Open access • Journal Article • DOI:10.2307/1914034

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Published on: 01 Nov 1973 - Econometrica (Center for Economic Research, Department of Economics, University of Minnesota)

Topics: Endogenous money, Velocity of money, Classical dichotomy, Money supply and Inflation

THE STABILITY OF MODELS OF MONEY AND GROWTH WITH PERFECT FORESIGHT

by

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Discussion Paper No. 16, May 1972

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Several economists[1/] have argued that if individuals correctly perceive the rate of inflation so that their expectations are "rational," then deterministic models of money and economic growth are unstable. On this view, points on the steady state equilibrium paths examined by Tobin [9] and others are "saddlepoints," there being a tendency to diverge more and more from such a path as time elapses if the system is not initially on the path. The source of instability is understood most easily in the context of a model in which money is "neutral," real growth and capital accumulation both being exogenous with respect to the money supply and price level and, moreover, both equaling zero. Time is continuous. The price level P and money supply M are assumed at each moment to satisfy the demand function for real balances

$$(1) \qquad \log_e \left(\frac{M(t)}{P(t)}\right) = \alpha\pi(t)$$

where α is a scalar and $\pi(t)$ is the expected rate of inflation, which in these "perfect foresight" models is equated to $d \log_e P(t)/dt$. Under that assumption, equation (1) can be solved to yield

$$(2) \qquad dp(t)/dt = (1/\alpha)[m(t) - p(t)]$$

where $p(t) = \log_e P(t)$ and $m(t) = \log_e M(t)$. Given an initial condition in the form of the price level at t = 0 and the requirement that p be a continuous function of time, the solution to equation (2) is

$$(3) \qquad p(t) = p(0)e^{-t/\alpha} + \frac{1}{\alpha}\int_0^t e^{-(t-s)/\alpha} m(s)ds \ .$$

The instability implied by this solution is exhibited in its simplest form if $m(s) = \bar{m}$ for $0 \leq s < t$, for then (3) becomes

(4) $$p(t) = \bar{m} + [p(0) - \bar{m}]e^{-t/\alpha}$$

If $\alpha < 0$ and $p(0) \neq \bar{m}$, the equilibrium value, then $p(t)$ diverges farther and farther from the stable-price equilibrium value as time passes.

The solution (3) can be interpreted as follows. Given m and p at a certain instant, the solution determines the rate of inflation, which is used to "update" p to get its value for the next instant, and so on. Since p is given at any instant, its derivative is what adjusts to satisfy equation (1). Essentially, the "next instant's" price is what adjusts to insure equality between the demand and supply of real balances at this instant.[2/] Thus, suppose that m(s) has been forever constant at $\bar{m}$ and that the system is initially at the stable-price equilibrium given by (4) with $p(t) = \bar{m}$. Now suppose that at some moment there occurs a sudden once-and-for-all increase in m produced, say, by a veterans' bonus.[3/] Then, although actual real balances temporarily exceed desired real balances, the price level does not jump to a new higher value; according to (3), the price level at t does not depend on the money stock at t. Instead, in order to satisfy equation (1), prices start to <u>fall</u> at a finite time rate so that people are just content to hold the new higher level of real balances. But the falling price level implies that actual real balances will be even larger in the "next instant," which will mean that prices will have to fall even faster in order to satisfy equation (1) at that time. Thus, according to solution (3) the once-and-for-all increase in the money supply will set off an ever accelerating deflation. While this is quite clearly what the mathematics behind equation (3) says, it is not at all clear what economic forces would cause prices to behave in this way. If actual real balances temporarily exceed desired real balances, what market forces cause prices to begin falling at the finite rate that increases desired real balances by just enough to satisfy equation (1)?

In this note, we suggest an alternative view of such models, one which retains the assumption of "rationality," but which leads to the conclusion that the models are stable even if $\alpha < 0$. Indeed, on our view, α must be negative in order for the equilibrium of the system to exist, so that we are comfortable, to say the least, with that assumption. In addition, on our view of the system, standard comparative static exercises retain their validity, while they do not under the view which asserts that the money and growth models are unstable where actual and expected inflation are equal.

