

The Standard Model on Non-Commutative Space-Time: Electroweak Currents and the Higgs Sector

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Abstract

In this article we review the electroweak charged and neutral currents in the Non-Commutative Standard Model (NCSM) and compute the Higgs and Yukawa parts of the NCSM action. With the aim to make the NCSM accessible to phenomenological considerations, all relevant expressions are given in terms of physical fields and Feynman rules are provided.

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1 Introduction

The approach to non-commutative field theory based on star products and Seiberg-Witten (SW) maps allows the generalization of the Standard Model (SM) of particle physics to the case of non-commutative space-time, keeping the original gauge group and particle content [1–8]. It provides a systematic way to compute Lorentz violating operators that could be a signature of a (hypothetical) non-commutative space-time structure [9–20].

In this article we carefully discuss the electroweak charged and neutral currents in the Non-Commutative Standard Model (NCSM) [6] and compute the Higgs and Yukawa parts of the NCSM action. Among the features which are novel in comparison with the SM is the appearance of additional gauge boson interaction terms and of interaction terms without Higgs boson which include additional mass dependent contributions. All relevant expressions are given in terms of physical fields and selected Feynman rules are provided with the aim to make the model more accessible to phenomenological considerations.

In the star product formulation of non-commutative field theory, one retains the ordinary functions (and fields) on Minkowski space, but introduces a new non-commutative product which encodes the non-commutative structure of space-time. For a constant antisymmetric matrix $\theta^{\mu\nu}$, the relevant product is the Moyal-Weyl star product

$$f * g = \sum_{n=0}^{\infty} \frac{\theta^{\mu_1\nu_1} \dots \theta^{\mu_n\nu_n}}{(-2i)^n n!} (\partial_{\mu_1} \dots \partial_{\mu_n} f) (\partial_{\nu_1} \dots \partial_{\nu_n} g) . \quad (1)$$

For coordinates: $x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$. More generally, a star product has the form

$$(f * g)(x) = f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}(x)\partial_\mu f(x)\partial_\nu g(x) + \mathcal{O}(\theta^2), \quad (2)$$

where the Poisson tensor $\theta^{\mu\nu}(x)$ may be x -dependent and satisfies the Jacobi identity. Higher-order terms in the star product are chosen in such a way that the overall star product is associative. In general, they involve derivatives of θ . For a discussion of the Seiberg-Witten approach to non-commutative field theory in the case of the space-time dependent $\theta^{\mu\nu}(x)$, see, for example [21–25].

Carefully studying non-commutative gauge transformations one finds that in general, non-commutative gauge fields are valued in the enveloping algebra of the gauge group [3, 4]. (Only for $U(N)$ in the fundamental representation it is possible to stick to Lie-algebra valued gauge fields.) A priori this would

imply an infinite number of degrees of freedom if all coefficient functions of the monomials that form an infinite basis of the enveloping algebra were independent. That is the place where the second important ingredient of gauge theory on non-commutative spaces comes into play: Seiberg-Witten maps [2,3] which relate non-commutative gauge fields and ordinary fields in commutative theory via a power series expansion in θ . Since higher-order terms are now expressed in terms of the zeroth-order fields, we do have the same number of degrees of freedom as in the commutative case. Non-commutative fermion and gauge fields read

$$\widehat{\psi} = \widehat{\psi}[V] = \psi - \frac{1}{2} \theta^{\alpha\beta} V_\alpha \partial_\beta \psi + \frac{i}{8} \theta^{\alpha\beta} [V_\alpha, V_\beta] \psi + \mathcal{O}(\theta^2), \quad (3)$$

$$\widehat{V}_\mu = \widehat{V}_\mu[V] = V_\mu + \frac{1}{4} \theta^{\alpha\beta} \{ \partial_\alpha V_\mu + F_{\alpha\mu}, V_\beta \} + \mathcal{O}(\theta^2), \quad (4)$$

where ψ and V_μ are ordinary fermion and gauge fields, respectively. Non-commutative fields throughout the paper are denoted by a hat. The Seiberg-Witten maps are not unique. The free parameters are chosen such that the non-commutative gauge fields are hermitian and the action is real. Still, there is some remaining freedom including the freedom of the classical field redefinition and the non-commutative gauge transformation. The noncommutative actions considered here are covariant under (global) Poincare transformations *provided that the Poisson tensor θ is transformed as well*. With respect to a fixed θ -background, however, the classical Lorentz symmetry is broken. What remains is a twisted Poincare symmetry [7,26], which can in principle be extended to SW expansions.

