*The Square-integrability of Operator-valued Functions with Respect to a Non-negative Operator-valued Measure and The Kolmogorov Isomorphism Theorem**

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1. Introduction. Let $\mathfrak{H}$ and $\mathfrak{K}$ be two separable Hilbert spaces, and $\mathfrak{J}$ be a δ-ring of subsets of a space Ω (a δ-ring is a ring closed under countable intersection). Let M be a countably additive function defined on $\mathfrak{J}$ with values in the class $T(\mathfrak{H}, \mathfrak{H})$ of all non-negative definite compact operators of finite trace on $\mathfrak{H}$ to $\mathfrak{H}$. In the case $\mathfrak{H}$ and $\mathfrak{K}$ are the Euclidean spaces R^q and R^p ($1 \leqq p, q < \infty$) respectively, the integral

$$\int_\Omega \Phi \, dM \, \Psi^* \tag{1.1}$$

is defined in such a way that the space $L_{2,M}$ of functions with values linear operators on R^q into R^p and square integrable with respect to M, *i.e.* the space

$$L_{2,M} = \left\{\Phi: \text{trace} \int_\Omega \Phi \, dM \, \Phi^* \text{ finite}\right\} \tag{1.2}$$

is a complete inner product space ([21]) under the norm

$$||\Phi||_M = \text{trace} \int_\Omega \Phi \, dM \, \Phi^*. \tag{1.3}$$

For the special case $p = q = 1$, the completeness of $L_{2,M}$ is the core of the celebrated Riesz–Fisher Theorem.

The importance of the completeness of $L_{2,M}$ along with the denseness of the simple functions in it in connection with multivariate prediction theory of

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stochastic processes has been emphasized by P. Masani in his interesting review paper [16].

Since in the study of prediction for infinite dimensional processes the countable additive measures with values in $T(\mathfrak{H}, \mathfrak{H})$ arise naturally (see [6], p. 909, and [11]) a solution to the following problem is fundamental in any further developments of the infinite dimensional prediction theory constituting an extension of Wiener–Masani work [26]:

(*) ***Problem.*** For Φ and Ψ measurable operators from $\mathfrak{H}$ into $\mathfrak{K}$ define the integral (1.1) in such a way that the space $L_{2,M}$ defined in (1.2) becomes a complete inner product space under the norm (1.3) and the simple functions remain dense in $L_{2,M}$.

The space $L_{2,M}$ has also come up in the study of the perturbation of self-adjoint operators (de Branges [1] and Kuroda [15]). Some workers in the field have restricted themselves to the measurable bounded operators on $\mathfrak{H}$ to $\mathfrak{K}$ following the finite dimensional case. This is the main deficiency of their work which forces $L_{2,M}$ to be incomplete ([15], p. 71). This necessitates the study of (1.1) for measurable *unbounded* operators.

In this paper we solve the Problem (*) and as an application, obtain a generalization of the celebrated isomorphism theorem of Kolmogorov ([14] 2.7) to the case of weakly stationary processes of Payen [18].

The main results are proved in Section 4 after establishing preliminary results of measure theoretic nature in Sections 2 and 3. In section 5 a slight generalization of the results of Section 4 to the measures with values non-negative definite operators of a special form is given. We define in Section 6, the stochastic integral $\int \Phi \, d\xi$ of $\Phi \in L_{2,M'}$ with respect to an operator-valued countably additive orthogonally scattered measure which generalizes the stochastic integrals of Rosenberg [21] and N. P. Kandelskii and N. N. Vaknaniya [13] and establish an isomorphism theorem between $L_{2,M}$ and the space of stochastic integrals. Finally we establish an analogue of the celebrated isomorphism between the time and spectral domain due to Kolmogorov [14] for the case of operator-valued stationary processes of Payen [18].

2. Measurability and Extension of Operator-Valued Measures. For any two separable (complex) Hilbert-spaces $\mathfrak{H}$, $\mathfrak{K}$ with inner products $(\cdot, \cdot)_{\mathfrak{H}}$, $(\cdot, \cdot)_{\mathfrak{K}}$ and norms $\| \ \|_{\mathfrak{H}}$, $\| \ \|_{\mathfrak{K}}$ we denote by

a) $O(\mathfrak{H}, \mathfrak{K})$, the class of all linear operators from $\mathfrak{H}$ into $\mathfrak{K}$;
b) $L(\mathfrak{H}, \mathfrak{K})$, the class of all operators in $O(\mathfrak{H}, \mathfrak{K})$ with domain $\mathfrak{H}$;
c) $B(\mathfrak{H}, \mathfrak{K})$, the class of all bounded operators in $L(\mathfrak{H}, \mathfrak{K})$;
d) $C(\mathfrak{H}, \mathfrak{H})$, the class of all non-negative definite compact operators in $B(\mathfrak{H}, \mathfrak{H})$;
e) $HS(\mathfrak{H}, \mathfrak{K})$, the class of all Hilbert–Schmidt operators ([1], p. 1010) in $B(\mathfrak{H}, \mathfrak{K})$;
f) $T(\mathfrak{H}, \mathfrak{H})$, the class of all operators in $C(\mathfrak{H}, \mathfrak{H})$ of finite-trace.

Let $\mathfrak{B}$ be a σ-ring of subsets of a space Ω. We shall need a concept of measurability of $O(\mathcal{H}, \mathcal{K})$-valued functions. For $B(\mathcal{H}, \mathcal{K})$-valued functions the notion of measurability (weak and strong) has been defined before [10]. In view of the separability of $\mathcal{H}$ and $\mathcal{K}$ the weak and strong measurability are equivalent. We shall refer to both notions as measurability. Following is our extension of the notion of measurability.

Definition 2.1. Let Φ be an $O(\mathcal{H}, \mathcal{K})$-valued function on Ω then Φ is said to be measurable if there exists a sequence of measurable $B(\mathcal{H}, \mathcal{K})$-valued functions (Φ_n) such that for each $\omega \in \Omega$ and for each x in the domain of $\Phi(\omega)$, we have $\lim_{n\to\infty} ||\Phi_n(\omega)x - \Phi(\omega)x|| = 0$.

Lemma 2.2. i) *Let Φ, Ψ be measurable $O(\mathcal{H}, \mathcal{K})$-valued functions, then $\Phi + \Psi$ is measurable.*

ii) *If A_1, A_2 and Φ are measurable $B(\mathcal{K}, \mathcal{K})$, $B(\mathcal{H}, \mathcal{H})$ and $O(\mathcal{H}, \mathcal{K})$ valued functions respectively then $A_1\Phi$ and ΦA_2 are measurable.*

In Section 4 we shall need some results concerning the measurability of functions associated with a $C(\mathcal{H}, \mathcal{H})$-valued function. We shall now obtain these results here. For this, we need the following generalization of an interesting theorem of H. Weyl ([4], p. 653). The proof being similar to the one in [4] is omitted.

Lemma 2.3. Let $C \in C(\mathcal{H}, \mathcal{H})$ and $\ell_1 \geqq \ell_2 \geqq \ell_3 \cdots \geqq \ell_q \geqq \cdots$ be the eigenvalues of C. Then for any positive integer q the sum

$$\sum_1^q \ell_i = \sup \sum_1^q (Cx_i, x_i),$$

where the supremum is taken over the family of all systems $(x_1, x_2, \cdots, x_q)$ of orthonormal vectors in $\mathcal{H}$.

Let A be a measurable $C(\mathcal{H}, \mathcal{H})$-valued function on Ω. Then we can write

$$A(\omega) = \sum_{i=1}^{\infty} \lambda_i(\omega)E_i(\omega) \qquad \omega \in \Omega. \tag{2.4}$$

where $\lambda_i(\omega)$ $(i = 1, 2, \cdots)$ are the distinct eigenvalues of $A(\omega)$ arranged in decreasing order (for each ω) and $E_i(\omega)$ (for each i) is the projection operator onto the eigensubspace corresponding to $\lambda_i(\omega)$. With this setting we have the following simple lemma.

