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The stable downward continuation of potential field data

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Key Words: FFT, edge effect, vertical continuation

ABSTRACT

Filtering methods based on the Fourier transform are routinely used in the processing of geophysical data. Because of the nature of the Fourier transform, the data must be prepared before the transform is calculated. This preparation usually takes the form of the removal of any trend from the data, combined with the padding of the data to 2^N points between the data edges. However, no data preparation procedure is perfect, and the result is that problems (in the form of edge effects) appear in the filtered data. When high-pass filters (such as derivatives or downward continuation) are subsequently used, then these edge effects become particularly apparent.

This paper suggests three methods for the stable downward continuation of geophysical data (two of which may be combined). The first method is applied to an integrated horizontal derivative of the data rather than to the data itself. Since the horizontal derivative can be calculated in the space domain where fast Fourier transform (FFT) edge effects are not present, this reduces the enhancement of the data at frequencies near the Nyquist, resulting in smaller edge effect problems. The second method measures the FFT-induced noise by comparing data that has been downward continued using both the space- and frequency-domain methods. The data is then compensated accordingly, and the compensated data may be downward continued to arbitrary distances that are not possible using space-domain operators. The final method treats downward continuation as an inverse problem, which allows the control of both FFT-induced noise and other noise that is intrinsic to the dataset. This method is computationally slow compared to the first two methods because of the inversion of large matrices that is required. The methods are demonstrated on synthetic models and on aeromagnetic data from the Bushveld igneous complex, South Africa.

INTRODUCTION

The Fourier transform *F(k)* of a dataset *f(x)* is given by

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \ , \qquad (1)$$

where k is wavenumber (Bracewell, 1978). The combination of the Fourier transform and modern computing power is extremely powerful, allowing many now indispensable filtering operations to be readily applied to geophysical data. However, there are certain practical problems that are associated with its application. Geophysical data is limited spatially, but the limits on the integral in equation (1) are $\pm\infty$. Processing a finite dataset using a discrete version of the Fourier transform is mathematically equivalent to processing an infinite number of copies of the finite dataset. If the finite dataset has any trend present then this will result in discontinuities at the copy boundaries that will introduce high-frequency edge effects into the processed data. Some form of trend removal is therefore usually undertaken before the calculation of the Fourier transform.

The fast Fourier transform (FFT) reduced the computational requirements of the Fourier transform from an $O(n^2)$ to an $O(n \log n)$ process (Press et al., 1992, p498) at the cost of requiring a dataset with 2^N data points. (Variations on the algorithm exist that are based on prime factors of n rather than 2, but these are not in common geophysical use.) This improvement was sufficient to make the Fourier transform a useful tool even on the primitive computers of the 1960s. Because of this restriction, datasets may be padded out to 2^N points prior to FFT processing. Edge effects can be further reduced by mirroring the data at the boundaries of the original dataset, while reducing the amplitude of the mirrored data to zero at the boundaries of the padded dataset (using a smooth function such as a cosine or exponential). Many variations on this process of data preparation exist, but none of them is able to produce a perfect result that yields a result exactly as would have been obtained with equation (1) and an infinite data series.

Many geophysical filters (such as derivatives or downward continuation) emphasise the high-frequency content of the data, and are used to bring out fine detail. Unfortunately, not only is any high-frequency noise originally present in the data also enhanced by this process, but new high-frequency noise is introduced by Fourier transform edge effect problems.

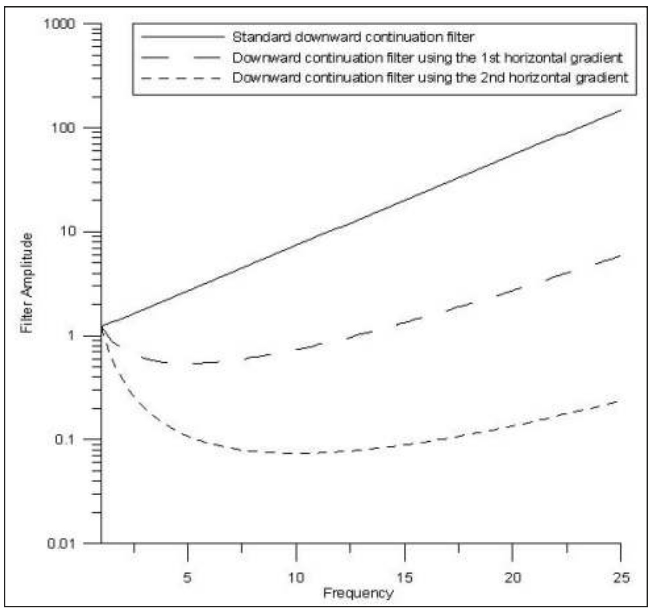

Fig. 1. The frequency responses of different downward continuation filters.

