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THE STABILITY OF ALMOST HOMOGENEOUS IN TIME MARKOV SEMIGROUPS OF OPERATORS

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M. V. KARTASHOV

Abstract. A homogeneous in time semigroup of Markov operators defined by its infinitesimal operator with a dense domain is considered. The operator is perturbed by another bounded operator that depends on time, and this results in a nonhomogeneous semigroup. Under certain assumptions, we prove that the perturbed semigroup is a unique solution of a weak integral equation determined by the initial semigroup and an operator perturbation function; this equation is an integral analog of the perturbed Kolmogorov equation. We find explicit estimates for the stability of the perturbed semigroup in the case where the perturbation operator is uniformly small.

The stability of perturbed homogeneous semigroups of operators is studied in the author monograph [1] for discrete time. The general questions of the perturbation theory of operators are discussed in the Kato monograph [2]. Problems concerning the stability of nonhomogeneous semigroups with continuous time become more important in view of the growing number of models in risk theory, insurance, and finance mathematics. These models are nonhomogeneous in time (in view of the season phenomena, say) and are not yet studied in detail.

1. Setting of the problem

1. Let (E, Ξ) be a measurable space. By $f\Xi$ and $m\Xi$ we denote the classes of measurable functions and finite measures that may attain negative values on (E, Ξ).

Let $\aleph \subset m\Xi$ be a Banach subspace of $m\Xi$ equipped with the norm $\|\cdot\|$ and such that

$$\text{(M)} \qquad \operatorname{Var}(\mu) \le c\|\mu\|, \qquad \|\mu\| \le \|\mu + \nu\| \qquad \text{for all } \mu, \nu \in \aleph,\ \nu \ge 0,$$

and some constant c.

Consider the dual space $\Im \subset f\Xi$ of $\aleph$ that consists of functions equipped with the norm

$$\|f\| = \sup(|\mu f|, \|\mu\| \le 1, \mu \in \aleph)$$

where the dual linear form μ is such that

$$\mu f = \int_E f(x)\, \mu(dx), \qquad \mu \in \aleph,\ f \in \Im,$$

and

$$\|\mu\| = \sup(|\mu f|, \|f\| \le 1, f \in \Im). \tag{1}$$

The space $\Im$ contains all measurable bounded functions if condition (M) holds. Some examples of such spaces and their dual counterparts are given in [1, Chapter 1].

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Any transition kernel $Q = (Q(x, B), x \in E, B \in \Xi)$ on (E, Ξ) generates the linear mappings

$$\mu Q(B) = \int_E \mu(dx) Q(x, B) \colon m\Xi \to m\Xi, \tag{2}$$

$$Qf(x) = \int_E Q(x, dy) f(y) \colon f\Xi \to f\Xi \tag{3}$$

(see [3]). The linear subclasses of these mappings equipped with the finite norms

$$\|Q\| = \sup(\|\mu Q\|, \|\mu\| \le 1) = \sup(\|Qf\|, \|f\| \le 1) < \infty \tag{4}$$

form Banach spaces of bounded linear operators and are denoted by $L(\aleph)$ and $L(\Im)$, respectively; the product of the corresponding operators is generated by the kernel

$$PQ(x, B) = \int_E P(x, dy) Q(y, B). \tag{5}$$

2. Let $(P(s, x, t, B), x \in E, B \in \Xi, 0 \le s \le t \le T)$ be a Markov transition function understood in the broad sense [4, Chapter 3]. Corresponding to this transition function are two linear mappings defined according to (2) and (3), namely

$$P_{st} \colon m\Xi \to m\Xi, \qquad P_{st} \colon f\Xi \to f\Xi.$$

These mappings form a semigroup with respect to multiplication if they are bounded (see [3, 4]). The mappings are bounded if $\aleph$ is the space of all bounded charges equipped with the total variation norm and $\Im$ is the space of all measurable functions equipped with the sup-norm.