Lemma 2.5. For each ω in $\Lambda_j = \{\omega : \lambda_j(\omega) \neq 0\}$ we have

$$E_j(\omega) = \lim_{m\to\infty} \left[\frac{1}{1 + \lambda_j(\omega)} \{I + A(\omega) - \sum_{i<j} (E_i(\omega) + \lambda_i(\omega)E_i(\omega))\}\right]^m. \tag{2.6}$$

Theorem 2.10. Let $A(\cdot)$ be a measurable $C(\mathcal{H}, \mathcal{H})$-valued function on Ω given by (2.4) then

a) *the eigenvalues $\lambda_i(\cdot)$ $(i = 1, 2, \cdots)$ are measurable, and*

b) *for each i, the function* $E_i(\cdot)$ *is measurable on* $\Lambda_i = \{\omega \mid \lambda_i(\omega) \neq 0\}$ $(i = 1, 2, \cdots)$.

Proof. a) Let us denote by $\ell_1(\omega) \geqq \ell_2(\omega) \geqq \cdots \geqq \ell_q(\omega) \geqq \cdots$ the eigenvalues of $A(\omega)$ counted as often as their multiplicities, then by Lemma 2.3 we have $\sum_1^q \ell_j(\omega)$ measurable for each q finite. This implies that ℓ_j is measurable for each j.

Now set $n_1(\omega) = 1$, then $\lambda_1(\omega) = \ell_{n_1(\omega)}(\omega) = \ell_1(\omega)$. Hence $\lambda_1(\cdot)$ is measurable. Let $n_2(\omega) = \{\text{first}; n > n_1(\omega) \text{ such that } \ell_{n_1(\omega)}(\omega) = \ell_i(\omega)\ i \leqq n - 1 \text{ and } \ell_{n_1(\omega)}(\omega) > \ell_n(\omega)\}$. Then clearly $n_2(\cdot)$ is measurable. Hence $\lambda_2(\cdot) = \ell_{n_2(\cdot)}(\cdot)$ is measurable. Similarly we can define $n_3(\omega)$, $n_4(\omega)$, $\cdots$ and prove inductively that for each k, $n_k(\cdot)$ and $\lambda_k(\cdot) = \ell_{n_k(\cdot)}(\cdot)$ are measurable.

b) Proof of b) follows from Lemma 2.5 and a).

We shall also be using the notion of generalized inverse of an operator in $B(\mathcal{H}, \mathcal{H})$.

Definition 2.11. If $A \in B(\mathcal{H}, \mathcal{H})$, then A^-, the generalized inverse of A, is defined to be $P_{\mathfrak{N}^\perp(A)} A^{-1} P_{\overline{\mathfrak{R}(A)}}$, where $\mathfrak{N}(A)$ and $\overline{\mathfrak{R}(A)}$ are respectively the null space and closure of the range of A and A^{-1} is the inverse relation to the operator A.

The above definition is essentially due to Hestenes [9], see also [20]. The following properties of A^- can be easily established.

(2.12)
a) If $A \in B(\mathcal{H}, \mathcal{H})$ and $A = A^*$ then $A^- = P_{\overline{\mathfrak{R}(A)}} A^{-1} P_{\overline{\mathfrak{R}(A)}}$
b) If $A \in C(\mathcal{H}, \mathcal{H})$ with $A = \sum_{i=1}^{\infty} \lambda_i E_i$ then $A^- = \sum_{i=1}^{\infty} \lambda_i^- E_i$ ([9] p. 1338) where λ_i^- is the generalized inverse of λ_i.
c) (i) $A^- A = P_{\mathfrak{N}^\perp(A)}$ (ii) The closure of $AA^- = P_{\overline{\mathfrak{R}(A)}}$.

From Definitions 2.1, 2.11 and Theorem 2.10 we have the following

Corollary 2.13. If Φ is a measurable $C(\mathcal{H}, \mathcal{H})$-valued function then Φ^- is measurable.

3. In this section we first study the extension of measures defined on a Pre-ring.

Definition 3.1. $\mathcal{P}$ is a pre-ring over a set Ω iff. $\mathcal{P}$ is a non-void family of subsets of Ω such that for all $A, B \in \mathcal{P}$,

(i) $A \cap B \in \mathcal{P}$
(ii) there is an integer $n \geqq 1$ and disjoint sets $C_1, \cdots, C_n \in \mathcal{P}$ such that $A - B = C_1 \cup C_2 \cup \cdots \cup C_n$.

Definition 3.2. Let $\mathcal{P}$ be a pre-ring over Ω, and $L(\mathcal{H}, \mathcal{H})$ be the class of all linear operators on $\mathcal{H}$ into $\mathcal{H}$. We say that M is an $L(\mathcal{H}, \mathcal{H})$-valued countably additive (c.a.) measure on $\mathcal{P}$ iff

(i) M is a function on $\mathcal{P}$ to $L(\mathcal{H}, \mathcal{H})$,
(ii) If $A_k \in \mathcal{P}$, A_k are disjoint and $\bigcup_{k=1}^{\infty} A_k \in \mathcal{P}$ then

$$(3.3) \qquad M\left(\bigcup_1^\infty A_k\right)x = \sum_{k=1}^\infty M(A_k)x \quad \text{for each} \quad x \in \mathcal{H},$$

where the series in (3.3) is convergent in $\mathcal{H}$.

Definition 3.4. An $L(\mathcal{H}, \mathcal{H})$-valued c.a. measure M is called pointwise bounded (p-bounded) if for each x there exists a non-negative finite-valued c.a. measure μ_x on $\mathcal{P}$ such that $\|M(A)x\| \leqq \mu_x(A)$ and is μ-bounded if $\mu_x = \mu$ for all x.

Remark. Clearly, if M is μ-bounded then M is $B(\mathcal{H}, \mathcal{H})$-valued and the countable additivity of M in (3.3) is equivalent to the countable additivity of M in the uniform norm. In fact, if M is defined on a δ-ring then μ-boundedness and countable additivity is equivalent to the countable additivity of M in the uniform norm.

For p-bounded measures the extension theorem below can be proved following P. Masani's extension of a Hilbert-space valued c.a. orthogonally scattered measure [17].

Theorem 3.5. *Let* (i) *M be a p-bounded $L(\mathcal{H}, \mathcal{H})$-valued* c.a. *measure on a pre-ring $\mathcal{P}$*

(ii) *$\tilde{\mu}_x$ be the (unique) non-negative* c.a. *extension of μ_x to the σ-ring $\mathfrak{B} = \sigma(\mathcal{P})$ generated by $\mathcal{P}$*

(iii) $\mathfrak{B}_{\mu_x} = \{B; B \in \mathfrak{B} \text{ and } \tilde{\mu}(B) < \infty\}$

Then

(a) *$\xi_x(B) = M(B)x$ has a (unique) vector-valued extension ξ_x to $\mathfrak{B}_{\mu_x}$;*

(b) *$\|\xi_x(A)\| \leqq \tilde{\mu}_x(A)$ for all $A \in \mathfrak{B}_{\mu_x}$ and $x \in \mathcal{H}$.*

Let $\mathfrak{B}_0 = \bigcap_{x \in \mathcal{H}} \mathfrak{B}_{\mu_x}$. Clearly $\mathfrak{B}_0$ is a δ-ring containing $\mathcal{P}$. For each $B \in \mathfrak{B}_0$ we define the operator $\tilde{M}(B)$ on $\mathcal{H}$ (not necessarily bounded) by

$$\tilde{M}(B)x = \xi_x(B) \quad \text{for} \quad x \in \mathcal{H}$$

We observe that $\tilde{M}(B)$ is linear. With the above definition and Theorem 3.5 we obtain

Theorem 3.6. *(Hahn extension of p-bounded $L(\mathcal{H}, \mathcal{H})$-valued measure). Let*

(i) *M be a p-bounded $L(\mathcal{H}, \mathcal{H})$-valued* c.a. *measure on a pre-ring.*

(ii) *Let $\mathfrak{B}_0$ be the δ-ring defined as before.*

Then

(a) *M has a (unique) $L(\mathcal{H}, \mathcal{H})$-valued* c.a. *extension $\tilde{M}$ to $\mathfrak{B}_0$.*

(b) *$\|\tilde{M}(B)x\| \leqq \tilde{\mu}_x(B)$, $B \in \mathfrak{B}_0$, where $\tilde{\mu}_x$ is the extension of μ_x.*

Corollary 3.7. *Let M be a μ-bounded $L(\mathcal{H}, \mathcal{H})$-valued (hence, $B(\mathcal{H}, \mathcal{H})$-valued)* c.a. *measure on $\mathcal{P}$ then M has a (unique) $B(\mathcal{H}, \mathcal{H})$-valued* c.a. *extension $\tilde{M}$ to δ-ring $\mathfrak{B}_\mu$.*

In case $\mathcal{H}$ is finite-dimensional the above corollary is obtained by H. Salehi [23].