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DOWNWARD CONTINUATION

Downward continuation calculates the potential field that would have been measured if the survey had been conducted closer to the sources of the field than it actually was. It is used to enhance subtle features, and to compare aeromagnetic and ground magnetic surveys. However, it is unstable and will enhance noise as well as detail. In the frequency domain the operator is

$$A'(k) = A(k) \cdot e^{-|k|\Delta z} , \tag{2}$$

where *A(k)* is the amplitude at a wavenumber *k*, and Δz is the downward continuation distance (a negative number) (Blakely, 1995, p320). As can be seen from equation (2) the power at high wavenumbers is boosted compared to that at low wavenumbers. One way to reduce the enhancement of FFT-induced noise is to perform as much as possible of the downward continuation in the space domain. High-pass filters that are easily calculated in the space domain include horizontal derivatives of any integer order, and the second vertical derivative. The derivatives may be downward continued in the frequency domain, and simultaneously integrated to yield an approximation to the downward-continued total field.

The operator is therefore

$$A'(k) = A(k)\frac{e^{|k|\Delta z}}{|ik|^n} , \tag{3}$$

because the frequency-domain horizontal derivative operator is

$$A'(k) = A(k)|ik|^n \tag{4}$$

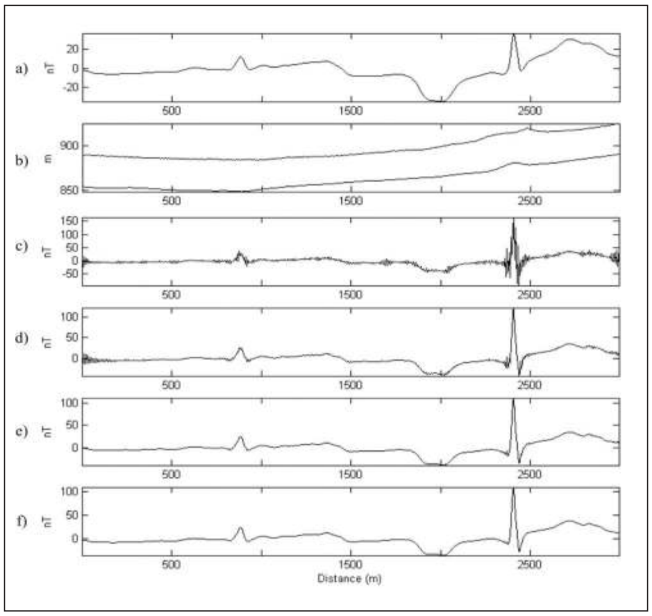

Fig. 2. Aeromagnetic data from the Eastern Bushveld complex, downward continued by 3 sample intervals
a) Aeromagnetic data. The sample interval is 5 m.
b) Digital terrain model (DTM) and aircraft flight height along the flight line
Data downward continued by 15 m using:
c) the standard algorithm (equation (2),
d) the first horizontal derivative,
e) the second horizontal derivative,
f) the third horizontal derivative.

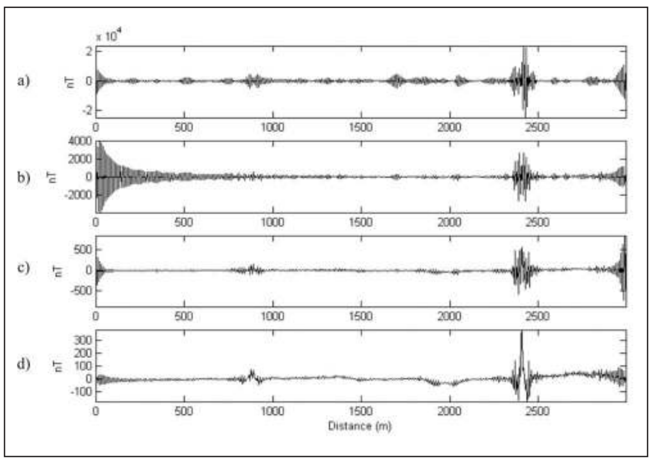

Fig. 3. Aeromagnetic data from the Eastern Bushveld complex, downward continued by 5 sample intervals (25 m) using:
a) the standard algorithm (equation (2),
b) the first horizontal derivative,
c) the second horizontal derivative,
d) the third horizontal derivative.