The semigroup of operators $(Q_{st}, 0 \le s \le t \le T)$ is homogeneous in time if

$$Q_{st} = Q_{t-s}, \qquad 0 \le s \le t \le T. \tag{6}$$

3. For some $0 \le T \le \infty$, let the homogeneous semigroup of bounded operators

$$(Q_t, t \in [0, T]) \subset L(\Im), \qquad Q_s Q_t = Q_{s+t}, \ s, t \in [0, T], \tag{7}$$

be defined on the spaces of measures $\aleph$ and functions $\Im$. Moreover we assume that

$$\text{there exists } h > 0\colon \quad \sup_{s \le h} \|Q_s\| \equiv q(h) < \infty. \tag{Q}$$

We also assume that

$$\Im_0 = \left\{ f \in \Im \colon \text{the limit } \lim_{h \to 0} \frac{1}{h}(Q_h f - f) \equiv Af \in \Im \text{ exists} \right\} \text{ is dense in } \Im. \tag{A}$$

The limit in condition (A) corresponds to strong convergence (that is, to the convergence in the norm of the space $\Im$). The linear operator A involved in condition (A) is densely defined, and, moreover, it is the strong infinitesimal operator of the semigroup Q_s. Note that condition (A) implies, in particular, that the semigroup given by

$$\text{for all } f \in \Im_0 : \qquad \text{there exists } \lim_{h \to 0} Q_h f = f \tag{8}$$

is strongly continuous and that the semigroup given by

$$\text{for all } \mu \in \aleph, \ f \in \Im\colon \quad \text{there exists } \lim_{h \to 0} \mu Q_h f = \mu f \tag{9}$$

is weakly continuous.

4. We also consider a nonhomogeneous semigroup given by

$$(P_{st}, s, t \in [0, T], s \le t) \subset L(\Im), \qquad P_{su} P_{ut} = P_{st}, \qquad 0 \le s \le u \le t \le T. \tag{10}$$

Assume that this semigroup is bounded, that is,

(P) $$\sup_{0\le s\le t\le T} \|P_{st}\| < \infty,$$

and that its perturbed infinitesimal operator is given by

(D) $$\text{the limit } \lim_{u\uparrow s, v\downarrow s} \frac{1}{v-u}(P_{uv}f - Q_{v-u}f) \equiv D_s f \in \Im \\ \text{exists for all } s\in[0,T] \text{ and } f\in\Im_0.$$

Moreover we assume that this operator is bounded; that is,

(T) $$\sup_{s\le T} \|D_s\| \equiv \varepsilon(T) < \infty \quad \text{for all } s\in[0,T], \quad D_s\in L(\Im).$$

Remark. Conditions (A), (D), and (T) imply that the infinitesimal operator of the semigroup P_{st} is defined on $\Im_0$ and is equal to $A + D_s$; that is,

(AD) $$\text{the limit } \lim_{u\uparrow s, v\downarrow s} \frac{1}{v-u}(P_{uv}f - f) \equiv (A+D_s)f \in \Im \\ \text{exists for all } s\in[0,T] \text{ and } f\in\Im_0.$$

2. Main results

The above assumptions yield that the perturbation of a nonhomogeneous semigroup $P_{st} - Q_{t-s}$ satisfies an integral analog of the Kolmogorov equation.

Theorem 1. *Assume that conditions* (Q), (A), (P), (D), *and* (T) *hold. Then*

(PQ) $$\mu P_{st} f = \mu Q_{t-s} f + \int_s^t \mu P_{su} D_u Q_{t-u} f\, du$$

for all $\mu \in \aleph$, $f\in\Im$, *and all* $0\le s\le t\le T$ *where*

(11) $$\varphi_u = \varphi_u(\mu, f) = \mu P_{su} D_u Q_{t-u} f, \qquad u\in[s,t],$$

is a real Borel bounded function and the integral is understood in the Lebesgue sense.

Similarly to the case of the Kolmogorov equations, equation (PQ) for the unknown operator function P_{st} uniquely determines the semigroup via Q_s and D_s in the case of bounded perturbations.

