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The Stability in L_p and W_p^1 of the L_2-Projection onto Finite Element Function Spaces

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Abstract. The stability of the L_2-projection onto some standard finite element spaces V_h, considered as a map in L_p and W_p^1, $1 \leqslant p \leqslant \infty$, is shown under weaker regularity requirements than quasi-uniformity of the triangulations underlying the definitions of the V_h.

0. Introduction. The purpose of this paper is to show the stability in L_p and W_p^1, for $1 \leqslant p \leqslant \infty$, of the L_2-projection onto some standard finite element subspaces. Special emphasis is placed on requiring less than quasi-uniformity of the triangulations entering in the definitions of the subspaces.

In the one-dimensional case, which is discussed in Section 1 below, we first give a new proof of a result of T. Dupont (cf. de Boor [2]) showing L_∞ stability without any restriction on the defining partitions, thus extending an earlier result by Douglas, Dupont and Wahlbin [6] for the quasi-uniform case. We then use the technique developed to show the stability in W_p^1, in the case $p > 1$, under a quite weak assumption on the partition, depending on p. We also show that some restriction on the partition is needed for stability if $p > 1$. We remark that the known L_p stability result has been extended to higher degrees of regularity of the subspaces; see de Boor [3] and references therein.

In the case of a two-dimensional polygonal domain, discussed in Section 2, we demonstrate L_p and W_p^1 stability results for the L_2-projection onto standard piecewise polynomial spaces of Lagrangian type. The requirements on the triangulations involved are more severe than in the one-dimensional case, but allow nevertheless a considerable degree of nonuniformity. The proofs are based on a technique used by Descloux [5] to show L_∞ stability in the quasi-uniform case (cf. also Douglas, Dupont and Wahlbin [7]).

Results such as the above are of interest, for instance, in the analysis of Galerkin finite element methods for parabolic problems. Thus Bernardi and Raugel [1] use the W_2^1 stability of the L_2-projection to prove quasi-optimality of the Galerkin solution with respect to the energy norm, and Schatz, Thomée and Wahlbin [8] apply the L_∞ stability in a similar way (in the quasi-uniform case).

1. The One-Dimensional Case. In this section we shall study the orthogonal projection $\pi = \pi_h$ with respect to $L_2(0,1)$ onto the subspace

$$V_h = \left\{\chi \in C(0,1);\ \chi|_{I_j} \in P_k,\ j = 0,\ldots,N;\ \chi(0) = \chi(1) = 0\right\},$$

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where $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ is a partition of $[0,1]$ and $I_j = (x_j, x_{j+1})$. We shall first demonstrate the following result, in which $\|\cdot\|_p$ denotes the norm in $L_p(0,1)$.

THEOREM 1. *There is a constant C depending only on k such that*

$$\|\pi u\|_p \leqslant C\|u\|_p \quad \forall\, u \in L_p(0,1),\, 1 \leqslant p \leqslant \infty.$$

We shall then turn to estimates in

$$\mathring{W}_p^1(0,1) = \left\{ v \in L_p(0,1);\ v' = dv/dx \in L_p(0,1);\ v(0) = v(1) = 0 \right\}$$

and show, with $h_i = x_{i+1} - x_i$,

THEOREM 2. *Let* $1 \leqslant p \leqslant \infty$ *and assume, for* $p > 1$, *that the partition is such that* $h_i/h_j \leqslant C_0\alpha^{|i-j|}$, *where* $1 \leqslant \alpha < (k+1)^{p/(p-1)}$. *Then*

$$\left\|(\pi u)'\right\|_p \leqslant C\|u'\|_p \quad \forall\, u \in \mathring{W}_p^1(0,1),$$

where C depends on k, and for $p > 1$ *also on* C_0, α, *and p.*

For the proofs of these results we introduce the spaces

$$V_h^2 = \{ \chi \in V_h;\ \chi(x_i) = 0,\ i = 1, \ldots, N \}$$

and V_h^1, the orthogonal complement of V_h^2 in V_h with respect to the usual inner product in $L_2(0,1)$. For $k = 1$ we have $V_h^2 = \{0\}$ and $V_h^1 = V_h$. We also introduce the orthogonal projections π_j onto V_h^j, $j = 1, 2$, and obtain at once

$$\pi = \pi_1 + \pi_2 \qquad (\pi = \pi_1 \text{ for } k = 1). \tag{1.1}$$

We note that π_2 is determined locally on each I_j by the equations

$$(\pi_2 v, q)_{I_j} = (v, q)_{I_j} \quad \text{for } q \in P_k^0(I_j) = \left\{ q \in P_k;\ q(x_j) = q(x_{j+1}) = 0 \right\}, \tag{1.2}$$

where $(\cdot,\cdot)_{I_j}$ is the standard inner product in $L_2(I_j)$, and that a function in V_h^1 is completely determined by its values at the interior nodes, so that $\dim V_h^1 = N$.