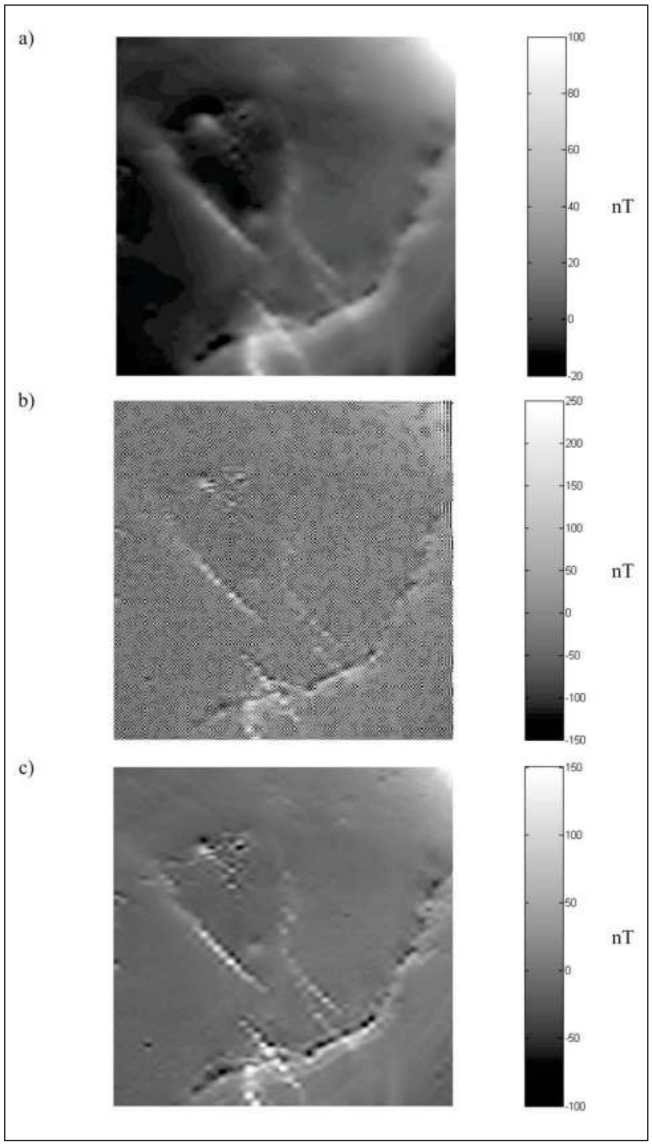

Fig. 4. Downward continuation of Bushveld aeromagnetic data.
a) TMI Data. The grid interval was 15 m, and the area shown covers 3 km by 3 km. The flight height varied from 40–50 m depending on the terrain.
b) Dataset in Figure 4a downward continued by 37.5 m using the standard method.
c) Dataset in Figure 4a downward continued by 37.5 m using the improved method.

where n is the order of horizontal derivative used (Blakely, 1995, p324). Equation (3) only produces an approximation to the downward-continued total field because the use of horizontal derivatives loses the long-wavelength information in the data. For example, the first horizontal derivative loses the average value of the data, since equation (3) cannot be evaluated for $k = 0$. This problem may be partly solved by comparing the output of equation (3) when $\Delta z = 0$ with the original data, and correcting the processed data accordingly. In this case the long-wavelength components of the data from zero frequency upwards (depending on the degree of the derivative used) are not downward continued. However, as can be seen from equation (2), this is not so much a problem as it may at first appear because the low-wavenumber components of the data are much less altered by the downward-continuation filter than are the high-wavenumber components.

The form of the modified downward-continuation filter in the wavenumber domain is shown in Figure 1 and it can be seen that the power at the high wavenumbers is boosted considerably less than it is by equation (2). If the horizontal derivative were initially calculated in the wavenumber domain using equation (4), then nothing would be gained by this process. However, as it is computed in the space domain, the result is to reduce FFT-induced edge effects in the filtered data.

Figure 2a shows an aeromagnetic flight line over a portion of the Eastern Bushveld igneous complex, South Africa. The data has been interpolated to a sample interval of 5 m. The flight height varied from 40–50 m depending on the terrain, as shown in Figure 2b. Figure 2c shows the result of downward continuing the data by three sample intervals (i.e., 15 m) using the standard method, and Figures 2d–2f show the result of using the first, second, and third horizontal derivative (computed in the space domain) with equation (3). As the order of derivative used is increased, the downward continuation becomes increasingly stable. Edge effects can clearly be seen in the standard downward-continued data. Figure 3 shows the result of downward continuing the data by five sample intervals (i.e., 25 m). In this case, even the downward continuation based on the third horizontal derivative (Figure 3d) has begun to get almost unusably noisy, but is still preferable to the output from the standard method (shown in Figure 3a).

