Mathematical Biosciences 168 (2000) 201–210

Mathematical Biosciences

www.elsevier.com/locate/mbs

The stage-structured predator–prey model and optimal harvesting policy ☆

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Received 29 October 1999; received in revised form 20 March 2000; accepted 13 April 2000

Abstract

In this paper, we establish a mathematical model of two species with stage structure and the relation of predator–prey, to obtain the necessary and sufficient condition for the permanence of two species and the extinction of one species or two species. We also obtain the optimal harvesting policy and the threshold of the harvesting for sustainable development. © 2000 Elsevier Science Inc. All rights reserved.

MSC: 34C05; 92D25

Keywords: Permanence; Extinction; Threshold of the harvesting; Maximum sustainable yield

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1. Introduction

The competitive cooperative, and predator–prey models have been studied by many authors (see monographs [1–8] and papers [9,10]). The permanence (or strong persistence) and extinction are significant concepts of those models which also show many interesting results and complicated phenomena. However, the stage structure of species has been considered very little. In the real world, almost all animals have the stage structure of immature and mature. Recently, papers [11–13] studied the stage structure of species with or without time delays.

In this paper, we intend to consider the stage structure of two species. For the simplicity of our model, we only consider the stage structure of immature and mature of the first species (their sizes

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☆ This work was supported by the National Natural Science Foundation of China.

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PII: S0025-5564(00)00033-X

of population are written as x_1, x_2, respectively), and do not consider the stage structure of the second species (its size of population is written as x_3), and two species satisfy the following assumptions:

H1: The birth rate of the immature population is proportional to the existing mature population with a proportionality constant α (cf. the term αx_2 in (1.1a)); for the immature population, the death rate and transformation rate of mature are proportional to the existing immature population with proportionality constants r_1 and β (cf. the terms $r_1 x_1$ and βx_1 in (1.1a)); the immature population is density restriction (cf. the term ηx_1^2 in (1.1a)).

H2: The death rate of the mature population is proportional to the existing mature population with a proportionality constant r_2 (cf. the term $r_2 x_2$ in (1.1b)).

H3: The second species is a predator of the immature population of the first species (cf. the terms $\beta_1 x_1 x_3$ and $k\beta_1 x_1 x_3$ in (1.1a) and (1.1c)); the second species satisfies the logistic predator–prey model.

According to H1, H2 and H3, we can set up the following stage-structured predator–prey model.

$$\dot{x}_1 = \alpha x_2 - r_1 x_1 - \beta x_1 - \eta x_1^2 - \beta_1 x_1 x_3, \tag{1.1a}$$

$$\dot{x}_2 = \beta x_1 - r_2 x_2, \tag{1.1b}$$

$$\dot{x}_3 = x_3(-r + k\beta_1 x_1 - \eta_1 x_3), \tag{1.1c}$$

where $\alpha,\ r_1,\ r_2,\ \beta,\ \beta_1,\ \eta,\ \eta_1,\ r, k$ are positive constants, k is a digesting constant and $\dot{x}_i = \mathrm{d}x_i/\mathrm{d}t$.

Let

$$y_1 = \frac{k\beta_1}{r_2} x_1, \quad y_2 = \frac{k\beta_1}{\beta} x_2, \quad y_3 = \frac{\eta_1}{r_2} x_3, \quad \mathrm{d}t = \frac{1}{r_2}\mathrm{d}\tau.$$

Then (1.1a)–(1.1c) can be turned into

$$\begin{aligned} \dot{y}_1 &= a y_2 - b y_1 - c y_1^2 - \mathrm{d} y_1 y_3, \\ \dot{y}_2 &= y_1 - y_2, \\ \dot{y}_3 &= y_3(-e + y_1 - y_3), \end{aligned} \tag{1.2}$$

where $\dot{y}_i = \mathrm{d}y_i/\mathrm{d}\tau,\ a = \alpha\beta/r_2^2,\ b = (r_1 + \beta)/r_2,\ c = \eta/(k\beta_1),\ d = \beta_1/\eta_1$ and $e = r/r_2$.