Theorem 2. *Let a homogeneous semigroup* Q_s *and perturbation* D_s *satisfy conditions of the boundedness and existence of the infinitesimal operator, that is, conditions* (Q) *and* (A), *and condition* (T) *of the boundedness of its perturbation.*

Then equation (PQ) *has a unique solution* $(P_{st}, 0\le s\le t\le T)\subset L(\Im)$ *that coincides with the Neuman series of the method of sequential iterations; that is,*

(12) $$\mu P_{st} f = \mu Q_{t-s} f + \sum_{n\ge1} \int\cdot\int_{s\le u_1\le\cdots\le u_n\le t} \mu Q_{u_1-s}\left(\prod_{k=1}^{n-1} D_{u_k} Q_{u_{k+1}-u_k}\right) \times D_{u_n} Q_{t-u_n} f\, du_1\ldots du_n.$$

Moreover, this solution is the semigroup defined by (10) *and satisfies conditions of the boundedness* (P) *and approximation* (D).

To state the result on the stability, we assume that the homogeneous semigroup Q_s is uniformly ergodic with respect to the norm of the space $\Im$ (see [1]); that is,

(13) $$\text{there exists } \Pi\in L(\Im) \text{ such that } \|Q_t - \Pi\|\to 0, \qquad t\to\infty,$$

where the operator norm is defined by (4).

In the case of a uniformly ergodic semigroup, the operator $\Pi \in L(\Im)$ is a stochastic projector [1]:

$$\Pi^2 = \Pi = \Pi Q_s = Q_s \Pi.$$

If, for some s, the transition kernel Q_s has a unique invariant probability $\pi \in \aleph$: $\pi = \pi Q_s$, then the above projector is generated by the kernel $\Pi(x, A) = \pi(A)$ that does not depend on x [1]. The uniqueness of the invariant probability holds for nonreducible Markov processes; the necessary and sufficient conditions for this property can be found in [5].

It is shown in [6, 7] that the integral ergodicity index

$$\sigma(T) = \int_0^T \|Q_t - \Pi\| \, dt \tag{14}$$

is finite for jump processes even if $T = \infty$.

Explicit bounds for $\sigma(T)$ are obtained in [7] in terms of the generalized potential of the corresponding process (this is the inverse operator to the infinitesimal operator A).

Theorem 1 implies the uniform estimate of the stability in the case of small perturbations D_s.

Theorem 3. *Assume that condition* (Q), (A), (P), (D), *and* (T) *hold. Let the integral ergodicity index* $\sigma(T)$ *in* (14) *be finite, and let the stochastic kernel* Q_s *have a unique invariant probability.*

If the norm of the perturbation in condition (T) *is such that*

$$\varepsilon(T) \equiv \sup_{s \le T} \|D_s\| < 1/\sigma(T), \tag{15}$$

then the following stability inequality holds for the operator norm (4)*:*

$$\sup_{0 \le s \le t \le T} \|P_{st} - Q_{t-s}\| \le \frac{\varepsilon(T)\sigma(T)}{1 - \varepsilon(T)\sigma(T)} q(T) \tag{16}$$

where $q(T)$ *is defined in condition* (Q).

The following estimate of the stability holds for a more general case. Note however that this estimate is weaker than the preceding one.

Theorem 4. *Assume that conditions* (Q), (A), (P), (D), *and* (T) *hold. Then*

$$\sup_{0 \le s \le t \le T} \|P_{st} - Q_{t-s}\| \le \big(\exp(T\varepsilon(T)q(T)) - 1\big) q(T). \tag{17}$$

3. Proofs

Note that the multiplicative property (7) and boundedness condition (Q) imply that $q(t) < \infty$ for all $t < T$. In what follows the symbol $I \in L(\Im)$ denotes the unit operator and $1 \in \Im$ is the function that equals 1 for all arguments.

The weak continuity of the semigroup (9) follows from the strong continuity of the semigroup (8) on a dense set $\Im_0$ in the same way as in the proof of Lemma 2(b) below.

