J Algebr Comb (2013) 37:289–312
DOI 10.1007/s10801-012-0367-z

The stable set of associated prime ideals of a polymatroidal ideal

Jürgen Herzog · Asia Rauf · Marius Vladoiu

Received: 29 September 2011 / Accepted: 21 March 2012 / Published online: 11 April 2012

Abstract The associated prime ideals of powers of polymatroidal ideals are studied, including the stable set of associated prime ideals of this class of ideals. It is shown that polymatroidal ideals have the persistence property and for transversal polymatroids and polymatroidal ideals of Veronese type the index of stability and the stable set of associated ideals is determined explicitly.

Keywords Associated prime ideals · Polymatroidal ideals · Analytic spread

1 Introduction

Let I be an ideal in a Noetherian ring R. It is customary to denote by $\operatorname{Ass}(I)$ the set of associated prime ideals of R/I. Brodmann [3] showed that $\operatorname{Ass}(I^k) = \operatorname{Ass}(I^{k+1})$ for all $k \gg 0$. One calls the smallest number k_0 for which this happens the *index of stability* and $\operatorname{Ass}(I^{k_0})$ is called the *stable set of associated prime ideals* of I. It is denoted by $\operatorname{Ass}^\infty(I)$. Several natural questions arise in the context of Brodmann's theorem.

J. Herzog
Fachbereich Mathematik, Universität Duisburg-Essen, Campus Essen, 45117 Essen, Germany
e-mail: juergen.herzog@uni-essen.de

A. Rauf
The Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy
e-mail: asia.rauf@gmail.com

A. Rauf
e-mail: arauf@ictp.it

M. Vladoiu (✉)
Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, Bucharest, 010014, Romania
e-mail: vladoiu@gta.math.unibuc.ro

(1) Is there an upper bound for the index of stability depending only on R?
(2) What can be said about the set $\mathrm{Ass}^\infty(I)$? Can $\mathrm{Ass}^\infty(I)$ be computed in case that R is a polynomial ring and I is a graded ideal?
(3) Is it true that $\mathrm{Ass}(I) \subset \mathrm{Ass}(I^2) \subset \cdots \subset \mathrm{Ass}(I^k) \subset \cdots$?

All these questions are widely open, even for monomial ideals, though in several interesting special cases, including edge ideals and vertex cover ideals of perfect graphs, these questions have been answered quite comprehensively, see [5, 9] and [18]. A nice survey on what is known about the stability of associated prime ideals of powers of edge ideals is given in [19]. Question (3) does not have a positive answer in general, see [13] and [18] for counterexamples. The ideals which provide these counterexamples are monomial ideals, but not squarefree. An ideal I for which (3) holds true is said to satisfy the *persistence property*. It is an open question whether all squarefree monomial ideals satisfy the persistence property. However, even in the case of vertex cover ideals of graphs the conjecture has not been settled.

Suppose now that $(R, \mathfrak{m})$ is local or a standard graded K-algebra with graded maximal ideal $\mathfrak{m}$. We say that an ideal $I \subset R$ has *non-increasing depth functions*, if for all prime ideals P in the support $V(I)$ of R/I, one sees that $\operatorname{depth} R_P/I^k R_P$ is a non-increasing function of k, and I is said to have *strictly decreasing depth functions*, if the depth functions of all its localizations are strictly decreasing until they reach their limit value. In the case that I is a graded ideal, respectively, a monomial ideal, we require the defining property of non-increasing (strictly decreasing) depth functions only for localizations with respect to prime ideals $P \in V^*(I)$, where $V^*(I)$ denotes the set of graded, respectively, monomial prime ideals containing I.

It is easily seen that for an ideal which has non-increasing depth functions the persistence property holds, see Proposition 2.1. Moreover, if an ideal has strictly decreasing depth functions, then its index of stability is bounded by $\dim R - 1$. We do not know of any example of a squarefree monomial ideal which does not have non-increasing depth functions. On the other hand it is shown in [13, Theorem 4.1] that, given any non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, there exists a monomial ideal in a polynomial ring S (with sufficiently many variables) such that $\operatorname{depth} S/I^k = f(k)$ for all k. This shows that among the monomial ideals, non-increasing depth functions can be expected in general only for squarefree monomial ideals.

There is at least one case known to us in which $\mathrm{Ass}^\infty(I)$ can be computed efficiently. Namely, if I is a monomial ideal in a polynomial ring $S = K[x_1, \ldots, x_n]$ whose Rees algebra $\mathcal{R}(I)$ is Cohen–Macaulay. By a result of Huneke [17] it follows that the associated graded ring of I is Cohen–Macaulay, and this implies that $\lim_k \operatorname{depth} S/I^k = n - \ell(I)$, where $\ell(I)$ denotes the analytic spread of I, that is, the Krull dimension of the fiber ring $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$, see Eisenbud and Huneke [8, Proposition 3.3]. This theorem allows us to identify the elements of $\mathrm{Ass}^\infty(I)$ in terms of the exponent matrix associated with the unique minimal monomial set $G(I)$ of generators of I. All this is explained in detail in Sect. 2. There we also define the invariants $\mathrm{dstab}(I)$ and $\mathrm{astab}(I)$. The first of them is the smallest integer k with the property that $\operatorname{depth} I^k = \operatorname{depth} I^\ell$ for all $\ell \geq k$, while the second is the smallest integer with $\mathrm{Ass}(I^k) = \mathrm{Ass}(I^\ell)$ for all $\ell \geq k$. One may ask whether there is any relation between these numbers. At the end of Sect. 2 we show that either one may be smaller than the other. However we show in Proposition 2.1 that $\mathrm{astab}(I)$ is bounded below and above by local data of dstab provided it has non-increasing depth functions, which is for example the case if all powers of I have a linear resolution, see Proposition 2.2. These facts are used to compute the index of stability for Stanley–Reisner ideal of the natural triangulation of the projective plane.

In Sect. 3 the strategies discussed in Sect. 2 are applied to study the associated prime ideals of powers of polymatroidal ideals. Two general properties of polymatroidal ideals are crucial: (1) all powers of polymatroidal ideals have a linear resolution, as shown in [7, Theorem 5.3], (2) localizations of polymatroidal ideals at monomial prime ideals are again polymatroidal, see Corollary 3.2. These two facts combined with Proposition 2.1 immediately yield that polymatroidal ideals have the persistence property, see Proposition 3.3. We recall in Theorem 3.4 the result of Villarreal [20, Proposition 3.11], which says that the Rees ring of a polymatroidal ideal is normal, and consequently Cohen–Macaulay. Applying then the Huneke–Eisenbud result, the limit depth of a polymatroidal ideal can be expressed by its analytic spread. From this one easily deduces an algorithm, described at the end of the section, to compute $\mathrm{Ass}^\infty(I)$ for any polymatroidal ideal I. All data required to compute $\mathrm{Ass}^\infty(I)$ are given by the exponent matrix of the minimal set of monomial generators of I.

In the remaining two sections we consider special classes of polymatroidal ideals where the questions concerning associated prime ideals of powers of ideals have complete answers. The ideals considered in Sect. 4 are the polymatroidal ideals of transversal polymatroids. Algebraically speaking, ideals of this type are simply arbitrary (finite) products of monomial prime ideals. In [7, Lemma 3.2] a primary decomposition of products of ideals generated by linear forms is given. However this decomposition is not at all irredundant and it is not easy to obtain an irredundant decomposition from that given in [7, Lemma 3.2].

Our first result (Lemma 4.1) asserts that the presentation of a transversal polymatroidal ideal as product of monomial prime ideals is unique. The key result of Sect. 4 is Theorem 4.3 where it is shown that the graded maximal ideal $\mathfrak{m}$ is associated to the transversal polymatroidal ideal $I = P_{F_1} \cdots P_{F_r}$ if and only if $\bigcup_{i=1}^r F_i = [n]$ and the intersection graph G_I is connected. Here G_I is the graph with vertex set $\{1, \dots, r\}$ and for which $\{i, j\}$ is an edge of G_I if and only if $F_i \cap F_j \neq \emptyset$. By using this result we conclude in Corollary 4.6 that $\mathrm{Ass}(I) = \mathrm{Ass}^\infty(I)$ for any transversal polymatroidal ideal. Furthermore we show in Theorem 4.7 that $\mathrm{Ass}(I)$ is determined by the trees of the graph G_I. As nice consequences of all this we classify in Corollary 4.9 all subsets $\mathcal{S} = \{F_1, \dots, F_r\}$ of $2^{[n]}$ for which there exists a transversal polymatroidal ideal I with $\mathrm{Ass}(I) = \{P_{F_1}, \dots, P_{F_r}\}$, and in Corollary 4.10 we give an irredundant primary decomposition of all powers I^k of I. We conclude this section with two results concerning the depth of S/I. In Theorem 4.12 it is shown that depth S/I is essentially determined by the number of components of G_I, and Corollary 4.14 says that $\mathrm{dstab}(I) = 1$. Thus for any transversal polymatroidal ideal, $\mathrm{dstab}(I) = \mathrm{astab}(I) = 1$.

The situation for ideals $I_{d;a_1,\dots,a_n}$ of Veronese type, which is the class of polymatroidal ideals considered in Sect. 5, is completely different. Here $\mathrm{Ass}^\infty(I) = V^*(I)$, as shown in Proposition 5.3, and the invariant $\mathrm{astab}(I)$ can be any number between 1 and $n-1$ determined by an explicit formula given in terms of the numbers d and $a_1, \dots, a_n$, see Corollary 5.6. Moreover it is shown in Corollary 5.7 that $\mathrm{astab}(I) = \mathrm{dstab}(I)$ and $\lim_{k\to\infty} \mathrm{depth}\, S/I^k$ and $\ell(I)$ are computed for any Veronese type ideal.

The common feature to transversal polymatroidal ideals and to ideals of Veronese type is that $\operatorname{astab}(I) = \operatorname{dstab}(I)$. It would be interesting to know whether this equality holds for any other polymatroidal ideal. As we have seen in Sect. 2, arbitrary monomial ideals, even when they are squarefree, do not satisfy this equality.

2 Generalities about the depth and the associated primes of powers of an ideal

Let $(R, \mathfrak{m})$ denote a Noetherian local ring or standard graded K-algebra with graded maximal ideal $\mathfrak{m}$, and $I \subset R$ an ideal. In the graded case we assume that I is graded ideal.

We are going to relate the index of stability and the persistence property of I to the property of I to have non-increasing depth functions. We say that $P \in V(I)$ is a *persistent prime ideal* of I, if whenever $P \in \operatorname{Ass}(I^k)$ for some exponent k, then $P \in \operatorname{Ass}(I^{k+1})$. If this happens to be so for k, then of course we have $P \in \operatorname{Ass}(I^\ell)$ for all $\ell \geq k$. The ideal I is said to have the *persistence property* if all prime ideals $P \in \bigcup_k \operatorname{Ass}(I^k)$ are persistent prime ideals.

By a famous theorem of Brodmann [2] it is known that depth R/I^k is constant for all $k \gg 0$. We call the smallest number k_0 such that depth $R/I^k =$ depth R/I^{k_0} for all $k \geq k_0$, the *index of depth stability* of I, and denote this number by $\operatorname{dstab}(I)$.