We now investigate an analogue of the Radon–Nikodym Theorem for μ-bounded $B(\mathcal{H}, \mathcal{H})$-valued c.a. measures defined on a δ-ring. This will be based on the validity of the Radon–Nikodym Theorem for (complex-valued) c.a. measure on a δ-ring. For this we need the following definition (See P. Halmos [8], p. 132, I. Segal [24]).

Definition 3.8. (Direct sum of finite measure spaces). Let $\mathfrak{F}$ be a δ-ring of subsets of Ω and μ be a non-negative finite-valued c.a. measure on $\mathfrak{F}$. We say that $(\Omega, \mathfrak{F}, \mu)$ is a direct sum of finite measure spaces if Ω is the union of a disjoint class D of elements of $\mathfrak{F}$ and every measurable set can be covered by countably many sets of D and a set of measure zero.

Let M be a μ-bounded $B(\mathcal{H}, \mathcal{H})$-valued c.a. measure on a δ-ring $\mathfrak{F}$ (in view of Remark following Def. 3.4 this is equivalent to the countable additivity of M in the uniform norm) such that $(\Omega, \mathfrak{F}, \mu)$ is a direct sum of finite measure spaces.

Following the proof of Theorem 4 ([2], p. 263) we obtain the Radon–Nikodym Theorem for μ-bounded measures.

Theorem 3.9. (Indefinite Integral Theorem). Let M be a μ-bounded non-negative definite $B(\mathcal{H}, \mathcal{H})$-valued c.a. *measure on a δ-ring $\mathfrak{F}$ and $(\Omega, \mathfrak{F}, \mu)$ be a direct sum of finite measure spaces. Then there exists (Note that the definition of $M'(\omega)$ depends on the choice of a CONS $\{e_i\}$) a measurable non-negative $B(\mathcal{H}, \mathcal{H})$-valued function M' on Ω such that $M'(\cdot)$ is locally Bochner integrable, i.e. $\int_B |M'(\omega)|_B \mu(d\omega)$ is finite and $M(B) = \int_B M'(\omega)\mu(d\omega)$ for all $B \in \mathfrak{F}$ ($|\cdot|_B$ will denote the Banach norm).*

Remark 3.10A. The condition that $(\Omega, \mathfrak{F}, \mu)$ is a direct sum of finite measure spaces does not severely restrict the measure space since a regular uniform locally compact measure space, which, in particular includes the Haar measure on locally compact groups satisfies this condition ([8], p. 256 and [24], p. 304).

B. It should be noted that once Theorem 3.9 is established one can further extend M to the class of sets A such that $\nu(A) < \infty$, where $\nu(A) = \int_A |M'_\mu(\omega)| \, d\mu$, *i.e.*, the class of sets on which M'_μ is Bochner integrable.

Remark 3.11. The indefinite integrals treated by Kuroda [15] constituting a generalization of de Branges [1] are special cases of the operator-valued measures satisfying the conditions of Theorem 3.9. Indeed we can take $\mu(A) = \int_A |M'_\sigma(\omega)|_B \sigma(d\omega)$ which under his assumptions become a totally finite measure and hence $(\Omega, \mathfrak{F}, \mu)$ is trivially a direct sum of finite measure spaces.

4. ***The space $L_{2,M}$.*** Let $\mathfrak{F}$ be a δ-ring of subsets of a space Ω and μ be a non-negative finite-valued c.a. measure on $\mathfrak{F}$ such that $(\Omega, \mathfrak{F}, \mu)$ is a direct sum of finite measure spaces. We will be interested in non-negative definite $B(\mathcal{H}, \mathcal{H})$-valued c.a. measures M on $\mathfrak{F}$ which are of the form

$$M(A) = \int_A M'_\mu(\omega)\mu(d\omega), \quad \text{for} \quad A \in \mathfrak{F}, \tag{4.1}$$

where $M'_\mu(\omega)$ is locally Bochner integrable. (Without loss of generality we assume $\mathfrak{F} = \{A : \int_A |M'_\mu(\omega)|_B \mu(d\omega) < \infty\}$).

Definition 4.2. Let Φ, Ψ be measurable $B(\mathcal{H}, \mathcal{K})$-valued functions on Ω and M be a non-negative $B(\mathcal{H}, \mathcal{H})$-valued c.a. measure and M' be as in (4.1). Then we say that (Φ, Ψ) is M-integrable if the $B(\mathcal{K}, \mathcal{K})$-valued function $\Phi M'_\mu \Psi^*$ is Bochner integrable μ and we write

$$\int \Phi \, dM \, \Psi^* = \int \Phi M'_\mu \Psi^* \, d\mu = (\Phi, \Psi)_M \tag{4.3}$$

Remark 4.4. It can be easily checked that if (Φ, Φ) and (Ψ, Ψ) are M-integrable, so is (Φ, Ψ).

The following lemma shows that Definition 4.3 does not depend on μ.

Lemma 4.5. *Let $(\Omega, \mathfrak{F}, \mu)$ and $(\Omega, \mathfrak{F}, \nu)$ be direct sums of finite measure spaces such that* (4.1) *holds with respect to* (w.r.t.) *μ and ν, then*

$$\int \Phi M'_\mu \Psi^* \, d\mu = \int \Phi M'_\nu \Psi^* \, d\nu,$$

in the sense that if either integral exists then so does the other and the two are equal.

Let V_M be the class of all measurable $B(\mathcal{H}, \mathcal{K})$-valued functions on Ω such that $\Phi M'_\mu \Phi^*$ is Bochner integrable and trace $\int \Phi M'_\mu \Phi^* \, d\mu$ is finite, *i.e.*

$$V_M = \left\{\Phi : \int \Phi \, dM \, \Phi^* \in T(\mathcal{K}, \mathcal{K})\right\} \tag{4.6}$$

Then we have the following result.

Theorem 4.7. *The space V_M has the following properties*

a) $\Phi, \Psi \in V_M$ *and* $A, B \in B(\mathcal{K}, \mathcal{K}) \Rightarrow A\Phi + B\Psi \in V_M$
b) *trace* $(\Phi, \Psi)_M$ *is an inner-product on* V_M *; i.e.,* V_M *is a pre-Hilbert-space over the ring* $B(\mathcal{K}, \mathcal{K})$ *with the norm* $\|\Phi\|_M^2 = $ *trace* $(\Phi, \Phi)_M$.

In general V_M will not be complete under $\|\cdot\|_M$ (See *e.g.* Kuroda [15]). The incompleteness of the pre-Hilbert space V_M stems from the fact that we only allow the $B(\mathcal{H}, \mathcal{K})$-valued functions Φ. In order to remove this deficiency one should include $0(\mathcal{H}, \mathcal{K})$-valued functions. However, as one can easily see the Definition 4.2 is not valid for this case. By a slight modification one can obtain the following definition.

Definition 4.8. Let Φ, Ψ be measurable $0(\mathcal{H}, \mathcal{K})$-valued functions (cf. 2.1) on Ω and M be a non-negative definite $B(\mathcal{H}, \mathcal{H})$-valued c.a. measure on $\mathfrak{F}$ and M' be as in (4.1). Then we say that (Φ, Ψ) is M-integrable if (i) $\Phi M'^{1/2}$ and $\Psi M'^{1/2}$ are $B(\mathcal{H}, \mathcal{K})$-valued and (ii) the $B(\mathcal{K}, \mathcal{K})$-valued function $(\Phi M'^{1/2}) \cdot (\Psi M'^{1/2})^*$ is Bochner integrable μ and we denote by

$$(4.9) \qquad (\Phi, \Psi)_M = \int \Phi \, dM \, \Psi^*$$
$$= \int (\Phi M'^{1/2})(\Psi M'^{1/2})^* \, d\mu \; (N.B. \; (\Psi M'^{1/2})^* \supseteq M'^{1/2}\Psi^*).$$

An analogue of Lemma 4.5 shows that the definition $(\Phi, \Psi)_M$ is independent of μ. We also observe that (Φ, Φ) and (Ψ, Ψ) are integrable so is (Φ, Ψ). Let us now define

$$(4.10) \qquad L_{2,M} = \left\{\Phi: \int \Phi \, dM \, \Phi^* \quad \text{exists and is in} \quad T(\mathcal{K}, \mathcal{K})\right\}.$$

In $L_{2,M}$ the functions Φ, Ψ with $(\Phi - \Psi)M_\mu'^{1/2} = 0$ a.e. $[\mu]$ are identified.