For $v \in C[0,1]$ with $v(0) = v(1) = 0$ we shall also use the piecewise linear interpolant $r_h v \in V_h$ and note that, for $1 \leqslant p \leqslant \infty$,

$$\left\|(r_h v)'\right\|_p \leqslant \|v'\|_p, \tag{1.3}$$

and, denoting the norm in $L_p(I_i)$ by $\|\cdot\|_{p,I_i}$,

$$\|v - r_h v\|_{p,I_i} \leqslant \tfrac{1}{2} h_i \|v'\|_p. \tag{1.4}$$

LEMMA 1. *There is a constant C depending only on k such that, for* $1 \leqslant p \leqslant \infty$,

$$\|\pi_2 u\|_p \leqslant C\|u\|_p, \qquad u \in L_p(0,1), \tag{1.5}$$

and

$$\left\|(\pi_2(u - r_h u))'\right\|_p \leqslant C\|u'\|_p, \qquad u \in \mathring{W}_p^1(0,1). \tag{1.6}$$

Proof. We consider first (1.5) for $p = 1$ and set $\tilde{u}_h = \pi_2 u$. It follows, by taking $q = \tilde{u}_h$ in (1.2), that

$$\|\tilde{u}_h\|_{2,I_i}^2 \leqslant \|u\|_{1,I_i} \|\tilde{u}_h\|_{\infty,I_i}.$$

Hence $\|\tilde{u}_h\|_{1,I_i} \leqslant C_1\|u\|_{1,I_i}$, where

$$C_1 = \max_{q \in P_k^0(I_i)} \frac{\|q\|_{1,I_i}\|q\|_{\infty,I_i}}{\|q\|_{2,I_i}^2}.$$

Using the change of variables $y = (x - x_i)/h_i$, it is easily seen that C_1 is independent of the interval I_i and thus depends only on k. Analogously, we obtain

$$\|\pi_2 u\|_{p,I_i} \leqslant C_1\|u\|_{p,I_i}, \tag{1.7}$$

for $p = \infty$, and then for general p by the Riesz-Thorin theorem [9]. The desired result now follows by taking pth powers and summing.

To prove (1.6), we note that

$$\left\|\left(\pi_2(u - r_h u)\right)'\right\|_{p,I_i} \leqslant \frac{C_2}{h_i}\|\pi_2(u - r_h u)\|_{p,I_i}, \quad \text{where } C_2 = \max_{q \in P_k^0(0,1)} \frac{\|q'\|_p}{\|q\|_p},$$

and, by (1.7) and (1.4),

$$\|\pi_2(u - r_h u)\|_{p,I_i} \leqslant C_1\|u - r_h u\|_{p,I_i} \leqslant \tfrac{1}{2}C_1 h_i\|u'\|_{p,I_i},$$

from which (1.6) follows with $C = \frac{1}{2}C_1C_2$.

In order to study the projection π_1, we shall construct a basis for V_h^1. For this purpose let us define $\psi \in P_k$ by

$$\psi(0) = 0, \quad \psi(1) = 1, \quad (\psi, q) = \int_0^1 \psi q \, dx = 0 \quad \forall q \in P_k^0.$$

For each nodal point x_i we associate the function ψ_i defined by

$$\begin{aligned} \psi_i(x) &= \psi\left(\frac{x - x_{i-1}}{h_{i-1}}\right) && \text{on } I_{i-1}, \\ &= \psi\left(\frac{x_{i+1} - x}{h_i}\right) && \text{on } I_i, \\ &= 0 && \text{on } \mathscr{C}(\overline{I_{i-1} \cup I_i}). \end{aligned}$$

It is then easily seen that $\{\psi_i\}_1^N \subset V_h^1$ and that these functions thus form a basis.

For u given, and $w = \pi_1 u = \sum_{i=1}^N w_i\psi_i$, we then have

$$\sum_{i=1}^N w_i(\psi_i, \psi_j) = (u, \psi_j) = u_j, \qquad j = 1, \ldots, N,$$

or in matrix form, with $G = ((\psi_i, \psi_j))$, $W = (w_1, \ldots, w_N)^T$ and $U = (u_1, \ldots, u_N)^T$,

$$GW = U. \tag{1.8}$$

We note that the Gram matrix G is tridiagonal. We shall need to compute its nonzero elements.

LEMMA 2. *We have*

$$\|\psi_i\|^2 = \frac{1}{k(k+2)}(h_{i-1} + h_i)$$

and

$$(\psi_i, \psi_{i+1}) = \frac{(-1)^{k-1}}{k(k+1)(k+2)}h_i.$$

Proof. By transformation of variables it suffices to show that

$$\int_0^1 \psi(x)^2\,dx = \frac{1}{k(k+2)}$$

and

$$\int_0^1 \psi(x)\psi(1-x)\,dx = \frac{(-1)^{k-1}}{k(k+1)(k+2)}.$$

The definition of ψ implies easily

$$\psi(x) = \frac{(-1)^{k-1}}{k!}\frac{1}{x(1-x)}\frac{d^{k-1}}{dx^{k-1}}\left[x^{k+1}(1-x)^k\right].$$