Brodmann also showed [3] that there exists an integer k_1 such that $\operatorname{Ass}(I^k) = \operatorname{Ass}(I^{k_1})$ for all $k \geq k_1$. The smallest such number is called the *index of stability* of I. We denote this number by $\operatorname{astab}(I)$.

At the end of this section we show by examples that the invariants $\operatorname{dstab}(I)$ and $\operatorname{astab}(I)$ are unrelated. In other words, either one of these numbers may be smaller than the other one or they may also be equal. However we have

Proposition 2.1 (a) *Suppose the depth function* depth R/I^k *is non-increasing, then* $\mathfrak{m}$ *is a persistent prime ideal.*

(b) *If I has non-increasing depth functions, then I satisfies the persistence property.*

(c) $\max_{P \in \operatorname{Ass}^\infty(I)}\{\operatorname{dstab}(IR_P)\} \leq \operatorname{astab}(I)$. *In addition, if I has non-increasing depth functions, then* $\operatorname{astab}(I) \leq \max_{P \in V(I)}\{\operatorname{dstab}(IR_P)\}$.

Proof (a) Let $\mathfrak{m} \in \operatorname{Ass}(I^k)$, then depth $R/I^k = 0$. Thus our assumption implies that depth $R/I^\ell = 0$ for all $\ell \geq k$. Hence $\mathfrak{m} \in \operatorname{Ass}(I^\ell)$ for all $\ell \geq k$.

(b) One has $P \in \operatorname{Ass}(I^k)$ if and only if $PR_P \in \operatorname{Ass}_{R_P}(I^k R_P)$. By part (a) this implies that $PR_P \in \operatorname{Ass}_{R_P}(I^\ell R_P)$ for all $\ell \geq k$. Thus $P \in \operatorname{Ass}(I^\ell)$ for all $\ell \geq k$.

(c) Let $r = \operatorname{astab}(I)$. Then, whenever $P \in \operatorname{Ass}^\infty(I)$, we have $P \in \operatorname{Ass}(I^\ell)$ for all $\ell \geq r$. This implies that depth $R_P/I^\ell R_P = 0$ for all $\ell \geq r$. Hence $\operatorname{dstab}(IR_P) \leq r$, which yields the first inequality.

Now let $s = \max_{P \in V(I)}\{\operatorname{dstab}(IR_P)\}$, and suppose that $r > s$. Then there exists $P \in \operatorname{Ass}^\infty(I)$ such that $P \in \operatorname{Ass}(I^r)$, but $P \notin \operatorname{Ass}(I^s)$. Indeed, otherwise we would find that depth $R_P/I^s R_P = 0$ for all $P \in \operatorname{Ass}^\infty(I)$. Since I has non-increasing depth functions it would follow that $\operatorname{astab}(I) \leq s < r$, a contradiction.

It follows that depth $R_P/I^s R_P >$ depth $R_P/I^r R_P = 0$, in contradiction to the definition of s. ☐

The next result generalizes [13, Proposition 2.1] and provides cases where we have non-increasing depth functions.

Proposition 2.2 *Suppose $I \subset S = K[x_1, \dots, x_n]$ is a graded ideal generated in degree d with the property that there exists an integer k_0 such that I^k has a linear resolution for all $k \geq k_0$. Then* $\operatorname{depth} I^k \geq \operatorname{depth} I^{k+1}$ *for all $k \geq k_0$.*

Proof Let $f \in I$ be a homogeneous polynomial of degree d. Then fI^k is generated in degree $(k+1)d$ and $fI^k \subset I^{k+1}$. The short exact sequence

$$0 \longrightarrow fI^k \longrightarrow I^{k+1} \longrightarrow I^{k+1}/fI^k \longrightarrow 0$$

induces the long exact sequence

$$\cdots \to \operatorname{Tor}_{i+1}\bigl(K, I^{k+1}/fI^k\bigr)_{i+1+(j-1)} \to \operatorname{Tor}_i\bigl(K, fI^k\bigr)_{i+j} \to \operatorname{Tor}_i\bigl(K, I^{k+1}\bigr)_{i+j} \to \cdots,$$

where for a graded S-module, $\operatorname{Tor}_i(K, M)_j$ denotes the jth graded component of $\operatorname{Tor}_i(K, M)$.

Both fI^k and I^{k+1} have a $(k+1)d$-linear resolution. Thus

$$\operatorname{Tor}_i\bigl(K, fI^k\bigr)_{i+j} = \operatorname{Tor}_i\bigl(K, I^{k+1}\bigr)_{i+j} = 0$$

for $j \neq (k+1)d$ and all i. Moreover, $\operatorname{Tor}_{i+1}(K, I^{k+1}/fI^k)_{i+1+(j-1)} = 0$ for $j = (k+1)d$, because the module I^{k+1}/fI^k is generated in degree $(k+1)d$. This shows that the natural maps $\operatorname{Tor}_i(K, fI^k) \longrightarrow \operatorname{Tor}_i(K, I^{k+1})$ are injective for all i. It follows that $\operatorname{proj\,dim} I^k = \operatorname{proj\,dim} fI^k \leq \operatorname{proj\,dim} I^{k+1}$, and consequently, $\operatorname{depth} S/I^{k+1} \leq \operatorname{depth} S/I^k$, by the Auslander–Buchsbaum formula (see for example [4, Theorem 1.3.3]). □

Let $I \subset S$ be a monomial ideal. Throughout this paper S stands for the polynomial ring $K[x_1, \dots, x_n]$ where K is a field. We denote by $G(I)$ the unique minimal set of monomial generators of I. In the case that $G(I) \subset T = K[x_{i_1}, \dots, x_{i_k}]$ we denote by an abuse of notation the ideal $G(I)T$ again by I. Observe that by using this notation it follows that $\operatorname{Ass}_S(I) = \operatorname{Ass}_T(I)$.

Let $u = \prod_{i \in L} x_i$ be a squarefree monomial in S. Then

$$(S/I)_u \cong S'\bigl[\bigl\{x_j^{\pm 1}\colon j \in L\bigr\}\bigr]/I_L S'\bigl[\bigl\{x_j^{\pm 1}\colon j \in L\bigr\}\bigr],$$

where $S' = K[\{x_i\colon i \notin L\}]$ and where $I_L \subset S'$ is the ideal which is obtained from I by applying the K-algebra homomorphism $S \to S'$ with $x_i \mapsto 1$ for all $i \in L$.

Let $P = (x_{i_1}, \dots, x_{i_r})$ be a monomial prime ideal, and $I \subset S$ any monomial ideal. We denote by $I(P)$ the monomial ideal in the polynomial ring $S(P) = K[x_{i_1}, \dots, x_{i_r}]$ where $I(P) = I_L$ with $L = [n] \setminus \{i_1, \dots, i_r\}$.

In the later proofs we need the following simple facts.

Lemma 2.3 *Let $I \subset S$ be a monomial ideal. Then*

(a) $P \in \operatorname{Ass}(I)$ *if and only if* $\operatorname{depth} S(P)/I(P) = 0$;

(b) $\mathrm{Ass}(I_L) = \{P \in \mathrm{Ass}(I)\colon x_i \notin P \text{ for all } i \in L\}$ *for all subsets* $L \subset [n]$.

Proof (a) has been observed in [9, Lemma 2.11].

(b) As before let $S' = K[\{x_i\colon i \notin L\}]$ and set $T = S'[\{x_j^{\pm 1}\colon j \in L\}]$. Then $T = S_u$ where $u = \prod_{i \in L} x_i$. Thus by using the basic rules concerning the behavior of associated prime ideals with respect to localization and polynomial ring extension we obtain

$$\mathrm{Ass}_T(I_L T) = \mathrm{Ass}_T(IT) = \bigl\{PT\colon P \in \mathrm{Ass}_S(I), x_i \notin P \text{ for all } i \in L\bigr\}.$$

On the other hand,

$$\mathrm{Ass}_T(I_L T) = \bigl\{PT\colon P \in \mathrm{Ass}_S(I_L)\bigr\}.$$

Since the assignment $P \mapsto PT$ establishes a bijection between the set $\mathrm{Ass}_{S'}(I_L)$ and $\{PT\colon P \in \mathrm{Ass}_S(I_L)\}$, the desired conclusion follows. □

If I is a monomial ideal, we say that I has non-increasing depth functions if depth $S(P)/I(P)^k$ is a non-increasing function of k for all $P \in V^*(I)$.

Since the associated prime ideals of a monomial ideal are monomial prime ideals, it follows (in analogy to Proposition 2.1(b)) that a monomial ideal has the persistence property if I has non-increasing depth functions as defined for monomial ideals.

For monomial ideals, the corresponding statement of Proposition 2.1(c) reads as follows:

Proposition 2.4 *Let* $I \subset S$ *be a monomial ideal which has non-increasing depth functions. Then*

$$\max_{P \in \mathrm{Ass}^\infty(I)} \bigl\{\mathrm{dstab}\bigl(I(P)\bigr)\bigr\} \le \mathrm{astab}(I) \le \max_{P \in V^*(I)} \bigl\{\mathrm{dstab}\bigl(I(P)\bigr)\bigr\}.$$

In particular, if $\mathrm{Ass}^\infty(I) = V^*(I)$, *one has* $\mathrm{astab}(I) = \max_{P \in V^*(I)}\{\mathrm{dstab}(I(P))\}$.

As a consequence, in the case of a monomial ideal which has non-increasing depth functions we need to compute the depth stability only for a finite number of monomial prime ideals in order to obtain bounds for its index of stability. The following example demonstrates this strategy.

Let I be the Stanley–Reisner ideal that corresponds to the natural triangulation of the projective plane. Then

$$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6,$$
$$x_3x_4x_5, x_3x_4x_6).$$

The following table displays in the jth row and the kth column the depth of $S(P)/I(P)^k$ where $P \in V^*(I)$ is of height j.

I	1	2	3	4
3	0	0	0	0
4	1	1	1	1
5	2	2	0	0
6	3	0	0	0

The ideal I is of height 3, so $\operatorname{depth} S(P)/I(P)^k = \dim S(P)/I(P)^k = 0$ for all k and all $P \in V^*(I)$ of height 3. For $P \in V^*(I)$ of height 4 the ideal $I(P)$ is a monomial ideal complete intersection of height 3. Therefore $\operatorname{depth} S(P)/I(P)^k = 1$ for all $k \geq 1$. This explains the first two rows of the table. If $P \in V^*(I)$ is of height 5, then $I(P)$ is the edge ideal of a 5-cycle. It follows from [5, Lemma 3.1] that $\operatorname{Ass}^\infty(I(P)) = \operatorname{Ass}(I(P)) \cup \{\mathfrak{m}\}$ and $\operatorname{astab}(I(P)) = 3$, which explains the third row of the table. In particular, we have $\operatorname{depth} S(P)/I(P)^k = 0$ for all $P \in V^*(I)$ of height 5 and for all $k \geq 3$. Finally, by using CoCoA [6] we find that $\operatorname{depth} S/I = 3$, $\operatorname{depth} S/I^2 = 0$ and $\operatorname{depth} S/I^3 = 0$. Borna [1, Corollary 3.3] has shown that I^k has a linear resolution for $k \geq 3$ and when $\operatorname{char}(K) = 0$. Applying Proposition 2.2, we see that $\operatorname{depth} S/I^k = 0$ for all $k \geq 3$. It follows also that I is an ideal with non-increasing depth functions and consequently I satisfies the persistence property, by Proposition 2.1(b). By applying the second inequality of Proposition 2.4 we obtain $\operatorname{astab} I \leq 3$. By using Singular [11], we find that all prime ideals of height 5 are in $\operatorname{Ass}(I^3)$. Since I satisfies persistence property we see that all prime ideals of height 5 are in $\operatorname{Ass}^\infty(I)$. It follows then that $\max_{P \in \operatorname{Ass}^\infty(I)}\{\operatorname{dstab}(IS_P)\} \geq 3$. Finally, by applying again Proposition 2.4 we obtain $\operatorname{astab}(I) = 3$. As a byproduct of computing the $\operatorname{astab}(I)$ we obtain $\operatorname{Ass}^\infty(I) = \operatorname{Ass}(I^3)$. Calculations with Singular show that $\operatorname{Ass}(I^3)$ consists of all prime ideals of height 3, 5 and 6 which belong to $V^*(I)$, altogether 17. Moreover we see that in this example, $\operatorname{dstab}(I) < \operatorname{astab}(I)$.