Remark 4.11. From the Definition 4.10 of $L_{2,M}$ we see that

(i) $\Phi M'^{1/2} \, \varepsilon \, HS(\mathcal{K}, \mathcal{K})$ a.e. $[\mu]$

(4.12) $\Phi \, \varepsilon \, L_{2,M}$ iff

(ii) $|\Phi M'^{1/2}|_E \, \varepsilon \, L_{2,\mu}$ ($| \; |_E$ denotes the Hilbert–Schmidt norm).

From (4.12) the following properties of $L_{2,M}$ are obvious.

(i) $\Phi, \Psi \, \varepsilon \, L_{2,M}$ implies $\Phi + \Psi \, \varepsilon \, L_{2,M}$;

(4.13)

(ii) $\Phi \, \varepsilon \, L_{2,M}, \; A \, \varepsilon \, B(\mathcal{K}, \mathcal{K})$ implies $A\Phi \, \varepsilon \, L_{2,M}$

i.e. $L_{2,M}$ is a vector space over the ring $B(\mathcal{K}, \mathcal{K})$.

Further $L_{2,M}$ has a Hilbertian structure with

a) the Gramian $(\Phi, \Psi)_M = \int \Phi \, dM\Psi^*$; and

(4.14) b) the inner-product $((\Phi, \Psi))_M = tr(\Phi, \Psi)_M$ for $\Phi, \Psi \, \varepsilon \, L_{2,M}$; and

c) the norm $||\Phi||_M = ((\Phi, \Phi))^{1/2}$.

From (4.13) and (4.14) we have

Theorem 4.15. *$L_{2,M}$ is a pre-Hilbert space over the ring $B(\mathcal{K}, \mathcal{K})$.*

Remark 4.16. We can prove that if M is given by (4.1) with $M_\mu'(\cdot) \, \varepsilon \, C(\mathcal{K}, \mathcal{K})$ then $L_{2,M}$ is complete and simple functions are dense in $L_{2,M}$. In view of the finite-dimensional case ([21], [22]) one would expect that the completeness should be true when $M_\mu'(\cdot)$ is a non-negative definite $B(\mathcal{K}, \mathcal{K})$-valued function. We are, at this time, not able to establish such a result as the analogue of Corollary 2.13 is not available. In fact if $[M_\mu']^-$ is a measurable $0(\mathcal{K}, \mathcal{K})$-valued function then the space $L_{2,M}$ is complete, the proof being identical with that of Theorem 4.19. If in addition $[M_\mu']^-$ is $B(\mathcal{K}, \mathcal{K})$-valued the Hilbert space $L_{2,M}$ will consist only of $HS(\mathcal{K}, \mathcal{K})$-valued functions, as by (4.10) each $\Phi \, \varepsilon \, L_{2,M}$ has the form $[\Phi(M_\mu')^{1/2}][(M_\mu')^{1/2}]^-$ with $\Phi(M_\mu')^{1/2} \, \varepsilon \, HS(\mathcal{K}, \mathcal{K})$. However if one assumes that M is of a special form, both the completeness of $L_{2,M}$ and denseness of simple functions in it are valid. But this will be proved by a special technique which heavily relies on a particular form of M (*cf.* §5). In the present section we shall establish the completeness of $L_{2,M}$ and denseness of simple functions in it for the case where $M'_\mu(\cdot) \in T(\mathcal{H}, \mathcal{H})$ where the bounding measure μ can be taken to be the trace M. However the proofs are valid for the case $M'_\mu(\cdot) \in C(\mathcal{H}, \mathcal{H})$. Our setting, henceforth, is as follows. $\mathfrak{F}$ is a δ-ring of subsets of a space Ω, M is a $T(\mathcal{H}, \mathcal{H})$-valued c.a. measure on $\mathfrak{F}$ such that

$$M(A) = \int_A M'_\tau(\omega)\tau(d\omega), \quad \text{for} \quad A \in \mathfrak{F},$$

$\tau(\cdot)$ is the trace of $M(\cdot)$ and $(\Omega, \mathfrak{F}, \tau)$ is a direct sum of finite measure spaces.

Remark 4.17. For $T(\mathcal{H}, \mathcal{H})$-valued set functions on a δ-ring $\mathfrak{F}$, the following statements are equivalent.

a) For $E_i \in \mathfrak{F}$, E_i disjoint, $\bigcup_{i=1}^\infty E_i \in \mathfrak{F}$ implies trace $[M(\bigcup_{i=1}^\infty E_i) - \sum_1^n M(E_i)] \to 0$ as $n \to \infty$, *i.e.* trace M is a c.a. measure on $\mathfrak{F}$.

b) For any CONS (e_j) in $\mathcal{H}$, the (complex-valued) set function $(M(\cdot)e_j, e_k)$ is c.a. for each pair j, k. This result can be obtained from Shatten ([25] p. 48) and Hille–Phillips ([10], p. 75).

For later reference we state the following theorem for $T(\mathcal{H}, \mathcal{H})$-valued measures.

Theorem 4.18. *Let M be a $T(\mathcal{H}, \mathcal{H})$-valued* c.a. *measure on a δ-ring $\mathfrak{F}$ with τ replacing μ in* (4.1). *Then*

a) *For all $A \in \mathfrak{F}$, $M(A) = \int_A M'_\tau(\omega)\tau(d\omega)$ where $M'(\cdot)$ is a measurable $T(\mathcal{H}, \mathcal{H})$-valued function;*

b) *$M'^{1/2}_\tau(\cdot)$ is a measurable $HS(\mathcal{H}, \mathcal{H})$-valued function;*

c) *$M'^{-1/2}_\tau = [M'^{1/2}_\tau]^-$ is a measurable $0(\mathcal{H}, \mathcal{H})$-valued function.*

We now proceed to the Main Theorem.

Main Theorem I 4.19. *The space $L_{2,M}$ is a Hilbert-space with inner-product $((,))_M$ and norm $\|\ \|_M$.*

By Theorem 4.15 we get that $((,))$ is an inner product. Hence in order to prove the main theorem it suffices to prove that $L_{2,M}$ is complete. The proof will depend on (4.12) and the following Lemma (see [2], p. 226).

Lemma 4.20. *Let μ be a σ-finite non-negative measure on $(\Omega, \mathfrak{F})$ and $L_{2,\mu}(\Omega, HS(\mathcal{H}, \mathcal{K}))$ be the space of functions Φ on Ω to $HS(\mathcal{H}, \mathcal{K})$ such that $|\Phi|_E$ is measurable and $\int_\Omega |\Phi|_E^2\, d\mu$ is finite. Then $L_{2,\mu}(\Omega, HS(\mathcal{H}, \mathcal{K}))$ is complete under the norm $(\int_\Omega |\Phi|_E^2\, d\mu)^{1/2}$.*

Proof of Main Theorem I. In view of (4.13) $L_{2,M}$ is a linear manifold over the ring $B(\mathcal{K}, \mathcal{K})$. To prove $L_{2,M}$ is complete under $\|\ \|_M$, consider a sequence

(Φ_n) in $L_{2,M}$ Cauchy under the norm $|| \ ||_M$. Then by (4.12) and (4.14) $\Phi_n[M'_\tau]^{1/2}$ is a Cauchy sequence in $L_{2,\tau}(\Omega, HS(\mathcal{H}, \mathcal{K}))$. Hence by Lemma 4.20 there exists a function $\Phi_0 \in L_{2,\tau}(\Omega, HS(\mathcal{H}, \mathcal{K}))$ such that

$$(4.21) \qquad \int_\Omega |\Phi_n[M'_\tau]^{1/2} - \Phi_0|_E^2 \, d\tau \to 0 \quad \text{as} \quad n \to \infty .$$

