m(\Gamma, c)} = \sum_{e\in E} c(e)(u_e, w_e)_{H^m((0,1))}\,.$$

Note that u is continuous on Γ if and only if u_e is continuous on $[0, 1]$ for every e in E, and $u_e(v) = u_{e'}(v)$ for all v in V and for all the edges e and e' having v as a common endpoint.

Consider the sesquilinear continuous form φ on $H^1(\Gamma, c)$ defined by

$$\varphi(u, w) = \sum_{e\in E} c(e)(u_e', w_e')_{L^2((0,1))}\,.$$

According to J. L. LIONS (see ref. [4]), we can associate to φ an operator Δ in the following way . Denote by $D(\Delta)$ the set of all the functions u in $H^1(\Gamma, c)$ such that $\varphi_u(w)$ is an antilinear continuous mapping on $H^1(\Gamma, c)$ with respect to $L^2(\Gamma, c)$ topology. If u is an element of $D(\Delta)$, then we can extend φ_u to an antilinear continuous mapping on $L^2(\Gamma, c)$, and so there exists only one element of $L^2(\Gamma, c)$, denoted by Δu, such that, for every w in $H^1(\Gamma, c)$

$$\varphi(u, w) = -(\Delta u, w)_{L^2(\Gamma, c)}\,.$$

It is easy to verify that Δ is a linear, unbounded, closed, self-adjoint, dissipative, nonpositive operator. Let e be in E and let y be an endpoint of e. Let $u = \{u_e\}_{e\in E}$ be a function on Γ. We denote by $(\partial u_e/\partial n_e)(y)$ the normal exterior derivative of u_e evaluated at y i.e.

$$\frac{\partial u_e}{\partial n_e}(y) = \begin{cases} -\lim\limits_{h\to 0^+} (u_e(y+h) - u_e(y))/h & \text{if } y = 0\,, \\ \lim\limits_{h\to 0^-} (u_e(y+h) - u_e(y))/h & \text{if } y = 1\,. \end{cases}$$

As in the case of a finite graph (see ref. [10]) we can prove the following results

LEMMA 2. – *The following properties are true.*

$$D(\Delta) = \left\{ u \in H^2(\Gamma, c) \colon \sum_{e \in E_v} c(e) \frac{\partial u_e}{\partial n_e}(v) = 0 \text{ for all } v \text{ in } V \right\}, \tag{12}$$

$$(\Delta u)_e = u_e'' \text{ for every } e \text{ in } E \text{ and for every } u \text{ in } D(\Delta). \tag{13}$$

The condition in (12) is called Kirchhoff type condition. Equation (13) justifies the name Laplacian for the operator Δ.

4. – Some properties of the fundamental solution.

Our aim is to determine the solution of the abstract Cauchy problem (3). By the theory of semigroups (see e.g. A. PAZY, ref. [12]), this solution can immediately be obtained by the solution of the abstract Cauchy problem (3′). Then we calculate the solution of (3′). As observed in introduction, in section 6 we will prove that, for every f in $L^2(\Gamma, c)$, the solution of (3′) is expressed by the integral of f against the fundamental solution K which we are going to construct. In order to do this, our strategy is the following. First we define abstractly a function $K(t, x, y)$ and study its regularity properties. Then, in section 5, using the results of this section, we will prove that K is actually the fundamental solution.

Let x and y be in Γ. Choose any e, e' in E such that $x \in e$ and $y \in e'$. Set

$$K(t, x, y) = c(e)^{-1} k(t, d(x, y))\, \delta_{e, e'} + L(t, x, y) \tag{14}$$

where k is as in (9),

$$\delta_{e, e'} = \begin{cases} 1 & \text{if } e' = e, \\ 0 & \text{otherwise}, \end{cases}$$

$$L(t, x, y) = c(e)^{-1} \sum_{m \geq \varrho(x, y)} \sum_{C \in C_{m+2}(e, e')} \varepsilon_C k(t, d(x, T(\pm e)) + m + d(y, I(\pm e'))), \tag{15}$$

($\varrho(x, y)$, $C_{m+2}(e, e')$, ε_C are as in (4), (5), (8″) respectively).

We observe that if x and/or y are vertices of Γ, then $L(t, x, y)$ depends on the choice of the edges e and e'. We will prove in Theorem 1 that $K(t, x, y)$ does not depend on the choice of the edges e and e'.

$L(t, x, y)$ has the following properties.

LEMMA 3. – *There exist $\eta > 0$ and $\nu > 0$ (independent of e and e') such that, for every (t, x, y) in $R_+ \times \Gamma \times \Gamma$*

$$|L(t, x, y)| \leq \frac{\varepsilon q}{c(e)\sqrt{\pi t}} \sum_{m \geq \varrho(x, y)} \exp\left(m(\ln(\varepsilon q) - m/4t)\right) \leq \frac{\eta}{c(e)\sqrt{t}}(1 + t)\exp(\nu t) \tag{16}$$

(ε *is as in* (8′)).

Moreover there exist $t_0 > 0$ *and* $\alpha > 0$ *(independent of e and e') such that, for all* (t, x, y) *in* $(0, t_0] \times \Gamma \times \Gamma$

$$|L(t, x, y)| \leqslant \frac{\alpha}{c(e)\sqrt{t}} \exp(-\varrho^2(x, y)\beta/t) \tag{17}$$

where $1/8 < \beta < 1/4$.

PROOF. – By (15) we have that

$$|L(t, x, y)| \leqslant c(e)^{-1} \sum_{m \geqslant \varrho(x, y)} \sum_{C \in C_{m+2}(e, e')} |\varepsilon_C| k(t, m).$$

Set $M = [4t \ln(\varepsilon q)]$. Then, by Lemma 1 and (8'), we obtain that

$$|L(t, x, y)| \leqslant c(e)^{-1} \sum_{m \geqslant \varrho(x, y)} 2(\varepsilon q)^{m+1} k(t, m) \leqslant$$

$$\leqslant \frac{\varepsilon q}{c(e)\sqrt{\pi t}} \sum_{m \geqslant 0} \exp(m(\ln(\varepsilon q) - m/4t)) \leqslant$$

$$\leqslant \frac{\varepsilon q}{c(e)\sqrt{\pi t}} \Big[\sum_{m \leqslant M} \exp(m(\ln(\varepsilon q) - m/4t)) + \sum_{m \geqslant M+1} \exp(m(\ln(\varepsilon q) - m/4t)) \Big] \leqslant$$

$$\leqslant \frac{\varepsilon q}{c(e)\sqrt{\pi t}} \Big[(M+1) \exp(M \ln(\varepsilon q)) + \sum_{j \geqslant 0} \exp((j+M+1)(\ln(\varepsilon q) - (j+M+1)/4t)) \Big] \leqslant$$

$$\leqslant \frac{\varepsilon q}{c(e)\sqrt{\pi t}} \Big[(4t \ln(\varepsilon q) + 1) \exp(4t \ln^2(\varepsilon q)) + \sum_{j \geqslant 0} \exp(-j(j + M + 1)/4t) \Big] \leqslant$$

$$\leqslant \frac{\varepsilon q}{c(e)\sqrt{\pi t}} \Big[(4t \ln(\varepsilon q) + 1) \exp(4t \ln^2(\varepsilon q))) + \sum_{j \geqslant 0} \exp(-j(M + 1)/4t) \Big] \leqslant$$

$$\leqslant \frac{\varepsilon q}{c(e)\sqrt{\pi t}} [(4t \ln(\varepsilon q) + 1) \exp(4t \ln^2(\varepsilon q)) + (\varepsilon q)/((\varepsilon q) - 1)] \leqslant \frac{\eta}{c(e)\sqrt{t}} (1+t) \exp(\nu t).$$

Fix β in $(1/8, 1/4)$. Clearly, if $t \leqslant t_0 = (1/4 - \beta)/\ln(\varepsilon q)$, then $(m \ln(\varepsilon q) - m^2/4t) \leqslant -m^2\beta/t$ for every $m \geqslant 0$. So, if $t \leqslant t_0$, then

$$|L(t, x, y)| \leqslant \frac{\varepsilon q}{c(e)\sqrt{\pi t}} \sum_{m \geqslant \varrho(x, y)} \exp(m(\ln(\varepsilon q) - m/4t))$$

$$\leqslant \frac{\varepsilon q}{c(e)\sqrt{\pi t}} \sum_{m \geqslant \varrho(x, y)} \exp(-m^2\beta/t) \leqslant \frac{\alpha}{c(e)\sqrt{t}} \exp(-\varrho^2(x, y)\beta/t)$$

i.e. (17). ■

REMARK 1. – We notice that a much better estimate was obtained in the case when all the conductances are equal to 1 (see e.g. B. GAVEAU, M. OKADA, T. OKADA, ref. [2], [11]). However our estimate is sufficient for our purposes. See Remark 5 below.