Lemma 1. *If* (Q) *and* (A) *hold, then*

$$Q_s f \in \Im_0 \quad \text{for all } s \in [0, T] \text{ and } f \in \Im_0, \tag{18}$$

$$A Q_s f = Q_s A f \in \Im \quad \text{for all } s \in [0, T] \text{ and } f \in \Im_0. \tag{19}$$

Proof. Statements (18) and (19) follow from the definition of the operator A in condition (A) by taking into account condition (Q):

$$A Q_s f \equiv \lim_{h \to 0} \frac{1}{h}(Q_h Q_s f - Q_s f) = Q_s \lim_{h \to 0} \frac{1}{h}(Q_h f - f) = Q_s A f. \qquad \square$$

Lemma 2. *If conditions* (Q) *and* (A) *hold, then*

(a) *for all* $f \in \Im_0$*, the function* $Q_s f\colon [0,T] \to \Im$ *is continuous,*
(b) *for all* $\mu \in \aleph$ *and* $f \in \Im$*, the function* $\mu Q_s f\colon [0,T] \to R$ *is continuous.*

Proof. Assertion (a) follows from (18) and from condition (Q):

$$\|Q_{s+h}f - Q_s f\| = \|Q_s(Q_h f - f)\| \le q(s)\|Q_h f - f\| \to 0, \qquad h \downarrow 0,$$
$$\|Q_{s-h}f - Q_s f\| = \|Q_{s-h}(Q_h f - f)\| \le q(s)\|Q_h f - f\| \to 0, \qquad h \downarrow 0.$$

Now we prove assertion (b). According to condition (A), for any $f \in \Im$ there exists a sequence f_n such that

$$f_n \in \Im_0\colon f_n \to f.$$

Then

$$\begin{aligned}
&\varlimsup_{h\to 0} |\mu Q_{s+h}f - \mu Q_s f| \\
&\quad \le \varlimsup_{h\to 0} |\mu Q_{s+h}f - \mu Q_{s+h}f_n| + \varlimsup_{h\to 0} |\mu Q_s f - \mu Q_s f_n| + \varlimsup_{h\to 0} |\mu Q_{s+h}f_n - \mu Q_s f_n| \\
&\quad \le 2\|\mu\| q(t)\|f - f_n\| + \varlimsup_{h\to 0} |\mu Q_{s+h}f_n - \mu Q_s f_n| = 2\|\mu\| q(t)\|f - f_n\|
\end{aligned}$$

for $s, s+h \in [0,t]$. The right-hand side of the latter relation tends to zero as $n \to \infty$. □

Lemma 3. *Let conditions* (A), (D), *and* (T) *hold. Then*

(a) *the infinitesimal operator of the semigroup* P_{st} *is defined on* $\Im_0$ *and is equal to*

$$A + D_s;$$

(b) *for all* $f \in \Im_0$*, the function* $P_{uv}f$ *is strongly continuous; that is,* $P_{uv}f \to f$ *as* $u \uparrow s$ *and* $v \downarrow s$*;*
(c) *for all* $\mu \in \aleph$ *and* $f \in \Im$*, the function* $\mu P_{uv}f$ *is left continuous with respect to* u *and right continuous with respect to* v *for all* $0 \le u \le v \le T$.

Proof. Assertion (a), as well as (AD), is a corollary of the obvious equality

$$\frac{1}{v-u}(P_{uv}f - f) = \frac{1}{v-u}(Q_{v-u}f - f) + \frac{1}{v-u}(P_{uv}f - Q_{v-u}f) \tag{20}$$

where the strong limits as $u \uparrow s$ and $v \downarrow s$ exist for $f \in \Im_0$ according to conditions (A) and (D).

The strong continuity in assertion (b) follows from the existence of the strong limits in (20).

Let $f \in \Im_0$. Then (AD) implies the right continuity of $\mu P_{uv}f$ with respect to v, since

$$\mu P_{u,v+h}f = (\mu P_{uv})P_{v,v+h}f \to (\mu P_{uv})f, \qquad h \downarrow 0.$$

In the general case, that is, in the case of $f \in \Im$, the continuity follows from the fact that $\Im_0 \subset \Im$ is dense; the proof is the same as that of Lemma 2(b).

Now we prove that $\mu P_{uv}f$ is left continuous with respect to u. Since $P_{u-h,u}f \to f$ as $h \downarrow 0$, we obtain weak convergence in the same way as in Lemma 2; that is, we prove that

$$\mu P_{u-h,u}f \to \mu f, \qquad h \downarrow 0,$$

for all $\mu \in \aleph$, $f \in \Im$, and for $f \in \Im_0$, since $\Im_0$ is dense.