Further, since $\psi(x) - x$ and $\psi(1-x) - (1-x) \in P_k^0$, we find

$$\int_0^1 \psi(x)(\psi(x) - x)\,dx = \int_0^1 \psi(x)(\psi(1-x) - (1-x))\,dx = 0.$$

Hence, integrating by parts $k-1$ times, we have

$$\begin{aligned}\int_0^1 \psi(x)^2\,dx &= \frac{(-1)^{k-1}}{k!}\int_0^1 \frac{1}{1-x}\frac{d^{k-1}}{dx^{k-1}}\left[x^{k+1}(1-x)^k\right]dx \\ &= \frac{1}{k!}\int_0^1 x^{k+1}(1-x)^k\frac{d^{k-1}}{dx^{k-1}}\frac{1}{1-x}\,dx \\ &= \frac{1}{k}\int_0^1 x^{k+1}\,dx = \frac{1}{k(k+2)}\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \psi(x)\psi(1-x)\,dx &= \frac{(-1)^{k-1}}{k!}\int_0^1 x^{k+1}(1-x)^k\frac{d^{k-1}}{dx^{k-1}}\frac{1}{x}\,dx \\ &= \frac{(-1)^{k-1}}{k}\int_0^1 x(1-x)^k\,dx = \frac{(-1)^{k-1}}{k(k+1)(k+2)},\end{aligned}$$

which completes the proof.

Let us introduce the diagonal matrix D with the same diagonal elements as G, i.e.,

$$d_i = \|\psi_i\|^2 = \frac{1}{k(k+2)}(h_{i-1} + h_i).$$

We may then write G in the form $G = D(I + K)$, where K is a tridiagonal matrix with diagonal elements zero and bidiagonal entries

$$\text{(1.9)}\qquad \begin{aligned}k_{i,i-1} &= \frac{(\psi_i, \psi_{i+1})}{\|\psi_i\|^2} = \frac{(-1)^{k-1}}{k+1}\frac{h_{i-1}}{h_{i-1}+h_i}, \\ k_{i,i+1} &= \frac{(-1)^{k-1}}{k+1}\frac{h_i}{h_{i-1}+h_i}.\end{aligned}$$

The equation (1.8) now takes the form

$$\text{(1.10)}\qquad (I + K)W = D^{-1}U.$$

We are now ready to prove Theorem 1. By Lemma 1 it remains only to prove

$$\|\pi_1 u\|_p \leqslant C\|u\|_p, \qquad u \in L_p(0,1), \tag{1.11}$$

and we begin by showing this for $p = \infty$. This will be done by showing (here and below we denote by $|\cdot|_p$ the standard l_p-norms for N-vectors)

$$\|\pi_1 u\|_\infty \leqslant C|W|_\infty, \tag{1.12}$$

then

$$|W|_\infty \leqslant C|D^{-1}U|_\infty, \tag{1.13}$$

and finally

$$|D^{-1}U|_\infty \leqslant C\|u\|_\infty.$$

To see that (1.12) holds, we note that, since for no x in $(0,1)$ more than two $\psi_i(x)$ are nonzero, we have

$$\|\pi_1 u\|_\infty = \max_x \left| \sum_{i=1}^N w_i \psi_i(x) \right| \leqslant 2\|\psi\|_\infty |W|_\infty.$$

In view of (1.10), in order to show (1.13), we only need to show that $(I + K)^{-1}$ is bounded in l_∞. But this follows at once from the fact that, by (1.9),

$$|K|_\infty = \max_i \sum_j |k_{ij}| = \frac{1}{k+1} < 1,$$

and hence

$$\left|(I + K)^{-1}\right|_\infty \leqslant \frac{1}{1 - 1/(k+1)} = \frac{k+1}{k}.$$

Finally,

$$|D^{-1}U|_\infty = \max_j \frac{|(u, \psi_j)|}{\|\psi_j\|^2} \leqslant C_1\|u\|_\infty,$$

where

$$C_1 = \max_j \frac{\|\psi_j\|_1}{\|\psi_j\|_2^2} = \frac{\|\psi\|_1}{\|\psi\|_2^2},$$

where the latter equation follows by transformation of the subintervals onto $[0,1]$.

This completes the proof of (1.11) for $p = \infty$. For $p = 1$ the result follows at once by duality and for $1 < p < \infty$ by the Riesz-Thorin theorem. The proof of Theorem 1 is now complete.

We now turn to the proof of Theorem 2. We may write

$$\pi u = \pi_1(u - r_h u) + \pi_2(u - r_h u) + r_h u.$$

In view of Lemma 1 and (1.3) the last two terms are bounded, as desired, and it remains to consider $w = \pi_1 \varepsilon$ where $\varepsilon = u - r_h u$. Letting $W = (w_1, \ldots, w_N)^T$ where $w_i = w(x_i)$, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)^T$ where $\varepsilon_i = (\varepsilon, \psi_i)$, we find that W solves (1.8) with U replaced by ε. We shall show, with D the diagonal matrix introduced above and $p' = p/(p-1)$,

$$\|w'\|_p \leqslant C|D^{-1/p'}W|_p,$$

then

$$(1.14) \qquad |D^{-1/p'}W|_p \leqslant C|D^{-1-1/p'}\varepsilon|_p,$$

and finally

$$(1.15) \qquad |D^{-1-1/p'}\varepsilon|_p \leqslant C\|u'\|_p,$$

which together complete the proof.