In [6], it was shown how to construct a model with non-commutative gauge invariance, which stays as close as possible to the regular Standard Model. The distinguishing feature of this *minimal* NCSM (mNCSM) is the absence of new triple neutral gauge boson interactions in the gauge sector. However, as shown here, triple Z coupling does appear from the Higgs action. Triple gauge boson interactions do quite naturally arise in the gauge sector of extended versions [6,9,10,15] of the NCSM and have been discussed in [9,10]. They also occur in an alternative approach to the non-commutative Standard Model given in [27,28]. Another interesting novel feature of NCSM, introduced by Seiberg-Witten (SW) maps, is the appearance of mixing of the strong and electroweak interactions already at the tree level [6,9,29].

We consider the θ -expanded NCSM up to first order in the non-commutativity parameter with an emphasis made on the electroweak interactions only. In Section 2, we give an introductory overview of the NCSM. In Section 3, we discuss different choices for representations of the gauge group which then yield minimal and non-minimal versions of the NCSM. In Section 4,

we carefully discuss electroweak charged and neutral currents of the NCSM. Explicit expressions for the NCSM corrections in the Higgs and Yukawa sectors are worked out in Section 5. These expressions can be used directly for further studies. The Feynman rules for the selected three- and four-field electroweak vertices are given in Section 6.

2 Non-commutative Standard Model

The action of the NCSM formally resembles the action of the classical SM: the usual point-wise products in the Lagrangian are replaced by the Moyal-Weyl product and (matter and gauge) fields are replaced by the appropriate Seiberg-Witten expansions. In the limit of vanishing non-commutativity one recovers the usual commutative theory. This limit is assumed to be continuous. If the transition is not continuous (compare, e.g. [16]), perturbative aspects of the theory under consideration can still be addressed. Problems with unitarity may occur in the non-expanded theory with non-trivial time-space commutation relations. These problems can be overcome by a careful analysis of perturbation theory in a Hamiltonian approach, cf. [20, 30] for scalar field theory. Other problems in non-commutative theories that are encountered already at the classical level are charge quantization in non-commutative QED, the definition of the tensor product of gauge fields, gauge invariance of the Yukawa couplings and ambiguities in the kinetic part of the action for gauge fields. As demonstrated in [6], all these problems can be overcome and do not affect the NCSM presented here. The action of the NCSM is

$$S_{\text{NCSM}} = S_{\text{fermions}} + S_{\text{gauge}} + S_{\text{Higgs}} + S_{\text{Yukawa}} , \quad (5)$$

where

$$S_{\text{fermions}} = \int d^4x \sum_{i=1}^3 \left(\overline{\widehat{L}}_L^{(i)} * (i\widehat{\mathcal{D}} \widehat{L}_L^{(i)}) + \overline{\widehat{Q}}_L^{(i)} * (i\widehat{\mathcal{D}} \widehat{Q}_L^{(i)}) \right. \\ \left. + \overline{\widehat{e}}_R^{(i)} * (i\widehat{\mathcal{D}} \widehat{e}_R^{(i)}) + \overline{\widehat{u}}_R^{(i)} * (i\widehat{\mathcal{D}} \widehat{u}_R^{(i)}) + \overline{\widehat{d}}_R^{(i)} * (i\widehat{\mathcal{D}} \widehat{d}_R^{(i)}) \right) , \quad (6)$$

$$S_{\text{Higgs}} = \int d^4x \left(h_0^\dagger(\widehat{D}_\mu \widehat{\Phi}) * h_0(\widehat{D}^\mu \widehat{\Phi}) - \mu^2 h_0^\dagger(\widehat{\Phi}) * h_0(\widehat{\Phi}) \right. \\ \left. - \lambda h_0^\dagger(\widehat{\Phi}) * h_0(\widehat{\Phi}) * h_0^\dagger(\widehat{\Phi}) * h_0(\widehat{\Phi}) \right) , \quad (7)$$