All that we propose to do is to abandon the requirement that $P(t)$ be a continuous function of time.[4/] At a particular instant t we do not view $p(t)$ as a parameter inherited from the past. What is inherited are values of $m(s)$ and $p(s)$ for $s < t$. But past values of m and p exert no influence on the current value of p, because at each moment the price level is free to jump discontinuously. On our view, the price level at each moment adjusts instantaneously in order to insure that the real balances people hold equal the amount they would like to hold. Costless trading of money for stocks of marketable real capital at each moment enables individuals to attempt to alter the composition of their portfolios and leads to instantaneous adjustments in the price of real capital, which is the price level in this one good model.

We assume that the anticipated rate of inflation is equal to the right-hand derivative of $p(t)$, which is the natural representation of the assumption of perfect foresight. We also assume that the public expects the money supply to follow some path m(s) for which right-hand time derivatives of all orders are defined. The price level at t is determined by equation (1), which requires that π, the right-hand derivative of p, also be determined at t. But to determine this derivative we in effect have to determine p for the "next instant," which, of course, depends in turn on p for the following instant. Thus to determine $\pi(t)$ we must pursue an infinite progression into the future, and determine the entire path of (expected) prices from t to forever, the expected prices being conditional on (expected) future values of the money supply.[5/] The solution is

$$(5) \qquad p(t) = -\frac{1}{\alpha}\int_t^\infty e^{(s-t)/\alpha} m(s)\,ds$$

The equilibrium value of the price level at the current moment is seen to depend on the (expected) path of the money supply from now until forever.[6/] Notice, for example, that for m fixed for all time, $\alpha < 0$ is a necessary and sufficient condition for the above integral to exist. Moreover, the solution for p(t) implies a time path of expected prices, which is obtained from (5) simply by replacing t by $\bar{t}$, $t \leq \bar{t} < \infty$.[7/]

To illustrate the characteristics of the system, suppose that initially M(s) was (expected to be) following a path $M_0(s)$ for which the equilibrium defined by (5) exists. Now suppose that at t the monetary authority produces an unanticipated once-and-for-all increase by a multiplicative factor of δ in the money supply so that from t on the money supply is (expected to be) given by $M(s) = \delta M_0(s)$, $s \geq t$. The once-and-for-all increase in M at time t leaves the right-hand derivative of m unaltered at t, which means that $\pi(t)$ is unaltered. For by differentiating equation (1) with respect to time and solving for $\pi(t)$, we have

$$\pi(t) = -\frac{1}{\alpha}\int_t^{\infty} e^{(s-t)/\alpha}\frac{d}{ds}m(s)ds ,$$

which is unaffected by the once-and-for-all jump in the money supply at t. According to (5) the price level jumps at t as follows:

$$p(t) - p_0(t) = -\frac{1}{\alpha}\int_t^{\infty} e^{(s-t)/\alpha}[\log_e\delta + m_0(s)]\, ds + \frac{1}{\alpha}\int_t^{\infty} e^{(s-t)/\alpha}m_0(s)ds = \log_e\delta$$

where $p_0(t)$ is the limit of p(s) as s approaches t from the left. Thus, the price level rises once and for all at t proportionally with the money supply.

This experiment suggests that the equilibrium that we have described is a "stable" one in the sense that the system will tend to return to the equilibrium instantaneously if it is disturbed. (The instantaneous character of the adjustments means that the "correspondence principle" must be interpreted with some care.) What makes the equilibrium stable is that the price level rises instantaneously when at an initial price level the supply of real balances exceeds the demand, thereby eliminating the excess supply of real balances without affecting π.

The characteristics of the system are further illustrated by considering the effects of a foreseen jump in the money stock. Thus, suppose that initially M(s) was (expected to be) fixed at M_0, but that at time t it is announced that the money stock will jump at a given date in the future so that for a known $\theta > 0$

$$M(s) = \begin{cases} M_0 , & s < t + \theta \\ \delta M_0 , & s \geq t + \theta , \quad \delta > 1 \end{cases}$$

From (5), for $t \leq \bar{t} \leq t + \theta$

$$p(\bar{t}) = -\frac{1}{\alpha}\int_{\bar{t}}^{t+\theta} e^{(s-\bar{t})/\alpha} m_0 ds - \frac{1}{\alpha}\int_{t+\theta}^{\infty} e^{(s-\bar{t})/\alpha}(\log_e\delta + m_0) ds = m_0 + e^{(t+\theta-\bar{t})/\alpha}\log_e\delta$$