We have first

$$\begin{aligned}\|w'\|_p^p &= \sum_{i=0}^{N} \int_{I_i} \left|w_i\psi_i' + w_{i+1}\psi_{i+1}'\right|^p dx \\ &\leqslant 2^{p/p'} \sum_{i=1}^{N} |w_i|^p \left(h_{i-1}^{-p+1} + h_i^{-p+1}\right)\|\psi'\|_p^p \\ &\leqslant C \sum_{i=1}^{N} d_i^{-p+1}|w_i|^p = C|D^{-1/p'}W|_p^p,\end{aligned}$$

where we have used

$$d_i^{p-1} \leqslant C(h_{i-1}+h_i)^{p-1} \leqslant C\left(h_{i-1}^{-p+1} + h_i^{-p+1}\right)^{-1}.$$

The proof of (1.15) is also straightforward. We have, by Hölder's inequality,

$$\begin{aligned}|\varepsilon_i| = |(\varepsilon, \psi_i)| &\leqslant \|\varepsilon\|_{p,I_{i-1}}\|\psi_i\|_{p',I_{i-1}} + \|\varepsilon\|_{p,I_i}\|\psi_i\|_{p',I_i} \\ &\leqslant C\left(h_{i-1}^{1/p'}\|\varepsilon\|_{p,I_{i-1}} + h_i^{1/p'}\|\varepsilon\|_{p,I_i}\right),\end{aligned}$$

and hence by (1.4),

$$\begin{aligned}|\varepsilon_i| &\leqslant C\left(h_{i-1}^{1+1/p'}\|u'\|_{p,I_{i-1}} + h_i^{1+1/p'}\|u'\|_{p,I_i}\right) \\ &\leqslant C d_i^{1+1/p'}\|u'\|_{p,I_{i-1}\cup I_i},\end{aligned}$$

whence (1.15) follows immediately.

It remains to show (1.14). Recalling that W satisfies (1.8), and hence (1.10), with U replaced by ε, we have

$$\left(D^{-1/p'}(I+K)D^{1/p'}\right)D^{-1/p'}W = D^{-1-1/p'}\varepsilon,$$

and it thus suffices to show that $I + D^{-1/p'}KD^{1/p'}$ has a bounded inverse in l_p under the assumptions of the theorem. For this purpose we estimate the powers of the second term. Since K^l is $(2l+1)$-diagonal and has nonnegative elements, we have

$$|D^{-1/p'}K^lD^{1/p'}|_p \leqslant \max_{|i-j|\leqslant 2l}\left(d_i/d_j\right)^{1/p'}|K^l|_p.$$

Here,

$$d_i/d_j = (h_{i-1}+h_i)/(h_{j-1}+h_j) \leqslant C_0^2\alpha^{2l+1} \quad \text{for } |i-j| \leqslant 2l.$$

Further, again since K^l is $(2l+1)$-diagonal, we have

$$|K^l|_1 \leqslant (2l+1)|K^l|_\infty \leqslant (2l+1)|K|_\infty^l \leqslant \frac{2l+1}{(k+1)^l},$$

and, using once more the Riesz-Thorin theorem,

$$|K^l|_p \leqslant (2l+1)^{1/p}\frac{1}{(k+1)^l} \quad \text{for } 1 \leqslant p \leqslant \infty.$$

Altogether we find, under the assumptions made,

$$\left|\left(I + D^{-1/p'}KD^{1/p'}\right)^{-1}\right|_p \leqslant 1 + \sum_{l=1}^{\infty} |D^{-1/p'}K^lD^{1/p'}|_p$$

$$\leqslant 1 + \left(C_0^2\alpha\right)^{1/p'} \sum_{l=1}^{\infty} (2l+1)^{1/p}\left(\frac{\alpha^{2/p'}}{k+1}\right)^l < \infty,$$

which completes the proof.

We conclude by remarking that in Theorem 1 and in the case $p = 1$ of Theorem 2 no restriction is made concerning the partitions used, and that quite strong mesh refinements are permitted for $p > 1$ in Theorem 2. The following example shows, however, that some restriction is needed in the latter case: Consider the partition with only one interior point $x_1 = 1 - \varepsilon$, so that $h_0/h_1 = (1-\varepsilon)/\varepsilon$. Let $k = 1$ and $u(x) = x(1-x)$. Then $\pi u = \beta\psi_1$, where β is determined by the equation $\beta\|\psi_1\|^2 = (u, \psi_1)$, or, after an easy calculation, $\beta = \frac{1}{4}(1 + \varepsilon(1-\varepsilon))$. In this case,

$$\left\|(\pi u)'\right\|_p = \beta\left\{\int_0^{1-\varepsilon} \varepsilon(1-\varepsilon)^{-p}\,dx + \int_{1-\varepsilon}^1 \varepsilon\varepsilon^{-p}\,dx\right\}^{1/p} \geqslant \frac{1}{4}\varepsilon^{-1/p'},$$

which tends to ∞ with $1/\varepsilon$ if $p > 1$.