As a second example we consider the ideal $I = (xyz, ytu, xzv, tuv, xtv) \subset S = K[x, y, z, t, u, v]$. Then $I = \bigcap_{F \in \mathcal{F}(\Delta)} P_F$ where Δ is the simplicial complex with facets $\mathcal{F} = \{\{z, t\}, \{x, t\}, \{x, u\}, \{y, v\}, \{z, u, v\}\}$ and where P_F is the monomial prime ideal whose generators correspond to the vertices of F. The simplicial complex Δ has no special odd cycles in the sense of [16]. Thus as a consequence of [16, Theorem 2.2.] it follows that the vertex cover algebra of Δ is standard graded which implies that $\operatorname{Ass}(I) = \operatorname{Ass}^\infty(I)$. Thus $\operatorname{astab}(I) = 1$. On the other hand one can check with CoCoA that $\operatorname{depth} S/I = \operatorname{depth} S/I^2 = 3$ and $\operatorname{depth} S/I^3 = 2$. Thus $\operatorname{astab}(I) < 3 \leq \operatorname{dstab}(I)$.

As a last topic of this section we want to recall a few facts about the limit depth of an ideal. As we mentioned already, the function $f(k) = \operatorname{depth} R/I^k$ is constant for $k \gg 0$. We call $\lim_{k\to\infty} \operatorname{depth} R/I^k$ the *limit depth of* I, see [13]. This limit depth can be computed under certain conditions that we are going to describe now.

Recall that the *analytic spread* of an ideal I is the Krull dimension of the fiber ring $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. It is known by Brodmann [2] that

$$\lim_{k\to\infty} \operatorname{depth} R/I^k \leq n - \ell(I).$$

Thus in particular, if the analytic spread of I is equal to n, then

$$\lim_{k\to\infty} \operatorname{depth} R/I^k = 0.$$

Eisenbud and Huneke [8, Proposition 3.3] showed that equality holds in Brodmann's inequality if the associated graded ring $\operatorname{gr}_I(R)$ is Cohen–Macaulay, which by Huneke [17] is the case if R and $\mathcal{R}(I)$ are Cohen–Macaulay.

In the case that I is a monomial ideal generated in a single degree, the analytic spread of I is the rank of the integer matrix whose rows correspond to the monomial generators of I.

3 Polymatroidal ideals and the persistence property

Discrete polymatroids were introduced in [12] and represent a natural generalization of matroids. In the following we recall some basic facts about discrete polymatroids (for more details see [12, 14]).

Let $\varepsilon_1, \dots, \varepsilon_n$ denote the canonical basis vectors of $\mathbb{R}^n$. Let $\mathbb{R}^n_+$ denote the set of vectors $u = (u(1), \dots, u(n)) \in \mathbb{R}^n$ with each $u(i) \geq 0$. If $u = (u(1), \dots, u(n))$ and $v = (v(1), \dots, v(n))$ are two vectors belonging to $\mathbb{R}^n_+$, then we write $u \leq v$ if all components $v(i) - u(i)$ of $v - u$ are nonnegative. Moreover, we write $u < v$ if $u \leq v$ and $u \neq v$. The modulus of $u = (u(1), \dots, u(n)) \in \mathbb{R}^n_+$ is $|u| = u(1) + \cdots + u(n)$. Also, let $\mathbb{Z}^n_+ = \mathbb{R}^n_+ \cap \mathbb{Z}^n$.

A *discrete polymatroid* on the ground set $[n]$ is a non-empty finite set $\mathcal{P} \subset \mathbb{Z}^n_+$ satisfying the following conditions:

(1) if $u \in \mathcal{P}$ and $v \in \mathbb{Z}^n_+$ with $v \leq u$, then $v \in \mathcal{P}$;
(2) if $u = (u(1), \dots, u(n)) \in \mathcal{P}$ and $v = (v(1), \dots, v(n)) \in \mathcal{P}$ with $|u| < |v|$, then there is $i \in [n]$ with $u(i) < v(i)$ such that $u + \varepsilon_i \in \mathcal{P}$.

A *base* of $\mathcal{P}$ is a vector $u \in \mathcal{P}$ such that $u < v$ for no $v \in \mathcal{P}$. The set of all bases of $\mathcal{P}$ is denoted by $B(\mathcal{P})$. It follows from (2) that if u_1 and u_2 are bases of $\mathcal{P}$, then $|u_1| = |u_2|$. The modulus of any base of $\mathcal{P}$ is called the *rank* of $\mathcal{P}$ and denoted by $\operatorname{rank} \mathcal{P}$. For later proofs it is very useful to have the following characterization of discrete polymatroids.

Let $\mathcal{P}$ be a non-empty finite set of integer vectors in $\mathbb{R}^n_+$ which contains with each $u \in \mathcal{P}$ all its integral subvectors, that is, vectors v with $v \leq u$, and let $B(\mathcal{P})$ be the set of vectors $u \in \mathcal{P}$ with $u < v$ for no $v \in \mathcal{P}$. Then (see [14, Theorem 12.2.4]) $\mathcal{P}$ is a discrete polymatroid with $B(\mathcal{P})$ its set of bases if and only if the following are satisfied:

(i) all $u \in B(\mathcal{P})$ have the same modulus;
(ii) if $u = (u(1), \dots, u(n)) \in B(\mathcal{P})$ and $v = (v(1), \dots, v(n)) \in B(\mathcal{P})$ with $u(i) > v(i)$, then there is $j \in [n]$ with $u(j) < v(j)$ such that $u - \varepsilon_i + \varepsilon_j \in B(\mathcal{P})$.

Let $\mathcal{P}$ be a discrete polymatroid on $[n]$ with $B(\mathcal{P})$ the set of bases. The *polymatroidal ideal* I attached to $\mathcal{P}$ is the monomial ideal of $S = K[x_1, \dots, x_n]$ whose set of minimal monomial generators is the set $G(I) = \{x^u : u \in B(\mathcal{P})\}$. Observe that I is generated in degree $\operatorname{rank} \mathcal{P}$.

Recall from Sect. 2 that for any monomial ideal $I \subset S$ and any $i \in [n]$ we have

$$I_{x_i} = I_{\{i\}} S_{x_i},$$

where $I_{\{i\}} \subset S_{\{i\}} = K[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$ is the monomial ideal which is obtained from I by applying the substitution $x_i \mapsto 1$.

Proposition 3.1 *Let $I \subset S$ be a polymatroidal ideal. Then for every $i \in [n]$ the ideal $I_{\{i\}}$ is again polymatroidal.*

Proof Let $\mathcal{P}$ be the polymatroid of rank d on the ground set $[n]$ defining the polymatroidal ideal I. Then $I = (x^u : u \in B(\mathcal{P}))$ is a monomial ideal in $S = K[x_1, \dots, x_n]$

which is generated in degree d. It follows that $I_{\{i\}}=(x^{u'}: u\in B(\mathcal{P}))\subset S_{\{i\}}$, where for all $u\in B(\mathcal{P})$ we set $x^{u'}=x^u/x_i^{u(i)}$.

We first show that $I_{\{i\}}$ is generated in one degree. More precisely, if

$$a_i=\max\{u(i): u\in B(\mathcal{P})\},$$

then we show that

$$G(I_{\{i\}})=\{x^u/x_i^{a_i}: u\in B(\mathcal{P}),\ u(i)=a_i\}.$$

Indeed, let $v\in B(\mathcal{P})$. Then $v(i)\le a_i$. We show that there exists $w\in B(\mathcal{P})$ with $w(i)=a_i$ and such that $x^{w'}$ divides $x^{v'}$. This will then yield the desired conclusion. To show this we proceed by induction on $a_i-v(i)$. If $a_i-v(i)=0$, then there is nothing to show. Suppose now that $v(i)<a_i$, and let $u\in B(\mathcal{P})$ with $u(i)=a_i$. Applying the symmetric exchange property (see [14, Theorem 12.4.1]), there exists an integer $j\in[n]$ with $u(j)<v(j)$ and such that $u-\varepsilon_i+\varepsilon_j\in B(\mathcal{P})$ and $v_1:=v-\varepsilon_j+\varepsilon_i\in B(\mathcal{P})$. Hence we see that $x^{v_1'}$ divides $x^{v'}$. Since $a_i-v_1(i)<a_i-v(i)$, our induction hypothesis implies that there exists $w\in B(\mathcal{P})$ with $w(i)=a_i$ and such that $x^{w'}$ divides $x^{v_1'}$. It follows that $x^{w'}$ divides $x^{v'}$ as well, as desired.

It remains to be shown that the set $B':=\{u': x^{u'}\in G(I_{\{i\}})\}$ is the set of bases of a discrete polymatroid $\mathcal{P}'$ of rank $d-a_i$ on the ground set $[n]\setminus\{i\}$. First notice that for all $u'\in B'$ we have $|u'|=d-a_i$. In order to verify the exchange property, let $u',v'\in B'$ with $u'(k)>v'(k)$. Then we have $k\neq i$. We may apply now the exchange property for $u,v\in B(\mathcal{P})$: $u(k)=u'(k)>v'(k)=v(k)$ then there exists $l\in[n]$ such that $u(l)<v(l)$ and such that the vector $t=u-\varepsilon_k+\varepsilon_l\in B(\mathcal{P})$. Since $u(i)=v(i)=a_i$, it follows that $l\neq i$ and $t(i)=a_i$. Therefore we obtain $t'\in B'$, where $t'=u'-\varepsilon_k+\varepsilon_l$, as desired. □

Corollary 3.2 *If I is a polymatroidal ideal, then $I(P)$ is a polymatroidal ideal for all $P\in V^*(I)$.*

Proposition 3.3 *Let $I\subset S$ be a polymatroidal ideal. Then I has the persistence property.*

Proof Let $k\ge 1$ be an integer. According to Lemma 2.3 we have $P\in\operatorname{Ass}(I^k)$ if and only if $\operatorname{depth} S(P)/I^k(P)=0$. Note that $I^j(P)=I(P)^j$ for all $j\ge 1$. Moreover, we know from Corollary 3.2 that $I(P)$ is again a polymatroidal ideal. Since powers of polymatroidal ideals are again polymatroidal, see [14, Theorem 12.6.3], and since by [14, Theorem 12.6.2] polymatroidal ideals have linear resolutions, we conclude that all powers of $I(P)$ have a linear resolution. Now we apply Proposition 2.2 and we obtain $\operatorname{depth} S(P)/I^j(P)=0$ for all $j\ge k$. But this implies that $P\in\operatorname{Ass}(I^j)$ for all $j\ge k$, as desired. □

Our next goal is to describe the stable set of associated prime ideals of a polymatroidal ideal. For that purpose we first recall the following result of Villarreal [20, Proposition 3.11]. For the convenience of the reader we present here an alternative proof of it.

