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THE STANDARD DOUBLE BUBBLE IS THE UNIQUE STABLE DOUBLE BUBBLE IN $\mathbf{R}^2$

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Abstract. We prove that the only equilibrium double bubble in $\mathbf{R}^2$ which is stable for fixed areas is the standard double bubble. This uniqueness result also holds for small stable double bubbles in surfaces, where it is new even for perimeter-minimizing double bubbles.

1. Introduction

In 1993 Foisy et al. [F] proved that the standard double bubble of Figure 1ab, consisting of three constant-curvature arcs meeting at 120 degrees, is the unique least-perimeter way to enclose and separate two planar regions of prescribed areas. The regions are not assumed to be connected. This proof left open the question of whether there might be other stable double bubbles (see Sullivan [SM, Prob. 2]). Our Theorem 3.2 proves there are no other stable double bubbles, except of course for two single bubbles.

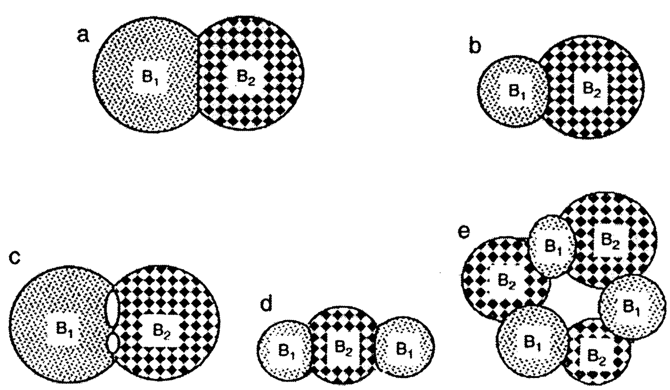

Figure 1. Foisy et al. [F] proved that the standard double bubble (a, b) is the least-perimeter way to enclose and separate two regions of prescribed areas in the plane, rather than any other crazy alternative.

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There *are* other *unstable* equilibrium double bubbles, like that of Figure 1d; shrinking one component of R_1 and enlarging the other reduces perimeter to second order.

Further results on bubbles with more than two regions will appear in Wichiramala's thesis [W].

1.1. **The proof.** The proof uses stability to show that the region R_1 of larger pressure has at most two components, which reduces the combinatorial possibilities for a connected bubble to the thirteen of Figure 2. Possibilities (6)–(13) are easily shown to be impossible, even for unstable equilibria. Then possibilities (2)–(5) are fairly easily shown to be unstable. The standard double bubble (1) is the only remaining possibility.

To prove that the region R_1 of larger pressure has at most two components, note that its components are convex, and hence have negative second variation under shrinking or expanding. If R_1 had three components, some nontrivial combination of shrinking and expanding would preserve both areas and yield a contradictory negative second variation.

1.2. **Bubbles in surfaces.** In Theorem 3.3 the uniqueness result Theorem 3.2 and its proof extend to small stable double bubbles in surfaces of bounded curvature, more than settling the conjecture of Cotton and Freeman [CF, Conj. 1.1] that a small perimeter-minimizing double bubble must be standard.

1.3. **Higher dimensions.** In $\mathbf{R}^3$ and above, it remains an open question whether every stable double bubble is standard. Indeed, in $\mathbf{R}^5$ and above, it remains an open question whether every *perimeter-minimizing* double bubble is standard, despite the recent proofs in $\mathbf{R}^3$ [HMRR] and $\mathbf{R}^4$ [RHLS]. For remarks on small perimeter-minimizing double bubbles in manifolds, see [M3, Sect. 1.2].

2. Soap bubbles and second variation

A planar *soap bubble* consists of finitely many constant-curvature arcs meeting in threes at 120 degrees, enclosing (not necessarily connected) regions R_i of prescribed areas; arcs separating the same two regions have the same curvature, the difference of the pressures of the two regions. These are just the equilibrium conditions. Usually the pressure of the exterior R_0 is taken to be 0.

