THE SQ-UNIVERSALITY OF SOME FINITELY PRESENTED GROUPS

Dedicated with love and gratitude to the memory of my mother

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1. SQ-universal groups

Following a suggestion of G. Higman we say that the group G is SQ-*universal* if every countable group is embeddable in some factor group of G. It is a well-known theorem of G. Higman, B. H. Neumann and Hanna Neumann that the free group of rank 2 is SQ-universal in this sense. Several different proofs are now available (see, for example, [1] or [9]). It is my intention to prove the

LEMMA. *If H is a subgroup of finite index in a group G, then G is* SQ-*universal if and only if H is* SQ-*universal.*

This fact has several applications. For example, it follows that the modular group $M = \mathrm{gp}(a, b \mid a^2 = b^3 = 1)$ is SQ-universal because the commutator subgroup M' has index 6 in M and, being free of rank 2, is SQ-universal by the original theorem of Higman, Neumann and Neumann. More generally it provides an answer to the question raised by Higman in his Oxford seminar in November 1966 whether the triangle groups

$$(l, m, n) = \mathrm{gp}(a, b \mid a^l = b^m = (ab)^n = 1)$$

are SQ-universal if $1/l + 1/m + 1/n < 1$. I shall prove that this is so, and slightly more,

THEOREM. *If G is a finitely generated Fuchsian group which does not have an abelian subgroup of finite index, then G is* SQ-*universal.*

2. Proofs of the lemma and the theorem

The proof of the lemma depends on the well-known fact that every countable group can be embedded in a countably infinite simple group. So, let X be any countable group and let S be a countable, infinite simple group which has a subgroup isomorphic to X.

Suppose first that G is SQ-universal and that H is a subgroup of finite index in G. There will exist a normal subgroup N of G such that $\bar{G} = G/N$ contains a subgroup $\bar{S}$ isomorphic to S. Now $\bar{H} = HN/N$ has finite index in $\bar{G}$. Therefore $\bar{H} \cap \bar{S}$ has finite index in $\bar{S}$ and consequently contains a subgroup which is normal and of finite index in $\bar{S}$. Since $\bar{S}$ is simple and infinite it follows that $\bar{H} \cap \bar{S} = \bar{S}$. Thus S, and consequently also X, is isomorphic to a subgroup of $\bar{H} = HN/N \cong H/H \cap N$. Hence H is SQ-universal.

To prove the converse suppose that H, a subgroup of finite index in G, is SQ-universal. The intersection of the conjugates of H is a subgroup K which is normal and of finite index in G and in H. What we have already proved applies and so we know that K must be SQ-universal. Let L_0 be a normal subgroup of K such that S is embeddable in K/L_0, say $S \cong T_0/L_0$ where $L_0 \leqq T_0 \leqq K$. The maximal principles of set theory guarantee the existence of a normal subgroup L of K containing L_0 and maximal subject to the condition that $L/L_0 \cap T_0/L_0 = 1$. If $T = T_0L$ then of course $T/L \cong T_0/L_0 \cong S$, and moreover, since S is simple, every normal subgroup of K which contains L properly also contains T, that is, every non-trivial normal subgroup of $A = K/L$ contains T/L.

The normaliser of L contains K and therefore has finite index in G. Let $g_1, \cdots, g_n$ be a transversal for this normaliser, and put $L_i = g_i^{-1}Lg_i$ for $1 \leqq i \leqq n$, so that $L_1, \cdots, L_n$ are all the conjugates of L in G. Since K is normal in G all of $L_1, \cdots, L_n$ are contained as normal subgroups in K and $K/L_i \cong K/L = A$. Now put $N = \bigcap_{i=1}^{n} L_i$. Then K/N is isomorphic to a subdirect product of the groups $K/L_1, \cdots, K/L_n$, that is, K/N is a subdirect power of A. I shall show by induction on the number of factors that any subdirect product B of finitely many copies of A has a subgroup isomorphic to S.

Suppose therefore that $B \leqq A_1 \times \cdots \times A_r$ $(r \geqq 1)$, where $A_i \cong A$, and that each of the r projections of B to $A_i (1 \leqq i \leqq r)$ is surjective. Let S_i denote the subgroup of A_i which corresponds to the subgroup T/L of A. If $r = 1$ then $B \cong A_1 \cong A$ and so B certainly has a subgroup isomorphic to S. Therefore as inductive hypothesis we may suppose that any subdirect product of $r-1$ groups isomorphic to A has a subgroup isomorphic to S. Since the projection of B to A_r is surjective, $B \cap A_r$ is normal in A_r. If $B \cap A_r = 1$ then B is isomorphic to a subdirect product of $A_1, \cdots, A_{r-1}$ and, by inductive hypothesis, contains a subgroup isomorphic to S. But if $B \cap A_r \neq 1$ then $B \cap A_r \geqq S_r$ and so again B contains a subgroup isomorphic to S.