Application to Map Data

When map data is to be filtered using this approach, the direction of the horizontal derivative used becomes important, and artefacts appear in the filtered data at 90° to the derivative direction. In this case it is better to use the second vertical derivative of the data, calculated in the space domain from the space-domain horizontal derivatives using Laplace's equation, as the basis for the downward continuation filter. So, equation (3) becomes

$$A'(k) = A(k)\frac{e^{|k|\Delta z}}{k^2} \ , \qquad (5)$$

because the second vertical derivative operator in the wavenumber domain is given by

$$A'(k) = A(k)\,k^2 \qquad (6)$$

(Blakely 1995, p.324).

Figure 4a shows a small portion of an aeromagnetic dataset over the Eastern Bushveld complex. Figure 4b shows the data downward continued by 37.5 m (the grid interval was 15 m) using the standard method, and Figure 4c shows the data downward continued by the same distance using the improved method. The FFT edge-effect induced noise which severely affects Figure 4b is almost completely absent from Figure 4c, enabling the dyke anomalies to become more clearly visible. The dykes are important in this area because their presence can affect mining.

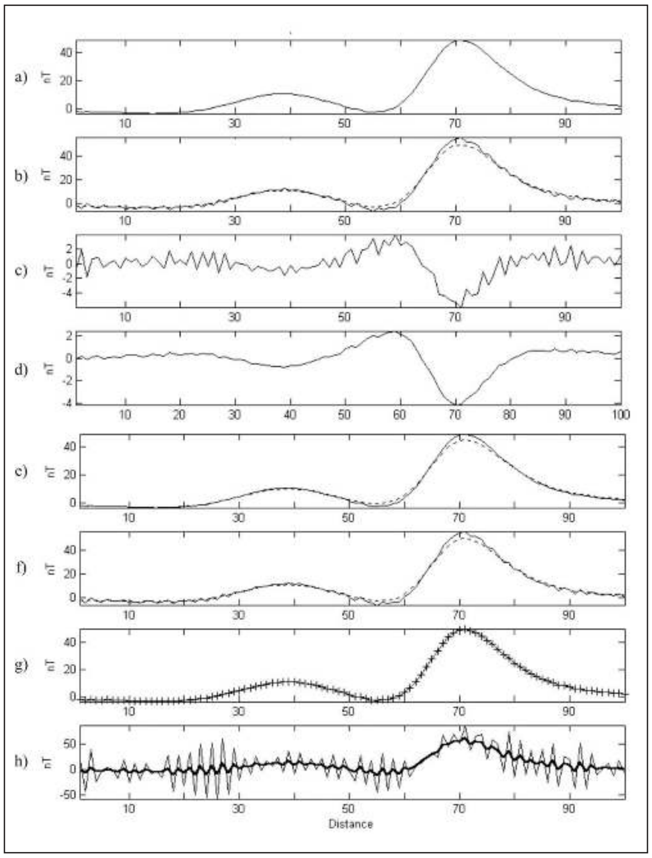

Fig. 5. A compensation approach to the minimisation of FFT edge effects.

- **a)** **Synthetic data from two dipole sources of depths 15 and 22.5 units, with uniformly distributed random noise of amplitude 1% of the maximum data amplitude added.**
- **b)** **A comparison of the downward continuation of the data in Figure 5a by one sample interval in the frequency domain (solid line) and the space domain (dashed line).**
- **c)** **Difference between the datasets in Figure 5b.**
- **d)** **Dataset in Figure 5c upward continued by one sample interval.**
- **e)** **Comparison of the original data and the compensated data (computed by adding the dataset in Figure 5d to the original data).**
- **f)** **A comparison of the downward continuation of the original data (solid line) and the compensated data (dashed line), by one sample interval. Both continuations were performed in the frequency domain.**
- **g)** **A comparison of the downward continuation of the original data (+ symbols) in the space domain and the compensated data (solid line) in the frequency domain, by one sample interval.**
- **h)** **The original dataset downward continued by 2.25 sample intervals (thin line), and the compensated data downward continued by the same distance (heavy line). Both continuations were performed in the frequency domain.**