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$U(1)_Q$	T_3
$e_R^{(i)}$	1	1	-1	-1	0
$L_L^{(i)} = \begin{pmatrix} \nu_L^{(i)} \\ e_L^{(i)} \end{pmatrix}$	1	2	-1/2	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$
$u_R^{(i)}$	3	1	2/3	2/3	0
$d_R^{(i)}$	3	1	-1/3	-1/3	0
$Q_L^{(i)} = \begin{pmatrix} u_L^{(i)} \\ d_L^{(i)} \end{pmatrix}$	3	2	1/6	$\begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$
$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	1	2	1/2	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$
W^+, W^-, Z	1	3	0	$(\pm 1, 0)$	$(\pm 1, 0)$
A	1	1	0	0	0
G^b	8	1	0	0	0

Table 1: The Standard Model fields. Here $i \in \{1, 2, 3\}$ denotes the generation index. The electric charge is given by the Gell-Mann-Nishijima relation $Q = (T_3 + Y)$. The physical electroweak fields A , W^+ , W^- and Z are expressed through the unphysical $U(1)_Y$ and $SU(2)$ fields \mathcal{A} and B_a ($a \in \{1, 2, 3\}$) in Eq. (26). The gluons G^b ($b \in \{1, 2, \dots, 8\}$) are in the octet representation of $SU(3)_C$.

$$\begin{aligned}
S_{\text{Yukawa}} = & - \int d^4x \sum_{i,j=1}^3 \\
& \left(G_e^{(ij)} (\overline{\widehat{L}}_L^{(i)} * h_e(\widehat{\Phi}) * \widehat{e}_R^{(j)}) + G_e^{\dagger(ij)} (\overline{\widehat{e}}_R^{(i)} * h_e(\widehat{\Phi})^\dagger * \widehat{L}_L^{(j)}) \right. \\
& + G_u^{(ij)} (\overline{\widehat{Q}}_L^{(i)} * h_u(\widehat{\Phi}_c) * \widehat{u}_R^{(j)}) + G_u^{\dagger(ij)} (\overline{\widehat{u}}_R^{(i)} * h_u(\widehat{\Phi}_c)^\dagger * \widehat{Q}_L^{(j)}) \\
& \left. + G_d^{(ij)} (\overline{\widehat{Q}}_L^{(i)} * h_d(\widehat{\Phi}) * \widehat{d}_R^{(j)}) + G_d^{\dagger(ij)} (\overline{\widehat{d}}_R^{(i)} * h_d(\widehat{\Phi})^\dagger * \widehat{Q}_L^{(j)}) \right). \tag{8}
\end{aligned}$$

The gauge part S_{gauge} of the action is given in the next section. The particle spectrum of the SM, as well as that of the NCSM, is given in Table 1. Analogously to the usual SM definitions for fermion fields, we define $\widehat{\psi} = \widehat{\psi}^\dagger \gamma^0$. (The γ matrix can be pulled out of the SW expansion because it commutes with the matrices representing internal symmetries.) The indices L and R denote the standard left and right components $\psi_L = 1/2(1 - \gamma_5)\psi$

and $\psi_R = 1/2(1 + \gamma_5)\psi$. For the conjugate Higgs field, we have $\Phi_c = i\tau_2\Phi^*$ (τ_2 is the usual Pauli matrix). In Eqs. (6) and (8) the generation index is denoted by $i, j \in \{1, 2, 3\}$. The matrices G_e , G_u and G_d are the Yukawa couplings.