REMARK 2. – The previous Lemma guarantees that the series which appears in the expression of $L(t, x, y)$ converges uniformly on the edge e' for every (t, x) in $R_+ \times \Gamma$.

Now we compute $(\partial K/\partial y)(t, x, y)$ and $(\partial L/\partial y)(t, x, y)$ and we study their regularity.

Let x and y be in Γ. Choose any e, e' in E such that $x \in e$ and $y \in e'$. Set

$$L_1(t, x, y) = c(e)^{-1} \sum_{m \geq \varrho(x, y)} \sum_{C \in C_{m+2}(e, e')} \times \varepsilon_C \frac{\partial k}{\partial y}(t, d(x, T(\pm e)) + m + d(y, I(\pm e'))) . \tag{18}$$

We note that if x and/or y are vertices of Γ, then $L_1(t, x, y)$ depends on the choice of the edges e and e'.

$L_1(t, x, y)$ has the following properties.

LEMMA 4. – *There exist $\eta > 0$ and $\nu > 0$ (independent of e and e') such that, for every (t, x, y) in $R_+ \times \Gamma \times \Gamma$*

$$|L_1(t, x, y)| \leq \frac{e^2(\varepsilon q)}{2c(e)\, t\sqrt{\pi t}} \sum_{m \geq \varrho(x, y)} \exp\left(m(\ln(\varepsilon q) + 1 - m/4t)\right) \leq \frac{\eta}{c(e)\, t\sqrt{t}}(1 + t) \exp(\nu t) \tag{19}$$

(ε is as in (8')).

For every (t, x) fixed in $R_+ \times \Gamma$ and for every y in e'

$$\frac{\partial L}{\partial y}(t, x, y) = L_1(t, x, y). \tag{20}$$

Moreover there exist $t_0 > 0$ and $\alpha > 0$ (independent of e and e') such that, for all (t, x, y) in $(0, t_0] \times \Gamma \times \Gamma$

$$\left| \frac{\partial L}{\partial y}(t, x, y) \right| \leq \frac{\alpha}{c(e)\, t\sqrt{t}} \exp(-\varrho^2(x, y)\, \beta/t) \tag{21}$$

where $1/8 < \beta < 1/4$.

PROOF. – We know by (9) that

$$\frac{\partial k}{\partial x}(t, x) = -\frac{x}{2t} k(t, x).$$

So by (18) we have that

$$|L_1(t, x, y)| \leqslant c(e)^{-1} \sum_{m \geqslant \varrho(x, y)} \sum_{C \in C_{m+2}(e, e')} |\varepsilon_C| \frac{m+2}{2t} k(t, m) \leqslant$$

$$\leqslant \frac{\varepsilon q}{2c(e)\, t\sqrt{\pi t}} \sum_{m \geqslant \varrho(x, y)} (\varepsilon q)^m (m+2) \exp(-m^2/4t) \leqslant$$

$$\leqslant \frac{e^2(\varepsilon q)}{2c(e)\, t\sqrt{\pi t}} \sum_{m \geqslant \varrho(x, y)} \exp(m(\ln(\varepsilon q) + 1 - m/4t)) \leqslant$$

$$\leqslant \frac{e^2(\varepsilon q)}{2c(e)\, t\sqrt{\pi t}} \sum_{m \geqslant 0} \exp(m(\ln(\varepsilon q) + 1 - m/4t)).$$

Now if we set $M = [4t(\ln(\varepsilon q) + 1)]$, then, arguing as in the proof of Lemma 3, we obtain that

$$|L_1(t, x, y)| \leqslant \frac{\eta}{c(e)\, t\sqrt{t}} (1+t) \exp(\nu t)$$

i.e. (19); while if we fix (t, x) in $R_+ \times \Gamma$, then we observe that we can derive term by term the series which appears in the expression of $L(t, x, \bullet)$. Thus we get (20).

Fix β in $(1/8, 1/4)$. Clearly, if $t \leqslant t_0 = (1/4 - \beta)/(\ln(\varepsilon q) + 1)$, then $(m(\ln(\varepsilon q) + 1) - m^2/4t) \leqslant -m^2\beta/t$ for every $m \geqslant 0$. So, if $t \leqslant t_0$, then

$$\left|\frac{\partial L}{\partial y}(t, x, y)\right| \leqslant \frac{e^2(\varepsilon q)}{2c(e)\, t\sqrt{\pi t}} \sum_{m \geqslant \varrho(x, y)} \exp(-m^2\beta/t) \leqslant \frac{\alpha}{c(e)\, t\sqrt{t}} \exp(-\varrho^2(x, y)\beta/t)$$

i.e. (21). ■

REMARK 3. – We obtain by (20) that $(\partial K/\partial y)(t, x, y)$ exists on $R_+ \times \Gamma \times (\Gamma \backslash V)$ and

$$\frac{\partial K}{\partial y}(t, x, y) = c(e)^{-1} \frac{\partial k}{\partial y}(t, d(x, y))\delta_{e, e'} + L_1(t, x, y).$$

REMARK 4. – We know by (19) and (20) that, for every (t, x) in $R_+ \times \Gamma$

$$\left|\frac{\partial L_{|e'|}}{\partial n_{|e'|}}(t, x, I(\pm e'))\right| \leqslant \frac{\eta}{c(e)\, t\sqrt{t}} (1+t) \exp(\nu t).$$

We can give an analogous estimate for the normal exterior derivative of $K(t, x, \bullet)$ on the edge $|e'|$ at $I(\pm e')$.

PROPOSITION 1. – *For every e' in E and x in Γ*

$$\frac{\partial K_{|e'|}}{\partial n_{|e'|}}(\bullet, x, I(\pm e')) \text{ is continuous on } R_+ .$$

PROOF. – We know that

$$\frac{\partial K_{|e'|}}{\partial n_{|e'|}}(\bullet, x, I(\pm e')) = c(e)^{-1}\frac{\partial k}{\partial n_{|e'|}}(\bullet, d(x, I(\pm e')))\,\delta_{e,e'} + \frac{\partial L_{|e'|}}{\partial n_{|e'|}}(\bullet, x, I(\pm e')) .$$

Hence it is enough to prove that $(\partial L_{|e'|}/\partial n_{|e'|})(\bullet, x, I(\pm e'))$ is a continuous function on R_+.

In order to do this, we observe that every element of the series which appears in the expression of $(\partial L_{|e'|}/\partial n_{|e'|})(\bullet, x, I(\pm e'))$ is a continuous function on R_+, and, for every $t > 0$, by (19) and (20), we can prove that there exists a neighborhood of t where the series converges uniformly.

So we can conclude that $(\partial L_{|e'|}/\partial n_{|e'|})(\bullet, x, I(\pm e'))$ is continuous on R_+. ∎

REMARK 5. – As a consequence of Remark 4 and Proposition 1, the Laplace transform of $(\partial K_{|e'|}/\partial n_{|e'|})(\bullet, x, I(\pm e'))$ is well defined for every e' in E and x in Γ (see e.g. P. K. F. KUHFITTIG, ref. [3]).

Now we compute $(\partial^2 K/\partial y^2)(t, x, y)$ and $(\partial^2 L/\partial y^2)(t, x, y)$ and we study their properties.

Let x and y be in Γ. Choose any e, e' in E such that $x \in e$ and $y \in e'$. Set

$$(22)\qquad L_2(t, x, y) = c(e)^{-1}\sum_{m \geq \varrho(x, y)}\ \sum_{C \in C_{m+2}(e, e')} \times$$

$$\times\, \varepsilon_C \frac{\partial^2 k}{\partial y^2}(t, d(x, T(\pm e)) + m + d(y, I(\pm e'))) .$$

We note that if x and/or y are vertices of Γ, then $L_2(t, x, y)$ depends on the choice of the edges e and e'.