Consider the function $\Phi = \Phi_0[M'^{1/2}_\tau]^-$. By Theorem 4.18 (c) and Lemma 2.2, Φ is measurable. Also by (2.12) (c) $\Phi_0[M'^{1/2}_\tau]^- M'^{1/2}_\tau = \Phi_0 P_{\overline{\mathcal{R}(M_\tau'^{1/2})}}$ and $|\Phi_0 P_{\overline{\mathcal{R}(M_\tau'^{1/2})}}|_E \leqq |\Phi_0|_E$ a.e. $[\tau]$. Hence by (4.12) we get $\Phi \in L_{2,M}$. Note that

$$\begin{aligned} ||\Phi_n - \Phi||_M^2 &= \int_\Omega |\Phi_n M'^{1/2}_\tau - \Phi_0 P_{\overline{\mathcal{R}(M_\tau'^{1/2})}}|_E^2 \, d\tau \\ &= \int_\Omega |\Phi_n M'^{1/2}_\tau P_{\overline{\mathcal{R}(M_\tau'^{1/2})}} - \Phi_0 P_{\overline{\mathcal{R}(M_\tau'^{1/2})}}|_E^2 \, d\tau \\ &\leqq \int_\Omega |\Phi_n M'^{1/2}_\tau - \Phi_0|_E^2 \, d\tau. \end{aligned}$$

From which it follows that $\Phi_n \to \Phi$ by (4.21).

We shall call $\Phi = \sum_{i=1}^n A_i \chi_{E_i}$ a simple function if $A_i \in B(\mathcal{H}, \mathcal{K})$ and χ_{E_i} is the indicator function of the measurable sets E_i in $\mathfrak{F}$. The following theorem gives an important characterization of the functions in $L_{2,M}$.

Main Theorem II 4.22. *Φ belongs to $L_{2,M}$ if and only if there exists a sequence Φ_n of simple functions in $L_{2,M}$ such that $\Phi_n \to \Phi$ in $L_{2,M}$; i.e. the simple functions are dense in $L_{2,M}$.*

The proof of the theorem will depend on a series of lemmas.

Lemma 4.23. *Let $\epsilon > 0$, $\Phi \in L_{2,M}$. Then there exists a $B(\mathcal{H}, \mathcal{K})$-valued function $\Psi \in L_{2,M}$ such that* dim $\mathcal{R}(\Psi(\cdot))$ *is finite a.e.* $[\tau]$ *and* $||\Phi - \Psi||_M \leqq \epsilon$.

Proof. Since $\Phi \in L_{2,M}$ by (4.12) we get that

$$(4.24) \qquad \Phi M'^{1/2}_\tau \in HS(\mathcal{H}, \mathcal{K}) \quad \text{a.e.} \quad [\tau] \quad \text{and} \quad |\Phi[M'_\tau]^{1/2}|_E \in L_{2,T} .$$

Now $\Phi M'^{1/2}_\tau$ is bounded implies that the domain of $\Phi(\cdot)$, $\mathfrak{D}(\Phi)$, contains $\mathcal{R}(M'^{1/2}_\tau(\cdot))$. Since $\mathcal{R}(M'^{1/2}_\tau(\cdot)) \supseteq \mathcal{R}(M'_\tau(\cdot))$, we get

$$\mathfrak{D}(\Phi) \supseteq \mathcal{R}(M'_\tau(\cdot))$$

and from the fact

$$M'_\tau = \sum_{i=1}^\infty (\text{sgn } \lambda_i)\lambda_i E_i$$

we get $\mathcal{R}(\sum_1^N (\text{sgn } \lambda_i)E_i) \subseteq \mathcal{R}(M'_\tau)$ for each N. Hence $\mathfrak{D}(\Phi) \supseteq \mathcal{R}(\sum_{i=1}^N \cdot (\text{sgn } \lambda_i)E_i)$. Define now the $B(\mathcal{H}, \mathcal{K})$-valued function Φ_N

$$\Phi_N(\cdot) = \Phi(\cdot) \sum_1^N (\text{sgn } \lambda_i(\cdot))E_i(\cdot).$$

Then $\dim \mathfrak{R}(\Phi_N(\cdot)) \leqq N$ a.e. $[\tau]$. We shall show that $\Phi_N(\cdot) \in L_{2,M}$ and $||\Phi_N(\cdot) - \Phi||_M \to 0$. Let $\varphi_1(\omega), \varphi_2(\omega), \cdots$ be the eigenvectors of $M'_\tau(\omega)$ counted as often as their multiplicity. Since $|\Phi M'^{1/2}_\tau|^2_E$ is finite we get ([7] p. 33)

$$(4.25)\qquad |\Phi M'^{1/2}_\tau|^2_E = \sum_1^\infty ||M'^{1/2}_\tau \varphi_i||^2_{\mathcal{K}} = \sum_{i=1}^\infty ||\Phi \lambda_i^{1/2} E_i \varphi_i||^2_{\mathcal{K}}$$
$$= \sum_{i=1}^\infty \lambda_i \, ||\Phi\varphi_i||^2_{\mathcal{K}} < \infty \quad \text{a.e.} \quad [\tau].$$

Since $\Phi_N M'^{1/2}_\tau$ is compact we get from (4.25)

$$|\Phi_N M'^{1/2}_\tau|^2_E = \sum_1^N \lambda_i \, ||\Phi\varphi_i||^2_{\mathcal{K}} \quad \text{is finite} \quad \text{a.e.} \quad [\tau].$$

This implies $\Phi_N M'^{1/2}_\tau \in HS(\mathcal{K}, \mathcal{K})$ a.e. $[\tau]$ and $|\Phi_N M'^{1/2}_\tau|^2_E \leqq |\Phi M'^{1/2}_\tau|^2_E$ which is in $L_{2,\tau}$. From this and (4.12) we get that $\Phi_N \in L_{2,M}$. Also

$$(4.26)\qquad ||\Phi_N - \Phi||_M = \int |\Phi_N M'^{1/2}_\tau - \Phi M'^{1/2}_\tau|^2_E \, d\tau = \int \sum_{N+1}^\infty \lambda_i \, ||\Phi\varphi_i||^2_{\mathcal{K}} \, d\tau.$$

Since $\lambda_i \geqq 0$ from (4.25) the integrand in (4.26) goes to zero giving from (4.24) and (4.26)

$$||\Phi_N - \Phi||_M \to 0 \quad \text{as} \quad N \to \infty$$

by Lebesgue Dominated Convergence Theorem.

Lemma 4.27. Let Ψ be a compact operator-valued function in $L_{2,M}$ then for every $\epsilon > 0$, there exists a Φ such that Φ is compact operator-valued function with $\int |\Phi|^2_B |M'^{1/2}_\tau|^2_E \, d\tau$ finite and $||\Psi - \Phi||_M < \epsilon$.

Proof. Since Ψ is $L_{2,M}$ by Theorem F ([8], §25), the set $\{\omega : |\Psi M'^{1/2}_\tau|_E \neq 0\}$ has σ-finite measure. Therefore there exists sets $A_1, A_2, \cdots$, with $\tau(A_i)$ finite such that

$$(4.28)\qquad \int |\Psi M'^{1/2}_\tau|^2_E \, d\tau = \sum_1^\infty \int_{A_i} |\Psi M'^{1/2}_\tau|^2_E \, d\tau.$$

For $\epsilon > 0$, choose an N such that

$$(4.29)\qquad \sum_{N+1}^\infty \int_{A_i} |\Psi M'^{1/2}_\tau|^2_E \, d\tau < \epsilon/2.$$

Define now, for each n

$$(4.30)\qquad \Phi_n(\omega) = \Psi(\omega) \quad \text{if} \quad \omega \in \bigcup_{i=1}^N A_i \quad \text{and} \quad |\Psi(\omega)|_B \leqq n$$
$$= 0 \quad \text{otherwise}$$

Clearly Φ_n are measurable compact operators. Also $\{\omega\colon |\Phi_n(\omega)|_B \, |M'^{1/2}_\tau(\omega)|_E \neq$