I have shown now that K/N has a subgroup isomorphic to S. But N is normal in G. Therefore G/N has a subgroup isomorphic to S, and so X is embeddable in G/N. Thus G is SQ-universal.

PROOF OF THE THEOREM. A finitely generated Fuchsian group may be presented by generators

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$$a_1, \cdots, a_g, b_1, \cdots, b_g, c_1, \cdots, c_h, e_1, \cdots, e_k \tag{F}$$

and relations

$$\left.\begin{aligned} &e_i^{n_i} = 1, \quad i = 1, \cdots, k \\ &[a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_h e_1 \cdots e_k = 1 \end{aligned}\right\} \tag{F}$$

where $n_i \geqq 2$ for $i = 1, \cdots, k$ and where $[a_i, b_i]$ denotes the commutator $a_i^{-1} b_i^{-1} a_i b_i$. For our purposes this may be taken as a definition of the term 'Fuchsian'. By a theorem of Nielsen, Bundgaard and Fox (see [12], p. 85 and [2], p. 68 for an elegant proof due to A. M. Macbeath and appropriate to this algebraic description of Fuchsian groups) there is a subgroup H of finite index in G, which is isomorphic to the fundamental group of a compact orientable surface. If the genus is g' and the surface has h' bounding circles, then H can be presented in terms of generators

$$\alpha_1, \cdots, \alpha_{g'}, \beta_1, \cdots, \beta_{g'}, \gamma_1, \cdots, \gamma_{h'}$$

and the one relation

$$[\alpha_1, \beta_1] \cdots [\alpha_{g'}, \beta_{g'}] \gamma_1 \cdots \gamma_{h'} = 1 .$$

If $g' = 0$ and $h' \leqq 2$, or if $g' = 1$ and $h' = 0$ then obviously H is abelian. Otherwise H is SQ-universal: if $h' \geqq 1$ then clearly H is free of rank $2g' + h' - 1$, while if $h' = 0$ and $g' \geqq 2$ then H has a free factor group of rank 2, as may be seen by substituting 1 for $\alpha_3, \cdots, \alpha_{g'}$ and $\beta_1, \cdots, \beta_{g'}$ in the above relation. Thus G either has an abelian subgroup of finite index or it has an SQ-universal subgroup of finite index and therefore, by the lemma, is itself SQ-universal.

The condition (see, for example, [2]) for the presentation (F) to describe a group having an abelian subgroup of finite index is that $\mu \leqq 0$, where

$$\mu = 2g - 2 + h + \sum_1^k (1 - n_i^{-1}) .$$

The triangle group (l,m,n) may be presented as $\mathrm{gp}(a, b, c \mid a^l = b^m = c^n = abc = 1)$ and this is of the form (F) with $g = h = 0$ and $k = 3$. In this case $\mu = 1 - 1/l - 1/m - 1/n$ and so (l, m, n) is SQ-universal if and only if $1/l + 1/m + 1/n < 1$.

3. Commentary

My interest in theorems of this kind has never arisen from any desire to know that, given an arbitrary countable group X, it may be embedded in a group whose generators satisfy certain special relations. Rather it is the deduction that the group G, if it is SQ-universal, must be enormously large. This may be taken as a definition of enormousness: but it is encouraging to note that an SQ-universal group must also be large in more traditional senses. For example, if G is SQ-universal then (*i*) G has a free subgroup of infinite rank; moreover either G is uncountable or, if countable, then (*ii*) G has $2^{\aleph_0}$ non-isomorphic factor groups, and (*iii*) G has infinite ascending chains of normal subgroups. If F is free of infinite rank there must exist N and K so that $N \lhd G, N \leqq K \leqq G$ and $K/N \cong F$. But then, since F is free, N is complemented in K and so G has a subgroup isomorphic to F. The second statement depends on the fact (see [7], or [4], p. 50) that there are $2^{\aleph_0}$ non-isomorphic finitely generated groups, G must have enough factor groups to embed all these, and yet each factor group has only countably many finitely generated subgroups. To see (*iii*), recall that if all well-ordered chains of normal subgroups in G were finite then every normal subgroup would be the normal closure of finitely many elements of G. There are only countably many finite sets of elements of G, hence in this case there would only be countably many normal subgroups, and by (*ii*) then G could not be SQ-universal.