$L_2(t, x, y)$ has the following properties.

LEMMA 5. – *Set $\tau = \min\{t, t^2\}$. There exist $\eta > 0$ and $\nu > 0$ (independent of e and e') such that, for every (t, x, y) in $R_+ \times \Gamma \times \Gamma$*

$$(23)\qquad |L_2(t, x, y)| \leq \frac{e^4(\varepsilon q)}{c(e)\,\tau\sqrt{\pi t}}\sum_{m \geq \varrho(x, y)} \exp(m(\ln(\varepsilon q) + 2 - m/4t)) \leq$$

$$\leq \frac{\eta}{c(e)\,\tau\sqrt{t}}(1 + t)\exp(\nu t)$$

(ε is as in (8')).

For every (t, x) *fixed in* $R_+ \times \Gamma$ *and for every* y *in* e'

$$\frac{\partial^2 L}{\partial y^2}(t, x, y) = L_2(t, x, y). \tag{24}$$

Moreover there exist $t_0 > 0$ *and* $\alpha > 0$ *(independent of* e *and* e'*) such that, for all* (t, x, y) *in* $(0, t_0] \times \Gamma \times \Gamma$

$$\left| \frac{\partial^2 L}{\partial y^2}(t, x, y) \right| \leqslant \frac{\alpha}{c(e)\, t^2 \sqrt{t}} \exp(-\varrho^2(x, y)\, \beta / t) \tag{25}$$

where $1/8 < \beta < 1/4$.

PROOF. – We know by (9) that

$$\frac{\partial^2 k}{\partial x^2}(t, x) = \frac{1}{4t^2}(-2t + x^2)\, k(t, x).$$

So by (22), we have that

$$|L_2(t, x, y)| \leqslant c(e)^{-1} \sum_{m \geqslant \varrho(x, y)} \sum_{C \in C_{m+2}(e, e')} |\varepsilon_C| \frac{(m+2)^2 + 1}{2\tau} k(t, m) \leqslant$$

$$\leqslant \frac{e^4(\varepsilon q)}{c(e)\, \tau \sqrt{\pi t}} \sum_{m \geqslant \varrho(x, y)} \exp(m(\ln(\varepsilon q) + 2 - m/4t)) \leqslant$$

$$\leqslant \frac{e^4(\varepsilon q)}{c(e)\, \tau \sqrt{\pi t}} \sum_{m \geqslant 0} \exp(m(\ln(\varepsilon q) + 2 - m/4t)).$$

Now if we set $M = [4t(\ln(\varepsilon q) + 2)]$, then, arguing as in the proof of Lemma 3, we obtain that

$$|L_2(t, x, y)| \leqslant \frac{\eta}{c(e)\, \tau \sqrt{t}} (1 + t) \exp(\nu t)$$

i.e. (23); while if we fix (t, x) in $R_+ \times \Gamma$, then we observe that we can derive term by term the series which appears in the expression of $L_1(t, x, y)$. Thus we get (24).

Fix β in $(1/8, 1/4)$. Clearly, if $t \leqslant t_0 = (1/4 - \beta)/(\ln(\varepsilon q) + 2)$, then $(m(\ln(\varepsilon q) + 2) - m^2/4t) \leqslant -m^2 \beta / t$ for every $m \geqslant 0$. So, if $t \leqslant t_0$, then

$$\left| \frac{\partial^2 L}{\partial y^2}(t, x, y) \right| \leqslant \frac{e^4(\varepsilon q)}{c(e)\, t^2 \sqrt{\pi t}} \sum_{m \geqslant \varrho(x, y)} \exp(-m^2 \beta / t) \leqslant \frac{\alpha}{c(e)\, t^2 \sqrt{t}} \exp(-\varrho^2(x, y)\, \beta / t)$$

i.e. (25). ■

REMARK. – 6. – It follows by (24) that $(\partial^2 K/\partial y^2)(t, x, y)$ exists on $R_+ \times \Gamma \times (\Gamma \backslash V)$ and

$$\frac{\partial^2 K}{\partial y^2}(t, x, y) = c(e)^{-1} \frac{\partial^2 k}{\partial y^2}(t, d(x, y))\, \delta_{e, e'} + L_2(t, x, y)$$

Now we verify that, for every (t, x) in $R_+ \times \Gamma$, $K(t, x, \bullet)$, $(\partial K/\partial y)(t, x, \bullet)$ and $(\partial^2 K/\partial^2 y)(t, x, \bullet)$ belong to $L^1(\Gamma, c) \cap L^2(\Gamma, c)$. We denote by $M(t, x, y)$ any one of the previous functions.

PROPOSITION 2. – *The function $M(t, x, y)$ has the following properties*

(i) $M(t, x, \bullet) \in L^1(\Gamma, c) \cap L^2(\Gamma, c)$ *for every* (t, x) *in* $R_+ \times \Gamma$,

(ii) *there exists* $\alpha_1(t) > 0$ *such that, for every* x *in* Γ $\|M(t, x, \bullet)\|_{L^1(\Gamma, c)} \leqslant \alpha_1(t)$,

(iii) *there exists* $\alpha_2(t) > 0$ *such that, for every* x *in* Γ $\|M(t, x, \bullet)\|_{L^2(\Gamma, c)} \leqslant \alpha_2(t)/c(e)$.

PROOF. – We prove Proposition for $M(t, x, y) = K(t, x, y)$, in the other cases the proof is similar.

Let x be in Γ. We remember that E_x is the set of all the edges containing x, while E_x' is the set of all the edges which are neighbours of the edges of E_x.

We begin with L^1. By (6) and (16) we have that

$$\int_\Gamma |K(t, x, y)|\, dy \leqslant \int_\Gamma c(e)^{-1} k(t, d(x, y)) \delta_{e, e'}\, dy +$$

$$+ \frac{\varepsilon q}{\sqrt{\pi t}} \int_\Gamma c(e)^{-1} \sum_{m \geqslant \varrho(x, y)} (\varepsilon q)^m \exp(-m^2/4t)\, dy \leqslant$$

$$\leqslant \frac{\text{const.}}{\sqrt{t}} + \frac{\text{const.}}{\sqrt{t}} \sum_{e' \in (E_x \cup E_x')} \frac{c(e')}{c(e)} \sum_{m \geqslant 0} (\varepsilon q)^m \exp(-m^2/4t) +$$

$$+ \frac{\text{const.}}{\sqrt{t}} \sum_{e' \in E \backslash (E_x \cup E_x')} \frac{c(e')}{c(e)} \sum_{m \geqslant \varrho(x, I(+e'))} (\varepsilon q)^m \exp(-m^2/4t) \leqslant$$

$$\leqslant \alpha(t) + \frac{\text{const.}}{\sqrt{t}} \sum_{n \geqslant 1} (\kappa q^2)^n \sum_{m \geqslant n} (\varepsilon q)^m \exp(-m^2/4t) \leqslant$$

$$\leqslant \alpha(t) + \frac{\text{const.}}{\sqrt{t}} \sum_{m \geqslant 1} (\varepsilon q)^m \exp(-m^2/4t) \sum_{n \leqslant m} (\kappa q^2)^n \leqslant$$

$$\leqslant \alpha(t) + \frac{\text{const.}}{\sqrt{t}} \sum_{m \geqslant 1} (\kappa (\varepsilon q)^3)^m \exp(-m^2/4t) = \alpha_1(t) < \infty\,.$$

Now we prove the L^2-estimate.