Finally, the boundedness condition (P) implies for all $\mu \in \aleph$ and $f \in \Im$ that

$$\mu P_{u-h,v}f = \mu P_{u-h,u}(P_{u,v}f) \to \mu P_{u,v}f, \qquad h \downarrow 0. \qquad \square$$

Lemma 4. *Let conditions* (Q), (A), (P), (D), *and* (T) *hold. Then the function*

$$\Phi_u = \Phi_u(\mu, f) = \mu P_{su}Q_{t-u}f, \qquad u \in [s,t], \tag{21}$$

is continuous for all $\mu \in \aleph$, $f \in \Im$, *and* $0 \le s \le t \le T$.

Proof. First we consider the case of $f \in \Im_0$. To check the left continuity, let v be fixed and $u \uparrow v$. Then

$$\begin{aligned}\Phi_v - \Phi_u &= \mu P_{sv} Q_{t-v} f - \mu P_{su} Q_{t-u} f = \mu P_{su}(Q_{t-v} - Q_{t-u}) f + \mu(P_{sv} - P_{su}) Q_{t-v} f \\ &= \mu P_{su} Q_{t-v}(I - Q_{v-u}) f + \mu P_{su}(P_{uv} - I) Q_{t-v} f.\end{aligned}$$

It follows from Lemma 1 that $Q_{t-v} f \in \Im_0$. Thus we derive from Lemmas 2 and 3 and conditions (Q), (T), and (P) that

$$|\Phi_v - \Phi_u| \le \sup_{u \le t} \|\mu P_{su}\| (q(t) \|(I - Q_{v-u}) f\| + \|(P_{uv} - I) Q_{t-v} f\|) \to 0, \qquad u \uparrow v.$$

To check the right continuity, fix u and write

$$\begin{aligned}\Phi_v - \Phi_u &= \mu P_{sv} Q_{t-v} f - \mu P_{su} Q_{t-u} f = \mu P_{sv}(Q_{t-v} - Q_{t-u}) f + \mu(P_{sv} - P_{su}) Q_{t-u} f \\ &= \mu P_{sv} Q_{t-v}(I - Q_{v-u}) f + \mu P_{su}(P_{uv} - I) Q_{t-u} f\end{aligned}$$

for $v > u$. Similarly, it follows from $Q_{t-u} f \in \Im_0$ by applying Lemmas 2 and 3 that

$$|\Phi_v - \Phi_u| \le \sup_{v \le t} \|\mu P_{sv}\| (q(t) \|(I - Q_{v-u}) f\| + \|(P_{uv} - I) Q_{t-u} f\|) \to 0, \qquad v \downarrow u.$$

Thus the function $\Phi_u(\mu, f)$ is continuous for all $f \in \Im_0$.

Now we consider the general case of $f \in \Im$. According to condition (A) we pick up a sequence $\{f_n\}$ such that $f_n \in \Im_0$ and $f_n \to f$. Then

$$(22) \qquad |\Phi_u(\mu, f) - \Phi_u(\mu, f_n)| \le \sup_{u \le t} \|\mu P_{su}\| \, q(t) \, \|f - f_n\| \to 0, \qquad n \to \infty,$$

uniformly with respect to $u \in [s, t]$. Since $\Phi_u(\mu, f_n)$ is continuous, $\Phi_u(\mu, f)$ is also continuous. □

Lemma 5. *Let conditions* (Q), (A), (P), (D), *and* (T) *hold. Then the function* Φ_u *defined in* (21) *is differentiable for all* $\mu \in \aleph$, $f \in \Im_0$, *and* $0 \le s \le t \le T$. *Moreover*

$$(23) \qquad \frac{d}{du} \Phi_u = \varphi_u \equiv \mu P_{su} D_u Q_{t-u} f$$

for all $u \in [s, t]$.

Proof. We prove that the right and left derivatives exist at every point and that they are equal to φ_u.