The question whether the modular group is SQ-universal was raised by B. H. Neumann ([8], p. 541) who mentions that the free product $C_m * C_n$ is SQ-universal if $m \geqq 2$ and $n \geqq 8$. F. Levin first answered the question in [5], and showed that $C_m * C_n$ is SQ-universal if $m \geqq 2$, $n \geqq 3$. His work is extended by J. McCool [6] who shows also that certain of the triangle groups are SQ-universal. Their arguments use small cancellation theory and need only very little modification to show the

PROPOSITION. *The free product* $A * B$ *is* SQ-*universal if* $|A| \geqq 2$ *and* $|B| \geqq 3$.

If A and B are both finite then this also follows from the main lemma above because the cartesian subgroup $[A, B]$, the kernel of the homomorphism of $A * B$ onto $A \times B$, is free and of rank $(|A| - 1)(|B| - 1)$. A similar application of the lemma proves the

PROPOSITION. *Suppose that* A, B *are groups, that* $H \leqq A, K \leqq B$ *and that* $\phi : H \to K$ *is an isomorphism. If* A *and* B *are finite, and* $|A : H| \geqq 2, |B : K| \geqq 3$, *then the generalised free product*

$$G = \underset{H=K}{A * B} = \mathrm{gp}(A \cup B \mid h = h\phi \text{ for all } h \in H)$$

is SQ-*universal.*

For, the amalgam of A and B is embeddable in a finite group G_0 (see [8], p.532), and if N is the kernel of the corresponding homomorphism of G to G_0 then $N \cap (A \cup B) = 1$. Therefore by a theorem of Hanna Neumann ([10], pp. 532, 540, or [3], pp. 228, 247) N is a free group. The rank of N is at least 2, for otherwise G would be cyclic-by-finite, whereas G certainly contains a free subgroup of rank 2 (for instance, the subgroup generated by ab_1 and ab_2 where $a \in A - H$ and $1, b_1, b_2$ lie in different cosets of K in B). Thus G has an SQ-universal subgroup N of finite index and the lemma shows that G is SQ-universal.

It seems reasonable to suppose that the statement remains true if the assumption that A and B are finite is replaced by the weaker hypothesis that the amalgamated subgroups H and K are finite, but I have not been able to prove this.

As these examples show, many finitely presented groups are SQ-universal. In an earlier draft of this note four years ago I conjectured that a 1-relator group would be either soluble (if it is cyclic or may be presented as $\mathrm{gp}(a, b \mid a^{-1}ba = b^k)$ for some integer k) or SQ-universal. A proof of this by G. Sacerdote seems to be almost complete now. A similar problem arose in conversation with A.M. Macbeath,

CONJECTURE. *If G can be presented in terms of $r + d$ generators and r relations, where $d \geqq 2$, then G is* SQ-*universal.*

At one time I had hoped that one might construct a finitely presented simple group as a generalised free product of two free groups A, B of finite rank amalgamating finitely generated subgroups H and K. Joan Landman-Dyer and I showed quite easily that if H has infinite index in A or K has infinite index in B then such a group G is not simple. Recently Schupp [11] has used small cancellation arguments to show that G is even SQ-universal under these conditions.

PROBLEM: *Let $G = A \underset{H=K}{*} B$ where A, B are non-abelian free groups of finite rank and $|A:H|, |B:K|$ are finite.*

(*a*) *Can it happen that G is simple?*

(*b*) *Is G always* SQ-*universal?*

4. Acknowledgements

The main lemma of this paper has been proved also by P. Hall. He proved this, and many other remarkable embedding theorems, a long time ago. His argument is a little different. He uses the fact that a countable group is embeddable in a countable simple group to show that G is SQ-universal if and only if every countable group is embeddable in a chief–factor of G. Then he uses his result that, if H is normal and of finite index in G and if C is a chief-factor of H, then G has a chief-factor isomorphic to a finite direct power of C.

My own ideas on the subject have benefitted from correspondence with P. Hall and conversations with G. Higman, H.B. Griffiths and A.M. Macbeath. I thank them all warmly.

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