$$\int_\Gamma |K(t,x,y)|^2 dy \leqslant 2\int_\Gamma |c(e)^{-1}k(t,d(x,y))\delta_{e,e'}|^2 dy +$$

$$+\frac{2(\varepsilon q)^2}{\pi t}\int_\Gamma \Big|c(e)^{-1}\sum_{m\geqslant \varrho(x,I(+e'))}(\varepsilon q)^m \exp(-m^2/4t)\Big|^2 dy \leqslant \frac{\text{const.}}{c(e)\,t}+$$

$$+\frac{\text{const.}}{t}\sum_{e'\in E} c(e')\Big(c(e)^{-1}\sum_{m\geqslant \varrho(x,I(+e'))}(\varepsilon q)^m\exp(-m^2/4t)\Big)^2 \leqslant$$

$$\leqslant \frac{\text{const.}}{c(e)t}+\frac{\text{const.}}{c(e)t}\sum_{e'\in E}\frac{c(e')}{c(e)}\Big(\sum_{m\geqslant \varrho(x,I(+e'))}(\varepsilon q)^m\exp(-m^2/4t)\Big)^2 \leqslant$$

$$\leqslant \frac{\text{const.}}{c(e)t}+\frac{\text{const.}}{c(e)t}\sum_{e'\in(E_x\cup E_x')}\frac{c(e')}{c(e)}\Big(\sum_{m\geqslant 0}(\varepsilon q)^m\exp(-m^2/4t)\Big)^2+$$

$$+\frac{\text{const.}}{c(e)\,t}\sum_{e'\in E\setminus(E_x\cup E_x')}\frac{c(e')}{c(e)}\Big(\sum_{m\geqslant \varrho(x,I(+e'))}(\varepsilon q)^m\exp(-m^2/4t)\Big)^2 \leqslant$$

$$\leqslant \frac{\text{const.}}{c(e)t}+\frac{\text{const.}}{c(e)t}\Big(\sum_{n\geqslant 1}(\kappa q)^n\sum_{m\geqslant n}(\varepsilon q)^m\exp(-m^2/4t)\Big)^2=\frac{\alpha_2(t)}{c(e)}<\infty\,. \qquad \blacksquare$$

REMARK 7. – If $t\leqslant t_0=((1/4)-\beta)/\ln q$ (see Lemma 3), by (17) we can prove that there exist $\alpha_1(t_0)>0$ and $\alpha_2(t_0)>0$ such that, for every x in Γ

$$\|K(t,x,\bullet)\|_{L^1(\Gamma,c)}\leqslant \alpha_1(t_0),$$

$$\|K(t,x,\bullet)\|_{L^2(\Gamma,c)}\leqslant \frac{\alpha_2(t_0)}{c(e)}\,.$$

REMARK 8. – We can repeat Proposition 2 and Remark 7 (with $c(e')$ in place of $c(e)$) substituting the function $M(t,x,\bullet)$ with the function $M(t,\bullet,y)$, where it is defined.

PROPOSITION 3. – *For all (t,x,y) in $R_+\times\Gamma\times(\Gamma\setminus V)$*

$$\frac{\partial K}{\partial t}(t,x,y)=\frac{\partial^2 K}{\partial y^2}(t,x,y)$$

PROOF. – We know by (14) that

$$K(t,x,y)=c(e)^{-1}k(t,d(x,y))\delta_{e,e'}+L(t,x,y)\,.$$

Moreover $k(t,x)$ is the source solution of the heat equation on R (see (9)), so by (22), we have that

$$L_2(t, x, y) = c(e)^{-1} \sum_{m \geq \varrho(x, y)} \sum_{C \in C_{m+2}(e, e')} \varepsilon_C \frac{\partial k}{\partial t}(t, d(x, T(\pm e)) + m + d(y, I(\pm e'))) .$$

By (23) it is easy to prove that, for every (t, x, y) in $R_+ \times \Gamma \times (\Gamma \backslash V)$, there exists a neighborhood of t where the above series converges uniformly. So, for every (t, x, y) in $R_+ \times \Gamma \times (\Gamma \backslash V)$

$$\frac{\partial L}{\partial t}(t, x, y) = L_2(t, x, y) = \frac{\partial^2 L}{\partial y^2}(t, x, y)$$

and hence, for every (t, x, y) in $R_+ \times \Gamma \times (\Gamma \backslash V)$

$$\frac{\partial K}{\partial t}(t, x, y) = c(e)^{-1} \frac{\partial k}{\partial t}(t, d(x, y))\delta_{e, e'} + \frac{\partial L}{\partial t}(t, x, y) =$$

$$= c(e)^{-1} \frac{\partial^2 k}{\partial y^2}(t, d(x, y))\delta_{e, e'} + \frac{\partial^2 L}{\partial y^2}(t, x, y) = \frac{\partial^2 K}{\partial y^2}(t, x, y). \quad ■$$

PROPOSITION 4. – *For all* (x, y) *in* $\Gamma \times (\Gamma \backslash V)$

$$\frac{\partial K}{\partial t}(\bullet, x, y) \text{ is continuous on } R_+ .$$

PROOF. – We know by the last equation in the proof of Proposition 3 that

$$\frac{\partial K}{\partial t}(\bullet, x, y) = c(e)^{-1} \frac{\partial k}{\partial t}(\bullet, x, y)\, \delta_{e, e'} + \frac{\partial L}{\partial t}(\bullet, x, y).$$

So it is enough to prove that $(\partial L/\partial t)(\bullet, x, y)$ is a continuous function on R_+. In order to do this, we observe that every element of the series which appears in the expression of $(\partial L/\partial t)(t, x, y)$ is a continuous function on R_+, and for every $t > 0$, by (23) again, we can prove that there exists a neighborhood of t where the series converges uniformly. Hence we conclude that $(\partial L/\partial t)(\bullet, x, y)$ is continuous on R_+. ■

5. – The fundamental solution of the heat equation on Γ.

THEOREM 1. – *The function K defined in* (14) *does not depend on e and e' and has the following properties*

(i) $(\partial K/\partial y)(t, x, y)$ *and* $(\partial^2 K/\partial y^2)(t, x, y)$ *exist on* $R_+ \times \Gamma \times (\Gamma \backslash V)$,

(ii) $(\partial K/\partial t)(\bullet, x, y)$ *exists continuous on* R_+ *for every* (x, y) *in* $\Gamma \times (\Gamma \backslash V)$,

(iii) $(\partial K/\partial t)(t, x, y) = (\partial^2 K/\partial y^2)(t, x, y)$ *on* $R_+ \times \Gamma \times (\Gamma \backslash V)$,

(iv) $K(t, x, \bullet) \in D(\Delta)$ *for every* (t, x) *in* $R_+ \times \Gamma$.

PROOF. – By (14) and (15), we know that

$$K(t, x, y) = c(e)^{-1} k(t, d(x, y))\delta_{e, e'} +$$

$$+ c(e)^{-1} \sum_{m \geq \varrho(x, y)} \sum_{C \in C_{m+2}(e, e')} \varepsilon_C k(t, d(x, T(\pm e)) + m + d(y, I(\pm e')))$$

where x is a point of the edge e, y a point of the edge e'.

We observe that if x and y are not vertices of Γ, then there exist only one edge e and only one edge e' such that $x \in e$, $y \in e'$, so there is not ambiguity in the definition of $K(t, x, y)$. On the contrary, if x and/or y are vertices of Γ, then we can choose the edge e containing x among q edges and the same holds for y. So, apparently, $K(t, x, y)$ is not well defined in the vertices. We will prove that the dependence on e and e' is illusory and that $K(t, x, y)$ is well defined everywhere.

By section 4, we know that K satisfies the properties (i), (ii), (iii) and moreover that $K(t, x, \bullet)$, $(\partial K/\partial y)(t, x, \bullet)$ and $(\partial^2 K/\partial y^2)(t, x, \bullet)$ belong to $L^2(\Gamma, c)$ for every (t, x) in $R_+ \times \Gamma$. Hence we must prove that $K(t, \bullet, \bullet)$ is well defined also on $V \times V$, $K(t, x, \bullet)$ is continuous on Γ and satisfies the conditions of connection of the weighted normal exterior derivatives at the vertices. In order to do this, we begin to prove the following conditions, then we will verify the continuity of K.

Let x be an interior point of the edge e (by the continuity the following conditions still hold when x is a vertex of the edge e), and let $e_1, e_2, \ldots, e_q$ be the q arcs having the same initial vertex i.e., $I(e_j) = I(e_1)$ for $j = 2, \ldots, q$, then we claim that

(26) there exists a constant depending on e such that, for $j = 1, \ldots, q$ one has $K(t, x, I(e_j)) = \text{const.}$,

$$\sum_{j=1}^{q} c(|e_j|) \frac{\partial K_{|e_j|}}{\partial n_{|e_j|}}(t, x, I(e_j)) = 0 . \tag{27}$$

Applying the Laplace transform, we obtain that (27) is equivalent to

$$\sum_{j=1}^{q} c(|e_j|) \frac{\partial K_{|e_j|}}{\partial n_{|e_j|}}(t, x, I(e_j)) \star \frac{1}{\sqrt{\pi t}} = 0 . \tag{28}$$

(By Remark 5, the Laplace transform of $(\partial K_{|e_j|}/\partial n_{|e_j|})(\bullet, x, I(e_j))$ is well defined for every x in Γ).