Let v be fixed and $u \uparrow v$. Put $h = v - u$. Then

$$\begin{aligned}\Phi_v - \Phi_u &= h \mu P_{su} D_v Q_{t-v} f + \mu P_{su}(P_{uv} - Q_{v-u} - h D_v) Q_{t-v} f \\ &= h \mu P_{sv} D_v Q_{t-v} f + o(h) \\ &= h \varphi_v + o(h), \qquad h \downarrow 0,\end{aligned}$$

since $\mu P_{su} D_v Q_{t-v} f \to \varphi_v$ by Lemma 3, $Q_{t-v} f \in \Im_0$ by Lemma 1, and

$$\|(P_{uv} - Q_{v-u} - h D_v) Q_{t-v} f\| = o(h), \qquad h \downarrow 0,$$

by condition (D). Therefore

$$\frac{d^-}{du} \Phi_u = \varphi_u.$$

Now let u be fixed and $v \downarrow u$. Put $h = v - u$. Then

$$\begin{aligned}\Phi_v - \Phi_u &= \mu P_{su}(P_{uv} - Q_{v-u}) Q_{t-v} f \\ &= h \mu P_{su} D_u Q_{t-u} f + \mu P_{su}(P_{uv} - Q_{v-u} - h D_u) Q_{t-u} f \\ &\quad + \mu P_{su}(P_{uv} - Q_{v-u}) Q_{t-v}(I - Q_{v-u} + h A) f \\ &\quad + h(\mu P_{su} Q_{t-u} A f - \mu P_{sv} Q_{t-v} A f).\end{aligned}$$

Since $Q_{t-u}f \in \Im_0$, we get

$$\begin{aligned}|\Phi_v - \Phi_u - h\varphi_u| &\le \sup_{u\le t} \|\mu P_{su}\| \cdot \|(P_{uv} - Q_{v-u} - hD_u)Q_{t-u}f\| \\ &\quad + \sup_{u\le t} \|\mu P_{su}\|(1+q(t))\|(I - Q_{v-u} + hA)f\| \\ &\quad + h|\mu P_{su}Q_{t-u}Af - \mu P_{sv}Q_{t-v}Af| \\ &= o(h), \qquad h \downarrow 0,\end{aligned}$$

in view of conditions (P), (D), (Q), and (A) and Lemma 4. Thus the derivative

$$\frac{d^+}{du}\Phi_u = \varphi_u$$

exists. □

Note that φ_u is a Borel function as a limit of continuous functions

$$(\Phi_v - \Phi_u)/(v-u).$$

The boundedness of φ_u obviously follows from (Q), (T), and (P):

$$\sup_{s\le u\le t} |\varphi_u| \le \sup_{s\le u\le t} \|\mu P_{su}\|\varepsilon(t)q(t)\|f\|.$$

Proof of Theorem 1. Let $f \in \Im_0$. According to Lemma 5

$$\Phi_t(\mu, f) - \Phi_s(\mu, f) = \int_s^t \varphi_u(\mu, f)\, du \tag{24}$$

for almost all s and t.

Since the left- and right-hand sides are continuous with respect to s and t, the latter equality holds for all s and t.

Given an arbitrary $f \in \Im$, consider $f_n \in \Im_0\colon f_n \to f$. According to (22),

$$\Phi_t = \Phi_t(\mu, f) = \lim_{n\to\infty} \Phi_t(\mu, f_n).$$

The convergence

$$\sup_{s\le u\le t} |\varphi_u(\mu, f) - \varphi_u(\mu, f_n)| \le \sup_{s\le u\le t} \|\mu P_{su}\|\varepsilon(t)q(t)\|f - f_n\| \to 0, \qquad n \to \infty,$$

is uniform with respect to u. Thus equality (24) for $f = f_n$ implies (24) for an arbitrary function $f \in \Im$. □

Proof of Theorem 2. Note that it is sufficient to prove the theorem for a finite T only, so that we assume below that $T < \infty$.

