THE STABILITY OF A ROTATING LIQUID MASS HELD TOGETHER BY SURFACE TENSION

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Summary

The stability of a drop of incompressible fluid held together by the action of surface tension and made to rotate rigidly about an axis is determined, the effect of gravity being neglected. Two distinct problems are investigated. In the first is considered an isolated drop in the form of a surface of revolution and the manner in which its stability changes with angular speed is investigated. At zero angular speed, where the drop is spherical, infinitesimal disturbances are shown to be stable and beyond a certain critical angular speed a new linear series of equilibrium forms emerges, the original series becoming unstable.

In the second problem the drop is rotating together with a denser medium. It appears that the drop tends to become a cylindrical thread with increasing angular speed and that it is stable to infinitesimal disturbances at high and low angular speeds. This suggests that the drop is stable at intermediate angular speeds as well.

I. Introduction

In recent years the behaviour of rotating liquid masses has attracted much attention because of their application to a number of practical problems including (1) the electromagnetic levitation of liquid metals, as discussed by Polonis, Butters, and Parr (1954), Comenetz and Salatka (1958), and Laithwaite (1965), (2) the study of the equilibrium configurations of fission barriers for liquid drop nuclei (see Bohr and Wheeler 1939; Sperber 1963), and (3) the measurement of the dynamic coefficient of surface tension, as suggested by Rosenthal (1962*a*) and carried out by Princen, Zia, and Mason (1967). In each of these problems the surface tension of the liquid is known to be a contributing factor but the details of the phenomena have only been worked out in the last case.

One question that is still open is the stability of the equilibrium configurations. Chandrasekhar (1965*a*), who was concerned with the stability of an isolated rotating liquid drop held together by surface tension, was the first to relate this problem to the stability analysis of self-gravitating liquids. He used a new technique involving an extension of the method of the tensor virial that he had developed earlier (Chandrasekhar 1960, 1961). This method has since been extended in a series of papers by Rosenkilde (1967*a*, 1967*b*, 1967*c*, 1967*d*), but this approach does not predict the

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existence of a point of bifurcation of the linear series of equilibrium forms. This fact is first mentioned in the preceding paper (pp. 823–35 of the present issue), where it is shown that a new series of equilibrium forms emerges when the angular speed increases from zero at a point where the angular speed is a maximum. With further increase in angular momentum the isolated drop tends to collapse onto its equator and the angular speed decreases until the drop finally breaks away from the axis of the rotation. Sperber (1963) suggested, incorrectly (at least, according to Chandrasekhar 1965*a*), that the point where collapse begins is the point at which the linear series of equilibrium forms becomes unstable. It is one of the purposes of the present paper to use some of the results of Basset (1888, 1892) and Lyttleton (1953) to show that the isolated drop is stable to certain types of small disturbance.

One of the major difficulties in this type of stability analysis is the fact that the equilibrium surfaces are so complicated that even infinitesimal perturbation theory is hard to apply. For isolated drops of rotating liquid held together by the action of surface tension Lord Rayleigh (1914) developed an approximate method for finding the equilibrium forms, while in the preceding paper it is shown that a solution in elliptic integrals is feasible; however, there a numerical method is used to describe the shape of the free surface. For a drop of liquid rotating with a denser medium and held together by surface tension the equilibrium forms have been obtained by Vonnegut (1942), Rosenthal (1962*a*), and in the preceding paper. Again, only the last paper shows that there is a point of bifurcation of the linear series of forms which emerges when the angular speed increases from zero. In fact, there is an angular speed at which the angular momentum of the liquid drop is a maximum. The existence of points of bifurcation for self-gravitating liquids has been known for some time (see Basset 1888, 1892; and the most recent papers of Chandrasekhar 1965*b*, 1966 for an up to date review of the subject).