Theorem 3.4 *Let $I \subset S$ be a polymatroidal ideal. Then $\mathcal{R}(I)$ is a normal ring.*

Proof It is a well-known fact that $\mathcal{R}(I)$ is a normal ring if and only if I is a normal ideal (see [15, Proposition 2.1.2]). By definition, I is normal if all powers of I are integrally closed. Since a product of polymatroidal ideals is again a polymatroidal ideal (see [7, Theorem 5.3]), it is enough to prove that polymatroidal ideals are integrally closed. Since I is in particular a monomial ideal, it follows from [14, Theorem 1.4.2] that I is integrally closed if and only if the following condition is satisfied: for every monomial $u \in S$ and every integer k such that $u^k \in I^k$ we have $u \in I$.

Let $u \in S$ be a monomial of degree t and k an integer such that $u^k \in I^k$. Since I is generated in one degree, say d, it follows from $u^k \in I^k$ that $tk \geq dk$, that is, $t \geq d$. Let I_l be the K-subspace of I spanned by all monomials of degree l. Then

$$\left(I^k\right)_{tk} = S_{tk-dk}\left(I^k\right)_{dk} = (S_{t-d})^k (I_d)^k = (S_{t-d} I_d)^k.$$

Observe that $S_{t-d} I_d = J_t$ where $J = \mathfrak{m}^{t-d} I$ is a polymatroidal ideal generated in degree t. Therefore, we obtain

$$u^k \in \left(I^k\right)_{tk} = (J_t)^k.$$

Consequently, we see that u belongs to the integral closure of the base ring $K[J]$. Applying now the normality of $K[J]$ (see [14, Theorem 12.5.1]) we obtain $u \in K[J]$. It follows that $u \in J$. Therefore $u \in I$, as desired. □

Corollary 3.5 *Let $I \subset S = K[x_1, \ldots, x_n]$ be a polymatroidal ideal. Then*

$$\lim_{k \to \infty} \operatorname{depth} S/I^k = n - \ell(I).$$

Combining Corollary 3.2 with the preceding corollary one obtains the following algorithm to determine $\operatorname{Ass}^\infty(I)$ for any polymatroidal ideal.

Algorithm 3.6 Let I be a polymatroidal ideal with $G(I) = \{x^{u_1}, \ldots, x^{u_m}\}$, and let A be the $m \times n$ integer matrix with entries $a_{ij} = u_i(j)$.

Let F be a non-empty subset of $[n]$, and $v_1, \ldots, v_m$ be the row vectors of the submatrix $(a_{ij})_{i \in [m],\, j \in F}$ of A. Furthermore, let $\{v_{i_1}, \ldots, v_{i_r}\}$ be the set of minimal elements among the vectors $v_1, \ldots, v_m$ with respect to the partial order given by componentwise comparison. Then $P_F \in \operatorname{Ass}^\infty(I)$ if and only if $\operatorname{rank}(a_{i_k, j})_{k=1,\ldots,r,\, j \in F} = |F|$.

Thus $\operatorname{Ass}^\infty(I)$ can be determined in finitely many steps.

4 Transversal polymatroids

Let F be a non-empty subset of $[n]$. As before we denote by P_F the monomial prime ideal $(\{x_i : i \in F\})$. A *transversal* polymatroidal ideal is an ideal I of the form

$$I = P_{F_1} P_{F_2} \cdots P_{F_r}, \tag{1}$$

where $F_1, \ldots, F_r$ is a collection of non-empty subsets of $[n]$ with $r \geq 1$. It follows from the definition that the product of transversal polymatroidal ideals is again a transversal polymatroidal ideal. By taking powers of the prime ideal factors of I which appear several times in (1), we get

$$I = \prod_{j=1}^{s} P_{G_j}^{a_j} \quad \text{with } a_j \geq 1, \tag{2}$$

where $G_j \neq G_k$ for $j \neq k$.

Lemma 4.1 *Let I be a transversal polymatroidal ideal. Then I has a unique presentation as in* (2).

Proof We proceed by induction on s, the number of different prime factors in the presentation of I. So let $s = 1$ and $I = P_{G_1}^{a_1}$. We identify G_1 as the set of all indices i for which $I_{x_i} = S_{x_i}$. The exponent a_1 is the degree of the generators of I.

Now let $s > 1$ and assume that I has a presentation as in (2). We may further assume that $\bigcup_{j=1}^{s} G_j = [n]$. Then for each $i = 1, \dots, n$ the ideal I_{x_i} determines

$$I_{\{i\}} = \prod_{\substack{j=1 \\ i \notin G_j}}^{s} P_{G_j}^{a_j}.$$

The transversal polymatroidal ideal $I_{\{i\}}$ has less different prime ideal factors than I since $\bigcup_{j=1}^{s} G_j = [n]$. Thus our induction hypothesis implies that the presentation of $I_{\{i\}}$ in the form (2) is unique. For each j such that $G_j \neq [n]$ there exists an integer $i \in [n]$ such that $P_{G_j}^{a_j}$ is a factor of $I_{\{i\}}$. Thus we identified all factors $P_{G_j}^{a_j}$ with $G_j \neq [n]$. The factor $P_{[n]}$ appears with the exponent

$$d - \sum_{\substack{j=1 \\ G_j \neq [n]}}^{s} a_j,$$

where d is the degree of the generators of I. □

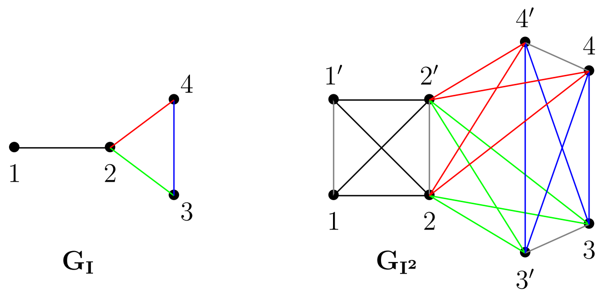
In order to characterize the set of associated prime ideals of I we will introduce a graph G_I associated with I as follows: the set of vertices $\mathcal{V}(G_I)$ is the set $\{1, \dots, r\}$ and $\{i, j\}$ is an edge of G_I if and only if $F_i \cap F_j \neq \emptyset$.

Example 4.2 Let $F_1 = \{1, 2\}$, $F_2 = \{1, 2, 3, 4\}$, $F_3 = \{3, 5\}$, $F_4 = \{4, 5\}$ and $I = P_{F_1} \cdots P_{F_4}$ be the transversal polymatroidal ideal of $K[x_1, \dots, x_5]$. Notice that $I = (x_1, x_2)(x_1, x_2, x_3, x_4)(x_3, x_5)(x_4, x_5)$. Then in Fig. 1 we have depicted the graphs G_I and G_{I^2}. One can notice that for any transversal polymatroidal ideal I the graph G_{I^k} is just the kth expansion of G_I (see [9, Definition 4.2]).

Now we are ready to decide whether the maximal ideal is an associated prime of the transversal polymatroidal ideal I from the connectedness of the graph G_I. More precisely, we have

Theorem 4.3 *Let $I = P_{F_1} \cdots P_{F_r} \subset S$ be a transversal polymatroidal ideal. Then $\mathfrak{m} \in \operatorname{Ass}(I)$ if and only if $\bigcup_{i=1}^{r} F_i = [n]$ and G_I is connected.*

Fig. 1 The graph of a transversal polymatroidal ideal



Proof Let us first assume that $\mathfrak{m} \in \mathrm{Ass}(I)$. Then it follows that $\bigcup_{i=1}^r F_i = [n]$. Indeed, let $F := \bigcup_{i=1}^r F_i \subsetneq [n]$. Then there exists an ideal $J \subset S(P_F)$ such that $I = JS$. Therefore we have

$$\operatorname{depth}_S S/I = \operatorname{depth}_{S(P_F)} S(P_F)/J + n - |F| > 0,$$

a contradiction.

Assume that G_I is disconnected. It follows from the definition of G_I that after an eventual relabeling of the vertices there exists an integer l such that $1 \le l < r$ and

$$\left(\bigcup_{i=1}^{l} F_i\right) \cap \left(\bigcup_{i=l+1}^{r} F_i\right) = \emptyset. \tag{3}$$

This implies that there exist integers $s, t \in [n]$ such that

$$s \in \left(\bigcup_{i=1}^{l} F_i\right) \quad \text{and} \quad t \in \left(\bigcup_{i=l+1}^{r} F_i\right). \tag{4}$$

Since $\mathfrak{m} \in \mathrm{Ass}(I)$ then there exists a monomial $z \in S \setminus I$ such that $\mathfrak{m} = I : (z)$. From this it follows in particular that $x_s z \in I$ and $x_t z \in I$. By using that $I = P_{F_1} \cdots P_{F_r}$ we obtain

$$x_s z = x_{i_1} \cdots x_{i_r} \quad \text{and} \quad x_t z = x_{j_1} \cdots x_{j_r}, \tag{5}$$

where $i_k, j_k \in F_k$ for all k. It follows now from (4) and (5) that $s \in \{i_1, \dots, i_l\}$ and $t \in \{j_{l+1}, \dots, j_r\}$. Hence $z \in P_{F_{l+1}} \cdots P_{F_r}$ and $z \in P_{F_1} \cdots P_{F_l}$. Consequently

$$z \in P_{F_1} \cdots P_{F_l} \cap P_{F_{l+1}} \cdots P_{F_r} = P_{F_1} \cdots P_{F_l} \cdot P_{F_{l+1}} \cdots P_{F_r} = I,$$

where the first equality is implied by (3). This yields $z \in I$, a contradiction.

Conversely, assume that $\bigcup_{i=1}^r F_i = [n]$ and G_I is connected. We will prove that $\mathfrak{m} \in \mathrm{Ass}(I)$ by explicitly constructing a monomial $z \in S \setminus I$ such that $I : z = \mathfrak{m}$. Since G_I is connected and has r vertices, we may consider a spanning tree $\mathcal{T}$ for G_I, that is, a collection of $r-1$ edges, say $e_1, \dots, e_{r-1}$ which cover all vertices of G_I (see the trees $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ from Example 4.4).

For $k = 1, \dots, r-1$ let $e_k = \{i_k, j_k\}$. Then by the definition of G_I we have $F_{i_k} \cap F_{j_k} \neq \emptyset$. For any such k we choose an element $l_k \in F_{i_k} \cap F_{j_k}$. We define now the monomial z as being

$$z = x_{l_1} \cdots x_{l_{r-1}}.$$

We claim that for any $i \in [r]$ we have $z \in \prod_{j \neq i} P_{F_j}$. If our claim is true then we obtain at once that $I : z = \mathfrak{m}$. Indeed, since $\deg(z) = r - 1$ we obtain first that $z \notin I$. It remains to be shown that $\mathfrak{m}z \subset I$. Let $i \in [n]$ be an arbitrary integer. By our assumption, we have $\bigcup_{i=1}^r F_i = [n]$, hence there exists an integer k such that $i \in F_k$. By the claim we have $z \in \prod_{j \neq k} P_{F_j}$. Therefore, we obtain

$$x_i z \in P_{F_k} \cdot \prod_{j \neq k} P_{F_j} = I,$$

as desired.