Before proving conditions (26) and (28), we observe that condition (26) guarantees that if y is a vertex, then the definition of $K(t, x, y)$ does not depend on the choice of the edge e' containing y. So $K(t, x, y)$ is well defined if x is an interior point of the edge e and y is a generic point of Γ. On the contrary, if x is a vertex, then apparently the expression of $K(t, x, y)$ still depends on the choice of the edge e containing x. We will also prove that the definition of $K(t, x, y)$ does not depend on the choice of the edge e.

Now we want to obtain for K another expression which will allow us to prove easily

(26) and (28). In order to do this, we introduce some matrices as follows. We set

$$\Lambda_{|e|}(t, x) = [\Lambda_{\tilde{e}, |e|}(t, x)]_{\tilde{e} \in A},$$

$$\Omega(t) = [\Omega_{\tilde{e}, \hat{e}}(t)]_{\tilde{e}, \hat{e} \in A},$$

$$\Gamma_{|e'|}(t, y) = [\Gamma_{|e'|, \tilde{e}}(t, y)]_{\tilde{e} \in A},$$

where

$$\Lambda_{\tilde{e}, |e|}(t, x) = (\varepsilon_{-e, \tilde{e}} k(t, d(x, I(+e))) + \varepsilon_{+e, \tilde{e}} k(t, d(x, I(-e)))),$$

$$\Omega_{\tilde{e}, \hat{e}}(t) = \varepsilon_{\hat{e}, \tilde{e}} h(t, 1),$$

$$\Gamma_{|e'|, \tilde{e}}(t, y) = \begin{cases} h(t, d(y, I(\tilde{e}))) & \text{if } \tilde{e} = \pm e', \\ 0 & \text{otherwise}. \end{cases}$$

(Remember that A is the set of all the arcs of Γ, $\varepsilon_{\hat{e}, \tilde{e}}$, k and h are as in (8), (9), (10) respectively, and e, e' are the edges containing x and y). Note that the elements of $\Lambda_{|e|}$ (column vector), Ω, $\Gamma_{|e'|}$ (row vector) belong to the convolution algebra formed by the functions that are continuous for $t \geq 0$ and that are 0 for $t \leq 0$. Moreover we know that the transfer coefficient between two arcs is 0 if the terminal vertex of the first arc is not equal to the initial vertex of the second arc (see (8)). So we can rewrite (14) as follows

$$(14') \quad K(t, x, y) = c(|e|)^{-1} k(t, d(x, y)) \delta_{|e|, |e'|} + c(|e|)^{-1} \Gamma_{|e'|}(t, y) \star \Lambda_{|e|}(t, x) + c(|e|)^{-1} \sum_{m \geq 1} \Gamma_{|e'|}(t, y) \star \Omega(t)^{\star m} \star \Lambda_{|e|}(t, x).$$

Set

$$(29) \qquad \theta_{|e|}(t, x) = \Lambda_{|e|}(t, x) + \sum_{m \geq 1} \Omega(t)^{\star m} \star \Lambda_{|e|}(t, x).$$

Observe that $\theta_{|e|}(t, x)$ is the solution of the following Volterra equation (see e.g. S. G. MIKHLIN, ref. [6])

$$(30) \qquad \theta_{|e|}(t, x) = \Omega(t) \star \theta_{|e|}(t, x) + \Lambda_{|e|}(t, x).$$

In fact, setting $\eta = \max_t h(t, 1)\, \varepsilon$ (ε is as in (8')), we can easily prove by induction that

$$|(\Omega(t)^{\star m})_{\tilde{e}, \hat{e}}| \leq \frac{q^{m-1} \eta^m t^{m-1}}{(m-1)!}$$

for every $m \geq 1$ and for every couple of arcs $\tilde{e}$, $\hat{e}$ in A (note that every row and every column of $\Omega(t)$ has exactly q elements not equal to 0, by the property of the transfer coefficient remembered above).

Then

$$|(\Omega(t)^{\star m}\star \Lambda_{|e|}(t,x))_{\tilde{e}}| = \left|\sum_{\hat{e}\in A}\int_0^t (\Omega(t-s)^{\star m})_{\tilde{e},\hat{e}}\Lambda_{\hat{e},|e|}(s,x)\,ds\right|$$

$$\leqslant \frac{2q^m\eta^m t^{m-1}}{(m-1)!}\int_0^t k(s,0)\,ds = \frac{2q^m\eta^m t^{m-1/2}}{\sqrt{\pi}(m-1)!}$$

for every $m \geqslant 1$ and for every arc $\tilde{e}$ in A.

So we can conclude that there exists the solution of the Volterra equation (30) and this solution is (29).

Moreover, again by (8) and (30), we can express every component of $\theta_{|e|}(t,x)$ as follows

$$\theta_{e_j,|e|}(t,x) = \varepsilon_{-e_j,e_j}k(t,d(x,I(e_j)))\,\delta_{|e|,|e_j|} + \sum_{i\neq j}\varepsilon_{-e_i,e_j}k(t,d(x,I(e_i)))\,\delta_{|e|,|e_i|} +$$

$$+\varepsilon_{-e_j,e_j}h(t,1)\star\theta_{-e_j,|e|}(t,x) + \sum_{i\neq j}\varepsilon_{-e_i,e_j}h(t,1)\star\theta_{-e_i,|e|}(t,x)$$

or equivalently

$$(31)\qquad k(t,d(x,I(e_j)))\,\delta_{|e|,|e_j|} + \theta_{e_j,|e|}(t,x) + h(t,1)\star\theta_{-e_j,|e|}(t,x) =$$

$$= \sum_{i=1}^{q}\frac{2c(|e_i|)}{c(I(e_i))}\Big(k(t,d(x,I(e_i)))\,\delta_{|e|,|e_i|} + h(t,1)\star\theta_{-e_i,|e|}(t,x)\Big)$$

where $e_1, e_2, \ldots, e_q$ are the q arcs having $I(e_j) = I(e_1)$ for $j = 2, \ldots, q$, and $c(I(e_i))$ is as in (7). (We will see that (31) is very important to prove (26) and (28)).

By (29), we can rewrite (14′) as follows

$$(14'')\qquad K(t,x,y) = c(|e|)^{-1}\Big(k(t,d(x,y))\,\delta_{|e|,|e'|} + h(t,d(I(+e'),y))\star\theta_{+e',|e|}(t,x) +$$

$$+h(t,d(I(-e'),y))\star\theta_{-e',|e|}(t,x)\Big)$$

where x is a point of the edge e and y a point of the edge e'. (To verify (26) and (28) we will always use equation (14″)).

We evaluate $K(t,x,\bullet)$ at $I(e_j)$. After some tedious computations, we have that

$$(32)\qquad K(t,x,I(e_j)) = c(|e|)^{-1}\Big(k(t,d(x,I(e_j)))\,\delta_{|e|,|e_j|} +$$

$$+\theta_{e_j,|e|}(t,x) + h(t,1)\star\theta_{-e_j,|e|}(t,x)\Big).$$

By (31), we get that

$$(32') \quad K(t, x, I(e_j)) = c(|e|)^{-1} \sum_{i=1}^{q} \frac{2c(|e_i|)}{c(I(e_i))} k(t, d(x, I(e_i)))\, \delta_{|e|, |e_i|} + \\ + c(|e|)^{-1} \sum_{i=1}^{q} \frac{2c(|e_i|)}{c(I(e_i))} h(t, 1) \star \theta_{-e_i, |e|}(t, x) = \text{const}.$$

i.e. (26).