Put $T_2 = \{(s,t)\colon 0 \le s \le t \le T\}$. Consider the Banach space of bounded linear operators

$$L(\Im, T_2) = \{P_{st}\colon T_2 \to L(\Im)\}$$

equipped with the norm

$$|||P_{\cdot\cdot}||| = \sup(\|P_{st}\|, (s,t) \in T_2).$$

Consider a linear operator $\pounds\colon L(\Im, T_2) \to L(\Im, T_2)$ acting for elements $\mu \in \aleph$ and $f \in \Im$ at a point (s,t) as follows:

$$\mu(\pounds P_{\cdot\cdot})_{st} f = \int_s^t \mu P_{su} D_u Q_{t-u} f\, du. \tag{25}$$

Since

$$\sup(|\mu(\pounds P_{\cdot\cdot})_{st} f|, (s,t) \in T_2) \le \|\mu\| \cdot |||P_{\cdot\cdot}|||\, \varepsilon(T)q(T)\, \|f\|\, T$$

in view of conditions (Q) and (T), relation (25) uniquely defines a bounded linear operator on $L(\Im, T_2)$. Iterating equality (25), we obtain for $n \ge 1$ that

$$\mu(\pounds^n P..)_{st} f = \int \dots \int_{s \le u_1 \le \dots \le u_n \le t} \mu P_{s,u_1} \prod_{k=1}^{n} D_{u_k} Q_{u_{k+1}-u_k} f \, du_1 \dots du_n \tag{26}$$

where $u_{n+1} = t$ by definition. Thus conditions (Q) and (T) imply

$$|\mu(\pounds^n P..)_{st} f| \le \|\mu\| \cdot |||P..||| \, \varepsilon^n(T) q^n(T) \|f\| \frac{T^n}{n!},$$

since $P..$ is bounded, whence

$$|||\pounds^n P..||| \le \frac{(\varepsilon(T) q(T) T)^n}{n!} \, |||P..||| \, . \tag{27}$$

The right-hand side of (27) tends to zero as $n \to \infty$. Thus the operator $\pounds$ is contractive for some n.

Equality (PQ), rewritten in the form

$$\mu P_{st} f = \mu Q_{t-s} f + \mu(\pounds P..)_{st} f,$$

has a unique solution in the space $L(\Im, T_2)$; this solution is the sum of the Neuman series

$$\mu P_{st} f = \sum_{n \ge 0} \mu(\pounds^n Q..)_{st} f \tag{28}$$

where $Q_{st} = Q_{t-s}$ by definition. Substituting equality (26) into (28), we complete the proof of (12). □

Proof of Theorem 3. Equalities for the projector Π follow from the operator convergence (13) and boundedness

$$\Pi = \lim_{t \to \infty} Q_{s+t} = Q_s \Pi = \Pi Q_s,$$

whence $\Pi = \Pi^2$ by passing to the limit as $s \to \infty$.

If an invariant probability measure π is unique, then we use the homogeneous equation

$$\mu \Pi = \mu \Pi Q_s$$

and obtain that $\mu \Pi = (\mu \Pi 1)\pi = (\mu 1)\pi$ for an arbitrary nonnegative measure μ. Thus the kernel $\Pi(x, A) = \pi(A)$ does not depend on x. Therefore

$$Q_s \Pi f = (Q_s 1) \pi f = \pi f, \qquad P_{uv} \Pi f = (P_{uv} 1) \pi f = \pi f$$

in this case for arbitrary Markov transition functions Q_s and P_{uv} and for all $f \in \Im$.

This together with condition (D) implies that

$$\begin{aligned} D_s \Pi f &= \lim_{u \uparrow s, v \downarrow s} \frac{1}{v-u} (P_{uv} \Pi f - Q_{v-u} \Pi f) \\ &= \lim_{u \uparrow s, v \downarrow s} \frac{1}{v-u} (\pi f - \pi f) = 0. \end{aligned} \tag{29}$$

□

Lemma 6. *Let the assumptions of Theorem 3 hold. If*

$$\alpha(T) \equiv \sup_{0 \le t \le T} \int_0^t \|D_u (Q_{t-u} - \Pi)\| \, du < 1, \tag{30}$$

then the operator perturbation

$$\Delta_{st} = P_{st} - Q_{t-s}, \qquad 0 \le s \le t \le T,$$

is such that

$$\sup_{0 \le s \le t \le T} \|\Delta_{st}\| \le q(T) \frac{\alpha(T)}{1 - \alpha(T)}. \tag{31}$$