In Section 2, we shall consider the condition of permanence and extinction of system (1.2). At first, we give the following notations and definitions

$$R_+^3 = \{y = (y_1, y_2, y_3) \in R^3:\ y_i \geqslant 0\}, \quad \mathrm{Int}\, R_+^3 = \{y = (y_1, y_2, y_3) \in R^3:\ y_i > 0\}.$$

Definition 1.1. System (1.2) is said to be *permanent* if there are positive constants m and M such that each positive solution $y(t, y_0)$ of (1.2) with initial condition $y_0 \in \mathrm{Int}\, R_+^3$ satisfies

$$m \leqslant \lim_{t\to\infty} \inf y_i(t, y_0) \leqslant \lim_{t\to\infty} \sup y_i(t, y_0) \leqslant M, \quad i = 1, 2, 3.$$

Definition 1.2. (i) The first species of system (1.2) is said to be *extinctive* if each positive solution $y(t, y_0)$ of (1.2) with initial condition $y_0 \in \mathrm{Int}\, R_+^3$ satisfies

$$\lim_{t\to\infty} y_i(t, y_0) = 0, \quad i = 1, 2.$$

(ii) The second species of system (1.2) is said to be *extinctive* if each positive solution $y(t, y_0)$ of (1.2) with initial condition $y_0 \in \operatorname{Int} R^3_+$ satisfies

$$\lim_{t\to\infty} y_3(t, y_0) = 0.$$

In Section 3, we shall consider the exploitation of the mature population. The optimal management of renewable resources, which has a direct relationship to sustainable development, has been studied extensively by many authors. Economic and biological aspects of renewable resources management have been considered by Clark [14] and other authors (cf. [15–17]). Generally speaking, the exploitation of population should be the mature population, which is more appropriate to the economic and biological views of renewable resources management. To our knowledge, there have been no results on the optimal harvesting policies of species with stage structure.

Since harvesting may lead to the extinction of one species of system (1.2), we give the definition of threshold of the harvesting.

Definition 1.3. A constant σ_0 is said to be a *threshold* of the harvesting if system (1.2) is permanent as the harvesting yield $h < \sigma_0$, and at least one species of system (1.2) will be extinctive as the harvesting $h > \sigma_0$.

In the end of this paper, we will depict the factors about the critically endangered animal (the Chinese Alligator or Yangtzi Alligator) and the conservation measures with our results.

2. Permanence and extinction of system (1.2)

The possible non-negative equilibria of system (1.2) are

$$O(0,0,0), \quad E_1((a-b)/c, (a-b)/c, 0), \quad E_2(\bar{y}_1, \bar{y}_2, \bar{y}_3),$$

where $\bar{y}_1 = \bar{y}_2 = (a-b+de)/(c+d)$ and $\bar{y}_3 = (a-b-ce)/(c+d)$.

Proposition 2.1. *System* (1.2) *has a positive equilibrium* $E_2(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ *if and only if* $a > b + ce$.

Proposition 2.2. R^3_+ *is invariant for system* (1.2).

Propositions 2.1 and 2.2 are very obvious, we omit their proof.

In order to discuss the permanence and extinction of system (1.2), at first, we analyze the local geometric properties of the non-negative equilibria of system (1.2).

The Jacobian matrix of the equilibrium $O(0,0,0)$ is

$$J(O) = \begin{pmatrix} -b & a & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -e \end{pmatrix}.$$

The characteristic equation of $J(O)$ is

$$(\lambda + e)[\lambda^2 + (1+b)\lambda + b - a] = 0.$$

Hence, $O(0,0,0)$ is a saddle with $\dim W^{u}(O) = 1$, $\dim W^{s}(O) = 2$ for $a > b$, that is the dimensions of the local unstable and stable manifold of the $O(0,0,0)$ are 1 and 2, respectively; $O(0,0,0)$ is locally asymptotically stable for $a < b$.

The Jacobian matrix of the equilibrium $E_1((a-b)/c, (a-b)/c, 0)$ is

$$J(E_1) = \begin{pmatrix} -2a+b & a & -d(a-b)/c \\ 1 & -1 & 0 \\ 0 & 0 & (a-b-ce)/c \end{pmatrix}.$$

The characteristic equation of $J(E_1)$ is

$$[\lambda - (a-b-ce)/c][\lambda^2 + (2a-b+1)\lambda + a - b] = 0.$$

Hence, $E_1((a-b)/c, (a-b)/c, 0)$ is a saddle with $\dim W^{u}(E_1) = 1$, $\dim W^{s}(E_1) = 2$ for $a > b + ce$; E_1 is locally asymptotically stable for $b < a < b + ce$.