Now we compute $(\partial K_{|e_j|}/\partial n_{|e_j|})(t, x, \bullet) \star 1/\sqrt{\pi t}$ and evaluate it at $I(e_j)$. By Remark 3, (11) and (32), we obtain that

$$(33) \quad \frac{\partial K_{|e_j|}}{\partial n_{|e_j|}}(t, x, I(e_j)) \star \frac{1}{\sqrt{\pi t}} = c(|e|)^{-1}\Big(-k(t, d(x, I(e_j)))\, \delta_{|e|, |e_j|} + \\ + \theta_{e_j, |e|}(t, x) - h(t, 1) \star \theta_{-e_j, |e|}(t, x)\Big).$$

To prove (28), we verify that summing the q equations (26) multiplied by suitable conductances and substracting (28) we again obtain the sum of the q weighted equations (26). In fact, by (32), (32′) and (33), we have that

$$\sum_{j=1}^{q} c(|e_j|)\, K(t, x, I(e_j)) - \sum_{j=1}^{q} c(|e_j|) \frac{\partial K_{|e_j|}}{\partial n_{|e_j|}}(t, x, I(e_j)) \star \frac{1}{\sqrt{\pi t}} = \\ = c(|e|)^{-1} \sum_{j=1}^{q} 2c(|e_j|)\Big(k(t, d(x, I(e_j)))\, \delta_{|e|, |e_j|} + h(t, 1) \star \theta_{-e_j, |e|}(t, x)\Big) = \\ = c(I(e_1))\, K(t, x, I(e_1)) = \sum_{j=1}^{q} c(|e_j|)\, K(t, x, I(e_j)).$$

To prove that the function $K(t, x, y)$ does not depend on the choice of the edge containing x, we verify that there exists a constant (depending on the conductance of e') such that, for $j = 1, \ldots, q$, one has $K(t, T(e_j), y) = \text{const}.$ (where $e_1, e_2, \ldots, e_q$ are the q arcs having the same terminal vertex and y is a generic point of Γ). In order to do this, we essentially repeat the procedure adopted to verify conditions (26) with the following matrices

$$\Lambda_{|e'|}(t, y) = [\Lambda_{\tilde e, |e'|}(t, y)]_{\tilde e \in A},$$

$$\Omega(t) = [\Omega_{\tilde e, \hat e}(t)]_{\tilde e, \hat e \in A},$$

$$\Gamma_{|e|}(t, x) = [\Gamma_{|e|, \tilde e}(t, x)]_{\tilde e \in A},$$

where

$$\Lambda_{\tilde e, |e'|}(t, y) = \big(\varepsilon_{\tilde e, +e'} k(t, d(y, I(+e'))) + \varepsilon_{\tilde e, -e'} k(t, d(y, I(-e')))\big),$$

$$\Omega_{\tilde e, \hat e}(t) = \varepsilon_{\tilde e, \hat e} h(t, 1),$$

$$\Gamma_{|e|, \tilde e}(t, x) = \begin{cases} h(t, d(x, T(\tilde e))) & \text{if } \tilde e = \pm e, \\ 0 & \text{otherwise}. \end{cases}$$

As before $\Lambda_{|e'|}(t, y)$ is a column vector and $\Gamma_{|e|}(t, x)$ is a row vector.

To finish the proof of Theorem, we verify that $K(t, x, \bullet)$ is continuous on Γ for every (t, x) in $R_+ \times \Gamma$. In fact, we know by (14) that

$$K(t, x, y) = c(e)^{-1} k(t, d(x, y))\delta_{e, e'} + L(t, x, y).$$

If y is not a vertex of Γ, then the continuity follows by the uniform convergence of the series which appears in the expression of $L(t, x, y)$ (see Remark 2). On the other hand, if y is a vertex, then the uniform convergence guarantees that $K(t, x, \bullet)$ is continuous on the edge e', but we have seen that the definition of $K(t, x, y)$ does not depend on the choice of the edge e' containing y. Hence $K(t, x, \bullet)$ is continuous on Γ. ∎

REMARK 9. – One can prove, essentially repeating the procedure adopted by J. P. ROTH (see ref. [17]), that if there exists a function $H(t, x, y)$ with the properties of Theorem 1, then, for every (t, x, y) in $R_+ \times \Gamma \times \Gamma$

$$H(t, x, y) = K(t, x, y).$$

Therefore we can consider the properties listed in Theorem 1 as the properties characterizing the fundamental solution of the heat equation on Γ.

6. – The solution of the abstract Cauchy problem.

Now we can determine the solution of the abstract Cauchy problem (3′) and so the solution of (3). Before introducing the function which will show to be the solution of system (3′), we prove the following

PROPOSITION 5. – *For every function f in $L^2(\Gamma, c)$ and for every $t > 0$ we have that $\int_\Gamma M(t, x, y) f(x)\,dx$ exists for every y in $(\Gamma \backslash V)$ and*

$$\int_\Gamma M(t, x, \bullet) f(x)\,dx \in L^2(\Gamma, c)$$

($M(t, x, y)$ denotes any one of three functions $K(t, x, y)$, $(\partial K/\partial y)(t, x, y)$, $(\partial^2 K/\partial y^2)(t, x, y)$).

PROOF. – Applying Proposition 2 and Remark 8, we obtain that

$$\int_\Gamma \Big|\int_\Gamma M(t, x, y) f(x)\,dx\Big|^2 dy \leq \int_\Gamma \left(\int_\Gamma |M(t, x, y)|\,dx \right) \left(\int_\Gamma |M(t, x, y)| |f(x)|^2 dx \right) dy \leq$$

$$\leq \alpha_1(t) \int_\Gamma \int_\Gamma |M(t, x, y)| |f(x)|^2 dx\,dy \leq$$

$$\leq \alpha_1(t) \int_\Gamma |f(x)|^2 \left(\int_\Gamma |M(t, x, y)|\,dy \right) dx \leq \alpha_1^2(t) \|f\|^2_{L^2(\Gamma, c)}.$$ ∎

REMARK 10. – We observe that $\int_\Gamma K(t, x, y) f(x)\,dx$ exists for every y in Γ.

DEFINITION 1. – For every f in $L^2(\Gamma, c)$ and for all y in Γ, set

$$P_t f(y) = \begin{cases} \int_\Gamma K(t, x, y) f(x)\, dx & \text{if } t > 0\,, \\ f(y) & \text{if } t = 0\,. \end{cases}$$

THEOREM 2. – *$P_t f$ is the solution of the abstract Cauchy problem* (3′).

PROOF. – We must verify that $P_t f$ has the following properties (see e.g. A. PAZY, ref. [12]):

(i) $P_t f(\bullet) \in D(\Delta)$ for $t > 0$,

(ii) $P_t f$ satisfies system (3′),

(iii) $P_t f$ is a continuous function of $L^2(\Gamma, c)$ for $t \geq 0$,

(iv) $P_t f$ is a continuously differentiable function of $L^2(\Gamma, c)$ for $t > 0$.

We prove property (i). We must verify that, for every fixed $t > 0$, $P_t f(\bullet)$ is an element of $H^2(\Gamma, c)$ satisfying the conditions of connection of the weighted normal exterior derivatives at every vertex of Γ (see (12)). In order to do this, we begin to verify that

$$P_t f(\bullet) \text{ is continuous on } \Gamma\,. \tag{34}$$

Fixed y_0 in Γ, let U be a (small) neighborhood of y_0 in the topology of CW-complex. We know by property (iv) of Theorem 1 that $K(t, x, \bullet) f(x)$ is continuous on U, for every (t, x) in $R_+ \times \Gamma$. We prove that there exists a function $g_t \in L^1(\Gamma, c)$ such that, for every y in U

$$|K(t, x, y) f(x)| \leq g_t(x)\,.$$

Therefore we obtain (34).

For every x in $(\Gamma \backslash V)$, we set

$$g_t(x) = \frac{1}{2c(e)\sqrt{\pi t}} \sum_{e' \in E_{y_0}} |f_{e'}(x)|\, \delta_{e, e'} + \frac{\varepsilon q}{c(e)\sqrt{\pi t}} |f_e(x)| \sum_{m \geq \varrho(x, y_0)} \exp\left(m(\ln(\varepsilon q) - m/4t)\right)$$

where e is the edge containing x.