Even the most technically general definition of soap bubbles as $(\mathbf{M}, \varepsilon, \delta)$-minimal sets ([M1, 11.3], [A]) reduces to our definition ([M2], [T]). The general existence of least-perimeter bubbles of prescribed volumes in $\mathbf{R}^n$ is well established ([M1, Chapt. 13], [M4], [A, Thm. VI.2]).

The following Second Variation Formula was provided in general dimension in the proof of the double bubble conjecture by Hutchings et al.

2.1. **Second Variation Formula** ([HMRR, Prop. 3.3, (3.11), Lemma 3.2])**.** Consider an (equilibrium) planar soap bubble and smooth variation vectorfield $\mathbf{u}$ which preserves areas to first order. Any such $\mathbf{u}$ is the initial velocity of many smooth flows which preserve areas. Let u_{ij} denote the associated normal variation of the interface from R_j into R_i, and κ_{ij} its curvature, nonnegative where R_i is convex.

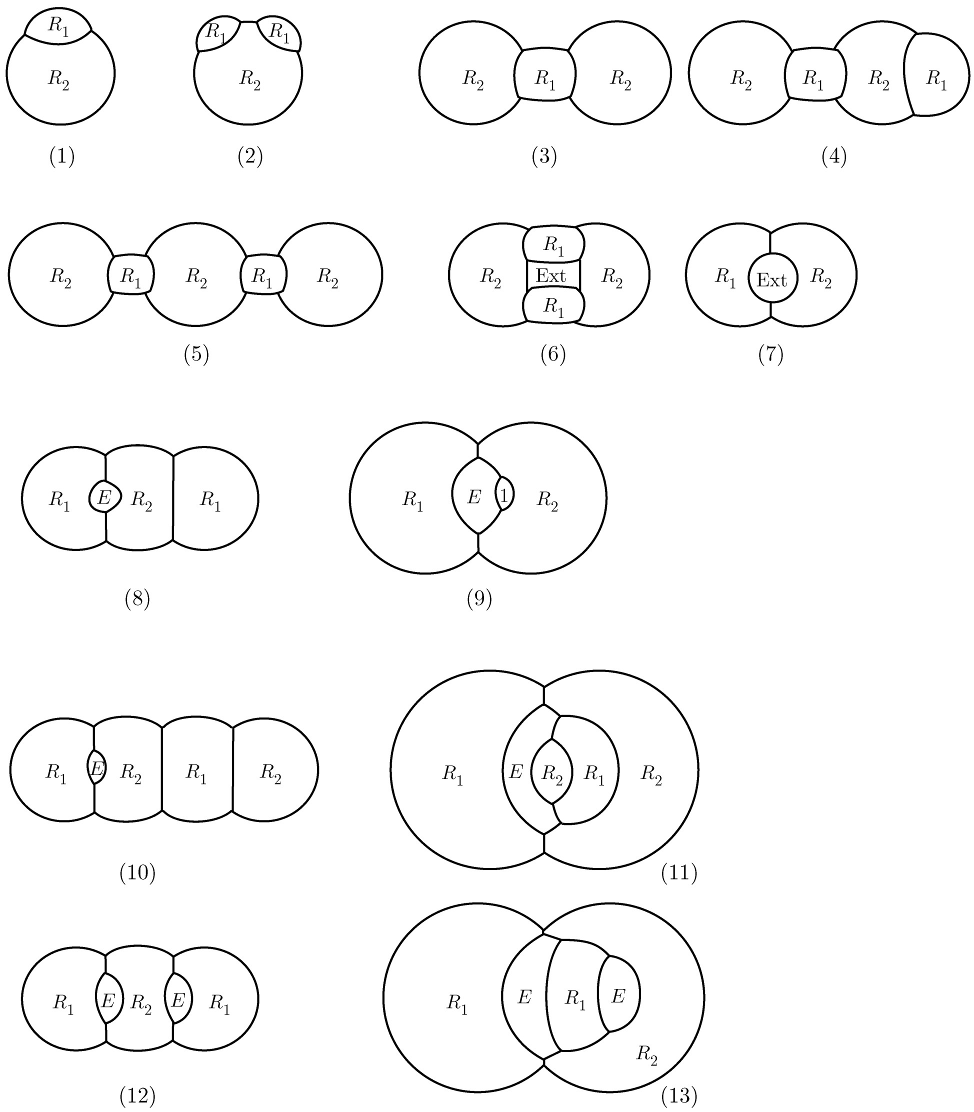

FIGURE 2. The thirteen initial combinatorial possibilities for a stable connected double bubble