The Jacobian matrix of the equilibrium $E_2(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is

$$J(E_2) = \begin{pmatrix} -a - c\bar{y}_1 & a & -d\bar{y}_1 \\ 1 & -1 & 0 \\ \bar{y}_3 & 0 & -\bar{y}_3 \end{pmatrix}.$$

The characteristic equation of $J(E_2)$ is

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{2.1}$$

where

$$\begin{aligned} A &= a + 1 + c\bar{y}_1 + \bar{y}_3 > 0, \\ B &= c\bar{y}_1 + \bar{y}_3(a + 1 + c\bar{y}_1 + d\bar{y}_1), \\ C &= (c+d)\bar{y}_1\bar{y}_3 > 0. \end{aligned}$$

Obviously, $AB - C > 0$. According to Routh–Hurwitz theorem [2], E_2 is locally asymptotically stable for $a > b + ce$.

In the following lemma, we shall discuss the global properties of the non-negative equilibria.

Lemma 2.1. (i) *If $a > b + ce$, then the positive equilibrium E_2 of system* (1.2) *is globally asymptotically stable.*

(ii) *If $b < a \leqslant b + ce$, then the non-negative equilibrium $E_1(\hat{y}_1, \hat{y}_2, 0)$ (where $\hat{y}_1 = \hat{y}_2 = (a-b)/c$) of system* (1.2) *is globally asymptotically stable.*

(iii) *If $a \leqslant b$, then the origin $O(0,0,0)$ of system* (1.2) *is globally asymptotically stable.*

Proof. (i) We make use of the general Liapunov function

$$V_1(y(t)) = \sum_{i=1}^{3} \alpha_i (y_i - \bar{y}_i - \bar{y}_i \ln(y_i/\bar{y}_i)),$$

where $\alpha_i,\ i = 1, 2, 3$ are positive constants.

Calculating the derivative along each solution of system (1.2), we have

$$\begin{aligned} \mathrm{d}V_1/\mathrm{d}t = & \sum_{i=1}^{3} \alpha_i [(y_i - \bar{y}_i)/y_i]\, \mathrm{d}y_i/\mathrm{d}t \\ = & -c\alpha_1 (y_1 - \bar{y}_1)^2 - \alpha_3 (y_3 - \bar{y}_3)^2 + (\alpha_3 - d\alpha_1)(y_1 - \bar{y}_1)(y_3 - \bar{y}_3) \\ & + a\alpha_1 (y_1 - \bar{y}_1)[(\bar{y}_1 y_2 - \bar{y}_2 y_1)/(y_1 \bar{y}_1)] + \alpha_2 (y_2 - \bar{y}_2)[(\bar{y}_2 y_1 - \bar{y}_1 y_2)/(y_2 \bar{y}_2)]. \end{aligned}$$

Let $\alpha_3 = d\alpha_1,\ \alpha_2 = a\alpha_1$.

$$\begin{aligned} \mathrm{d}V_1/\mathrm{d}t = & -c\alpha_1 (y_1 - \bar{y}_1)^2 - \alpha_3 (y_3 - \bar{y}_3)^2 - [\alpha_2 y_2/(y_1 \bar{y}_1)](y_1 - \bar{y}_1)^2 \\ & - [\alpha_2 y_1/(y_2 \bar{y}_1)](y_2 - \bar{y}_2)^2 + (2\alpha_2/\bar{y}_1)(y_1 - \bar{y}_1)(y_2 - \bar{y}_2) \\ = & -c\alpha_1 (y_1 - \bar{y}_1)^2 - \alpha_3 (y_3 - \bar{y}_3)^2 \\ & - (\alpha_2/\bar{y}_1)\left[\sqrt{y_2/y_1}(y_1 - \bar{y}_1) - \sqrt{y_1/y_2}(y_2 - \bar{y}_2)\right]^2 \leqslant 0. \end{aligned}$$

Set $D_1 = \{y \in \mathrm{Int}\, R_+^3 :\ \mathrm{d}V_1/\mathrm{d}t = 0\} = \{y \in \mathrm{Int}\, R_+^3 :\ y_1 = \bar{y}_1,\ y_3 = \bar{y}_3,\ y_1 = y_2\} = E_2$. According to LaSalle theorem [18], E_2 is globally asymptotically stable for $a > b + ce$.