2. The Two-Dimensional Case. In this section we shall consider the orthogonal projection onto a finite element subspace of $L_2(\Omega)$ where Ω is a bounded domain in R^2. For simplicity we assume that Ω is polygonal and consider a family of triangulations $\mathcal{T}_h$ of $\bar{\Omega}$ into closed triangles K with disjoint interiors such that no vertex of any triangle lies on the interior of an edge of another triangle. We shall use the approximating spaces

$$V_h = \left\{v \in C(\bar{\Omega});\ v|_K \in P_k,\ v|_{\partial\Omega} = 0\right\}.$$

In order to express our assumptions concerning the partition of Ω, we shall introduce some notation. For a given K_0 we let $R_j(K_0)$ be the set of triangles which are "j triangles away from K_0", defined by setting $R_0(K_0) = K_0$ and then, recursively, for $j \geqslant 1$, $R_j(K_0)$ the union of the closed triangles in $\mathcal{T}_h$ which are not in $\bigcup_{i<j} R_i(K_0)$, but which have at least one vertex in $R_{j-1}(K_0)$. Thus $R_j(K_0)$ is the union of the triangles which may be reached by a connected path $Q_1, \ldots, Q_j$ with Q_1 a vertex of K_0, Q_j a vertex of K and Q_iQ_{i+1} an edge of the triangulation for $1 \leqslant i < j$, and not by any shorter such path. Setting $l(K_0, K) = j$ for $K \in R_j(K_0)$ it follows, in particular, that $l(K_0, K)$ is symmetric in K and K_0. We also define $n_j(K_0)$ to be the number of triangles in $R_j(K_0)$.

Letting a_K denote the area of K, we shall assume below that, with some positive constants C_1, C_2, α, β, r with $\alpha \geqslant 1$, $\beta \geqslant 1$, we have uniformly for small h,

$$a_K/a_{K_0} \leqslant C_1\alpha^{l(K,K_0)} \qquad \forall K, K_0 \in \mathcal{T}_h, \tag{2.1}$$

and

$$n_j(K) \leqslant C_2 j^r\beta^j \qquad \forall K \in \mathcal{T}_h,\ j \geqslant 1. \tag{2.2}$$

When all triangles have angles bounded below, independently of h, then a_K is bounded above and below by ch_K^2, where h_K is the diameter of K. The case when the triangulations are quasi-uniform then corresponds to $\alpha = 1$. Note that by (2.1) we have

$$\operatorname{area}(R_j(K_0)) \geqslant cn_j(K_0)a_{K_0}\alpha^{-j},$$

and, if the angles are bounded below,

$$\operatorname{area}(R_j(K_0)) \leqslant \operatorname{area}\Big(\bigcup_{i\leqslant j} R_i(K_0)\Big) \leqslant C\Big(\sum_{i=0}^{j} h_{K_0}\alpha^{i/2}\Big)^2,$$

whence

$$\begin{aligned} n_j(K_0) &\leqslant Cj^2 && \text{if } \alpha = 1,\\ &\leqslant C\alpha^{2j} && \text{if } \alpha > 1. \end{aligned}$$

In particular, if the angles are bounded below, (2.1) with $\alpha > 1$ implies (2.2) with $r = 0$, $\beta = \alpha^2$. However, in practice this is a very crude estimate. In fact, for any triangulation which is a deformation of a quasi-uniform one, (2.2) holds with $\beta = 1$, $r = 2$.

The results of this section are based on the following variant of a lemma by Descloux [5] concerning the orthogonal projection π in $L_2(\Omega)$ onto V_h.

LEMMA 3. *Let $1 \leqslant p \leqslant \infty$. There are positive constants $\gamma < 1$ and C such that, if* $\operatorname{supp} v_0 \subset K_0$,

$$\|\pi v_0\|_{2,K} \leqslant C\gamma^{l(K,K_0)}a_{K_0}^{1/2-1/p}\|v_0\|_p \quad \forall K,\ K_0 \in \mathcal{T}_h, \tag{2.3}$$

where γ depends only on k and C only on k and p.