As pointed out by Lyttleton (1953) the existence of a point of bifurcation does not, in itself, ensure that there is a loss of stability of the original series of equilibrium forms, as is so often assumed. Certainly what happens at such a point should be examined in detail. One possibility is that the free surface ceases to be a surface of revolution. This is precisely what happens to the self-gravitating liquid when the angular momentum is sufficiently high. In that case, Poincaré (1895) and Jeans (1919) showed that a "pear-shaped" series could emerge. On the other hand, Basset (1888) showed that the rotation also ceased to be a rigid one. This is a point that seems to have been overlooked in the literature.

Now according to Lyttleton (1953), Schwarzchild (1897) was the first to show that an equilibrium form may be unstable when the angular speed is constant, yet may be stable when the angular momentum remains constant. It is with this point in mind that the present paper examines the stability of rotating liquid drops held together by surface tension. Essentially the method employed here makes use of the potential and kinetic energies of the drops since, according to Orr (1906, 1907), Lyttleton (1953), and Pozharitiskii (1964), these quantities may be related to a discussion of the stability of rigid body rotations.

II. The Isolated Drop

It is well known that an isolated spherical drop held together by the action of the capillary force is stable to infinitesimal disturbances that depend exponentially upon time. Landau and Lifshitz (1959) expressed the most general small disturbance of this kind as a Fourier series in the azimuthal angle and so were able to define a perturbation from the equilibrium configuration in terms of the stream function

$$\Psi = \sum_{(m,n)} A_{m,n} r^n \mathrm{P}_n^m(\cos\theta) \exp\{\mathrm{i}(\sigma_{m,n} t + m\phi)\},$$

where (r, θ, ϕ) are the spherical polar coordinates, $\sigma_{m,n}$ is the complex frequency of the appropriate normal mode, $\mathrm{P}_n^m(\cos\theta)$ are the associated Legendre polynomials, and $A_{m,n}$ are the complex amplitude factors that are indeterminate in this linearized stability analysis. Landau and Lifshitz showed that such disturbances are stable and that in fact the frequency of the normal modes is given by

$$\sigma_{m,n}^2 = \alpha n(n-1)(n+2)/\rho_1 d^3, \tag{1}$$

where n is a positive integer. We see that to each value of n there correspond $2n+1$ different modes; for $n = 0$ the oscillations must be radial but, if the liquid is assumed to be incompressible, then their amplitudes must vanish; for $n = 1$ the motion is a translation of the liquid as a whole and so the smallest possible frequency of oscillation corresponds to $n = 2$ in which case

$$\sigma_{\min}^2 = 8\alpha/\rho_1 d^3.$$

This result is not unexpected because any disturbance of the surface must lead to an increase in the total surface area and hence to an increase in the potential energy. Now in any motion for which the sum of the kinetic and potential energies is constant, the potential energy cannot exceed the initial value of this sum and so any small disturbances must be stable. Of course this system may oscillate finitely, but in a real fluid this motion would be damped by the action of the viscous stresses.

When the liquid is made to rotate the situation is much more complicated as the surface is no longer spherical and it may happen that a small disturbance will lead to a reduction in the surface area and hence to an initial decrease in the potential energy U. Alternatively, the disturbance may extract energy from the source of the angular momentum and so be maintained.

Lyttleton (1953) showed that the positions of relative equilibrium, for a fluid rotating with constant angular speed ω, are obtained by finding stationary values of the total mechanical potential $U - \frac{1}{2}I\omega^2$, where I is the moment of inertia of the system about the axis of rotation. Thus, in order to obtain the equilibrium configurations we may proceed as follows.