(ii) We construct the following Liapunov function

$$V_2(y(t)) = \sum_{i=1}^{2} \alpha_i (y_i - \hat{y}_i - \hat{y}_i \ln(y_i/\hat{y}_i)) + \alpha_3 y_3.$$

Calculating the derivative of $V_2(y(t))$ along each solution of system (1.2), we have

$$\begin{aligned} \mathrm{d}V_2/\mathrm{d}t = & \sum_{i=1}^{2} \alpha_i [(y_i - \hat{y}_i)/y_i]\, \mathrm{d}y_i/\mathrm{d}t + \alpha_3\, \mathrm{d}y_3/\mathrm{d}t \\ = & -c\alpha_1 (y_1 - \hat{y}_1)^2 - \alpha_3 (y_3 - \hat{y}_3)^2 + (\alpha_3 - d\alpha_1)(y_1 - \hat{y}_1)(y_3 - \hat{y}_3) \\ & - (e - \hat{y}_1) y_3 + a\alpha_1 (y_1 - \hat{y}_1 [(\hat{y}_1 y_2 - \hat{y}_2 y_1)/(y_1 \hat{y}_1)] \\ & + \alpha_2 (y_2 - \hat{y}_2)[(\hat{y}_2 y_1 - \hat{y}_1 y_2)/(y_2 \hat{y}_2)]. \end{aligned}$$

Let $\alpha_3 = d\alpha_1$, $\alpha_2 = a\alpha_1$.

$$\begin{aligned} \mathrm{d}V_2/\mathrm{d}t = & -c\alpha_1 (y_1 - \bar{y}_1)^2 - \alpha_3 (y_3 - \hat{y}_3)^2 - (e - \hat{y}_1) y_3 \\ & - (\alpha_2/\bar{y}_1)\left[\sqrt{y_2/y_1}(y_1 - \bar{y}_1) - \sqrt{y_1/y_2}(y_2 - \bar{y}_2)\right]^2 \leqslant 0, \end{aligned}$$

where $e - \hat{y}_1 = (ce + b - a)/c \geqslant 0$. Hence, $D_2 = \{y \in R_+^3 :\ \mathrm{d}V_2/\mathrm{d}t = 0\} = E_1$. According to LaSalle theorem, E_1 is globally asymptotically stable for $b < a \leqslant b + ce$.

(iii) We construct the following Liapunov function

$$V_3 = \sum_{i=1}^{3} \alpha_i y_i.$$

Calculating th derivative of $V_2(y(t))$ along each solution of system (1.2), we have

$$\mathrm{d}V_3/\mathrm{d}t = (a\alpha_1 - \alpha_2)y_2(\alpha_2 - b\alpha_1)y_1 - c\alpha_1 y_1^2 - e\alpha_3 y_3 - \alpha_3 y_3 - \alpha_3 y_3^3 + (\alpha_3 - d\alpha_1)y_1 y_3.$$

Let $\alpha_3 = d\alpha_1, \ \alpha_2 = a\alpha_1$.

$$\mathrm{d}V_3/\mathrm{d}t = -(b-a)\alpha_1 y_1 - e\alpha_3 y_3 - c\alpha_1 y_1^2 - \alpha_3 y_3^2 \leqslant 0.$$

Hence,

$$D_3 = \{y \in R_+^3 : \ \mathrm{d}V_3/\mathrm{d}t = 0\} = \{y \in R_+^3 : \ y_1 = y_3 = 0, \ y_2 \geqslant 0\}.$$

If D_3 is an invariant set of system (1.2), by the first equation of system (1.2), we have $y_2 = 0$, $D_3 = O(0,0,0)$. Hence, $O(0,0,0)$ is globally asymptotically stable for $a \leqslant b$.

Remark 2.1. In Lemma 2.1:
(i) Condition $a > b + ce \iff \alpha > [1 + r_1/\beta + (\eta r)/(k\beta_1\beta)]r_2$.
(ii) Condition $b < a \leqslant b + ce \iff (1 + r_1/\beta)r_2 < \alpha \leqslant [1 + r_1/\beta + (\eta r)/(k\beta_1\beta)]r_2$.
(iii) Condition $a \leqslant b \iff \alpha \leqslant (1 + r_1/\beta)r_2$.

Therefore, the birth rate of the mature determines the persistence and extinction of two species. We have the following theorem.

Theorem 2.1. (i) *Two species of system* (1.1a)–(1.1c) *are permanent if and only if the reproduction rate of the first mature species satisfies*

$$\alpha > [1 + r_1/\beta + (\eta r)/(k\beta_1\beta)]r_2.$$

(ii) *The second species* (*predator*) *of system* (1.1a)–(1.1c) *is extinctive and the first species is not extinctive if and only if the reproduction rate of the first mature species satisfies*

$$(1 + r_1/\beta)r_2 < \alpha \leqslant [1 + r_1/\beta + (\eta r)/(k\beta_1\beta)]r_2.$$

(iii) *The two species of system* (1.1a)–(1.1c) *are extinctive if and only if the reproduction rate of the first mature species satisfies*

$$\alpha \leqslant (1 + r_1/\beta)r_2.$$

Proof. By Definitions 1.1 and 1.2, and Lemma 2.1, we can easily prove Theorem 2.1.