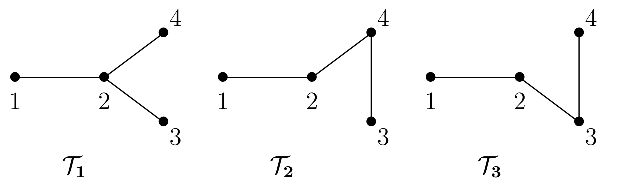
In order to prove our claim let $i \in [r]$ be an integer. We will reformulate our claim in terms of certain numerical functions on trees and prove it by induction on r. Indeed, it follows from the definition of z that $z \in \prod_{j \neq i} P_{F_j}$ if there exists a function $f : \{e_1, \dots, e_{r-1}\} \to [r]$ with $\operatorname{Im} f = [r] \setminus \{i\}$ and such that $f(e_k) \in \{i_k, j_k\}$.

The case $r = 2$ is obvious. Since $\mathcal{T}$ is a tree, there exists a vertex of degree 1. For simplicity, we may assume that this vertex is 1, his only adjacent vertex is 2 and $e_1 = \{1, 2\}$. The graph $\mathcal{T} \setminus \{1\}$ is a tree with the $r - 1$ vertices $\{2, \dots, r\}$ and edges $\{e_2, \dots, e_{r-1}\}$. It follows from the induction hypothesis that for every $i \in \{2, \dots, r\}$ there exists a function $f_i : \{e_2, \dots, e_{r-1}\} \to \{2, \dots, r\}$ such that $\operatorname{Im} f_i = \{2, \dots, r\} \setminus \{i\}$. We may extend these functions to

$$\tilde{f_i} : \{e_1, \dots, e_{r-1}\} \mapsto \{1, \dots, r\},$$

by setting $\tilde{f_i}(e_1) = 1$ and $\tilde{f_i}(e_j) = f_i(e_j)$ for all $j \geq 2$ and obtain $\operatorname{Im} \tilde{f_i} = [r] \setminus \{i\}$ for all $i \geq 2$. Finally consider $\tilde{f_1} : \{e_1, \dots, e_{r-1}\} \mapsto \{1, \dots, r\}$ to be the function defined by $\tilde{f_1}(e_1) = 2$ and $\tilde{f_1}(e_j) = f_2(e_j)$ for all $j \geq 2$. It follows that $\operatorname{Im} \tilde{f_1} = [r] \setminus \{1\}$ and we are done. □

Example 4.4 In the case that $\mathfrak{m} \in \operatorname{Ass}(I)$ the proof of Theorem 4.3 allows us to compute a monomial z such that $I : z = \mathfrak{m}$. However we may have several possibilities for choosing z since its choice depends on the spanning tree of G_I and on the intersections of the sets F_i corresponding to the adjacent vertices of the tree. Indeed, let us return to the ideal I from the Example 4.2. The graph G_I is connected with the three spanning trees $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ depicted below.



The spanning tree $\mathcal{T}_1$ gives rise to two such monomials, $x_1x_3x_4$ and $x_2x_3x_4$, since $F_1 \cap F_2 = \{1, 2\}$, $F_2 \cap F_3 = \{3\}$ and $F_2 \cap F_4 = \{4\}$. Analogously $\mathcal{T}_2$ and $\mathcal{T}_3$ determine the monomials $x_1x_4x_5$, $x_2x_4x_5$, respectively, $x_1x_3x_5$, $x_2x_3x_5$.

Corollary 4.5 *Let $I \subset S$ be a transversal polymatroidal ideal. Then $\mathfrak{m} \in \operatorname{Ass}(I)$ if and only if $\mathfrak{m} \in \operatorname{Ass}^\infty(I)$.*

Proof By Proposition 3.3, I satisfies the persistence property. Therefore $\mathfrak{m} \in \mathrm{Ass}^\infty(I)$ if $\mathfrak{m} \in \mathrm{Ass}(I)$. For the converse, let $I = P_{F_1} \cdots P_{F_r}$ and $\mathfrak{m} \in \mathrm{Ass}^\infty(I)$. Then there exists an integer $k \geq 1$ such that $\mathfrak{m} \in \mathrm{Ass}(I^k)$. Since I^k is again a transversal polymatroidal ideal, it follows from Theorem 4.3 that G_{I^k} is connected and $\bigcup_{i=1}^r F_i = [n]$. One easily notices that G_{I^k} is connected if and only if G_I is connected. Applying again Theorem 4.3 we get the desired conclusion. □

By using the fact that the localization of a transversal polymatroidal ideal is again a transversal polymatroidal ideal we obtain the following.

Corollary 4.6 *Let $I \subset S$ be a transversal polymatroidal ideal. Then* $\mathrm{astab}(I) = 1$, *that is*,

$$\mathrm{Ass}(I) = \mathrm{Ass}^\infty(I).$$

Proof It follows from Proposition 3.3 that $\mathrm{Ass}(I) \subset \mathrm{Ass}^\infty(I)$. For the converse inclusion, let $P \in \mathrm{Ass}^\infty(I)$. Then there exists $k \geq 1$ such that $P \in \mathrm{Ass}(I^k)$. Applying Lemma 2.3 we obtain $P \in \mathrm{Ass}(S(P)/I^k(P))$. Notice now that $I^k(P) = I(P)^k$ and that $I(P)$ is also a transversal polymatroidal ideal. Since P is the maximal ideal of $S(P)$, it follows from Corollary 4.5 that $P \in \mathrm{Ass}(S(P)/I(P))$. Therefore, by applying again Lemma 2.3 we obtain $P \in \mathrm{Ass}(I)$. □

In particular, in the case when $I = P_{F_1} \cdots P_{F_r}$ is the transversal polymatroidal ideal such that the sets $F_1, \ldots, F_r$ are pairwise disjoint we recover the previously known fact that $\mathrm{Ass}(I) = \mathrm{Ass}^\infty(I)$, see [10, Theorem 4.6] and [10, Corollary 4.26]. Next we want to describe the set of associated prime ideals of a transversal polymatroidal ideal I. In [7, Lemma 3.2] the authors gave a primary decomposition of such a transversal polymatroidal ideal, but unfortunately this primary decomposition is in general far from being irredundant, see also [7, Proposition 3.4]. Therefore we cannot read off from their primary decomposition the set of associated prime ideals of a transversal polymatroidal ideal. However, by using the graph G_I this can be done. For this, to each subgraph $\mathcal{H}$ of G_I we associate the prime ideal $P_{\mathcal{H}} = \sum_{i \in \mathcal{V}(\mathcal{H})} P_{F_i}$.

Theorem 4.7 *Let $I \subset S$ be a transversal polymatroidal ideal. Then*

$$\mathrm{Ass}(I) = \{P_{\mathcal{T}} \colon \mathcal{T} \text{ is a tree in } G_I\}.$$

Proof Let $I = P_{F_1} \cdots P_{F_r}$. We prove the statement by induction on r. The case $r = 1$ is trivial, since in that case G_I is just a vertex. We may assume that G_I is connected. Indeed, let $G_1, \ldots, G_k$ be the connected components of G_I, with $k \geq 2$. Then $I = I_1 \cdots I_k$, where

$$I_j = \prod_{i \in \mathcal{V}(G_j)} P_{F_i}.$$

Furthermore, $G_j = G_{I_j}$ for all j. Notice that $I = I_1 \cdots I_k = I_1 \cap \cdots \cap I_k$, since the ideals I_j are generated in pairwise disjoint sets of variables. Hence we obtain $\mathrm{Ass}(I) = \mathrm{Ass}(I_1) \cup \cdots \cup \mathrm{Ass}(I_k)$, where I_j is a transversal polymatroidal ideal with the associated connected graph G_{I_j}.

Obviously we may assume that $\bigcup_{j=1}^r F_j = [n]$.

Let $P \in \operatorname{Ass}(I)$. If $P = \mathfrak{m}$, then by Theorem 4.3 we have $P = P_{\mathcal{T}}$, where $\mathcal{T}$ is a spanning tree of G_I. Otherwise there exists an integer $i \in [n]$ such that $x_i \notin P$. Then $P \in \operatorname{Ass}(I_{\{i\}})$ where

$$I_{\{i\}} = \prod_{\substack{j=1 \\ i \notin F_j}}^{r} P_{F_j}.$$

The number of prime factors appearing in $I_{\{i\}}$ is less than r, since $\bigcup_{j=1}^r F_j = [n]$. Applying now the induction hypothesis, we obtain $P = P_{\mathcal{T}}$ for some tree $\mathcal{T}$ in $G_{I_{\{i\}}}$. Since $G_{I_{\{i\}}}$ is a subgraph of G_I, we obtain the desired conclusion.

Conversely, let $\mathcal{T}$ be a tree in G_I. If $\mathcal{T}$ is a spanning tree, then we know from the proof of Theorem 4.3 that $P_{\mathcal{T}} = \mathfrak{m} \in \operatorname{Ass}(I)$. Therefore we may assume that $\mathcal{T}$ is a tree in G_I with $|\mathcal{V}(\mathcal{T})| < r$ and $P_{\mathcal{T}} \neq \mathfrak{m}$. This implies that there exists an integer $i \in [n]$ such that $x_i \notin P_{\mathcal{T}}$. Then $\mathcal{T}$ remains a tree in $G_{I_{\{i\}}}$ since all vertices of $\mathcal{T}$ belong to $G_{I_{\{i\}}}$. Moreover, the number of prime factors appearing in $I_{\{i\}}$ is less than r. Therefore, by induction hypothesis we obtain $P_{\mathcal{T}} \in \operatorname{Ass}(I_{\{i\}})$ and consequently $P_{\mathcal{T}} \in \operatorname{Ass}(I)$. □

Example 4.8 Consider again the ideal I given in the Example 4.2, that is,

$$I = (x_1, x_2)(x_1, x_2, x_3, x_4)(x_3, x_5)(x_4, x_5).$$

The trees of G_I have one, two, three, or four vertices. The one-vertex trees, that is, the vertices, correspond to the associated primes $P_{F_1}, \dots, P_{F_4}$. The two-vertex trees correspond to the associated primes $P_{F_1} + P_{F_2}$, $P_{F_2} + P_{F_3}$, $P_{F_2} + P_{F_4}$, $P_{F_3} + P_{F_4}$. All trees with three and four vertices generate the same associated prime $\mathfrak{m}$. Consequently we obtain

$$\operatorname{Ass}(I) = \bigl\{(x_1, x_2), (x_1, x_2, x_3, x_4), (x_3, x_5), (x_4, x_5), (x_3, x_4, x_5), (x_1, x_2, x_3, x_4, x_5)\bigr\}.$$

In particular, we also find that the minimal associated primes correspond to vertices of G_I. However, as this example shows, in general not all the vertices give rise to minimal prime ideals.