$0\} \subseteq \bigcup_{i=1}^{N} A_i$ and hence

$$(4.31) \qquad \int |\Phi_n(\omega)|_B^2 \, |M_\tau'^{1/2}(\omega)|_E^2 \, d\tau = \int_{\bigcup_{i=1}^{N} A_i} |\Phi_n(\omega)|_B^2 \, |M_\tau'^{1/2}(\omega)|_E^2 \, d\tau \leqq n^2 \tau\Big(\bigcup_{i=1}^{N} A_i\Big) \quad \text{finite.}$$

Now,

$$(4.32) \qquad \int |(\Psi(\omega) - \Phi_n(\omega))M_\tau'^{1/2}(\omega)|_E^2 \, d\tau = \sum_{N}^{\infty} \int_{A_i} |\Psi(\omega)M_\tau'^{1/2}(\omega)|_E^2 \, d\tau + \int_{F} |\Psi(\omega)M_\tau'^{1/2}(\omega)|_E^2 \, d\tau$$

where $F_n = \bigcup_{i=1}^{N} A_i \cap \{\omega: |\Psi(\omega)|_B > n\}$. The first member of the right hand side of (4.32) is less than $\epsilon/2$ by (4.29). Also by choosing n sufficiently large one can make the second term on the right hand side less than $\epsilon/2$ since $F_n \downarrow \phi$ and $|\Psi(\omega)M_\tau'^{1/2}(\omega)|_E \in L_{2,\tau}$.

Let Ψ be a compact operator a.e. $[\tau]$. Let P_k, Q_k be the projection operators onto the subspaces generated by the first k elements of the CONS in $\mathcal{H}$, $\mathcal{K}$ respectively. From an argument similar to the one given on p. 204 of [19], it follows that the sequence of operators $\Psi_k = Q_k \Psi P_k$ converges to Ψ uniformly. More precisely, $|\Psi - \Psi_k|_B \to 0$ as $k \to \infty$.

Lemma 4.33. Let Ψ be a compact operator valued function such that $\int |\Psi|_B^2 |M_\tau'^{1/2}|_E^2 \, d\tau$ is finite and Ψ_k be as defined above. Then

(i) *$\int |\Psi_k|_B^2 \, |M_\tau'^{1/2}|_E^2 d\tau$ is finite*
(ii) *$||\Psi - \Psi_k||_M \to 0$ as $k \to \infty$.*

Lemma 4.34. Let Ψ and Ψ_k be as in Lemma 4.33. Then

(i) *$\Psi_k \in L_{2,M_k}$ where $M_k = P_k M P_k$ is a $T(\mathcal{H}, \mathcal{H})$-valued* c.a. *measure of dimension k.*
(ii) *Given $\epsilon > 0$, for each k, there exists a simple function Φ_k such that $\Phi_k \in L_{2,M_k}$ and $||\Psi_k - \Phi_k||_{M_k} < \epsilon$.*
(iii) *$\Phi_k P_k \in L_{2,M}$ and $||\Psi_k - \Phi_k P_k||_M < \epsilon$.*

Proof. From Lemma (4.33) (i) $\Psi_k \in L_{2,M}$, *i.e.* trace $\int \Psi_k M_\tau' \Psi_k^* \, d\tau$ is finite. But

$$(4.35) \qquad \int \Psi_k M_\tau' \Psi_k^* \, d\tau = \int \Psi_k P_k M_\tau' P_k \Psi_k^* \, d\tau = \int \Psi_k M_k' \Psi_k^* \, d\tau, \quad \text{where} \quad M_k' = P_k M_\tau' P_k,$$

hence

$$(4.36) \qquad \Psi_k \in L_{2,M_k} \quad \text{and} \quad ||\Psi_k||_{M_k} = ||\Psi_k||_M .$$

Since $\int |\Psi_k M_k'^{1/2}|_E^2 \, d\tau$ is finite, by an argument similar to the one preceding (4.28), there exists a sequence $\{A_i\}$ of disjoint sets with finite τ-measure such that

$$\int |\Psi_k M_k'^{1/2}|_E^2 \, d\tau = \sum_1^\infty \int_{A_i} |\Psi_k M_k'^{1/2}|^2 \, d\tau \tag{4.37}$$

Let $\epsilon > 0$. There exists an integer N such that

$$\sum_{N+1}^\infty \int |\Psi_k M_k'^{1/2}|_E^2 \, d\tau < \epsilon/2 \tag{4.38}$$

Consider now $(F, \mathfrak{J} \cap F, M_k)$ where $F = \bigcup_1^N A_i$. Then since $\int_F |\Psi_k M_k'^{1/2}|_E^2 \, d\tau$ is finite, there exists a simple function Φ_k defined on F ([21], p. 296) such that

$$\int_F |\Phi_k M_k'^{1/2}|_E^2 \, d\tau < \infty \quad \text{and} \quad \int_F |(\Psi_k - \Phi_k) M_k'^{1/2}|_E^2 \, d\tau < \epsilon/2. \tag{4.39}$$

Set $\Phi_k = 0$ on the complement of F. Note that from (4.38) and (4.39)

$$\int_\Omega |\Phi_k M_k'^{1/2}|_E^2 \, d\tau < \infty \quad \text{and} \quad \int_\Omega |(\Psi_k - \Phi_k) M_k'^{1/2}|_E^2 \, d\tau|| < \epsilon. \tag{4.40}$$

From (4.35) it can be deduced that $\Phi_k P_k \in L_{2,M}$ and further from (4.36) and (4.40)

$$||\Psi_k - \Phi_k P_k||_M = ||\Psi_k - \Phi_k||_{M_k} < \epsilon.$$

Proof of Main Theorem II. Let Φ be in $L_{2,M}$ and $\epsilon > 0$, from Lemma 4.23, there exists a compact operator-valued function Ψ in $L_{2,M}$ such that

$$||\Phi - \Psi||_M < \epsilon/2. \tag{4.41}$$

By Lemmas 4.27, 4.33 and 4.34 there exists a simple function χ in $L_{2,M}$ such that

$$||\Psi - \chi||_M < \epsilon/2. \tag{4.42}$$

From (4.41) and (4.42), the necessity follows. The proof of sufficiency is obvious from Theorem 4.19 and hence is omitted.

5. Let $\mathfrak{J}$ be a δ-ring of subsets of a space Ω. Let $\mu_i (i = 1, 2, \cdots)$ be a sequence of finite-valued c.a. measures on $\mathfrak{J}$ and P_i $(i = 1, 2, \cdots)$ be a sequence of orthogonal projection operators on $\mathcal{H}$ with $\sum_{i=1}^\infty P_i = I$.

Lemma 5.1. *Suppose for each $A \in \mathfrak{J}$, $\sup_i \mu_i(A)$ is finite, then*

(i) *$\sum_{i=1}^\infty \mu_i(A) P_i x$ converges for $A \in \mathfrak{J}$, $x \in \mathcal{H}$.*
(ii) *$M(A) = \sum_{i=1}^\infty \mu_i(A) P_i$ is a $B(\mathcal{H}, \mathcal{H})$-valued* c.a. *measure on $\mathfrak{J}$ (cf.* 3.2).

Hence $M(A) = \sum_{i=1}^\infty \mu_i(A) P_i$ is a $B(\mathcal{H}, \mathcal{H})$-valued measure. However it *need not be* ν-bounded for any finite-valued measure ν on $\mathfrak{J}$. Therefore $M(A)$ is *not* an indefinite integral of a locally Bochner integrable function. In fact, $M(A)$ need not be representable as a weak integral of a bounded operator. In spite of this, we can define $L_{2,M}$ without any further assumptions on M because of its special form. Consider first $M_i(A) = \mu_i(A)P_i$. We first define L_{2,M_i} analogous to section 4.

Definition 5.2. We say that a measurable $0(\mathcal{K}, \mathcal{K})$-valued function Φ is

a) M_i-integrable if (i) ΦP_i is $B(\mathcal{K}, \mathcal{K})$-valued and (ii) $(\Phi P_i)(\Phi P_i)^*$ is Bochner integrable with respect to μ_i and define $\int \Phi \, dM_i \Phi^* = \int (\Phi P_i)(\Phi P_i)^* \, d\mu_i$

b) M integrable if (i) Φ is M_i-integrable for each i and (ii) $\sum_{i=1}^{\infty} \int \Phi \, dM_i \Phi^*$ converges absolutely in $B(\mathcal{K}, \mathcal{K})$ and define $\int \Phi \, dM \Phi^* = \sum_{i=1}^{\infty} \int \Phi \, dM_i \Phi^*$.