The non-commutative Higgs field $\widehat{\Phi}$ is given by the hybrid SW map

$$\begin{aligned}\widehat{\Phi} &\equiv \widehat{\Phi}[\Phi, V, V'] \\ &= \Phi + \frac{1}{2}\theta^{\alpha\beta}V_\beta \left(\partial_\alpha\Phi - \frac{i}{2}(V_\alpha\Phi - \Phi V'_\alpha) \right) \\ &\quad + \frac{1}{2}\theta^{\alpha\beta} \left(\partial_\alpha\Phi - \frac{i}{2}(V_\alpha\Phi - \Phi V'_\alpha) \right) V'_\beta + \mathcal{O}(\theta^2),\end{aligned}\tag{9}$$

which generalizes the Seiberg-Witten maps of both gauge bosons and fermions. $\widehat{\Phi}$ is a functional of two gauge fields V and V' and transforms covariantly under gauge transformations:

$$\delta\widehat{\Phi}[\Phi, V, V'] = i\widehat{\Lambda} * \widehat{\Phi} - i\widehat{\Phi} * \widehat{\Lambda}',\tag{10}$$

where $\widehat{\Lambda}$ and $\widehat{\Lambda}'$ are the corresponding gauge parameters. Hermitian conjugation yields $\widehat{\Phi}[\Phi, V, V']^\dagger = \widehat{\Phi}[\Phi^\dagger, V', V]$. The covariant derivative for the non-commutative Higgs field $\widehat{\Phi}$ is given by

$$\widehat{D}_\mu\widehat{\Phi} = \partial_\mu\widehat{\Phi} - i\widehat{V}_\mu * \widehat{\Phi} + i\widehat{\Phi} * \widehat{V}'_\mu.\tag{11}$$

As explained in [6], the precise representations of the gauge fields V and V' in the Yukawa couplings are inherited from the fermions on the left ($\bar{\psi}$) and on the right side (ψ) of the Higgs field found in (8), respectively. The following notation was introduced in Eqs. (7) and (8)

$$\begin{aligned}h_0(\widehat{\Phi}) &= \widehat{\Phi}[\Phi, \frac{1}{2}g'\mathcal{A} + gB^aT_L^a, 0], \\ h_\psi(\widehat{\Phi}) &= \widehat{\Phi}[\Phi, \mathcal{R}_{\psi_L}(V), \mathcal{R}_{\psi_R}(V)], \\ h_\psi(\widehat{\Phi}_c) &= \widehat{\Phi}[\Phi_c, \mathcal{R}_{\psi_L}(V), \mathcal{R}_{\psi_R}(V)].\end{aligned}\tag{12}$$

The representations \mathcal{R}_ψ , determined by the multiplet ψ , are listed in Table 2. Note that $\mathcal{R}_\psi(f(V_\mu)) = f(\mathcal{R}_\psi(V_\mu))$ for any function f . Gauge invariance does not restrict the choice of representation for the Higgs field in S_{Higgs} . The simplest choice for h_0 which is adopted in the NCSM closely follows the SM representation for the Higgs field. For a better understanding of the gauge

ψ	$\mathcal{R}_\psi(V_\mu)$
$e_R^{(i)}$	$-g' \mathcal{A}_\mu$
$L_L^{(i)} = \begin{pmatrix} \nu_L^{(i)} \\ e_L^{(i)} \end{pmatrix}$	$-\frac{1}{2} g' \mathcal{A}_\mu + g B_\mu^a T_L^a$
$u_R^{(i)}$	$\frac{2}{3} g' \mathcal{A}_\mu + g_s G_\mu^b T_S^b$
$d_R^{(i)}$	$-\frac{1}{3} g' \mathcal{A}_\mu + g_s G_\mu^b T_S^b$
$Q_L^{(i)} = \begin{pmatrix} u_L^{(i)} \\ d_L^{(i)} \end{pmatrix}$	$\frac{1}{6} g' \mathcal{A}_\mu + g B_\mu^a T_L^a + g_s G_\mu^b T_S^b$

Table 2: The gauge fields in the covariant derivatives of the fermions and in the Seiberg-Witten maps of the fermions in the Non-Commutative Standard Model. The matrices $T_L^a = \tau^a/2$ and $T_S^b = \lambda^b/2$ correspond to the Pauli and Gell-Mann matrices respectively, and the summation over the indices $a \in \{1, 2, 3\}$ and $b \in \{1, \dots, 8\}$ is understood.