A COMPENSATION APPROACH TO DOWNWARD CONTINUATION

If the high-frequency content of the data is primarily noise, and if its frequency content does not overlap with that of the geophysical signal of interest, then it may be smoothed with a low-pass filter of some type prior to downward continuation. However, this will not help with noise that is introduced into the data by the necessarily imperfect data preparation used prior to the application of the FFT, as discussed briefly above. An alternative approach to that of the minimisation of edge effect problems is to accept that they will never be perfectly removed, and instead try to

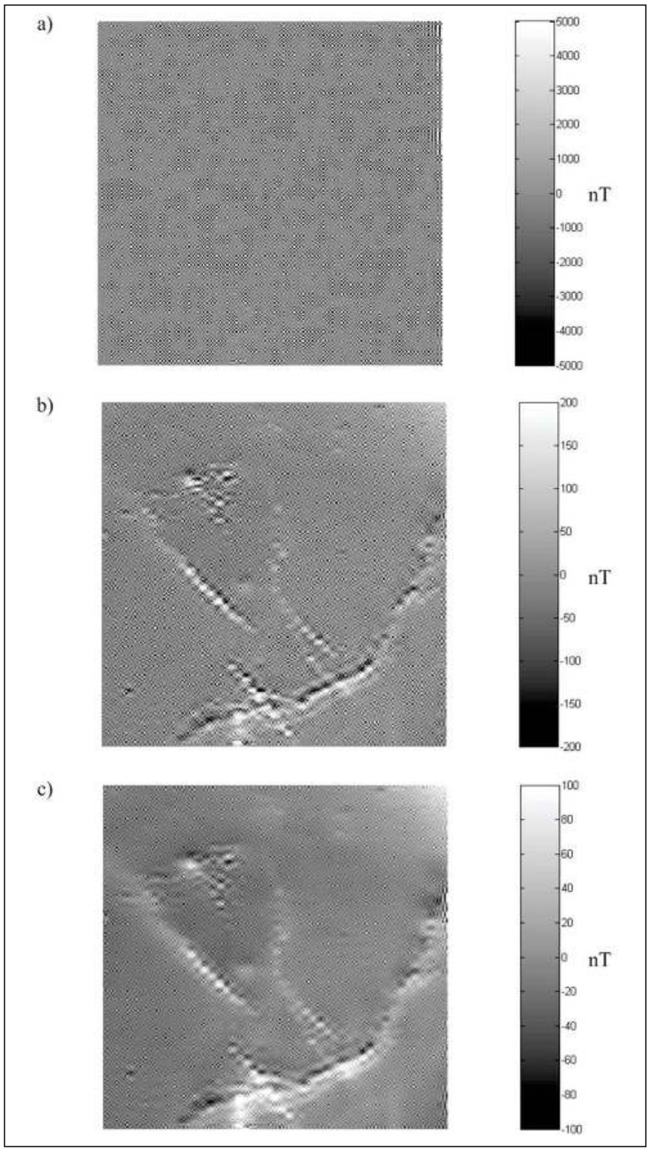

Fig. 6. Comparison of different downward continuation methods:
a) Result of downward continuing the data in Figure 4a by 50 m (3.25 sample intervals) using the standard method.
b) Result of downward continuing the data in Figure 4a by 50 m using the integrated second vertical derivative method (equation (5).
c) Result of downward continuing the data in Figure 4a by 50 m using the compensated integrated second vertical derivative method.

measure and then compensate for them. This can be done if there is a space-domain analogue to the desired frequency-domain filter. Of course, if the appropriate space-domain filter can be used in all of the same circumstances in which the frequency-domain filter can be used then the compensation method is not needed, and the data may simply be filtered in the space domain. However in many cases it is only simple to apply a space-domain filter in a limited set of circumstances: e.g., integer-order horizontal derivatives may be calculated trivially in the space domain, but the filter size becomes very large (several hundred coefficients) if non-integer derivatives are required.