Let us fix an edge $\hat{e}$ of E_{y_0}, and let us denote by $\overset{(n)}{\sum}$ the sum extended to the edges of Γ having one endpoint at geodesic distance n from y_0, and by $\gamma(t)$ a suitable function depending on t and y_0 (and, possibly on $\hat{e}$), which may vary from line to line.

$$\int_\Gamma g_t(x)\, dx \leq \frac{\text{const.}}{\sqrt{t}} \sum_{e \in E_{y_0}} \|f_e\|_{L^1((0,1))} +$$

$$+ \frac{\text{const.}}{\sqrt{t}} \sum_{e \in (E_{y_0} \cup E'_{y_0})} \|f_e\|_{L^1((0,1))} \left(\sum_{m \geq 0} \exp\left(m(\ln(\varepsilon q) - m/4t)\right) \right) +$$

$$+\frac{\text{const.}}{\sqrt{t}}\sum_{n\geq 1}\left[\left(\sum^{(n+1)}\|f_e\|_{L^1((0,1))}\right)\times\sum_{m\geq n}\exp\left(m\left(\ln(\varepsilon q)-\frac{m}{4t}\right)\right)\right]\leq$$

$$\leq\gamma(t)\sup_{e\in(E_{y0}\cup E'_{y0})}\|f_e\|_{L^1((0,1))}+\gamma(t)\sum_{n\geq 1}\left[\kappa^n\left((\kappa q^2)^n\sum^{(n+1)}\frac{c(e)}{c(\hat{e})}\|f_e\|^2_{L^2((0,1))}\right)^{1/2}\times\right.$$

$$\left.\times\sum_{m\geq n}\exp\left(m(\ln(\varepsilon q)-m/4t)\right)\right]\leq\gamma(t)\sup_{e\in(E_{y0}\cup E'_{y0})}\|f_e\|_{L^1((0,1))}+$$

$$+\gamma(t)\|f\|_{L^2(\Gamma,c)}\sum_{n\geq 1}(\kappa^2 q)^n\sum_{m\geq n}\exp\left(m(\ln(\varepsilon q)-m/4t)\right)<\infty$$

(the convergence of the last series is proved as in the proof of (*ii*) of Proposition 2).

Now we verify that

$$\frac{\partial}{\partial y}(P_t f(\bullet))\in L^2(\Gamma,c). \tag{35}$$

By (19) and Lebesgue theorem, we can prove that, for all y in $(\Gamma\backslash V)$

$$\frac{\partial}{\partial y}\left(\int_\Gamma K(t,x,y)f(x)\,dx\right)=\int_\Gamma\frac{\partial K}{\partial y}(t,x,y)f(x)\,dx. \tag{36}$$

Applying Proposition 5 with $M(t,x,y)=(\partial K/\partial y)(t,x,y)$, we obtain (35) for every $t>0$.

Likewise we can prove that, for every $t>0$

$$\frac{\partial^2}{\partial y^2}(P_t f(\bullet))\in L^2(\Gamma,c), \tag{37}$$

So, by (34), (35) and (37), we can conclude that $P_t f(\bullet)$ is an element of $H^2(\Gamma,c)$ for every $t>0$.

Now we show that

$$\sum_{e\in E_v}c(e)\frac{\partial}{\partial n_e}(P_t f)_e(v)=0.$$

Remembering that $K(t,x,y)$ satisfies the conditions of connection of the weighted normal exterior derivatives at every vertex y (see Theorem 1), and arguing as in the proof of (36), we can observe that, for every (t,v) in $R_+\times V$

$$\sum_{e\in E_v}c(e)\frac{\partial}{\partial n_e}(P_t f)_e(v)=\int_\Gamma\sum_{e\in E_v}c(e)\frac{\partial K_e}{\partial n_e}(t,x,v)f(x)\,dx=0. \tag{38}$$

Hence, for every $t>0$, $P_t f$ is an element of $H^2(\Gamma,c)$ satisfying the conditions of connection (38). So $P_t f$ is an element of $D(\Delta)$ (see (12)) i.e., $P_t f$ satisfies property (i).

It is also easy to see that $P_t f$ satisfies system (3′). Indeed, by Definition 1 $P_0 f = f$. Next

$$\frac{\partial}{\partial t}(P_t f) = \Delta(P_t f)$$

since, by (23), Proposition 3 and Lebesgue theorem

$$\frac{\partial}{\partial t}(P_t f)(y) = \int_\Gamma \frac{\partial K}{\partial t}(t, x, y) f(x)\, dx \tag{39}$$

Hence, for every y in $(\Gamma \backslash V)$, we obtain that

$$\frac{\partial}{\partial t}(P_t f)(y) - \Delta(P_t f)(y) = \int_\Gamma \left(\frac{\partial K}{\partial t}(t, x, y) - \frac{\partial^2 K}{\partial y^2}(t, x, y) \right) f(x)\, dx = 0 .$$

Therefore (ii) holds.

Now we prove that, for every $t \geqslant 0$

$$\|P_{t+s} f(\bullet) - P_t f(\bullet)\|_{L^2(\Gamma, c)} \to 0 \qquad \text{if } s \to 0 .$$

We distinguish two cases

(1.iii) $t > 0$,

(2.iii) $t = 0$.

(1.iii) By (39) we have that, for every y in $(\Gamma \backslash V)$

$$P_{t+s} f(y) \to P_t f(y) \qquad \text{if } s \to 0 .$$

Then using Propositions 2 and 5, Remarks 7 and 8 and Lebesgue theorem, we can verify that there exists a function g_t in $L^1(\Gamma, c)$ such that, for all s sufficiently small

$$|P_{t+s} f(y) - P_t f(y)|^2 \leqslant g_t(y) .$$

So $P_t f$ is a continuous function of $L^2(\Gamma, c)$ for $t > 0$.

(2.iii) By (14) and Definition 1 we have that

$$\|P_s f(\bullet) - f(\bullet)\|_{L^2(\Gamma, c)} =$$

$$= \left[\sum_{e' \in E} c(e') \int_{e'} \left| \int_{e'} k(s, d(x, y)) f_{e'}(x)\, dx - f_{e'}(y) + \sum_{e \in E} c(e) \int_e L(s, x, y) f_e(x)\, dx \right|^2 dy \right]^{1/2} \leqslant$$

$$\leqslant \left[\sum_{e' \in E} c(e') \int_{e'} \left| \int_{e'} k(s, d(x, y)) f_{e'}(x)\, dx - f_{e'}(y) \right|^2 dy \right]^{1/2} +$$

$$+ \left[\sum_{e' \in E} c(e') \int_{e'} \left| \sum_{e \in E} c(e) \int_e L(s, x, y) f_e(x)\, dx \right|^2 dy \right]^{1/2} .$$

We begin to prove that the first term in the last inequality tends to 0 when s tends to 0^+. In order to do this, we identify every function f_e on $[0, 1]$ with a function on R which is 0 everywhere outside $[0, 1]$, and we observe that

$$\left[\sum_{e'\in E} c(e')\int_{e'}\left|\int_{e'} k(s, d(x, y)) f_{e'}(x)\, dx - f_{e'}(y)\right|^2 dy\right]^{1/2} =$$

$$= \left[\sum_{e'\in E} c(e')\int_0^1\left|\int_{-\infty}^{+\infty} k(s, v) f_{e'}(v+y)\, dv - f_{e'}(y)\right|^2 dy\right]^{1/2} \leqslant$$

$$\leqslant \int_{-\infty}^{+\infty} k(s, v)\left[\sum_{e'\in E} c(e')\int_0^1 |f_{e'}(y+v) - f_{e'}(y)|^2 dy\right]^{1/2} dv$$

and we argue as in the proof of

$$\|k_\varepsilon * f - f\|_{L^2(R)} \to 0 \qquad \text{if } \varepsilon \to 0$$

where $\{k_\varepsilon\}$ is a summability kernel on R and f is a function of $L^2(R)$ (see e.g. E. M. STEIN-G. WEISS, ref. [18]). (Note that the space $C_c(\Gamma)$ of all the continuous functions on Γ whose supports are compact is dense in $L^2(\Gamma, c)$). Now we study the behaviour of the second term. We begin to observe that

$$\left[\sum_{e'\in E} c(e')\int_{e'}\left|\sum_{e\in E} c(e)\int_e L(s, x, y) f_e(x)\, dx\right|^2 dy\right]^{1/2} \leqslant$$

$$\leqslant \left[\sum_{e'\in E} c(e')\int_{e'}\left|\sum_{e\in(e'\cup E'_{e'})} c(e)\int_e L(s, x, y) f_e(x)\, dx\right|^2 dy\right]^{1/2} +$$

$$+ \left[\sum_{e'\in E} c(e')\int_{e'}\left|\sum_{e\in(E\setminus(e'\cup E'_{e'}))} c(e)\int_e L(s, x, y) f_e(x)\, dx\right|^2 dy\right]^{1/2}.$$