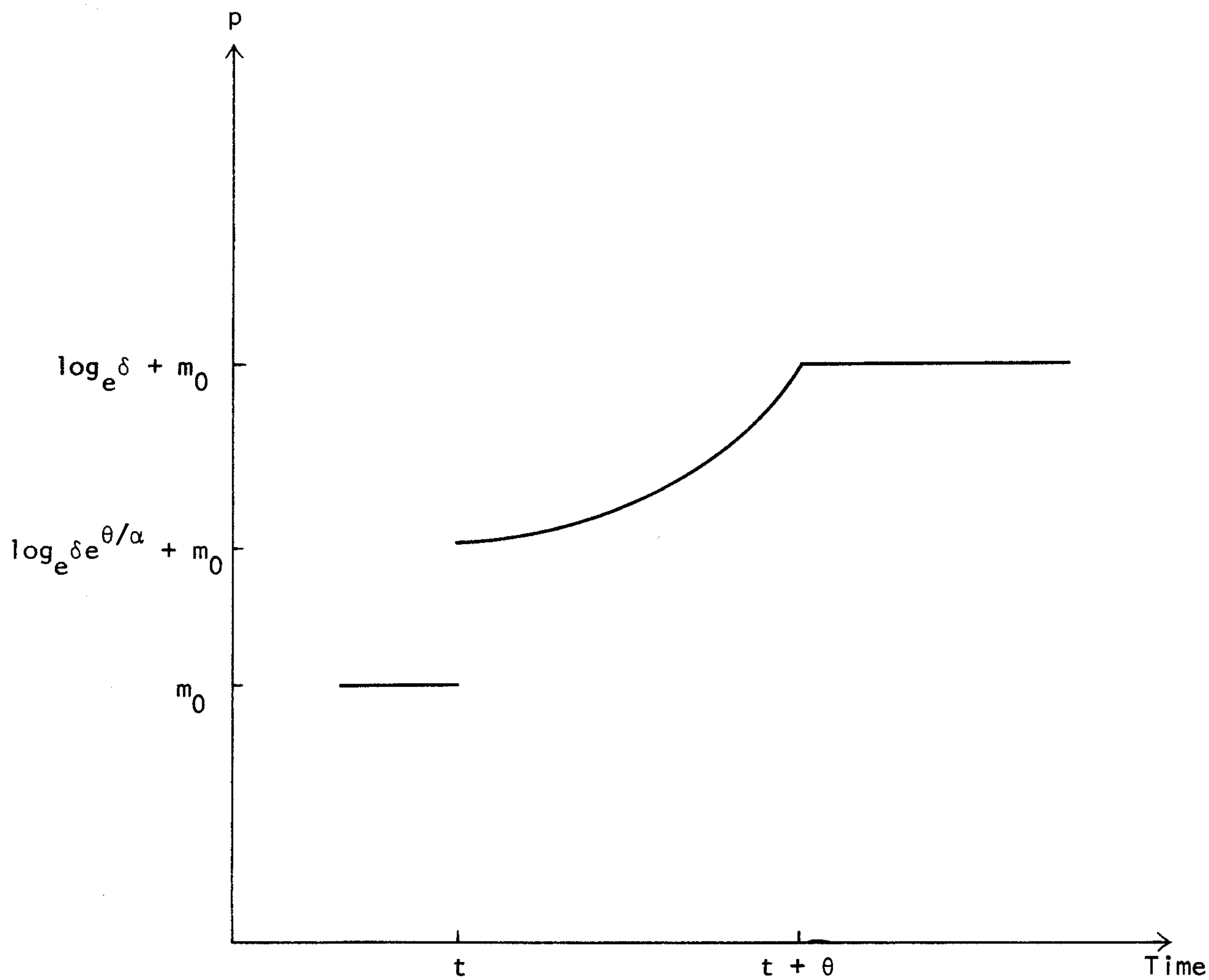
The path of p is shown in Figure 1.

[INSERT FIGURE 1]

The initial jump at t is larger, the larger in absolute value is α. Note that the part of the path to the right of t is an expected path. It will be the actual path if the path of the money stock turns out as expected.