For downward continuation, the situation is similar. If the data is approximated as the sum of a set of sinc functions, then for multiples of the data spacing interval it may be vertically continued in the space domain using

$$g(z-\Delta z) = g(z)_i \int_0^1 \int_0^1 \cos m\pi ra \cos n\pi sa e^{\pi\sqrt{m^2+n^2}\frac{\Delta z}{a}} dm dn \quad (7)$$

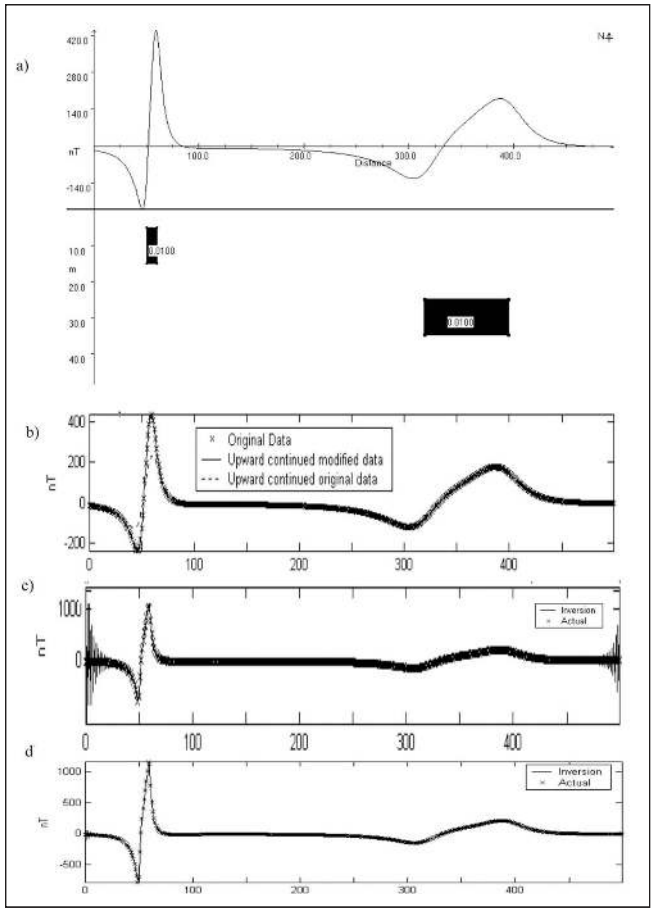

Fig. 7. A comparison of downward continuation using the Fourier and the inverse approaches with a noise-free synthetic dataset.
a) 2.5D synthetic model. The susceptibilities of the bodies are shown upon them (in c.g.s. units).
b) Original data overlain with the upward-continued downward-continued data (solid line), and the result of upward continuing the original data (dashed line).
c) Result of downward continuing the data using the standard FFT method (solid line) overlain on the correct result (× symbols) produced from the model in Figure 7a. Note the edge effects.
d) Result of downward continuing the data using the inverse approach (solid line) overlain on the correct result (× symbols).

where the data spacing is *a*, g_i is the *i*th measured potential field value, and Δz is the continuation distance (Oldham, 1967). The compensation procedure is then as follows, and is illustrated in Figure 5 for a synthetic data profile:

a) Downward continue the data by an integer number of data sample intervals (one in this case) in both the space and frequency domains (Figure 5b),
b) Subtract one dataset from the other (Figure 5c), and upward continue the difference (Figure 5d),
c) Add the upward continued difference to the original dataset (Figure 5e), and use the compensated dataset as the basis for further frequency-domain filtering.

The frequency-domain downward-continued original data (still only by one sample interval) and the frequency-domain downward-continued compensated data are compared in Figure 5f, and the latter is clearly less noisy. In fact, the frequency-domain downward-continued compensated dataset is now almost exactly the same as the space-domain downward-continued original dataset (see Figure 5g). Figure 5h compares the result of downward continuing the original and the compensated datasets (in the frequency domain) by 2.25 sample intervals. The downward-continued compensated data clearly suffers less from edge-effect noise than does the downward-continued original data. Note that this compensation is not necessarily a smoothing process; the modifications that are applied to the original data can take any form. All that is required is that the FFT-induced noise be minimised.

Combination of the Derivative and Compensation Approaches to Downward Continuation

The two methods described above are not exclusive, and can be combined to achieve optimal results. That is to say, the data after downward continuation in the space domain (by an integer number of data grid intervals) is compared with the result of downward continuing the integrated second vertical derivative in the frequency domain. The compensated data is then used as a basis for further filtering, in the manner described above. Figure 6 compares the results of downward continuing the Bushveld aeromagnetic survey from Figure 4a by 50 m (3.33 sample intervals) using the standard method (Figure 6a) with the integrated second vertical derivative method (Figure 6b), and the compensated integrated second vertical derivative method (Figure 6c). It should be noted that the computations required by the compensation process do slightly more than double the processing time involved in the downward continuation of the data, but it is felt that a comparison of the signal-to-noise ratio in Figures 6a and 6c more than justifies the time spent.