For any such flow, the initial second variation of perimeter equals the sum over $0 \le i < j$ of

$$\int (u_{ij}'^2 - \kappa_{ij}^2 u_{ij}^2) - \sum_p (u_{ij}^2 q_{ij})|_p$$

where e.g. at a point p where regions R_1, R_2, and R_3 meet

$$q_{12}(p) = \left. \frac{\kappa_{13} + \kappa_{23}}{\sqrt{3}} \right|_p .$$

Remark. By approximation, nonnegative second variation for smooth variations is equivalent to nonnegative second variation for variations which are piecewise C^1.

3. Stable double bubbles must be standard

Theorem 3.2 provides our main result that the standard double bubble is essentially the unique stable double bubble. Proposition 3.1, first formulated in [W], where it is used in the study of minimizing triple bubbles, provides an estimate on the number of components of a stable bubble cluster. Here we give a proof using the Second Variation Formula 2.1, as for certain double bubbles in $\mathbf{R}^3$ by [HMRR, Prop. 6.5].

3.1. Proposition ([W]). *Consider a planar bubble cluster B of m regions, of nonnegative second variation for fixed areas. Then B has at most m nonpolygonal nonadjacent convex components.*

Proof. Let $\mathbf{v}_k$ be a variation vectorfield which just shrinks the kth nonpolygonal convex component at unit rate; i.e. on that component, $u_{1j} = 1$ and elsewhere u_{ij} vanishes. If there are more than m nonpolygonal convex components, then some nontrivial linear combination $\mathbf{v}$ of the $\mathbf{v}_k$ preserves the m areas to first order. By the convexity hypothesis and the Second Variation Formula 2.1, the second variation is negative, a contradiction.

3.2. Theorem. *For prescribed areas in the plane, the standard double bubble is the unique stable double bubble, indeed, the unique* (*equilibrium*) *double bubble with nonnegative second variation, except of course for two single bubbles* (*one possibly inside the other*).

Proof. Let B be a double bubble of nonnegative second variation for fixed areas. First we consider the case of B connected. Let R_1 denote the region of larger (or equal) pressure. Consideration of the outer boundary of B shows that R_1 has greater pressure than the exterior (much greater for small bubbles). Since every component of R_1 must interface the exterior, every component is nonpolygonal as well as convex. Therefore by Proposition 3.1, R_1 has at most two components.

Let C be a component of R_1. C has an even number of edges because R_2 and the exterior alternate around C. Since C is convex and nonpolygonal, with interior angles of 120 degrees, it has fewer than six edges. Therefore C is a curvilinear digon or quadrilateral.

We claim that there are just twelve combinatorial possibilities for B, as pictured in Figure 2. If R_1 consists of one digon, two digons, or one quad, then B must be (1), (2), or (3) or (7). If R_1 consists of one digon and one quad, it comes from inserting a digon in (3) or (7) and hence B must be (4), or (8) or (9). Finally we consider the case that R_1 consists of two quads, sharing four edges with R_2, which therefore has eight edges and at most four components, actually at most three components because B is connected. If R_2 has one component, B must be (12) or (13). So we may assume that R_2 has at least two components. If R_2 consists of two quads, B is (6). Otherwise one component of R_2 is a digon, and B comes from inserting that digon into an R_1 digon of (4), (8), or (9), yielding (5), (10), or (11).

We claim that (6)–(11) cannot occur even as equilibrium bubbles. Both regions and the exterior have bounded components with at most six edges as well as interior angles of 120 degrees. Such a bounded component of a region of least pressure must be convex and hence a hexagon (with straight edges), so that all three regions have the same pressure. But we already have that R_1 has greater pressure than the exterior, a contradiction.

Likewise we claim that (12) cannot occur. The bounded components of the exterior, with less pressure than R_1, must have more pressure than R_2, and the interface with R_2 must have more than 240 degrees. Since all the edges of R_2 are concave and the interior angles are 120 degrees, this yields a contradiction.