The view that we have adopted here has a number of merits as applied to the literature on money and growth.[8/] First, the fact that on our interpretation most money and growth models are stable even with "rational" expectations means that the analysis of steady states retains its interest. Second, the approach makes it possible to perform "static" or point-in-time exercises with those models by subjecting policy variables to once-and-for-all shocks at a particular moment. Such exercises either aren't permitted or are uninteresting on the alternative view. Finally, our approach avoids some of the more bizarre conclusions that spring from its alternative. For example, some authors[9/] have argued that while money and growth models are unstable when money is exogenous or "active," they can be stable if the monetary authorities permit money to be "passive," i.e., to be determined somehow by the price level. That letting money be "passive" can be stabilizing in a full-employment, neutral-money system is a view sharply at variance with "static" macroeconomic analysis.[10/]

FIGURE 1. The Response of p to an Announcement at Time t of a Jump in m to Occur at Time $t + \theta$.



1/ For example, see Nagatani [6], Olivera [4], Burmeister and Dobell [2], Sidrauski [8], and Hadjimichalakis [3].

2/ Burmeister and Dobell [2, p. 157] give the following description of the structure of such models:

> It should be noted that our analysis is based on a differential equation structure. We take a price level and capital stock to be given by the past. Then all that we may choose is the rate of change of the price level and of the capital stock, and we choose these so as to satisfy a money market equilibrium condition and a saving hypothesis. These choices determine the new values of the price level and the capital stock for the "next instant," and the process is repeated.
>
> Our procedure thus appears to say nothing directly about the level of prices, as opposed to the rate of price inflation. It is therefore in marked contrast to the usual theorizing about money market equilibrium that purports to determine the equilibrium price level.

3/ If $dp(t)/dt$ in (2) is interpreted as both the left- and right-hand time derivatives of $p(t)$ at t, then (3) is a solution only if the limit of $m(s)$ as s approaches t from the right and the left equals $m(t)$, a requirement that rules out jumps in m at t. If, however, $dp(t)/dt$ in (2) is interpreted as a right-hand derivative, which is consistent with perfect foresight, then (3) is a solution to (2) if the limit of $m(s)$ as s approaches t from the right (i.e., over values where $s > t$) equals $m(t)$. On the latter interpretation, jumps in m at t are not ruled out. (These statements can be proved by making use of the kind of limiting process described in footnote 6.)

4/ The view that the price level responds instantaneously to the imposition of disturbances at a point in time is required to rationalize the assumption that the model is continuously operating at full employment. Thus, our assumption seems the natural one to employ in the context of full-employment models of money and growth.

5/ We could also characterize this process of looking farther and farther into the future in terms of looking at higher and higher order right-hand derivatives of m at t. By repeatedly differentiating (1) with respect to time in the right-hand direction and then eliminating derivatives of π, we can obtain

$$\pi(t) = \left(1 - \alpha\frac{d}{dt} + \alpha^2\left(\frac{d}{dt}\right)^2 - \alpha^3\left(\frac{d}{dt}\right)^3 + \ldots\right)\frac{d}{dt}m(t)$$

where $\frac{d}{dt}$ is the right-hand derivative operator. We assume that some finite-order derivative of m(t) vanishes, which guarantees that the sum above is finite.

6/ To verify that (5) is a solution to (2), compute the right-hand time derivative of (5) by way of the following limiting process: for $\varepsilon > 0$

$$\lim_{\varepsilon\to 0} \frac{p(t+\varepsilon) - p(t)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{-1}{\alpha\varepsilon}\left[\int_{t+\varepsilon}^{\infty} e^{(s-t-\varepsilon)/\alpha} m(s)\,ds - \int_{t}^{\infty} e^{(s-t)/\alpha} m(s)\,ds\right]$$

7/ Since the models of money and growth assume that the marginal product of capital equals the cost of capital at each instant, they can be regarded as containing a perfect market in capital, the accumulated stock of the one good in the model. Were we to assume the existence of perfect futures markets in capital for $\bar{t}$, $t \leq \bar{t} < \infty$, the futures prices at t would be given by $p(\bar{t})$ calculated from (5).

8/ In their analysis of the stability of models with heterogeneous capital goods, Shell and Stiglitz [7] permit instantaneous jumps in relative prices and emphasize the role of future markets in guaranteeing stability. Our analysis is in the same spirit as theirs.

9/ See Olivera [4] and Black [1].

10/ See Olivera [5].

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