Proof. Letting $D_j = \bigcup_{i>j} R_i(K_0)$ denote the union of triangles which may only be reached by paths of length at least j, we shall want to show that for some $\kappa > 0$,

$$\|\pi v_0\|_{2,D_j}^2 \leqslant \kappa\|\pi v_0\|_{2,R_j}^2 \quad \text{for } j \geqslant 1. \tag{2.4}$$

Assuming this for a moment, we denote the left side by q_j and thus find

$$q_j \leqslant \kappa(q_{j-1} - q_j) \quad \text{for } j \geqslant 1,$$

whence

$$q_j \leqslant \frac{\kappa}{1+\kappa}q_{j-1} \leqslant \Big(\frac{\kappa}{1+\kappa}\Big)^j q_0 \leqslant \gamma^{2j}\|\pi v_0\|_2^2,$$

where $\gamma = (\kappa/(1+\kappa))^{1/2}$. Here, since $\operatorname{supp} v_0 \subset K_0$, we find, with $(\cdot,\cdot)_R$ the standard inner product in $L_2(R)$ with R omitted for $R = \Omega$, and p' the conjugate exponent $p' = p/(p-1)$,

$$\|\pi v_0\|_2 = \max_{\chi\in S_h}\frac{(v_0,\chi)}{\|\chi\|_2^2} \leqslant \max_{q\in P_k}\frac{(v_0,q)_{K_0}}{\|q\|_{2,K_0}^2} \leqslant \|v_0\|_{p,K_0}\max_{q\in P_k}\frac{\|q\|_{p',K_0}}{\|q\|_{2,K_0}^2},$$

and hence by the standard transformation to a reference triangle, with δ depending on p and k,

$$\|\pi v_0\|_2 \leqslant \delta a_{K_0}^{1/p'-1/2}\|v_0\|_{p,K_0}.$$

Altogether, we have, if $j = l(K, K_0)$,

$$\|\pi v_0\|_{2,K} \leqslant \|\pi v_0\|_{2,R_j} \leqslant q_{j-1}^{1/2} \leqslant \delta\gamma^j a_{K_0}^{1/2-1/p}\|v_0\|_{p,K},$$

which is the desired result with $C = \delta/\gamma$.

It remains to show (2.4). Since $\operatorname{supp} v_0 \subset K_0$ we have

$$(2.5) \qquad (\pi v_0, \chi) = 0 \quad \text{for } \chi \in V_h,\ \operatorname{supp}\chi \subset D_{j-1} = D_j \cup R_j, \text{ if } j \geqslant 1.$$

Let $\omega = \pi v_0$ and define for any $\omega \in S_h$ a new function $\tilde{\omega}_j$ in S_h by setting $\tilde{\omega}_j = \omega$ on D_j and $\tilde{\omega}_j = 0$ on $\Sigma_{j-1} = \bigcup_{K \in \mathscr{T}_h, K \cap D_j = 0} K$, the union of triangles, all vertices of which may be reached from K_0 by paths of length at most $j-1$. To define $\tilde{\omega}_j$ on the remaining triangles K, which are then included in $R_j(K_0)$ but not in Σ_{j-1}, we introduce for such a K the Lagrangian nodes (having barycentric coordinates $(i_1/k, i_2/k, i_3/k)$ with i_1, i_2 and i_3 nonnegative integers) and set $\tilde{\omega}_j = \omega$ at all such nodes which do not belong to Σ_{j-1} or to an edge joining two vertices in Σ_{j-1}, and $\tilde{\omega}_j = 0$ at the other nodes. With $\chi = \tilde{\omega}_j$, (2.5) takes the form

$$(\omega, \tilde{\omega}_j) = \|\omega\|_{2,D_j}^2 + (\omega, \tilde{\omega}_j)_{R_j} = 0,$$

whence

$$(2.6) \qquad \|\omega\|_{2,D_j}^2 \leqslant -(\omega, \tilde{\omega}_j)_{R_j}.$$

In order to estimate the latter quantity, we consider again a triangle $K \subset R_j$ with K not included in Σ_{j-1} and note that K has either one or two vertices in Σ_{j-1} and the remaining vertices in D_j. For $q \in P_k$ we let $\tilde{q}_K$ be the polynomial in P_k which vanishes at the nodal points that are in Σ_{j-1} or on an edge joining two vertices in Σ_{j-1} and agrees with q at the other Lagrangian nodes. We thus have

$$-(\omega, \tilde{\omega}_j)_K \leqslant \|\omega\|_{2,K}^2 \max_{q \in P_k} \frac{-(q, \tilde{q}_K)_K}{\|q\|_{2,K}^2}.$$

By transformation to a reference triangle we find that the latter maximum is independent of K in the two possible cases for the location of its vertices, so that, after summation,

$$-(\omega, \tilde{\omega}_j)_{R_j} = -\sum_{K \subset R_j} (\omega, \tilde{\omega}_j)_K \leqslant \kappa \|\omega\|_{2,R_j}^2.$$

Together with (2.6), this completes the proof of (2.4) and hence of the lemma.

The constant κ may thus be expressed in terms of the reference triangle $\hat{K}$ with vertices Q_1, Q_2 and Q_3 as

$$\kappa = \max_{j=1,2} \max_{q \in P_k} \frac{-(q, \tilde{q}_{\hat{K},j})_{\hat{K}}}{\|q\|_{2,\hat{K}}^2},$$

where $\tilde{q}_{\hat{K},1} = 0$ at Q_1, $\tilde{q}_{\hat{K},1} = q$ at the other nodes and $\tilde{q}_{\hat{K},2} = 0$ at the vertices of $\overline{Q_1Q_2}$ and $= q$ at the other vertices.