Remark 2.2. Theorem 2.1 depicts a very intuitive biological phenomenon. We can regard α/r_2 as a relative birth rate of the first mature species, $\beta/(\beta + r_1)$ as a relative transformation rate of the first immature species. Conditions (i)–(iii) of Theorem 2.1 are, respectively, equivalent to

1. $\dfrac{\alpha}{r_2}\dfrac{\beta}{\beta + r_1} > 1 + \dfrac{\eta r}{k\beta_1(r_1 + \beta)},$

2. $1 < \frac{\alpha}{r_2}\frac{\beta}{\beta + r_1} \leqslant \frac{\eta r}{k\beta_1(r_1+\beta)}$,

3. $\frac{\alpha}{r_2}\frac{\beta}{\beta + r_1} \leqslant 1.$

Hence, the product of relative birth rate and relative transformation rate is very crucial to the permanence of the stage-structured predator–prey species.

3. The optimal harvesting policy of system (1.2)

In order to study the optimal harvesting yield of system (1.2), usually, we need only consider the following system harvested at a consistent per-captia rate qE

$$\begin{aligned} \dot{y}_1 &= ay_2 - by_1 - cy_1^2 - dy_1y_3, \\ \dot{y}_2 &= y_1 - y_2 - h, \\ \dot{y}_3 &= y_3(-e + y_1 - y_3), \end{aligned} \tag{3.1}$$

where $h = qEy_2$ is the harvesting yield, q is the catachability coefficient and E is the harvesting effort.

The possible non-negative equilibria of system (3.1) are

$$O(0,0,0), \quad P_1(\hat{y}_1, \hat{y}_2, 0), \quad P_2(\bar{y}_1, \bar{y}_2, \bar{y}_3),$$

where

$$\begin{aligned} \hat{y}_1 &= [a - b(1+qE)]/[c(1+qE)], \\ \hat{y}_2 &= [a - b(1+qE)]/[c(1+qE)^2], \\ \bar{y}_3 &= [a - (b+ce)(1+qE)]/[(c+d)(1+qE)], \\ \bar{y}_1 &= \bar{y}_3 + e, \\ \bar{y}_2 &= (\bar{y}_3 + e)/(1+qE). \end{aligned}$$

Similar to the proof of Lemma 2.1, we have the following lemma.

Lemma 3.1. (i) *System* (3.1) *is permanent if and only if*

$$a > (b+ce)(1+qE). \tag{3.2}$$

(ii) *The second species* (*predator*) *of system* (3.1) *is extinctive and the first species is not extinctive if and only if*

$$b(1+qE) < a \leqslant (b+ce)(1+qE).$$

(iii) *The two species of system* (3.1) *are extinctive if and only if*

$$a \leqslant b(1+qE).$$

In the following we shall consider the maximum sustainable yield of system (3.1). Theorem 3.1 is the optimal harvesting policy of system (3.1).

Theorem 3.1. (i) *If* $b + ce < a \leqslant de - b$ *or* $b + ce < a \leqslant b + 2ce + de$, *the maximum sustainable yield in system* (1.2) *is*

$$h_{\mathrm{MSY}} = h(E^*) = e(a - b - ce)/a,$$

where $E^* = (a/(b + ce) - 1)/q$, *which is a threshold of the harvesting of system* (1.2)

(ii) *If* $a > b + 2ce + de$, *then, the maximum sustainable yield in system* (1.2) *is*

$$h_{\mathrm{MSY}} = h(\bar{E}) = (a - b + de)^2/[4a(c + d)],$$

where $\bar{E} = (a + de - b)/[q(a + b - de)]$.