As a consequence of the above theorem we obtain a description of all possible sets of associated prime ideals of a transversal polymatroidal ideal. More precisely we have

Corollary 4.9 *Let $\mathcal{F}$ be a subset of $2^{[n]}$ such that $\emptyset \notin \mathcal{F}$. Assume that $\mathcal{F}$ satisfies the following condition*:

$$A \cup B \in \mathcal{F} \quad \text{for all } A, B \in \mathcal{F} \text{ with } A \cap B \neq \emptyset. \tag{6}$$

Then there exists a transversal polymatroidal ideal I such that

$$\operatorname{Ass}(I) = \{P_A \colon A \in \mathcal{F}\}.$$

Conversely, given any transversal polymatroidal ideal I, the set

$$\bigl\{A \colon A \subset [n] \text{ and } P_A \in \operatorname{Ass}(I)\bigr\}$$

satisfies condition (6).

Proof Consider $\mathcal{F} = \{A_1, \ldots, A_r\}$ to be a set of non-empty subsets of $[n]$ which satisfies condition (6). We define the transversal polymatroidal ideal $I \subset S$ to be $I = \prod_{i=1}^{r} P_{A_i}$. Then since we consider the vertices of G_I as trees as well, we obtain from Theorem 4.7 that

$$\mathrm{Ass}(I) = \{P_{\mathcal{T}}\colon \mathcal{T} \text{ is a tree in } G_I\} \supset \{P_{A_1}, \ldots, P_{A_r}\}.$$

We prove the converse inclusion by showing that $P_{\mathcal{T}} \in \{P_{A_1}, \ldots, P_{A_r}\}$ for any tree $\mathcal{T}$ of G_I. This will be shown by induction on k, the number of vertices of a tree of G_I. The case $k = 1$ is obvious, since the vertices of G_I correspond to all P_{A_i} with $i = 1, \ldots, r$. Assume now that $\mathcal{T}$ is a tree of G_I with the set of vertices $\mathcal{V}(\mathcal{T}) = \{i_1, \ldots, i_k\}$. Since $\mathcal{T}$ is a tree, there exists a vertex of degree 1. We may assume that this vertex is i_1 and furthermore that $\{i_1, i_2\}$ is an edge of $\mathcal{T}$. Therefore $A_{i_1} \cap A_{i_2} \neq \emptyset$. For the tree $\mathcal{T}' = \mathcal{T} \setminus \{i_1\}$ we apply the induction hypothesis and obtain $P_{\mathcal{T}'} = P_A$, where $A = \bigcup_{j=2}^{k} A_{i_j} \in \mathcal{F}$. Hence $A_{i_1} \cap A \neq \emptyset$, and by using the fact that $A_{i_1}, A \in \mathcal{F}$ we obtain via (6) that $A_{i_1} \cup A \in \mathcal{F}$. The conclusion follows at once from the equality $P_{\mathcal{T}} = P_B$, where $B = A_{i_1} \cup A$.

Conversely, let $I = P_{F_1} \cdots P_{F_r}$ be a transversal polymatroidal ideal. Consider now two subsets A, B of $[n]$ such that $A \cap B \neq \emptyset$ and $P_A, P_B \in \mathrm{Ass}(I)$. By Theorem 4.7 we know that there exist two trees $\mathcal{T}, \mathcal{T}'$ of G_I such that $P_A = P_{\mathcal{T}}$ and $P_B = P_{\mathcal{T}'}$. Therefore, we obtain $A = \bigcup_{i \in \mathcal{V}(\mathcal{T})} F_i$ and $B = \bigcup_{i \in \mathcal{V}(\mathcal{T}')} F_i$. Thus $A \cap B \neq \emptyset$ implies that there exist two vertices $i \in \mathcal{V}(\mathcal{T})$ and $j \in \mathcal{V}(\mathcal{T}')$ such that $F_i \cap F_j \neq \emptyset$. Consequently, the subgraph $\mathcal{H}$ of G_I, whose set of vertices $\mathcal{V}(\mathcal{H})$ is $\mathcal{V}(\mathcal{T}) \cup \mathcal{V}(\mathcal{T}')$ and the edges of $\mathcal{H}$ are the edges of $\mathcal{T}$ and $\mathcal{T}'$, is connected. A spanning tree $\mathcal{T}''$ of $\mathcal{H}$ is a tree of G_I and has the property that

$$P_{\mathcal{T}''} = \sum_{i \in \mathcal{V}(\mathcal{T}'')} P_{F_i} = \sum_{i \in \mathcal{V}(\mathcal{H})} P_{F_i} = \sum_{i \in \mathcal{V}(\mathcal{T}) \cup \mathcal{V}(\mathcal{T}')} P_{F_i} = P_{A \cup B}.$$

Therefore, by applying again Theorem 4.7 we obtain $P_{A \cup B} \in \mathrm{Ass}(I)$, as desired. □

We obtain also from the Theorem 4.7 an irredundant primary decomposition for any power of a transversal polymatroidal ideal. This improves [7, Lemma 3.2], where the authors could give only a primary decomposition of a transversal polymatroidal ideal, which in general was far from being irredundant. Our proof uses their primary decomposition.

Corollary 4.10 *Let $I \subset S$ be a transversal polymatroidal ideal with the set of associated prime ideals* $\mathrm{Ass}(I) = \{P_1, \ldots, P_l\}$. *Consider $\mathcal{T}_1, \ldots, \mathcal{T}_l$ maximal trees of G_I such that $P_j = P_{\mathcal{T}_j}$ for all $j = 1, \ldots, l$. Then*

$$I^k = \bigcap_{j=1}^{l} P_j^{k a_j},$$

is an irredundant primary decomposition of I^k for any $k \geq 1$, where $a_j = |\mathcal{V}(\mathcal{T}_j)|$ for all j.

Proof First we recall that for the transversal polymatroidal ideal $I = P_{F_1} \cdots P_{F_r}$ we have the following primary decomposition (see [7, Lemma 3.2]):

$$I=\bigcap_{\substack{A\subset[r]\\ A\neq\emptyset}}\Big(\sum_{i\in A}P_{F_i}\Big)^{|A|}. \tag{7}$$

Since $(\sum_{i\in A}P_{F_i})^{|A|}$ is $\sum_{i\in A}P_{F_i}$-primary and $\operatorname{Ass}(I)=\{P_1,\ldots,P_l\}$ it follows that

$$I=\bigcap_{j=1}^{l}\Big(\bigcap_{\substack{A\subset[r]\\ \sum_{i\in A}P_{F_i}=P_j}}P_j^{|A|}\Big).$$

In order to obtain the desired irredundant primary decomposition it remains to be shown that $a_j\geq|A|$ for all A with $\sum_{i\in A}P_{F_i}=P_j$. To see this, let $\mathcal{H}$ be the induced subgraph of G_I with vertex set $\mathcal{V}(\mathcal{H})=\{i\colon i\in\mathcal{V}(G_I)$ and $P_{F_i}\subset P_j\}$. Then we have $A\subset\mathcal{V}(\mathcal{H})$. Since $P_j=\sum_{i\in\mathcal{V}(\mathcal{T}_j)}P_{F_i}$ it follows that $\mathcal{T}_j$ is also a tree of $\mathcal{H}$.

We show that $\mathcal{H}$ is connected. Indeed, let i,s be two vertices of $\mathcal{H}$. Since $P_{F_i}\subset\sum_{i\in\mathcal{V}(\mathcal{T}_j)}P_{F_i}$, it follows that there exists an integer i_0 such that $F_i\cap F_{i_0}\neq\emptyset$. Therefore, $\{i,i_0\}$ is an edge of $\mathcal{H}$. Similarly, we see that there exists an integer s_0 such that $\{s,s_0\}$ is an edge of $\mathcal{H}$. Hence there exists a path from i to s and this yields the desired conclusion.

It follows that $\mathcal{V}(\mathcal{T}_j)=\mathcal{V}(\mathcal{H})$. Therefore, $|A|\leq|\mathcal{V}(\mathcal{T}_j)|=a_j$.

The irredundant primary decomposition for I^k follows at once, since the maximal trees in G_{I^k} that realize P_j have ka_j vertices. □

Example 4.11 For the ideal I introduced in Example 4.2 we have computed in Example 4.8 the set of associated prime ideals

$$\begin{aligned}\operatorname{Ass}(I)&=\{P_1,\ldots,P_6\}\\ &=\big\{(x_1,x_2),(x_1,x_2,x_3,x_4),(x_3,x_5),(x_4,x_5),(x_3,x_4,x_5),(x_1,x_2,x_3,x_4,x_5)\big\}.\end{aligned}$$

We noticed there that different trees may determine the same associated prime ideal. For example, $P_6=\mathfrak{m}$ is determined by all spanning trees of G_I, all trees of G_I with three vertices and the following trees with two vertices: $\{2,4\}$, $\{2,3\}$. Therefore $a_6=4$, and for $\mathcal{T}_6$ we can choose any spanning tree of G_I. For each of the trees $\mathcal{T}_1,\ldots,\mathcal{T}_5$ there is only one choice. These trees are determined by their sets of vertices:

$$\begin{aligned}&\mathcal{V}(\mathcal{T}_1)=\{1\}, &&\mathcal{V}(\mathcal{T}_2)=\{1,2\}, &&\mathcal{V}(\mathcal{T}_3)=\{3\},\\ &\mathcal{V}(\mathcal{T}_4)=\{4\}, &&\mathcal{V}(\mathcal{T}_5)=\{3,4\}.\end{aligned}$$

Associated with these trees we have $a_1=1$, $a_2=2$, $a_3=1$, $a_4=1$ and $a_5=2$. It follows now from Corollary 4.10 that the corresponding irredundant primary decomposition of I is

$$I=P_1\cap P_2^2\cap P_3\cap P_4\cap P_5^2\cap P_6^4.$$

By using the graph G_I of a transversal polymatroidal ideal I we get also a formula for depth S/I. More precisely we have

Theorem 4.12 *Let $I = P_{F_1} \cdots P_{F_r} \subset S$ be a transversal polymatroidal ideal. Then*

$$\operatorname{depth} S/I = c(G_I) - 1 + n - \left|\bigcup_{i=1}^{r} F_i\right|,$$

where by $c(G_I)$ we denote the number of connected components of the graph G_I.

Proof We may assume that $\bigcup_{i=1}^{r} F_i = [n]$. Indeed, let $A = \bigcup_{i=1}^{r} F_i$. Since $I = JS$ for the polymatroidal ideal $J \subset S(P_A)$ with $G(J) = G(I)$, we have

$$\operatorname{depth} S/I = \operatorname{depth} S(P_A)/J + n - |A|.$$

The graphs G_I and G_J are identical, therefore $c(G_I) = c(G_J)$. Consequently, $\operatorname{depth} S(P_A)/J = c(G_J) - 1$ implies the desired formula for $\operatorname{depth} S/I$.

Let $k = c(G_I)$. We prove the statement by induction on k. If $k = 1$, then G_I is connected. By applying Theorem 4.3 we obtain $\mathfrak{m} \in \operatorname{Ass}(I)$. Hence $\operatorname{depth} S/I = 0$, as desired. Assume now that $k \geq 2$, and let $G_1, \ldots, G_k$ be the connected components of G_I. As in the proof of Theorem 4.7 we denote by $I_1, \ldots, I_k$ the transversal polymatroidal ideals for which the associated graphs are the connected components of G_I. Hence $I = I_1 \cdots I_k = I_1 \cap \cdots \cap I_k$. Without loss of generality we may assume that $1 \leq l_1 \leq \cdots \leq l_k$, where for all j

$$l_j = \left|\bigcup_{i \in \mathcal{V}(G_j)} F_i\right|,$$

and that $\bigcup_{i \in \mathcal{V}(G_1)} F_i = \{1, \ldots, l_1\}$. Observe that $l_1 + \cdots + l_k = n$.