DOWNWARD CONTINUATION AS AN INVERSE PROBLEM

Equation (8) relates three functions f_A, f_B, and f_C of a dataset y together:

$$f_A\left(f_B(y)\right) = f_C(y) \quad (8)$$

In this application, f_A and f_C are known functions while f_B is to be established. For example, if f_C is the third horizontal derivative of the data and f_A is the first horizontal derivative, then f_B must be the second horizontal derivative since the first horizontal derivative of the second horizontal derivative is the third horizontal derivative. More usefully however, if f_A represents the upward continuation of the data by a distance h, and f_C leaves the data unchanged, then f_B is the downward continuation of the data by h. Since the upward continuation process is stable, as discussed above, there are no problems with its computation using equation (2). The downward continuation is then achieved by least squares inversion using equation (9):

$$f_B(y) = \left(\mathbf{A}^{\mathbf{T}}\mathbf{A} + k\mathbf{I}\right)^{-1}\mathbf{A}^{\mathbf{T}} e \quad (9)$$

where e is the misfit between the original data y and the upward continued modified data. **A** is the gradient matrix, **I** is the identity matrix, and the constant k is a damping factor. **A** is calculated by a numerical differencing operation; i.e., each point on the profile is perturbed slightly and the change in the upward continued profile is noted. The inversion process (which is linear for $k = 0$) then modifies the data so that when upward continued it becomes the original measured data. The inversion process is even-determined since each data point is a parameter. This means that the downward continuation of an N point dataset requires the inversion of an $N \times N$ matrix, and so the process is slower than the FFT based methods described previously.

Figure 7 compares the downward continuation of noise-free profile data over a synthetic 2D model (shown in Figure 7a) using this method and the standard frequency-domain approach. Figure 7b shows the original data after upward continuation by four data sample intervals and the result of upward continuing the modified

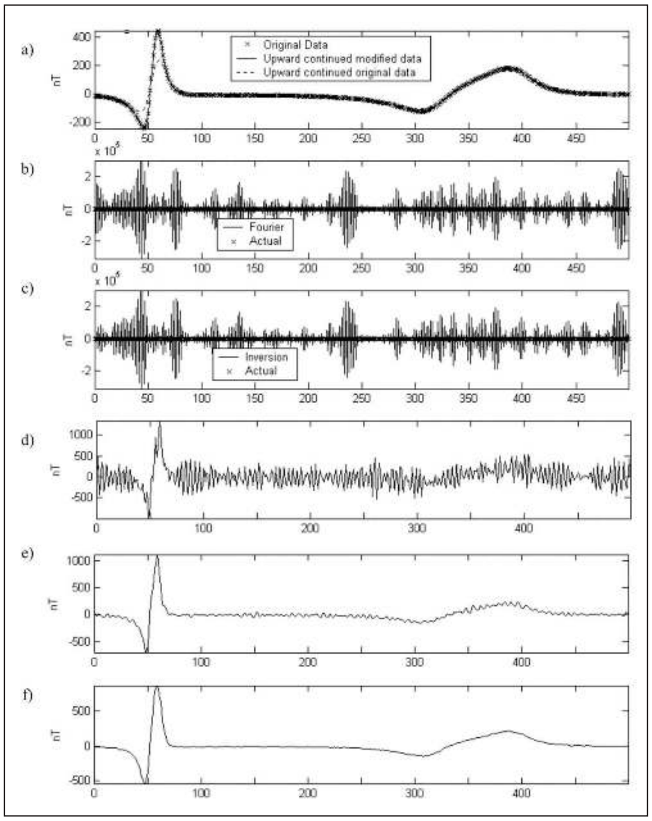

Fig. 8. Downward continuation of noisy data using the inversion method.

a) Data from the model in Figure 7a with 1% of uniformly distributed random noise added to it (× symbols), overlain with the upward-continued downward-continued data (solid line), and the result of upward continuing the original data (dashed line)

Result of downward continuing the data 4 sample intervals using:

b) the standard FFT method (solid line) overlain on the correct result (× symbols) produced from the model in Figure 7a

c) the inverse approach (solid line) with k = 0 overlain on the correct result (× symbols).

d) the inverse approach using k = 10^{-5} .The correct magnetic response is not overlain, for clarity.