invariance, let us consider the hypercharges in two examples:

$$\begin{aligned}
& \overline{L}_L[V] * \widehat{\Phi}[\Phi, V, V'] * \widehat{e}_R[V'] \\
Y : & \quad 1/2 \quad \underbrace{-1/2; 1}_{1/2} \quad -1, \\
& \overline{Q}_L[V] * \widehat{\Phi}[\Phi, V, V'] * \widehat{d}_R[V'] \\
Y : & \quad -1/6 \quad \underbrace{1/6; 1/3}_{1/2} \quad -1/3.
\end{aligned} \tag{13}$$

The choice of representation allows us to assign separate left and right hypercharges to the noncommutative Higgs field $\widehat{\Phi}$, which add up to Higgs usual hypercharge [6]. Because of the minus sign in (10), the right hypercharge attributed to the Higgs is effectively $-Y_{\psi_R}$.

In Grand Unified Theories (GUT) it is more natural to first combine the left-handed and right-handed fermion fields and then contract the resulting expression with Higgs fields to obtain a gauge-invariant Yukawa term. Consequently, in NC GUTs we need to use the hybrid SW map for the left-handed fermion fields and then sandwich them between the NC Higgs on the left and the right-handed fermion fields on the right [31].

3 Gauge Sector of the NCSM Action

The general form of the gauge kinetic terms is [31]

$$S_{\text{gauge}} = -\frac{1}{2} \int d^4x \sum_{\mathcal{R}} c_{\mathcal{R}} \text{Tr} \left(\mathcal{R}(\widehat{F}_{\mu\nu}) * \mathcal{R}(\widehat{F}^{\mu\nu}) \right), \quad (14)$$

where the non-commutative field strength $\widehat{F}_{\mu\nu}$

$$\begin{aligned} \widehat{F}_{\mu\nu} &= \partial_{\mu} \widehat{V}_{\nu} - \partial_{\nu} \widehat{V}_{\mu} - i[\widehat{V}_{\mu} * \widehat{V}_{\nu}] \\ &= F_{\mu\nu} + \frac{1}{2} \theta^{\alpha\beta} \{F_{\mu\alpha}, F_{\nu\beta}\} - \frac{1}{4} \theta^{\alpha\beta} \{V_{\alpha}, (\partial_{\beta} + D_{\beta}) F_{\mu\nu}\} + \mathcal{O}(\theta^2), \end{aligned} \quad (15)$$

was obtained from the SW map for the non-commutative vector potential (4). Ordinary field strength $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_{\mu} V_{\nu} - \partial_{\nu} V_{\mu} - i[V_{\mu}, V_{\nu}], \quad (16)$$

while its covariant derivative reads

$$D_{\beta} F_{\mu\nu} = \partial_{\beta} F_{\mu\nu} - i[V_{\beta}, F_{\mu\nu}]. \quad (17)$$

Here V_{μ} represents the whole of the gauge potential for the SM gauge group,

$$V_{\mu}(x) = g' \mathcal{A}_{\mu}(x) Y + g \sum_{a=1}^3 B_{\mu}^a(x) T_L^a + g_s \sum_{b=1}^8 G_{\mu}^b(x) T_S^b. \quad (18)$$

The sum in (14) is over all unitary, irreducible and inequivalent representations \mathcal{R} of a gauge group. The freedom in the kinetic terms is parametrized by real coefficients $c_{\mathcal{R}}$ that are subject to the constraints

$$\frac{1}{g_I^2} = \sum_{\mathcal{R}} c_{\mathcal{R}} \text{Tr} \left(\mathcal{R}(T_I^a) \mathcal{R}(T_I^a) \right), \quad (19)$$

where g_I are the usual ‘‘commutative’’ coupling constants g' , g , g_s and T_I^a are generators of $U(1)_Y$, $SU(2)_L$, $SU(3)_C$, respectively. Equations (14) and (19) can also be written more compactly as

$$S_{\text{gauge}} = -\frac{1}{2} \int d^4x \text{Tr} \frac{1}{\mathbf{G}^2} \widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu}, \quad \frac{1}{g_I^2} = \text{Tr} \frac{1}{\mathbf{G}^2} T_I^a T_I^a, \quad (20)$$

where the trace Tr is again over all representations and \mathbf{G} is an operator that commutes with all generators T_I^a and encodes the coupling constants [9]. The trace in the kinetic terms for gauge bosons is not unique, it depends on the

choice of representation. This would not be of importance if the gauge fields were Lie algebra valued, but in the noncommutative case they live in the enveloping algebra. The possibility of new parameters in gauge theories on non-commutative space-time is a consequence of the fact that the gauge fields can take any value in the enveloping algebra of the gauge group.