We have two cases to analyze. First we treat the case $l_1 = 1$. This implies that the ideal I_1 is generated by x_1. If for all $j = 1, \ldots, k$ we have $l_j = 1$ then $k = n$ and the ideal I is principal. Therefore, $\operatorname{depth} S/I = n - 1$, as desired. Otherwise $l_k \geq 2$ and $k \leq n - 1$. Consider the short exact sequence

$$0 \longrightarrow S/\bigl(I : (x_1)\bigr) \longrightarrow S/I \longrightarrow S/\bigl(I + (x_1)\bigr) \longrightarrow 0.$$

Since $S/(I + (x_1)) = S/(x_1)$ it follows that $\operatorname{depth} S/(I + (x_1)) = n - 1$. We also have $I : (x_1) = I_2 \cdots I_k$. By induction hypothesis $\operatorname{depth} S/(I : (x_1)) = k - 1 - 1 + n - (n - 1) = k - 1 \leq n - 2$. By applying Depth Lemma (see [4, Proposition 1.2.9]) we obtain $\operatorname{depth} S/I = k - 1$, as desired.

Consider now the second case, that is, $l_1 \geq 2$. We use the following short exact sequence:

$$0 \longrightarrow S/I \longrightarrow S/I_1 \oplus S/(I_2 \cap \cdots \cap I_k) \longrightarrow S/(I_1 + I_2 \cap \cdots \cap I_k) \longrightarrow 0.$$

By induction hypothesis we have $\operatorname{depth} S/I_1 = n - l_1$ and $\operatorname{depth} S/(I_2 \cap \cdots \cap I_k) = k - 2 + l_1$, since $I_2 \cap \cdots \cap I_k = I_2 \cdots I_k$. It follows from $\bigcup_{i=1}^{r} F_i = [n]$ that $l_1 + \cdots + l_k = n$ and consequently that $kl_1 \leq n$. Therefore we have $n - l_1 \geq k$ and $k - 2 + l_1 \geq k$. This implies that

$$\operatorname{depth}\bigl(S/I_1 \oplus S/(I_2 \cap \cdots \cap I_k)\bigr) = \min\bigl\{\operatorname{depth} S/I_1, \operatorname{depth} S/(I_2 \cap \cdots \cap I_k)\bigr\} \geq k.$$

Since the ideals I_1 and $I_2 \cdots I_k = I_2 \cap \cdots \cap I_k$ are generated in disjoint sets of variables we obtain

$$S/(I_1 + I_2 \cdots I_k) \cong S_1/I_1 \otimes_K S_2/(I_2 \cdots I_k),$$

where $S_1 = K[x_1, \dots, x_{l_1}]$ and $S_2 = K[x_{l_1+1}, \dots, x_n]$. Therefore, by the additivity of depth and using the induction hypothesis we have

$$\operatorname{depth} S/(I_1 + I_2 \cdots I_k) = \operatorname{depth} S_1/I_1 + \operatorname{depth} S_2/(I_2 \cdots I_k) = 0 + (k-2) = k-2.$$

By applying again Depth Lemma we find that $\operatorname{depth} S/I = k-1$, as desired. □

Remark 4.13 In [14, Theorem 12.6.7] the authors classify all Cohen–Macaulay polymatroidal ideals, which turn out to be the principal ideals, the Veronese ideals and the squarefree Veronese ideals. In the special case of transversal polymatroidal ideal one may derive as a consequence of Theorem 4.12 and Theorem 4.7 the above result and obtains that the Cohen–Macaulay transversal polymatroidal ideals are the principal ideals and the Veronese ideals. Indeed, notice that Theorem 4.7 implies

$$\dim S/I = n - \min\{|F_i|\colon i = 1, \dots, r\}.$$

We denote by a the minimal cardinality of a set F_i, where $i = 1, \dots, r$. Hence $\dim S/I = n - a$. We may assume that $\bigcup_{i=1}^r F_i = [n]$, and then it follows from Theorem 4.12 that S/I is Cohen–Macaulay if and only if $n - a = k - 1$, where k represents the number of connected components of G_I. Since $n \geq ka$ it follows that $k-1 = n-a \geq ka-a = (k-1)a$. Therefore we see that this inequality is valid either if $k = 1$ or $a = 1$. If $k = 1$, then $a = n$ and consequently $|F_i| = n$, for all $i = 1, \dots, r$. This implies that $F_i = [n]$ for all i and hence $I = \mathfrak{m}^r$, the Veronese ideal. Otherwise $a = 1$ and then $k = n$. In this case G_I has n connected components. Hence we obtain $r = n$ and $|F_i| = 1$ for all $i = 1, \dots, n$. This yields that I is a principal ideal.

As a consequence of Theorem 4.12 we find that $\operatorname{depth} S/I^k$, as a function of k, is constant for any transversal polymatroidal ideal and hence we may also compute the analytic spread of I.

Corollary 4.14 *Let $I \subset S$ be a transversal polymatroidal ideal. Then* $\operatorname{depth} S/I = \operatorname{depth} S/I^k$ *for all $k \geq 1$. In particular, we have* $\operatorname{depth} S/I = \lim_{k\to\infty} \operatorname{depth} S/I^k$ *and* $\ell(I) = n - \operatorname{depth} S/I$.

Proof Since $c(G_I) = c(G_{I^k})$ for any $k \geq 1$, then by applying Theorem 4.12 we obtain $\operatorname{depth} S/I = \operatorname{depth} S/I^k$ for all $k \geq 1$. Therefore we also have

$$\lim_{k\to\infty} \operatorname{depth} S/I^k = \operatorname{depth} S/I.$$

By Corollary 3.5 we get the desired formula for $\ell(I)$. □

5 Ideals of Veronese type

Fix a positive integer d and non-negative integers $a_1, \dots, a_n$ with $a_1 + \cdots + a_n \geq d$. Let $B \subset \mathbb{Z}_+^n$ be the set of vectors $u \in \mathbb{Z}_+^n$ with $u(i) \leq a_i$ for all $i = 1, \dots, n$ and with $|u| = d$. Then B represents the set of bases of a discrete polymatroid $\mathcal{P}$ on the ground set $[n]$, of rank d, which is called a *discrete polymatroid of Veronese type*. Its polymatroidal ideal $I \subset S$ is called an *ideal of Veronese type* and will be denoted by $I_{d;a_1,\dots,a_n}$. The following result is an immediate consequence of the definition of an ideal of Veronese type and of Proposition 3.1.

Lemma 5.1 *Let $I = I_{d;a_1,\dots,a_n} \subset S$ be an ideal of Veronese type. Then we have*

(a) $I^k = I_{kd;ka_1,\dots,ka_n}$ *for every integer* $k \geq 1$;
(b) $I_{\{i\}} = I_{d-b_i;a_1,\dots,0,\dots,a_n} \subset K[\{x_j\colon j \neq i\}]$, *where b_i is the maximal degree of the variable x_i in a minimal generator of I.*

There are three particular cases of ideals of Veronese type that we will consider first. The first case is when $d = \sum_{i=1}^n a_i$, that is, $I = I_{d;a_1,\dots,a_n}$ is a principal ideal. Then $\operatorname{Ass}^\infty(I) = \operatorname{Ass}(I) = \{(x_{i_1}),\dots,(x_{i_r})\}$, where $a_{i_1},\dots,a_{i_r}$ are all the nonzero integers from $a_1,\dots,a_n$. Moreover, we have $\operatorname{depth} S/I = n-1$, $\lim_{k\to\infty} \operatorname{depth} S/I^k = n-1$, $\ell(I) = 1$ and $\operatorname{dstab}(I) = \operatorname{astab}(I) = 1$.

The second case is when $d = 1$. Then $I = I_{1;a_1,\dots,a_n}$ is a monomial prime ideal. Therefore $\operatorname{Ass}^\infty(I) = \operatorname{Ass}(I) = \{I\}$. Furthermore, we have $\operatorname{depth} S/I = n - \operatorname{height} I$, $\lim_{k\to\infty} \operatorname{depth} S/I^k = n - \operatorname{height} I$, $\ell(I) = \operatorname{height} I$ and $\operatorname{dstab}(I) = \operatorname{astab}(I) = 1$.

The third case to be considered is when there exists $i \in [n]$ such that $a_i = 0$. Let A be the subset of $[n]$ defined as $A = \{j\colon a_j \neq 0\}$. Then $A \neq \emptyset$ and $G(I) \subset S(P_A)$, where $I = I_{d;a_1,\dots,a_n}$. By the convention made before Lemma 2.3 we identify I with $G(I)S(P_A)$. For simplicity of notation we denote by $J \subset S(P_A)$ the ideal of Veronese type $G(I)S(P_A)$. It follows then that

$$\operatorname{Ass}_S^\infty(I) = \operatorname{Ass}_{S(P_A)}^\infty(J).$$

Furthermore, we have $\operatorname{astab}(I) = \operatorname{astab}(J)$ and $\operatorname{depth} S/I = \operatorname{depth} S(P_A)/J + n - |A|$. In addition, since I^k can be identified with J^k, we also have $\lim_{k\to\infty} \operatorname{depth} S/I^k = \lim_{k\to\infty} \operatorname{depth} S(P_A)/J^k + n - |A|$, $\ell(I) = \ell(J)$ and $\operatorname{dstab}(I) = \operatorname{dstab}(J)$.

Due to these considerations we may assume throughout the rest of this section that $I = I_{d;a_1,\dots,a_n} \subset S$ is a Veronese type ideal satisfying

$$d < \sum_{i=1}^n a_i \quad \text{and} \quad d > 1 \quad \text{and} \quad a_1,\dots,a_n \geq 1. \tag{8}$$

We recall that for such ideals of Veronese type there is a precise description of the associated prime ideals given in [21, Proposition 3.1].

Proposition 5.2 *Let $I = I_{d;a_1,\dots,a_n} \subset S$ be an ideal of Veronese type satisfying (8) and A a subset of $[n]$. Then*

$$P_A \in \operatorname{Ass}(I) \Longleftrightarrow \sum_{i=1}^n a_i \geq d - 1 + |A| \quad \text{and} \quad \sum_{i \notin A} a_i \leq d - 1.$$

By using this result we prove the following.