Consider a drop of fluid of density ρ_1 which rotates, at constant angular speed ω, together with a fluid of density ρ_2. If we introduce the cylindrical polar coordinates (r, θ, z) with the z axis as the axis of rotation and assume that the interface is the surface of revolution $r = f(z)$, then the equilibrium configurations are given by finding the functions $f = f(z)$ for which

$$L = U - T, \tag{2}$$

with

$$U = 4\pi \int \alpha f (1+f'^2)^{\frac{1}{2}}\, dz, \qquad \text{and} \qquad T = \tfrac{1}{2}\pi \int (\rho_1 - \rho_2)\omega^2 f^3\, dz, \tag{2a}$$

is stationary but subject to the restriction that the volume

$$V = 2\pi \int f^2\, dz$$

is known beforehand. Here α is the coefficient of surface tension appropriate to the two fluids, and the prime denotes differentiation with respect to z. This is an interesting result for we may now apply the Legendre test (see Fox 1950) to the integrands of equations (2a). We find that $\partial L/\partial f' < 0$ and hence that the quantity L is a minimum with respect to all neighbouring configurations rotating with the same angular speed ω and having the same volume $V = 4\pi d^3/3$, say.

Now the extremal problem can be solved by using a first integral of the Euler–Lagrange equation. Thus, for a drop of liquid that meets the axis of rotation it is shown in the preceding paper that the equation of the interface is given by

$$\pm(1+f'^2)^{-\frac{1}{2}} = (1-e)f/a + e(f/a)^3, \tag{3}$$

where

$$e = \omega^{*2}(a/d)^3, \qquad \text{with} \qquad \omega^{*2} = (\rho_1 - \rho_2)\omega^2 d^3/8\alpha, \tag{4}$$

and $2a$ is the width of the drop corresponding to the curve on the surface for which $f' = 0$. For $e \geqslant 0$ equations (3) and (4) correspond to the isolated drop of density ρ_1 (with $\rho_2 = 0$) but, for $-\frac{1}{2} < e \leqslant 0$, these equations describe the interface between the drop of density ρ_1 and the order medium of density ρ_2 ($\rho_2 > \rho_1$) with which it rigidly rotates.

Now Lyttleton (1953) proved that, when there are no external forces other than the couple maintaining the constant rotation, the energy of this system is

$$T_R + U - \tfrac{1}{2}I\omega^2 = \text{constant},$$

where T_R is the kinetic energy of the liquid relative to the rotating axes. Thus, if the equilibrium configuration is such that $U - \frac{1}{2}I\omega^2$ is an absolute minimum and since $T_R \geqslant 0$, it follows that $U - \frac{1}{2}I\omega^2$ cannot increase indefinitely and that it remains small if the initial disturbance is infinitesimal. In the absence of frictional forces, the drop must therefore oscillate in the neighbourhood of the equilibrium position, but if there is friction proportional to the relative velocities, as there must always be to some extent in a natural system, then these oscillations die away. Under these conditions the equilibrium configuration is secularly stable.

In order to discuss the stability of the isolated drop it is necessary to evaluate the change in the potential energy U and the moment of inertia I for a general deformation of the surface. Clearly this is difficult and moreover the rotation can then no longer be assumed to remain rigid; however, if we restrict the displacements to those for which the free surface is an equilibrium form corresponding to a neighbouring value of the parameter e then the two energy terms U and $\frac{1}{2}I\omega^2$ are given by the expressions we already have. Of course, this method can only be used to show instability, for a general displacement may be assumed to contain contributions from all possible types of deformation. At zero angular speed the drop is spherical and hence stable to infinitesimal disturbances. Now it was shown in the preceding paper that

$$\mathrm{d}(U-T)/\mathrm{d}e < 0 \qquad \text{for} \qquad 0 < \omega^* \leqslant \omega_{\max},$$

where

$$\omega_{\max} = 0\cdot7540\{8\alpha/(\rho_1-\rho_2)d^3\}^{\frac{1}{2}},$$

and this suggests that the drop is stable at angular speeds less than $\omega_{\max}$ and that it becomes unstable for greater values for which a new equilibrium or linear series of forms emerges. At $\omega_{\max}$ there must therefore be a loss of stability of the original linear series. Of course, for $\omega < \omega_{\max}$, any displacement of the drop from the axis of rotation results in an interchange of denser fluid near the axis with less dense fluid away from the axis. This lowers the potential energy of the rotation and leads to instability. This kind of argument was also developed by Orr (1906, 1907) and Pozharitiskii (1964), who showed that, for a system with a finite number of degrees of freedom, there is a close relation between the energy of the disturbance and the frequencies of the normal modes into which it can be resolved.