It is instructive to provide the general form of S_{gauge} , (14), in terms of SM fields:

$$S_{\text{gauge}} = -\frac{1}{2} \int d^4x \text{Tr} \frac{1}{\mathbf{G}^2} F_{\mu\nu} F^{\mu\nu} + \theta^{\rho\sigma} \int d^4x \text{Tr} \frac{1}{\mathbf{G}^2} \left[\left(\frac{1}{4} F_{\rho\sigma} F_{\mu\nu} - F_{\rho\mu} F_{\sigma\nu} \right) F^{\mu\nu} \right] + \mathcal{O}(\theta^2). \quad (21)$$

3.1 Minimal NCSM

In the minimal Non-Commutative Standard Model (mNCSM) which adopts the whole of the gauge potential (18) for the SM gauge group, the mNCSM gauge action is given by

$$S_{\text{gauge}}^{\text{mNCSM}} = -\frac{1}{2} \int d^4x \left(\frac{1}{g'^2} \text{Tr}_1 + \frac{1}{g^2} \text{Tr}_2 + \frac{1}{g_s^2} \text{Tr}_3 \right) \widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu}. \quad (22)$$

Here the simplest choice was taken, i.e, a sum of three traces over the $U(1)$, $SU(2)$, $SU(3)$ sectors with

$$Y = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (23)$$

in the definition of Tr_1 and the fundamental representations for $SU(2)$ and $SU(3)$ generators in Tr_2 and Tr_3 , respectively. In terms of physical fields, the action then reads

$$S_{\text{gauge}}^{\text{mNCSM}} = -\frac{1}{2} \int d^4x \left(\frac{1}{2} \mathcal{A}_{\mu\nu} \mathcal{A}^{\mu\nu} + \text{Tr} B_{\mu\nu} B^{\mu\nu} + \text{Tr} G_{\mu\nu} G^{\mu\nu} \right) + \frac{1}{4} g_s d^{abc} \theta^{\rho\sigma} \int d^4x \left(\frac{1}{4} G_{\rho\sigma}^a G_{\mu\nu}^b - G_{\rho\mu}^a G_{\sigma\nu}^b \right) G^{\mu\nu,c} + \mathcal{O}(\theta^2), \quad (24)$$

where $\mathcal{A}_{\mu\nu}$, $B_{\mu\nu}(= B_{\mu\nu}^a T_L^a)$ and $G_{\mu\nu}(= G_{\mu\nu}^a T_S^a)$ denote the $U(1)$, $SU(2)_L$ and $SU(3)_c$ field strengths, respectively:

$$\begin{aligned} \mathcal{A}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \\ B_{\mu\nu}^a &= \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + g \epsilon^{abc} B_\mu^b B_\nu^c, \\ G_{\mu\nu}^a &= \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c. \end{aligned} \quad (25)$$

Note that in order to obtain the above result¹, one makes use of the following symmetry properties of the group generators $T_L^a = \tau^a/2$ and $T_S^a = \lambda^a/2$:

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}, \quad \text{Tr}(\tau^a \tau^b \tau^c) = 2i \epsilon^{abc}, \quad \text{Tr}(\lambda^a \lambda^b \lambda^c) = 2(d^{abc} + i f^{abc}),$$

where ϵ^{abc} is the usual antisymmetric tensor, while f^{abc} and d^{abc} are totally antisymmetric and totally symmetric structure constants of the $SU(3)$ group.