(Remember that $E'_{e'}$ is the set of all the edges which are neighbours of e'). We analyse the first term. We suppose that f_e is a positive function for every edge e (in the other cases the proof is similar) and we observe that

$$\left[\sum_{e'\in E} c(e')\int_{e'}\left|\sum_{e\in(e'\cup E'_{e'})} c(e)\int_e L(s, x, y) f_e(x)\, dx\right|^2 dy\right]^{1/2} \leqslant$$

$$\leqslant \text{const.}\left[\sum_{e'\in E} c(e')\int_{e'}\left(\sum_{e\in(e'\cup E'_{e'})}\sum_{m\geqslant 0} 2(\varepsilon q)^{m+1}\exp(-m^2/4s) \times\right.\right.$$

$$\times \int_e k(s, d(x, T(\pm e)) + d(y, I(\pm e')))f_e(x)\,dx\Bigg)^2 dy\Bigg]^{1/2} \leqslant$$

$$\leqslant \text{const.}\left[\sum_{m \geqslant 0} 2(\varepsilon q)^{m+1} \exp(-m^2/4t_0)\right]\cdot$$

$$\cdot\left[\sum_{e' \in E} c(e') \int_{e'} \left(\sum_{e \in (e' \cup E'_{e'})} \int_e k(s, d(x, T(\pm e)) + d(y, I(\pm e')))f_e(x)\,dx\right)^2 dy\right]^{1/2} \leqslant$$

$$\leqslant \text{const.}\left[\sum_{e' \in E} c(e') \int_{e'} \left(\sum_{e \in (e' \cup E'_{e'})} \int_e k(s, d(x, T(\pm e)) + d(y, I(\pm e')))f_e(x)\,dx\right)^2 dy\right]^{1/2}.$$

(We have supposed $s \leqslant t_0 = (1/4 - \beta)/\ln(\varepsilon q)$, see Lemma 3). We note that the previous term is less or equal to a sum of $(2q-1)$ series of the form

$$\left[\sum_{e' \in E} c(e') \int_{e'} \left(\int_e k(s, d(x, T(\pm e)) + d(y, I(\pm e')))f_e(x)\,dx\right)^2 dy\right]^{1/2},$$

where the edge e is an edge of the set $(e' \cup E'_{e'})$. We observe that $d(x, T(\pm e))$ may be either x or $(1-x)$. We argue for the case x, the other case is analogous. Thus we have

$$\left[\sum_{e' \in E} c(e') \int_{e'} \left(\int_e k(s, d(x, T(\pm e)) + d(y, I(\pm e')))f_e(x)\,dx\right)^2 dy\right]^{1/2} =$$

$$= \left[\sum_{e' \in E} c(e') \int_{e'} \left(\int_0^1 k(s, x + d(y, I(\pm e')))f_e(x)\,dx\right)^2 dy\right]^{1/2} =$$

$$= \left[\sum_{e' \in E} c(e') \int_{e'} \left(\int_{-\infty}^{+\infty} k(s, v) f_e(v - d(y, I(\pm e')))\,dv\right)^2 dy\right]^{1/2} \leqslant$$

$$\leqslant K\left[\sum_{e' \in E} c(e') \int_{e'} \left(\int_{-\infty}^{+\infty} k(s, v) f_{e'}(v - d(y, I(\pm e')))\,dv\right)^2 dy\right]^{1/2} \leqslant$$

$$\leqslant K \int_{-\infty}^{+\infty} k(s, v)\left[\sum_{e' \in E} c(e') \int_{e'} \big(f_{e'}(v - d(y, I(\pm e')))\big)^2 dy\right]^{1/2} dv$$

and the last term tends to 0 since $f_{e'}(-d(y, I(\pm e'))) = 0$ a.e. y in e'. (Note that we have used condition (6)).

Now we analyse the second term. In order to do this we begin to observe that, for every y in $(\Gamma \setminus V)$

$$\left| \sum_{e \in E \setminus (e' \cup E'_{e'})} c(e) \int_e L(s, x, y) f_e(x)\, dx \right| \to 0 \quad \text{if } s \to 0^+ .$$

In fact, as above we denote by $\overset{(n)}{\sum}$ the sum extended to the edges of Γ having one end-point at geodesic distance n from y. We use (17) and have that

$$\left| \sum_{e \in E \setminus (e' \cup E'_{e'})} c(e) \int_e L(s, x, y) f_e(x)\, dx \right| \leqslant$$

$$\leqslant \frac{\alpha}{\sqrt{s}} \sum_{n \geqslant 1} \exp(-n^2 \beta / s) \left(\overset{(n+1)}{\sum} \|f_e\|_{L^1((0,1))} \right) \to 0 \quad \text{if } s \to 0^+ .$$

In fact

$$\frac{\alpha}{\sqrt{s}} \sum_{n \geqslant 1} \exp(-n^2 \beta / s) \left(\overset{(n+1)}{\sum} \|f_e\|_{L^1((0,1))} \right) \leqslant$$

$$\leqslant \frac{\alpha}{\sqrt{t_0}} \sum_{n \geqslant 1} \exp(-n^2 \beta / t_0) \left(\overset{(n+1)}{\sum} \|f_e\|_{L^2((0,1))} \right) \leqslant$$

$$\leqslant \frac{\alpha}{\sqrt{t_0}} \sum_{n \geqslant 1} \exp(-n^2 \beta / t_0) \left[(\kappa^2 q)^{n+1} \left(\overset{(n+1)}{\sum} \frac{c(e)}{c(e')} \|f_e\|^2_{L^2((0,1))} \right)^{1/2} \right] \leqslant$$

$$\leqslant \frac{\alpha}{\sqrt{c(e') t_0}} \|f\|_{L^2(\Gamma, c)} \sum_{n \geqslant 1} (\kappa^2 q)^{n+1} \exp(-n^2 \beta / t_0) < \infty .$$

(We have again supposed $s \leqslant t_0$). Finally, using Propositions 2 and 5, Remarks 7 and 8 and Lebesgue theorem, we can verify that there exists a function g_0 in $L^1(\Gamma, c)$ such that, for all s sufficiently small

$$\left| \sum_{e \in E \setminus (e' \cup E'_{e'})} c(e) \int_e L(s, x, y) f_e(x)\, dx \right|^2 \leqslant g_0(y).$$

Then, we can conclude that $P_t f$ is a continuous function of $L^2(\Gamma, c)$ for $t \geqslant 0$ i.e., property (iii). To finish the proof of Theorem, we verify that $P_t f$ is a continuously differentiable function of $L^2(\Gamma, c)$ for $t > 0$.

Fixed y in $(\Gamma \setminus V)$, using property (ii) of Theorem 1, (39) and Lebesgue theorem, we can prove that $(\partial/\partial t)(P_t f)(y)$ is continuous for $t > 0$.

Arguing as above, we can construct a function g_t in $L^1(\Gamma, c)$ such that, for all s sufficiently small

$$\left| \frac{\partial}{\partial t}(P_{t+s} f)(y) - \frac{\partial}{\partial t}(P_t f)(y) \right|^2 \leq g_t(y).$$

Then, by Lebesgue theorem, we can conclude that $P_t f$ is a continuously differentiable function of $L^2(\Gamma, c)$ for $t > 0$ i.e., property (iv). ■

REMARK 11. – By the general theory (see e.g. K. YOSIDA, ref. [20]) and by Theorem 2, we can conclude that the mapping on $(R_+ \cup \{0\}) \times L^2(\Gamma, c)$ defined as above by

$$P_t f(y) = \begin{cases} \int_\Gamma K(t, x, y) f(x)\, dx & \text{if } t > 0, \\ f(y) & \text{if } t = 0, \end{cases}$$

is the semigroup on $L^2(\Gamma, c)$ having Δ as infinitesimal generator.

REMARK 12. – By the theory of semigroups (see e.g. A. PAZY, ref. [12]), we obtain that the solution of the Cauchy problem associated to the spread equation of the potential (i.e. the solution of (3)) is $(\exp(-t))P_t f$.

REMARK 13. – Note that we can obtain all the previous results substituting the homogeneous tree Γ by a tree Γ such that the degrees of its vertices are uniformly bounded and equal to 3 at least.

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