Proposition 5.3 *Let $I = I_{d;a_1,\dots,a_n} \subset S$ be an ideal of Veronese type satisfying (8). Then $\operatorname{Ass}^\infty(I) = V^*(I)$.*

Proof It is obvious that $\mathrm{Ass}^\infty(I) \subset V^*(I)$. Conversely, let $P_A \in V^*(I)$ for some subset A of $[n]$. Then there exists a minimal prime ideal $P_B \in \mathrm{Ass}(I)$ such that $P_B \subset P_A$. This implies that $B \subset A$ and furthermore, by applying Proposition 5.2, we have

$$\sum_{i \notin A} a_i \leq \sum_{i \notin B} a_i \leq d - 1.$$

Consequently we get for any integer $l \geq 1$

$$\sum_{i \notin A} l a_i \leq l(d-1) \leq ld - 1.$$

Since I satisfies (8) we have $\sum_{i=1}^n a_i \geq d + 1$. Then for $k = |A| - 1$ we have

$$k\left(\sum_{i=1}^n a_i - d\right) \geq |A| - 1.$$

Therefore, we get $\sum_{i=1}^n k a_i > kd - 1 + |A|$. Combining this inequality with $\sum_{i \notin A} k a_i \leq kd - 1$ and applying Proposition 5.2, we obtain $P_A \in \mathrm{Ass}(I_{kd;ka_1,\dots,ka_n})$. Therefore, by Lemma 5.1(a), we have $P_A \in \mathrm{Ass}(I^k)$. Thus we get $P_A \in \mathrm{Ass}^\infty(I)$, by the persistence property, as desired. □

It follows immediately from Proposition 5.3 that $\mathrm{Ass}^\infty(I)$ is determined by the minimal prime ideals of I. According to Proposition 5.2 these minimal prime ideals can be determined as follows: P_F is a minimal prime ideal of I if and only if F is a minimal subset of $[n]$ with respect to inclusion satisfying the following inequalities:

$$\sum_{i \notin F} a_i + \sum_{i \in F} (a_i - 1) \geq d - 1 \quad \text{and} \quad \sum_{i \notin F} a_i \leq d - 1.$$

We can say somewhat more about the set $V^*(I)$ for $I = I_{d;a_1,a_2,\dots,a_n}$. Without any loss of generality we may assume that $a_1 \geq a_2 \geq \cdots \geq a_n$. We will use the following facts:

(i) ([21, Lemma 2.1]) $\sqrt{I}$ is squarefree strongly stable, that is, for all monomials $x_F \in I$ and all integers $1 \leq i < j \leq n$ such that $j \in F$ and $i \notin F$ it follows that $x_{(F \setminus \{j\}) \cup \{i\}} \in I$. Here $x_F = \prod_{i \in F} x_i$ for $F \subset [n]$.
(ii) ([21, Lemma 2.3]) Let J be a squarefree strongly stable ideal, then the Alexander dual $J^\vee$ of J is also squarefree strongly stable.
(iii) $P_F \in V^*(J)$ if and only if $x_F \in J^\vee$ for any monomial ideal J.

Combining (i), (ii) and (iii) we obtain

Proposition 5.4 *Let $I = I_{d;a_1,\dots,a_n}$ be an ideal of Veronese type with $a_1 \geq a_2 \geq \cdots \geq a_n$. Then for all $P_F \in V^*(I)$ and $1 \leq i < j \leq n$ with $j \in F$ and $i \notin F$ it follows that $P_{(F \setminus \{j\}) \cup \{i\}} \in V^*(I)$.*

It is not the case, as one might expect, that any set $\mathcal{F}$ of incomparable monomial prime ideals can be realized as the set of minimal prime ideals of an ideal of Veronese type. For example, let $\mathcal{F} = \{(x_1, x_2), (x_3, x_4)\}$. No matter which order of the variables we choose, the ideal $(x_1, x_2)(x_3, x_4)$ is never squarefree strongly stable with respect to the given order of the variables.

We now characterize the ideals of Veronese type $I \subset S$ satisfying (8) for which $\operatorname{astab}(I) = 1$.

Corollary 5.5 *Let $I = I_{d;a_1,\dots,a_n} \subset S$ be an ideal of Veronese type satisfying* (8). *Then the following conditions are equivalent*:

(a) $\mathfrak{m} \in \operatorname{Ass}(I)$;
(b) $\operatorname{Ass}(I) = \operatorname{Ass}^\infty(I)$;
(c) $\sum_{i=1}^n a_i \ge d - 1 + n$.

Proof By applying Proposition 5.2 we obtain (a) ⇔ (c), since $\mathfrak{m} = P_{[n]}$. The implication (b) ⇒ (a) follows from Proposition 5.3. For (a) ⇒ (b), let $P_A \in \operatorname{Ass}^\infty(I)$ for some subset A of $[n]$. Since $\mathfrak{m} \in \operatorname{Ass}(I)$ we obtain $\sum_{i=1}^n a_i \ge d - 1 + n$. Therefore $\sum_{i=1}^n a_i \ge d - 1 + |A|$. The inequality $\sum_{i \notin A} a_i \le d - 1$ follows from the proof of Proposition 5.3. Hence, by applying again Proposition 5.2, we obtain $P_A \in \operatorname{Ass}(I)$, as desired. □

In the following we give an upper bound for the index of stability of any prime $P \in \operatorname{Ass}^\infty(I)$ which we define to be the smallest integer k such that $P \in \operatorname{Ass}(I^k)$.

Corollary 5.6 *Let $I = I_{d;a_1,\dots,a_n} \subset S$ be an ideal of Veronese type satisfying* (8) *and A a subset of $[n]$ with $P_A \in \operatorname{Ass}^\infty(I)$. Then the index of stability of P_A is equal to $\lceil \frac{|A|-1}{\sum_{i=1}^n a_i - d} \rceil$. In particular,*

$$\operatorname{astab}(I) = \left\lceil \frac{n-1}{\sum_{i=1}^n a_i - d} \right\rceil,$$

and $\operatorname{astab}(I) \le n - 1$.

Proof Let k be the smallest integer such that $P_A \in \operatorname{Ass}(I^k)$. Then we have $P_A \in \operatorname{Ass}(I^k) \setminus \operatorname{Ass}(I^{k-1})$. Therefore, by applying Lemma 5.1 and Proposition 5.2 this is equivalent to saying that the following inequalities are fulfilled:

$$k\left(\sum_{i=1}^n a_i - d\right) \ge |A| - 1 > (k-1)\left(\sum_{i=1}^n a_i - d\right) \quad \text{and} \quad \sum_{i \notin A} a_i \le d - 1.$$

The first two inequalities imply the desired formula for the index of stability of P_A. For the second equality, it is enough to observe that $\operatorname{astab}(I)$ is equal to the index of stability of $\mathfrak{m} = P_{[n]}$. The last inequality of the statement is obvious. □

The upper bound given in Corollary 5.6 is sharp since for every integer $n \ge 2$ the ideal of Veronese type $I = I_{n-1;1,\dots,1}$ has $\operatorname{astab}(I) = n - 1$. Moreover, for an ideal of Veronese type I satisfying (8) we have $\operatorname{astab}(I_{d;a_1,\dots,a_n}) = n - 1$ if and only if $\sum_{i=1}^n a_i = d + 1$. In addition, it follows from the discussion of the third particular case (before Proposition 5.2) that, for a fixed integer k with $1 \le k \le n-1$, the ideal of Veronese type $I = I_{d;a_1,\dots,a_k,0,\dots,0} \subset S$ with $\sum_{i=1}^k a_i = d + 1$ satisfies $\operatorname{astab}(I) = k$.

Therefore the index of stability of a Veronese type ideal can be any integer between 1 and $n-1$.

It follows from [13, Theorem 3.3] and Lemma 5.1 that for an ideal I of Veronese type we can compute depth S/I, the limit depth and dstab I. More precisely, we have

Corollary 5.7 *Let* $I = I_{d;a_1,\dots,a_n} \subset S$ *be an ideal of Veronese type satisfying* (8). *Then we have*

(a) $\operatorname{depth} S/I = \max\{0, d+n-1-\sum_{i=1}^n a_i\}$;
(b) $\operatorname{depth} S/I^k = \max\{0, kd+n-1-\sum_{i=1}^n ka_i\}$;
(c) $\operatorname{dstab}(I) = \lceil \frac{n-1}{\sum_{i=1}^n a_i - d} \rceil$.

In particular, $\operatorname{astab}(I) = \operatorname{dstab}(I)$, $\lim_{k\to\infty} \operatorname{depth} S/I^k = 0$ *and* $\ell(I) = n$.

Proof (a) was observed in [13, Theorem 3.3]. Notice that (b) follows at once from (a) and Lemma 5.1(a). Finally, one can immediately see that (b) implies (c). The last equalities are obvious. □

Acknowledgements This paper was partially written during the visit of the second and third author at the Universität Duisburg-Essen, Campus Essen. The second author wants to thank The Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy for supporting her. The third author was supported by a Romanian grant awarded by UEFISCDI, project number 83/2010, PNII-RU code TE_46/2010, program Human Resources, "Algebraic modeling of some combinatorial objects and computational applications".

References

1. Borna, K.: On linear resolution of powers of an ideal. Osaka J. Math. **46**, 1047–1058 (2009)
2. Brodmann, M.: The asymptotic nature of the analytic spread. Math. Proc. Camb. Philos. Soc. **86**, 35–39 (1979)
3. Brodmann, M.: Asymptotic stability of Ass$(M/I^n M)$. Proc. Am. Math. Soc. **74**, 16–18 (1979)
4. Bruns, W., Herzog, J.: Cohen–Macaulay Rings, Revised edn. Cambridge University Press, Cambridge (1996)
5. Chen, J., Morey, S., Sung, A.: The stable set of associated primes of the ideal of a graph. Rocky Mt. J. Math. **32**, 71–89 (2002)
6. CoCoATeam: CoCoA: a system for doing computations in commutative algebra. Available at http://cocoa.dima.unige.it
7. Conca, A., Herzog, J.: Castelnuovo–Mumford regularity of products of ideals. Collect. Math. **54**, 137–152 (2003)
8. Eisenbud, D., Huneke, C.: Cohen–Macaulay Rees algebras and their specialization. J. Algebra **81**, 202–224 (1983)
9. Francisco, C., Tai Ha, H., Van Tuyl, A.: Coloring of hypergraphs, perfect graphs and associated primes of powers of monomial ideals. J. Algebra **331**, 224–242 (2011)
10. Gitler, I., Reyes, E., Villarreal, R.H.: Blowup algebras of squarefree monomial ideals and some links to combinatorial optimization problems. Rocky Mt. J. Math. **39**, 71–102 (2009)
11. Greuel, G.M., Pfister, G., Schönemann, H.: Singular 2.0. A computer algebra system for polynomial computations. Centre for Computer Algebra, University of Kaiserslautern (2001). http://www.singular.uni-kl.de
12. Herzog, J., Hibi, T.: Discrete polymatroids. J. Algebr. Comb. **16**, 239–268 (2002)
13. Herzog, J., Hibi, T.: The depth of powers of an ideal. J. Algebra **291**, 534–550 (2005)
14. Herzog, J., Hibi, T.: Monomial Ideals. GTM, vol. 260. Springer, Berlin (2010)
15. Herzog, J., Simis, A., Vasconcelos, W.: Arithmetic of normal Rees algebras. J. Algebra **143**, 269–294 (1991)

16. Herzog, J., Hibi, T., Trung, N.V., Zheng, X.: Standard graded vertex cover algebras, cycles and leaves. Trans. Am. Math. Soc. **360**, 6231–6249 (2008)
17. Huneke, C.: On the associated graded ring of an ideal. Ill. J. Math. **26**, 121–137 (1982)
18. Martinez-Bernal, J., Morey, S., Villarreal, R.H.: Associated primes of powers of edge ideals. arXiv:1103.0992v3
19. Morey, S., Villarreal, R.H.: Edge ideals: algebraic and combinatorial properties. arXiv:1012.5329v3
20. Villarreal, R.H.: Rees cones and monomial rings of matroids. Linear Algebra Appl. **428**, 2933–2940 (2008)
21. Vladoiu, M.: Equidimensional and unmixed ideals of Veronese type. Commun. Algebra **36**, 3378–3392 (2008)