There are no new electroweak gauge boson interactions in Eq. (24) nor the vertices already present in SM, like $W^+ W^- \gamma$ and $W^+ W^- Z$, do acquire any corrections. This is a consequence of our choice of the hypercharge (23) and of the antisymmetry in both the Lorentz and the group representation indices. However, new couplings, like ZZZ , and θ corrections to SM vertices enter from the Higgs kinetic terms as elaborated in Section 5.1.

For the convenience of the reader, we list some usual definitions that we use in the analysis of the electroweak sector. The physical fields for the electroweak gauge bosons (W^\pm , Z) and the photon (A) are given by

$$\begin{aligned} W_\mu^\pm &= \frac{B_\mu^1 \mp i B_\mu^2}{\sqrt{2}}, \\ Z_\mu &= \frac{-g' \mathcal{A}_\mu + g B_\mu^3}{\sqrt{g^2 + g'^2}} = -\sin \theta_W \mathcal{A}_\mu + \cos \theta_W B_\mu^3, \\ A_\mu &= \frac{g \mathcal{A}_\mu + g' B_\mu^3}{\sqrt{g^2 + g'^2}} = \cos \theta_W \mathcal{A}_\mu + \sin \theta_W B_\mu^3, \end{aligned} \quad (26)$$

where electric charge $e = g \sin \theta_W = g' \cos \theta_W$.

3.2 Non-Minimal NCSM

We can use the freedom in the choice of traces in kinetic terms for gauge fields to construct non-minimal versions of the mNCSM (nmNCSM). Since the fermion-gauge boson interactions remain the same regardless on the choice of traces in the gauge sector, the matter sector of the action is not affected, i.e. it is the same for both versions of the NCSM.

The expansion in θ is at the same time an expansion in the momenta. The θ -expanded action can thus be interpreted as a low-energy effective action. In such an effective low-energy description it is natural to expect that all representations that appear in commutative theory (matter multiplets and adjoint representation) are important. All representations of gauge fields that appear in the SM then have to be considered in the definition of the

¹Note that hereby we correct Eq. (56) of Ref. [6].

trace (20). In [9] the trace was chosen over all particles on which covariant derivatives act and which have different quantum numbers. In the SM, these are, five multiplets of fermions for each generation and one Higgs multiplet. The operator \mathbf{G} , which determines the coupling constants of the theory, must commute with all generators (Y, T_L^a, T_S^b) of the gauge group, so that it does not spoil the trace property of \mathbf{Tr} . This implies that \mathbf{G} takes on constant values g_1, \dots, g_6 on the six multiplets (Table 1). The operator \mathbf{G} is in general a function of Y and of the Casimir operators of $SU(2)$ and $SU(3)$. The action derived from (21) for such nmNCSM takes the following form:

$$\begin{aligned}
S_{\text{gauge}}^{\text{nmNCSM}} &= S_{\text{gauge}}^{\text{mNCSM}} \\
&+ g'^3 \kappa_1 \theta^{\rho\sigma} \int d^4x \left(\frac{1}{4} \mathcal{A}_{\rho\sigma} \mathcal{A}_{\mu\nu} - \mathcal{A}_{\mu\rho} \mathcal{A}_{\nu\sigma} \right) \mathcal{A}^{\mu\nu} \\
&+ g' g^2 \kappa_2 \theta^{\rho\sigma} \int d^4x \left[\left(\frac{1}{4} \mathcal{A}_{\rho\sigma} B_{\mu\nu}^a - \mathcal{A}_{\mu\rho} B_{\nu\sigma}^a \right) B^{\mu\nu,a} + c.p. \right] \\
&+ g' g_s^2 \kappa_3 \theta^{\rho\sigma} \int d^4x \left[\left(\frac{1}{4} \mathcal{A}_{\rho\sigma} G_{\mu\nu}^b - \mathcal{A}_{\mu\rho} G_{\nu\sigma}^b \right) G^{\mu\nu,b} + c.p. \right] \\
&+ \mathcal{O}(\theta^2), \tag{27}
\end{aligned}$$

where *c.p.* denotes cyclic permutations of field strength tensors with respect to Lorentz indices. The constants κ_1 , κ_2 and κ_3 represent parameters of the model given in [9, 10]. In the following we comment only the pure triple electroweak gauge-boson interactions.