Definition 5.3. (a) $\Phi \in L_{2,M_i}$ if $\int \Phi \, dM_i \Phi^*$ exists and is in $T(\mathcal{K}, \mathcal{K})$ (b) $\Phi \in L_{2,M}$ if $\int \Phi \, dM \Phi^*$ exists and is in $T(\mathcal{K}, \mathcal{K})$.

Analogous to Main Theorem I we have

Lemma 5.4. The space L_{2,M_i} is complete with the inner product and norm

$$((\Phi, \Psi))_{M_i} = tr \int \Phi \, dM_i \Psi^*, \quad ||\Phi||_{M_i} = (((\Phi, \Phi)))^{\frac{1}{2}} \tag{5.5}$$

Theorem 5.6. The space $L_{2,M}$ is complete with inner

$$((\Phi, \Psi))_M = \sum_{i=1}^{\infty} ((\Phi, \Psi))_{M_i}, \quad ||\Phi||_M = \left(\sum_{i=1}^{\infty} ||\Phi||_{M_i}^2\right)^{1/2}. \tag{5.7}$$

Proof. Consider the direct sum $\sum_1^{\infty} \oplus L_{2,M_i}$ of L_{2,M_i}. Define an operator S by $S\Phi = (\Phi P_1, \Phi P_2, \cdots)$ for $\Phi \in L_{2,M}$. Then S is an isometry on $L_{2,M}$ into $\sum_{i=1}^{\infty} \oplus L_{2,M_i}$. In fact, it is onto since if $(\Phi_1, \Phi_2, \cdots) \in \sum_{i=1}^{\infty} \oplus L_{2,M_i}$ one can take $\Phi(\omega) = \Phi_i(\omega)$ for $x \in \mathcal{R}(P_i)$. Since $\sum_{i=1}^{\infty} \oplus L_{2,M_i}$ is complete, ([7], p. 116), so is $L_{2,M}$.

Theorem 5.8. (i) *Simple functions are dense in L_{2,M_i}.* (ii) *Simple functions are dense in $L_{2,M}$.*

Proof. (i) Let $\Phi \in L_{2,M_i}$. Then $\int |\Phi P_i|_E^2 \, d\mu_i < \infty$. This implies that $\Psi = \Phi P_i$ is $HS(\mathcal{K}, \mathcal{K})$-valued function and hence compact. Note that $(\Phi - \Psi)P_i = \Phi P_i - \Psi P_i = \Phi P_i - \Phi P_i^2 = \Phi P_i - \Phi P_i = 0$. Therefore by identification of functions in L_{2,M_i} without any loss of generality we may assume that Φ is compact. Arguments similar to the ones given in Lemma 4.27, Lemma 4.33, Lemma 4.34 show that simple functions are dense in L_{2,M_i}.

(ii) Let $\Phi \in L_{2,M}$. Then $\sum_{i=1}^{\infty} \int |\Phi P_i|_E^2 \, d\mu_i < \infty$. Let $\epsilon > 0$. Choose N sufficiently large so that

$$\sum_{i=N+1}^{\infty} \int |\Phi P_i|_E^2 \, d\mu_i < \epsilon/2. \tag{5.9}$$

By part (i) for each $i = 1, 2, \cdots, N$, there exists a simple function $\chi_1, \chi_2, \cdots, \chi_N$ such that

$$\chi_i \in L_{2,M_i} \quad \text{and} \quad \int |(\Phi - \chi_i)P_i|_E^2 \, d\mu_i < \epsilon/2N. \tag{5.10}$$

Let $\chi = \chi_1 P_1 + \cdots + \chi_N P_n$. Then obviously χ is a simple function and for each $i = 1, 2, \cdots$

$$\chi P_i = \begin{cases} \chi_i P_i & \text{if } i \leqq N \\ 0 & \text{if } i \geqq N + 1 \end{cases} \tag{5.11}$$

Hence by (5.9), (5.10) and (5.11)

$$\sum_{i=1}^{\infty} \int |(\Phi - \chi)P_i|_E^2 \, d\mu_i = \sum_{i=1}^{N} \int |(\Phi - \chi_i)P_i|_E^2 \, d\mu_i + \sum_{i=N+1}^{\infty} \int |\Phi P_i|_E^2 \, d\mu_i < \epsilon$$

6. Stochastic Integrals and Isomorphism Theorem In this section we define stochastic integrals with respect to an operator-valued countably additive orthogonally scattered (c.a.o.s.) measure (the notion of integration with respect to a c.a.o.s. measure is different from the concept of integration defined in §4. For further details on this see [17] (pp. 72–75).) and prove an isomorphism theorem concerning these measures. This theorem is applied in the next section to establish a generalization of Kolmogorov's Isomorphism Theorem ([14] 2.7) between the time and spectral domain to an operator-valued stationary process [18]. The generalization is basic in any further development of infinite-dimensional prediction theory constituting an extension of Wiener Masani work [26].

Let $\mathfrak{F}$ be a δ-ring of subsets of a space Ω and M be a $T(\mathcal{H}, \mathcal{H})$-valued c.a. measure on $\mathfrak{F}$.

Definition 6.1. Let W and $\mathcal{H}$ be two separable Hilbert-spaces. We say that ξ is an $HS(\mathcal{H}, W)$-valued c.a.o.s. measure on $\mathfrak{F}$ iff (i) for all $A \in \mathfrak{F}$, $\xi(A) \in HS(\mathcal{H}, W)$ and (ii) for all $A, B \in \mathfrak{F}$,

$$\xi^*(A)\xi(B) = M(A \cap B) \tag{6.2}$$

where M is a $T(\mathcal{H}, \mathcal{H})$-valued c.a. measure on $\mathfrak{F}$.

Remark 6.3. From 6.2 it follows that $A, B \in \mathfrak{F}$ and $A \cap B = \emptyset$ implies $\xi^*(A)\xi(B) = 0$ and since M is a $T(\mathcal{H}, \mathcal{H})$-valued c.a. measure on $\mathfrak{F}$, (6.2) implies that for any $(A_i)_1^\infty$, A_i disjoint with $\bigcup_{i=1}^\infty A_i \in \mathfrak{F}$, $|\xi(\bigcup_{i=1}^\infty A_i) - \sum_1^N \xi(A_i)|_E^2 \to 0$ as $N \to \infty$.

We shall define the stochastic integral $\int \Phi \, d\xi$ where ξ is an $HS(\mathcal{H}, W)$-valued c.a.o.s. measure with associated measure M and Φ is an $0(\mathcal{H}, \mathcal{K})$-valued function in $L_{2,M}$. As is shown in §4, $L_{2,M}$ coincides with the closure of simple functions in it.

Definition 6.4. For a simple function $\sum_{i=1}^n A_i \chi_{E_i}$ with $A_i \in B(\mathcal{H}, \mathcal{K})$ and $E_i \in \mathfrak{F}$ $(i = 1, 2, \cdots, n)$

$$\int \Phi \, d\xi = \sum_{i=1}^{n} \xi(E_i) A_i^* \in HS(\mathcal{K}, W)$$

For Φ, Ψ simple functions and A, $B \varepsilon B(\mathcal{K}, \mathcal{K})$ a direct computation gives

(6.5) $$\int (A\Phi + B\Psi)\, d\xi = \int \Phi\, d\xi\, A^* + \int \Psi\, d\xi\, B^*$$

and for the Gramian $(\int \Phi\, d\xi, \int \Psi\, d\xi)$ defined by $(\int \Phi\, d\xi)^*(\int \Psi\, d\xi)$ we have

(6.6) $$\left(\int \Phi\, d\xi, \int \Psi\, d\xi\right) = \int \Phi\, dM\, \Psi^*$$

Since by (6.6)

$$\left|\int \Phi_n\, d\xi - \int \Phi_m\, d\xi\right|_E^2 = ||\Phi_n - \Phi_n||_M^2$$

if Φ_m, Φ_n are simple functions, the following definition is unambiguous.