We are now ready for our stability estimate for π in $L_p(\Omega)$. Here and below, α, β and γ are the parameters in (2.1), (2.2) and (2.3).

THEOREM 3. *Let $1 \leqslant p \leqslant \infty$ and assume that the numbers α, β and γ are such that*

$$\gamma\beta\alpha^{|1/2-1/p|} < 1. \tag{2.7}$$

Then

$$\|\pi u\| \leqslant C\|u\|_p \quad \forall u \in L_p(\Omega),$$

where C depends only on C_1, C_2, α, β, r, k and p.

Proof. We have in the usual way, for each $K \in \mathscr{T}_h$,

$$\|\pi u\|_{p,K} \leqslant C a_K^{-1/2+1/p}\|\pi u\|_{2,K}. \tag{2.8}$$

Here, writing $u = \sum_{K' \in \mathscr{T}_h} u|_{K'}$, and using Lemma 3, we find

$$\|\pi u\|_{2,K} \leqslant \sum_{K' \in \mathscr{T}_h} \|\pi(u|_{K'})\|_{2,K} \leqslant C \sum_{K' \in \mathscr{T}_h} \gamma^{l(K,K')} a_{K'}^{1/2-1/p}\|u\|_{p,K'},$$

so that, using also (2.8) and (2.1),

$$\begin{aligned}\|\pi u\|_{p,K} &\leqslant C \sum_{K' \in \mathscr{T}_h} \gamma^{l(K,K')}(a_{K'}/a_K)^{1/2-1/p}\|u\|_{p,K'} \\ &\leqslant C \sum_{K' \in \mathscr{T}_h} \left(\gamma\alpha^{|1/2-1/p|}\right)^{l(K,K')}\|u\|_{p,K'}.\end{aligned}$$

Introducing the vectors $X = \{x_K = \|\pi u\|_{p,K};\ K \in \mathscr{T}_h\}$ and $Y = \{y_K = \|u\|_{p,K};\ K \in \mathscr{T}_h\}$ and the symmetric matrix $M = (m_{K,K'})$ with $m_{K,K'} = \delta^{l(K,K')}$, where $\delta = \gamma\alpha^{|1/2-1/p|}$, we conclude for the corresponding l_p-vector and associate matrix norms $|\cdot|_p$

$$\|\pi u\|_p = |X|_p \leqslant |M|_p |Y|_p = |M|_p \|u\|_p.$$

It remains to bound the matrix norm $|M|_p$. We have by the Riesz-Thorin theorem and the symmetry of M,

$$|M|_p \leqslant |M|_1^{1/p}|M|_\infty^{1-1/p} = |M|_\infty = \max_K \sum_{K'} \delta^{l(K,K')}.$$

Using now also the hypothesis (2.2) we find

$$|M|_p \leqslant \max_K \sum_{j=0}^{\infty} n_j(K)\delta^j \leqslant C\sum_{j=0}^{\infty} j^r(\beta\delta)^j,$$

where the latter sum is finite under assumption (2.7). This completes the proof.

We now show a stability estimate for the gradient of the L_2-projection.

THEOREM 4. *Let $1 \leqslant p \leqslant \infty$ and assume that the angles of $\mathscr{T}_h$ are bounded below, uniformly in h, and that α, β, and γ are such that*

$$\gamma\beta\alpha^{1-1/p} < 1. \tag{2.9}$$

Then

$$\|\nabla\pi u\|_p \leqslant C\|\nabla u\|_p \quad \textit{for } u \in \mathring{W}_p^1(\Omega).$$

Proof. There exists a linear operator r_h: $\mathring{W}_p^1(\Omega) \to V_h$ such that for $u \in \mathring{W}_p^1(\Omega)$,

$$\|\nabla r_h u\|_p \leqslant C\|\nabla u\|_p \tag{2.10}$$

and

$$\|u - r_h u\|_{p,K} \leqslant Ch_K \|\nabla u\|_{p,\hat{K}} \leqslant Ca_K^{1/2} \|\nabla u\|_{p,\hat{K}}. \tag{2.11}$$

For $p > 2$, $u \in \mathring{W}_p^1(\Omega)$ implies $u \in C(\overline{\Omega})$, and $r_h u$ may be chosen as an interpolant of u and $\hat{K}$ as K, whereas for $p \geqslant 2$ a preliminary local regularization as in Clément [4] is needed and $\hat{K}$ may be chosen as $K \cup R_1(K)$.