Likewise (13) cannot occur. Consideration of the exterior region digon shows that the exterior has more pressure than R_2, while consideration of the outer boundary shows that R_2 has more pressure than the exterior, a contradiction.

Now we will show that (2)–(5) cannot be stable. Consider possibility (2). Of course the digons are identical, determined by their curvatures and 120-degree angles. Shrinking one (increasing curvature dP/dA) and enlarging the other (reducing curvature) at the same rate reduces perimeter to second order while maintaining areas to first order, a contradiction. More explicitly, choose $u_{1j} = 1$ on one component and $u_{1j} = -1$ on the other. (These come from a piecewise C^1 variation vectorfield $\mathbf{v}$.) Then by the Second Variation Formula 2.1, the second variation is less than the sum over four points of

$$-\frac{\kappa_{12} + \kappa_{02} + \kappa_{10} + \kappa_{20}}{\sqrt{3}} = -\frac{\kappa_{12} + \kappa_{10}}{\sqrt{3}} \leq -\frac{\kappa_{10}}{\sqrt{3}} < 0.$$

(This is a trivial case of an instability argument in the proof of the double bubble conjecture in $\mathbf{R}^3$ [HMRR, Prop. 6.5].)

In possibilities (3) and (5), the exterior boundary of R_2 contains two long arcs greater than 180 degrees (actually at least 240 degrees). Shrinking one through circular arcs (increasing curvature) and enlarging the other (decreasing curvature), while maintaining area, decreases perimeter to second order, a contradiction.

Consider possibility (4). Some linear combination of shrinking the two, convex components of R_1 and the long arc of R_2 preserves area, with contradictory negative second variation as before.

Only possibility (1), the standard double bubble, remains for a connected bubble B.

For a disconnected bubble, each connected component is therefore a standard double bubble or single bubble, of the same curvatures, possibly nested. There can be no bounded components of the exterior, since pressure continues to drop as you move outward. If R_1 occurs in more than one component of the bubble, shrinking it in one and enlarging it in the other provides contradictory negative second variation as before. If R_2 occurs in more than one component of the bubble, shrinking a long exterior arc through circular arcs in one and enlarging a long exterior arc in the other provides contradictory negative second variation as before. Two possibilities remain: two single bubbles (possibly nested) and the standard double bubble, as the theorem asserts.

Two single bubbles are of course stable. The standard double bubble, with less perimeter, must be the minimizer.

Theorem 3.2 and its proof generalize easily to other ambient surfaces.

3.3. Theorem. *Let M be a smooth Riemannian surface with bounded Gauss curvature G. For small prescribed areas, a double bubble of nonnegative second variation is standard, i.e., consists of three constant-curvature arcs meeting at* 120 *degrees.*

Remark. For perimeter-minimizing double bubbles, Theorem 3.3 appears as a conjecture by Cotton and Freeman [CF, Conj. 1.1]. For small perimeter-minimizing double bubbles in compact surfaces, there is an alternative limit argument proof [M3], which requires proving bounds on curvature and numbers of components.

Proof. The proof requires only minor modifications of the Euclidean case of Theorem 3.2. In the small, M is nearly Euclidean, and most of the estimates have plenty of leeway. The second variation formula has an additional $-\int u_{ij}^2 G$ term, which is small for small bubbles. A modified Proposition 3.1 says that a region has at most m convex components on which some edge has curvature at least 1.

In eliminating possibility (2) of Figure 2, the components of R_1 need not be quite identical, so that shrinking one and enlarging the other at rates to preserve the area of R_1 may not preserve the area of R_2 to first order. A compensating variation of the exterior boundary of R_2, one arc of which is at least 60 degrees, could have positive second variation. To compare these effects for small bubbles, renormalize by scaling M to give the exterior boundary arcs of R_2 unit curvature and M small Gauss curvature. Now the components of R_1 are nearly identical, the compensating variation of the larger exterior arc of R_2 is small, its effect on the second variation is small, and the net second variation remains negative, the desired contradiction.

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