III. A Drop at the Axis of a Rotating Body of Liquid

The shape of a drop of liquid that is less dense than its surroundings was investigated by Rosenthal (1962*a*). He was able to show that the liquid takes on a form that differs very little from an ellipse when the angular speed is such that the parameter e is small, and that it tends to become a cylindrical thread as $e \leqslant -\frac{1}{2}$. Actually, in this case, the spherical form is stable, as can easily be verified by using the method of Landau and Lifshitz described earlier. In fact the frequencies of the normal modes can be obtained from equation (1) by replacing n by $-n-1$ and ρ_1 by $-\rho_2$.

As a preliminary investigation into the stability of these rather complicated surfaces we can examine the growth or decay of disturbances on an infinitely long column of liquid that rotates as a rigid body together with its denser surroundings. Here we neglect the effect of the ends when the drop is sufficiently long since they are more likely to have a stabilizing rather than a destabilizing influence. To carry out this analysis we may divide infinitesimal disturbances into three kinds: (1) plane disturbances that are confined to cross sectional planes and are the same in all such parallel planes, (2) axisymmetrical disturbances for which the varicosity is of a given wavelength, and (3) axisymmetrical disturbances in the form of torsional oscillations. Investigations along such lines have been carried out by numerous authors. Thus, Hocking and Michael (1959) carried out an analysis on an inviscid column of liquid and found that it is stable to plane disturbances of wave number S if the coefficient of surface tension satisfies

$$\alpha \geqslant (\rho_1-\rho_2)\omega^2\bar{a}^3/S(S+1), \tag{5}$$

where $\bar{a}$ is the mean radius of the column, which differs from the undisturbed radius a by a term of the second order in the perturbation amplitudes. This fact was mentioned by Lord Rayleigh (1879). Thus, when we take into account the denser outer liquid it is clear that we have $(\rho_1-\rho_2) < 0$ so that condition (5) is satisfied for all wave numbers S. Therefore in the present problem we can infer that the column is stable to infinitesimal plane disturbances. This is not surprising for it is clear that any plane disturbance must increase the length of the boundary of the plane section of the column and hence produce an increase in the surface energy, and, moreover, the effect of rotation is stabilizing.

Hocking (1960) examined the influence of viscosity on the stability of the column and showed that, at high and low viscosities, it is stable to plane disturbances for which

$$\alpha \geqslant (\rho_1-\rho_2)\omega^2\bar{a}^3/(S^2-1). \tag{6}$$

This criterion differs from the inviscid case although it does not involve the viscosity explicitly. It follows from (5) and (6) that, when account is taken of a small but finite viscosity, a previously neutral disturbance can become unstable, i.e. viscosity induces a measure of instability. The curiosity lies in the fact that the inviscid liquid should be stable for all wave numbers satisfying (5) but not (6). Hocking explains this as due to the fact that the disturbed part of the pressure at the surface is exactly 180° out of phase with the increase in the radius of the column, so that it can provide the necessary inward force to balance the difference between the capillary and the centrifugal forces. This balance is upset once criterion (5) fails and then instabilities arise. Gillis (1961) used a computational method to obtain (6) at intermediate values of the viscosity, while Gillis and Suh (1962) came to the same conclusion although they included the effects of a rotating concentric solid core. In the present problem $\rho_1-\rho_2 < 0$ and so we can infer stability.