We may write

$$\nabla \pi u = \nabla \pi \varepsilon + \nabla r_h u, \quad \text{where } \varepsilon = u - r_h u,$$

and, in view of (2.10), it suffices to estimate $\nabla \pi u$. We have the inverse estimate

$$\|\nabla \pi \varepsilon\|_{p,K} \leqslant C a_K^{-1+1/p} \|\pi \varepsilon\|_{2,K},$$

and, as in the proof of Theorem 3,

$$\|\pi \varepsilon\|_{2,K} \leqslant C \sum_{K' \in \mathcal{T}_h} \gamma^{l(K,K')} a_{K'}^{1/2-1/p} \|\varepsilon\|_{p,K'}.$$

Hence, using also (2.1) and (2.11),

$$\begin{aligned} \|\nabla \pi \varepsilon\|_{p,K} &\leqslant C \sum_{K' \in \mathcal{T}_h} \gamma^{l(K,K')} \left(a_{K'}/a_K\right)^{1-1/p} a_K^{-1/2} \|\varepsilon\|_{p,K'} \\ &\leqslant C \sum_{K' \in \mathcal{T}_h} \left(\gamma \alpha^{1-1/p}\right)^{l(K,K')} \|\nabla u\|_{p,K'}. \end{aligned}$$

The proof is now completed as in Theorem 3.

It is clear that the assumptions (2.7) and (2.9) are satisfied in the quasi-uniform case. In order to see that they permit severely nonuniform triangulations, it is necessary to know that the constant γ is not too close to 1. For this purpose we recall that $\gamma = (\kappa/(1+\kappa))^{1/2}$ with $\kappa = \kappa_k = \max(\kappa_{1k}, \kappa_{2k})$, where with the notation of the proof of Lemma 3,

$$\kappa_{jk} = \max_{q \in P_k} \frac{-\left(q, \tilde{q}_{\hat{K},j}\right)_{\hat{K}}}{\|q\|_{2,\hat{K}}^2}, \qquad j = 1, 2,\ k \geqslant 1. \tag{2.12}$$

Introducing the Lagrangian basis functions $\{\psi_j\}_1^{N_k}$ corresponding to the Lagrangian nodes $\{Q_j\}_1^{N_k}$ in $\hat{K}$, so that $\psi_i(Q_j) = \delta_{ij}$, we have

$$\|q\|_{2,\hat{K}}^2 = (A\xi, \xi), \qquad q = \sum_{i=1}^{N_k} \xi_i \psi_i \in P_k,$$

where A is the matrix with elements $a_{ij} = (\psi_i, \psi_j)$. Correspondingly, the quadratic form in the numerator in (2.12) may be obtained as

$$\left(q, \tilde{q}_{\hat{K},j}\right) = \left(B_j \xi, \xi\right), \qquad j = 1, 2,$$

where B_j is a symmetric matrix obtained from A as follows: Let S be the set of indices i such that $\tilde{q}_{\hat{K},j}$ is forced to vanish at Q_i, $i \in S$, and let

$$S' = \{1, 2, \ldots, N_k\} \setminus S.$$

Then $\tilde{q}_{\hat{K},j} = \sum_{j \in S'} \xi_j \psi_j$ and hence $(q, \tilde{q}_{\hat{K}}) = (B\xi, \xi)$, with $B = (b_{ij})$, where

$$\begin{aligned} b_{ij} &= 0 && \text{if } i, j \in S, \\ &= \tfrac{1}{2} a_{ij} && \text{if } i \in S,\ j \in S' \text{ or } j \in S,\ i \in S', \\ &= a_{ij} && \text{if } i, j \in S'. \end{aligned}$$

For $i = 1$, $S = \{1\}$, and for $i = 2$, S consists of the indices for which Q_i are on $\overline{Q_1Q_2}$. With this notation, κ_{jk} is the largest eigenvalue of the eigenvalue problem

$$-B_j\xi = \lambda A\xi. \tag{2.13}$$

For $k = 1$ we have $N_1 = 3$ and

$$(A\xi,\xi) = \left(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3\right)a_K/6,$$

$$(B_1\xi,\xi) = \left(\xi_2^2 + \xi_3^2 + \xi_2\xi_3 + \tfrac{1}{2}\xi_1\xi_2 + \tfrac{1}{2}\xi_1\xi_3\right)a_K/6,$$

$$(B_2\xi,\xi) = \left(\xi_3^2 + \tfrac{1}{2}\xi_1\xi_3 + \tfrac{1}{2}\xi_2\xi_3\right)a_K/6.$$

By completing squares we find easily that for both $j = 1$ and 2, $\lambda = (\sqrt{6} - 2)/4$ is the smallest number such that

$$\lambda(A\xi,\xi) + (B_j\xi,\xi) \geqslant 0 \quad \forall \xi \in R^3.$$

Hence,

$$\kappa_1 = \kappa_{11} = \kappa_{12} = (\sqrt{6} - 2)/4 = .112, \qquad \gamma_1 = \sqrt{3} - \sqrt{2} = .318.$$

For $k = 2$ and $k = 3$ we have $N_2 = 6$ and $N_3 = 10$ nodal points, respectively. By numerical computation we have determined the largest eigenvalues of (2.13) in these cases and found

$$\kappa_{12} = .048, \quad \kappa_{22} = .165, \quad \kappa_2 = .165, \quad \gamma_2 = .376,$$

and

$$\kappa_{13} = .032, \quad \kappa_{23} = .142, \quad \kappa_3 = .142, \quad \gamma_3 = .353.$$

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