e) the inverse approach using k = 10^{-3}

f) the inverse approach using k = 10^{-1}

data by the same distance. The upward-continued modified data is almost identical to the original data (the average error is approximately 10^{-16} of the data amplitude in magnitude). A value of $k = 0.0$ (equation (9)) was used, and a single inversion iteration was required. Figure 7c compares the result of the Fourier transform based downward-continuation method (using equation (8)) with the correct magnetic response of the model at that altitude. Noise is visibly present, particularly at the edges of the dataset where any problems in the data preparation prior to the application of the fast Fourier transform have been accentuated. Figure 7d shows the result of the new downward continuation filter (i.e., the data which when upward continued by four sample intervals is almost identical to the original data) overlain on the correct magnetic response. The two are almost identical. The same approach may also be used to calculate stable horizontal and vertical gradients; in this case, f_C would be the original data and f_A becomes the horizontal or vertical integral of the same order as that of the required derivative.

The use of equation (8) is successful in reducing noise in downward-continued datasets that is due to FFT edge effects. Unfortunately, noise that is present in the original data is 'real' and will be increased in amplitude (perhaps significantly) by downward continuation both by the standard FFT method and by the inversion method. The inversion method however allows the stability of

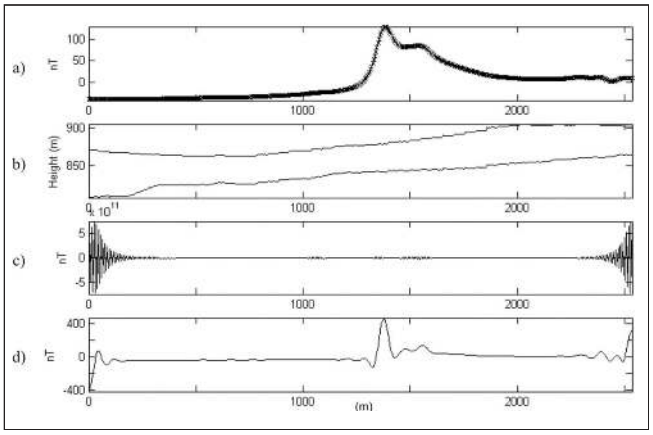

Fig. 9. Downward continuation of data from the Bushveld igneous complex using the inversion method.
a) Original data (× symbols, interpolated to 5-m sample interval) overlain with upward-continued downward-continued data (solid line).
b) Terrain and flight path of the aircraft.
c) Data downward continued by 50 m (10 sample intervals) using the standard Fourier method. Note the scale on the y axis.
d) Data downward continued by 50 m (10 sample intervals) using the inversion method.

the downward continuation process to be controlled. The greater the value of the constant k in equation (8), the more the inversion process will be damped, and the smoother the resulting downward-continued data will be. The damped inversion process will now take several iterations to converge (depending on the value of k), compared to the single iteration required when $k = 0$.

Figure 8 compares the results of the downward continuation of the data from the same model as shown in Figure 7a, which this time has had uniformly distributed random noise of amplitude equal to 1% of the data added to it. The noise demonstrates the instability of the downward continuation process quite clearly. The overlain actual response from the model has no noise added. The value of k used in Figures 8c to 8f increases from 0.0 to 10^{-1} (the elements of the matrix $\mathbf{A}^T\mathbf{A}$ had values ranging from 0.18 to 1.1×10^{-6}). Attempts at smoothing the results of the frequency-domain downward continuation process produced unsatisfactory results in this case, as did the smoothing of the data before the frequency-domain downward continuation operator was applied.

Figure 9 shows how effective the inversion method of downward continuation can be. Figure 9a shows an aeromagnetic data profile from the Eastern Bushveld igneous complex (× symbols) overlain with the upward-continued downward-continued data (solid line). Figure 9b shows the aircraft flight path and the terrain. The flight height is approximately 50 m. Figure 9c shows the data when downward continued by 50 m (10 sample intervals) using the standard Fourier method. The amplitude at the edges of the profile is 10^{11} nT, and the result is unusable. Figure 9d shows the data when downward continued by 50 m using the inverse method ($k = 10^{-1}$). While significant edge effects are present, the anomalies in the centre of the profile are well resolved and still interpretable.

CONCLUSIONS

Three new methods for the reduction of FFT-induced noise in the downward continuation process were introduced, and the results of their application were shown to be more robust with respect to noise than the standard algorithm. A combination of the first two techniques was particularly effective, and was demonstrated both on synthetic and real data. The third method not only minimised the FFT-induced noise, but allowed the stabilising of the downward continuation process and the smoothing of high-frequency noise that was present in the original data. Because the latter method involved the inversion of large matrices, it is slow when applied to large datasets.

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