Hocking (1960) also considered the case of axisymmetric disturbances with varicosity of wave number k and found that, at zero or high viscosity, infinitesimal oscillations are stable whenever

$$\alpha \geqslant (\rho_1-\rho_2)\omega^2\bar{a}^3/(k^2\bar{a}^2-1).$$

Again, Gillis and Kaufman (1961) showed that this criterion is valid at all viscosities. If we now write the above inequality in the form

$$k^2\bar{a}^2-8e(\bar{a}/a)^3 \geqslant 1,$$

then this condition is satisfied for all wave numbers k when the thread of liquid is long because $e \to -\frac{1}{2}$; so the drop of liquid is stable to all disturbances of varicosity k if $e < -\frac{1}{8}$, provided that the ends do not have a destabilizing influence on the drop.

The work of Pekeris (1948) on laminar flow in a circular pipe suggests that axisymmetric infinitesimal disturbances can be of two types which are not coupled when there is no rotation or when the Reynolds number is sufficiently small. In the one case motion is in concentric circles in planes perpendicular to the axis of the column (torsional) and in the other the motion is in planes passing through the axis (meridional). For the case of flow in a pipe he shows that torsional oscillations are stable at all Reynolds numbers directly from the differential equation. The problem with which we are concerned is to determine the effect on the stability of having a fluid interface in place of a rigid boundary but with no mean flow. Hocking (1960) showed that, for an isolated column of liquid at zero viscosity, the torsional and meridional oscillations are proportional and hence that these two types of infinitesimal disturbances have exactly the same stability characteristics; while at high viscosities there can be no interaction between the disturbances to the rotation and the motion in the meridian planes. In this special case the torsional oscillations satisfy a Bessel-type differential equation for which it is easily shown that the eigenvalues are such that the disturbances do not make the column unstable. When the effect of an outer medium is considered there again exists a simple solution which implies stability to torsional oscillations when both media have high viscosities. This is proved by Rosenthal (1962*b*) but does not seem to occur in any earlier literature.

The above arguments suggest that infinitesimal disturbances of the rigid rotation of a column of less dense liquid are stable at all Reynolds numbers because of the stabilizing effect of the rigid rotation of the outer medium. These results are in agreement with those of Ponstein (1959) and Pedley (1967), who considered a similar problem and found that the column is unstable when the outer medium has the smaller density. In fact they were able to show that the most unstable modes are not necessarily axisymmetric, although for a nonrotating column it is known that only the axially symmetric disturbances can be unstable.

So far we have only considered the stability of two limiting forms. Thus, a drop of less dense liquid is stable when there is no rotation and when the angular speed is sufficiently large. This suggests that the drop is stable at all angular speeds since it is unlikely that there are two or more critical speeds at which there is an exchange of stability.

A discussion of the stability of intermediate configurations can be given in terms of the energy of the system. Let U_0 denote the potential energy that the drop would have if there were no rotation (that is, $U_0 = 4\pi\alpha d^2$) then the quantity $E = U-U_0+\frac{1}{2}H^2/I$ measures the capacity of the drop to do work on itself. Thus, if the system is assumed to rotate freely with constant angular momentum then its capacity to do work on itself is least when E is a minimum and in this case the drop will be in stable equilibrium. Suppose that the interface is slightly disturbed so that E is increased. Then the system will tend to return to its relative equilibrium position and may oscillate although the action of frictional forces will be to dampen this motion. This criterion was obtained by Basset (1888, 1892) in relation to self-gravitating masses but it works equally well in this context. Again for a freely rotating system Lyttleton (1953) proved that

$$T_{\mathrm{R}}+U+\tfrac{1}{2}H^2/I = \text{constant}$$

and hence that any coordinate measured relative to the equilibrium position remains bounded by a small quantity related to the initial infinitesimal disturbance. He thereby arrived at the same criterion.

If we apply this result here then the above suggests that the drop may become secularly unstable where $d(U+T)/de = 0$, i.e. at the point of bifurcation where the angular momentum is a maximum. However, as Basset pointed out, a surface of bifurcation does not necessarily involve an exchange of stabilities. We conclude that this drop of liquid is always stable to infinitesimal disturbances.

IV. References

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