Definition 6.7. For $\Phi \varepsilon L_{2,M}$ and (Φ_n), a sequence of simple functions, such that $\Phi_n \to \Phi$ in $L_{2,M}$ we define

(6.8) $$\int \Phi\, d\xi = \lim_{n\to\infty} \int \Phi_n\, d\xi,$$

where $\lim_{n\to\infty}$ refers to limit in $|\ |_E$ norm.

The integral in (6.8) is defined for all $\Phi \varepsilon L_{2,M}$ by Main Theorem II. Let us denote by S_ξ the (closed) subspace of the Hilbert space $HS(\mathcal{K}, W)$ over $B(\mathcal{K}, \mathcal{K})$ spanned by $\{\sum_{i=1}^n \xi(E_i)A_i^*,\ A_i \varepsilon B(\mathcal{K}, \mathcal{K}),\ E_i \varepsilon \mathfrak{F},\ n \text{ an integer}\}$. Then we have the following theorem.

Theorem 6.9. The correspondence $\Phi \to \int \Phi\, d\xi$ is an isomorphism from $L_{2,M}$ onto S_ξ such that (6.5) and (6.6) are satisfied. Hence in particular $||\Phi||_M = |\int \Phi\, d\xi|_E$.

7. In this section we show that Theorem 6.9 provides us with the basic isomorphism between the time and spectral domain of an operator-valued weakly stationary process. Let $\mathcal{K}$, W be two separable Hilbert spaces and $(G, +)$ be a locally compact abelian group. Following Payen [18] we say that $\{X_t, t \varepsilon G\}$ is an $HS(\mathcal{K}, W)$-valued weakly stationary process if $X_t^* X_s$ is a function of $s - t$.

Under the condition that $t \to X_t$ is a continuous map we get

(7.1) $$X_t = \int_{\hat{G}} \overline{\langle t, \lambda\rangle} E\,(d\lambda) X_0$$

from Payen ([18], p. 363) where $\hat{G}$ is the character group of G and E is a spectral measure on the Borel algebra $\mathfrak{B}$ of subsets of $\hat{G}$ (algebra generated by open-subsets of $\hat{G}$) with values being projection operators of W into W. If we now define

(7.2) $$\xi(A) = E(A)X_0,$$

then

$$\xi^*(A)\xi(B) = X_0^*E(A \cap B)X_0 = M(A \cap B). \tag{7.3}$$

Since X_0 is in $HS(\mathcal{H}, W)$ and E is in $B(W, W)$ we get that ξ is $HS(\mathcal{H}, W)$-valued and by (7.3) it is a c.a.o.s. measure. We call M *the spectral measure of the process.*

Remark 7.4. Since M is defined on a σ-algebra and, by (7.3), τM is finite on it, we get that the space $(\hat{G}, \mathfrak{B}, \tau)$ is a finite measure space.

From (7.1) and (7.2) we obtain

$$X_t = \int_{\hat{G}} \overline{\langle t, \lambda\rangle} I\xi(d\lambda), \tag{7.5}$$

where the integral is taken in the sense of §6.

Let us denote by S_X the (closed) subspace of $HS(\mathcal{H}, W)$ spanned by $\{\sum_{i=1}^n X_{t_i}A_i, A_i \in B(\mathcal{H}, \mathcal{H}), t_i \in G, n \text{ an integer}\}$. From Proposition 1 ([18], p. 335) we deduce that $\xi(B) = E(B)X_0$ is in S_X. Hence S_ξ, the (closed) subspace of $HS(\mathcal{H}, W)$ spanned by $\{\sum_{i=1}^n \xi(B_i)A_i, A_i \in B(\mathcal{H}, \mathcal{H}), B_i \in \mathfrak{B}\}$ is included in S_X. Clearly $S_X \subseteq S_\xi$ and hence from (7.5),

$$S_X = S_\xi . \tag{7.6}$$

From (7.6) and Theorem 6.9 we obtain the following theorem on the isomorphism between the time and spectral domain of a stationary process.

Theorem 7.8. *Let $\mathcal{H}$, W be two separable Hilbert spaces and $(G, +)$ be a locally compact abelian group. Let $\{X_t, t \in G\}$ be an $HS(\mathcal{H}, W)$-valued weakly stationary stochastic process with the associated spectral measure M and let S_X be the Hilbert-space spanned by the process. Then the space $L_{2,M}$ of $\mathcal{O}(\mathcal{H}, \mathcal{H})$-valued functions square-integrable M, is isomorphically isometric to the space S_X. The isomorphism takes $\langle t, \cdot\rangle I$ to X_t.*

Remark. If W is the space of square integrable functions with respect to a probability measure then the $HS(\mathcal{H}, W)$-valued stationary process is a Hilbert space valued stationary process in the sense of ([12] §9). Hence Theorem 7.8 is valid for these processes. Also if we consider $H = \ell_2$ then we get $HS(\ell_2, W)$-valued stationary processes which constitute a generalization of multivariate stationary processes studied by M. Rosenberg [21] and Yu A. Rozanov [22] in the sense that ℓ_2 is replaced by $R^q (q < \infty)$. Our Isomorphism theorem is thus a direct generalization of the corresponding one in [21], [22].

Appendix. Let (i) $\mathfrak{B}$ be a σ-algebra over Ω, (ii) μ is c.a. probability measure on $\mathfrak{B}$, (iii) $S_i \in \mathfrak{B}$ and $\Omega = S_1 \supseteq S_2 \supseteq \cdots$, $\mu(S_i) > 0$, (iv) $\{P_i\}$ is a sequence of one-dimensional orthogonal projection operators on $\mathcal{H}$ into $\mathcal{H}$, (v) $M(B) = \sum_i \mu(S_i \cap B)/\mu(S_i)P_i$.

This measure M arises in the Hellinger–Hahn Theorem (see *e.g.* [3] p. 914). In this interesting case $L_{2,M}$ consists only of $HS(\mathcal{H}, \mathcal{K})$-valued functions, nevertheless it is complete, and the manifold of $HS(\mathcal{H}, \mathcal{K})$-valued, $\mathfrak{B}$-simple functions is everywhere dense in it. This result in vectorial case along with the proof was pointed out to us by P. Masani after our manuscript was completed. It plays an important part in the explicit form of the spectral representation; (*cf.* P. Masani, Bull. Amer. Math. Soc., **76**, (1970) 427-528) and is included here for the general case, at his request.

In view of Theorem 5.8(ii) we have to only justify the completeness part. Let $S_\infty = \bigcap_{i \in J} S_i$. In case $\mu(S_\infty) \neq 0$ the result follows by Remark 4-16. If $\mu(S_\infty) = 0$, the proof is as follows: Let $\{\Phi_n\}$ be a Cauchy sequence in $L_{2,M}$. Then for every $\epsilon > 0$, there exists an N_ϵ such that

$$(A.1) \qquad \|\Phi_n - \Phi_m\|_M^2 = \sum_{i \in J} \int |\Phi_n P_i - \Phi_m P_i|_E^2 \, d\mu_i < \epsilon^2 (m, n \geqq N_\epsilon).$$

From (A.1), Lemma 4.20, definition of μ_i, and $\mu(S_\infty) = 0$, we get

There exist a $\Phi_i \in L_{2,\mu_i}(\Omega, HS(\mathcal{H}, \mathcal{K}))$ such that

$$(A.2) \qquad \int_\Lambda |\Phi_n P_i - \Phi_i|_E^2 \, d\mu_i \to 0 (n \to \infty) \quad \text{and} \quad \sum_{i=1}^{\infty} |\Phi_i(w)|_E^2 < \infty \quad \text{a.e.} \quad [\mu].$$

Define $\Phi = \sum_i \Phi_i P_i$. Then Φ is $HS(\mathcal{H}, \mathcal{K})$-valued. Now by (A.1) and (A.2) we have

$$\|\Phi_n - \Phi\|_M \leqq \epsilon \quad \text{for} \quad n \geqq N_\xi .$$

Note added in proof. Since this paper was completed, we have been able to show that for a non-negative $B(\mathcal{H}, \mathcal{H})$-valued function Φ, the generalized inverse Φ^- is a measurable $0(\mathcal{H}, \mathcal{H})$-valued function.

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