Proof. Let $y_2 = \bar{y}_2$, the harvesting of system (3.1) is

$$h(E) = qE\bar{y}_2 = qE[a + (de - b)(1 + qE)]/[(c + d)(1 + qE)^2], \tag{3.3}$$

where $E \in [0, E^*]$, $(E^* = (a/(b + ce) - 1)/q)$. Calculating the derivative of $h(E)$ for E, we have

$$\frac{\mathrm{d}h}{\mathrm{d}E} = \frac{q[a + de - b - (a + b - de)qE]}{(c + d)(1 + qE)^3}. \tag{3.4}$$

(i) If $a + b - de \leqslant 0$, then $\mathrm{d}h/\mathrm{d}E > 0$ for $\forall E \in [0, E^*]$. The maximum sustainable yield for $h(E)$ is

$$h_{\mathrm{MSY}} = h(E^*) = e(a - b - ce)/a.$$

(ii) If $a + b - de > 0$, then the solution of $\mathrm{d}h/\mathrm{d}E = 0$ is $\bar{E} = (a + de - b)/[q(a + b - de)]$. Comparing the two numbers E^* and $\bar{E}$, we have the following results:

(a) $\bar{E} \in [0, E^*]$ as $a > b + 2ce + de$. The corresponding maximum sustainable yield for $h(E)$ is

$$h_{\mathrm{MSY}} = h(\bar{E}) = (a - b + de)^2/[4a(c + d)].$$

(b) $\bar{E} > E^*$ as $b + ce < a \leqslant b + 2ce + de$. The corresponding maximum sustainable yield for $h(E)$ is

$$h_{\mathrm{MSY}} = h(E^*) = e(a - b - ce)/a.$$

Summarizing the above discussion. If $b + ce < a \leqslant de - b$ or $b + ce < a \leqslant b + 2ce + de$, then the maximum sustainable yield for $h(E)$ is $h_{\mathrm{MSY}} = h(E^*) = e(a - b - ce)/a$. If the harvesting $h_{\mathrm{MSY}} = h(E^*) = e(a - b - ce)/a$, then the non-negative equilibria P_2 and P_3 of system (3.1) coincide, the non-negative equilibrium P_2 is globally asymptotically stable by Lemma 3.1. Hence, the second species (predator) will be extinctive eventually. By Definition 1.3, $h_{\mathrm{MSY}} = h(E^*)$ is a threshold of the harvesting.

If $a > b + 2ce + de$, then the maximum sustainable yield for $h(E)$ is $h_{\mathrm{MSY}} = h(\bar{E}) = (a - b + de)^2/[4a(c + d)]$. If the harvesting $h_{\mathrm{MSY}} = h(\bar{E}) = (a - b + de)^2/[4a(c + d)]$, then the unique positive equilibrium P_3 of system (3.1) is globally asymptotically stable by Lemma 3.1. Hence, $h_{\mathrm{MSY}} = h(\bar{E}) = (a - b + de)^2/[4a(c + d)]$ is the optimal harvesting for system (3.1).

Remark 3.1. If $a > b + 2ce + de$, then $h(\bar{E}) > h(E^*)$. Therefore, the maximum sustainable yield depends on the reproduction rate of the mature population.

4. Discussion

Theorems 2.1 and 3.1 should provide a useful insight into the conservation of critically endangered animals. As an example, we depict the case of the Chinese Alligator or Yangtzi Alligator. This animal is only endemic to the north of China; the range is limited only to the waters of a hilly region below 200 m altitude to the north of the southern Anhui maintains in China (see Ref. [19]). The Chinese Alligator was listed in Appendix I of the Convention on International Trade in Endangered Species of Wild Fauna and Flora by the International Union for the Conservation of Nature and Natural Resources. The Chinese Alligator can be regarded as a stage-structured species since the mature is more than 10 years old, and can also be regarded as a predator because almost all acquatic animals are the chief food of the Chinese Alligator. The primary endangering factors can be summarized as follows:

1. The habitat environments of the Chinese Alligator have been destroyed since the human population has greatly expanded. Those environments enable the Chinese Alligator not only to procure food, build holds and mate in water, but also to build nests and reproduce on land. Thus, the factor makes the product of the relative birth rate and the relative transformation rate to become very small.
2. The excessive, indiscriminate catching or slaughtering makes the catching quantity larger than the threshold of the harvesting.
3. The application of large chemical fertilizers and insecticides decreases the number of the natural food of the Chinese Alligator.

In order to assure that the product of the relative birth rate and the relative transformation rate is larger than one about the Chinese Alligator, we suggest to take the following conservation measures:

1. Techniques and theories concerning rearing and cultivating the mature alligators, captive propagation, artificial hatching of eggs, rearing of the immature alligators etc. should be resolved.
2. To establish some new conservation regions in which the alligators had distributed formerly and where the human population, at present, is less dense. Economic activity will be forbidden in those regions. The new conservation regions should be protected by the local government such that the alligators can freely reproduce their offspring.

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