Proof of Lemma 6. It follows from (PQ) and (29) that

$$\mu\Delta_{st}f = \int_s^t \mu Q_{u-s}D_u(Q_{t-u}-\Pi)f\,du + \int_s^t \mu\Delta_{su}D_u(Q_{t-u}-\Pi)f\,du \tag{32}$$

for all $\mu \in \aleph$, $f \in \Im$, and $0 \le s \le t \le T$.

Now condition (Q) and definition (30) imply that

$$|\mu\Delta_{st}f| \le \|\mu\| q(t)\alpha(T) + \sup_{0\le s\le u\le T} \|\mu\Delta_{su}\|\,\alpha(T) \tag{33}$$

uniformly on the unit sphere $\{f \in \Im \colon \|f\| \le 1\}$. Taking the supremum of both sides of (33) with respect to f and $0 \le s \le t \le T$ and then with respect to $\mu\colon \|\mu\| \le 1$, we obtain

$$\sup_{0\le s\le t\le T}\|\Delta_{st}\| \le q(T)\alpha(T) + \sup_{0\le s\le t\le T}\|\Delta_{st}\|\,\alpha(T) \tag{34}$$

in view of (1) and (4).

Inequality (34) and condition (30) imply that the left-hand side of (34) is finite and satisfies inequality (31). Since

$$\alpha(T) \le \varepsilon(T)\int_0^T \|Q_u - \Pi\|\,du = \varepsilon(T)\sigma(T) < 1 \tag{35}$$

under conditions (T), (14), and (15), we derive bound (16) from Lemma 6 and inequality (31). □

Proof of Theorem 4. We use equality (28) and bound (27) obtained in the proof of Theorem 2:

$$\mu P_{st}f - \mu Q_{st}f = \sum_{n\ge1}\mu(\pounds^n Q_{\cdot\cdot})_{st}f$$

where $Q_{st} = Q_{t-s}$ by definition. Thus

$$\begin{aligned}|||P_{\cdot\cdot} - Q_{\cdot\cdot}||| &\le \sum_{n\ge1}|||\pounds^n Q_{\cdot\cdot}||| \le \sum_{n\ge1}\frac{(\varepsilon(T)q(T)T)^n}{n!}\,|||Q_{\cdot\cdot}|||\\ &\le q(T)(\exp(\varepsilon(T)q(T)T)-1).\end{aligned}$$

□

Bibliography

1. N. V. Kartashov, *Strong Stable Markov Chains*, VSP/TBiMC, Utrecht/Kiev, 1996. MR1451375 (99e:60150)
2. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1965. MR0203473 (34:3324)
3. D. Revuz, *Markov Chains*, North-Holland Elsevier, Amsterdam, 1975. MR0415773 (54:3852)
4. E. B. Dynkin, *Markov Processes*, Fizmatlit, Moscow, 1962; English transl., Springer-Verlag, Berlin–Göttingen–Heidelberg, 1965.
5. O. Hernandez-Lerma and J. B. Lasser, *Markov Chains and Invariant Probabilities*, Birkhäuser, Basel–Boston–Berlin, 2003. MR1974383 (2004b:60002)

6. M. V. Kartashov, *Uniformly ergodic jump Markov processes with bounded intensities*, Teor. Imovir. Matem. Statist. **52** (1995), 86–98; English transl. in Theory Probab. Math. Statist. **52** (1996), 91–103. MR1445541 (97m:60104)
7. M. V. Kartashov, *Computation and estimation of the exponential ergodicity exponent for general Markov processes and chains with recurrent kernels*, Teor. Imovir. Matem. Statist. **54** (1996), 47–57; English transl. in Theory Probab. Math. Statist. **54** (1997), 49–60. MR1644643 (99e:60199)

Department of Probability Theory and Mathematical Statistics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue 6, Kyiv 03127, Ukraine

E-mail address: `winf@carrier.kiev.ua`

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