New anomalous triple-gauge boson interactions that are usually forbidden by Lorentz invariance, angular momentum conservation and Bose statistics (Landau-Pomeranchuk-Yang theorem) can arise within the framework of the nmNCSM [9, 10], but also in the alternative approach to the NCSM given in [27]. Neutral triple-gauge boson terms which are not present in the SM Lagrangian can be extracted from the action (27). In terms of physical fields (A, Z) they are

$$\begin{aligned}
\mathcal{L}_{\gamma\gamma\gamma} &= \frac{e}{4} \sin 2\theta_W K_{\gamma\gamma\gamma} \theta^{\rho\sigma} A^{\mu\nu} (A_{\mu\nu} A_{\rho\sigma} - 4A_{\mu\rho} A_{\nu\sigma}), \\
\mathcal{L}_{Z\gamma\gamma} &= \frac{e}{4} \sin 2\theta_W K_{Z\gamma\gamma} \theta^{\rho\sigma} [2Z^{\mu\nu} (2A_{\mu\rho} A_{\nu\sigma} - A_{\mu\nu} A_{\rho\sigma}) \\
&\quad + 8Z_{\mu\rho} A^{\mu\nu} A_{\nu\sigma} - Z_{\rho\sigma} A_{\mu\nu} A^{\mu\nu}], \\
\mathcal{L}_{ZZ\gamma} &= \mathcal{L}_{Z\gamma\gamma}(A_\mu \leftrightarrow Z_\mu), \\
\mathcal{L}_{ZZZ} &= \mathcal{L}_{\gamma\gamma\gamma}(A_\mu \rightarrow Z_\mu), \tag{28}
\end{aligned}$$

where

$$\begin{aligned}
K_{\gamma\gamma\gamma} &= \frac{1}{2} gg'(\kappa_1 + 3\kappa_2), \\
K_{Z\gamma\gamma} &= \frac{1}{2} \left[g'^2 \kappa_1 + (g'^2 - 2g^2) \kappa_2 \right], \\
K_{ZZ\gamma} &= \frac{-1}{2gg'} \left[g'^4 \kappa_1 + g^2 (g^2 - 2g'^2) \kappa_2 \right], \\
K_{ZZZ} &= \frac{-1}{2g^2} (g'^4 \kappa_1 + 3g^4 \kappa_2),
\end{aligned} \tag{29}$$

and here we have introduced the shorthand notation $X_{\mu\nu} \equiv \partial_\mu X_\nu - \partial_\nu X_\mu$ for $X \in \{A, Z\}$. Details of the derivations of neutral triple-gauge boson terms and the properties of the coupling constants in (27) are explained in [9, 10].

Additionally, in contrast to the mNCSM (24), electroweak triple-gauge boson terms already present in the SM acquire θ corrections in the nmNCSM. Such contributions which originate from (27) read

$$\begin{aligned}
\mathcal{L}_{WW\gamma} &= \mathcal{L}_{WW\gamma}^{\text{SM}} + \mathcal{L}_{WW\gamma}^\theta + \mathcal{O}(\theta^2), \\
\mathcal{L}_{WWZ} &= \mathcal{L}_{WWZ}^{\text{SM}} + \mathcal{L}_{WWZ}^\theta + \mathcal{O}(\theta^2), \\
\mathcal{L}_{WW\gamma}^\theta &= \frac{e}{2} \sin 2\theta_W K_{WW\gamma} \theta^{\rho\sigma} \left\{ A^{\mu\nu} \left[2 (W_{\mu\rho}^+ W_{\nu\sigma}^- + W_{\mu\rho}^- W_{\nu\sigma}^+) \right. \right. \\
&\quad \left. \left. - (W_{\mu\nu}^+ W_{\rho\sigma}^- + W_{\mu\nu}^- W_{\rho\sigma}^+) \right] + 4A_{\mu\rho} [W^{+\mu\nu} W_{\nu\sigma}^- + W^{-\mu\nu} W_{\nu\sigma}^+] \right. \\
&\quad \left. - A_{\rho\sigma} W_{\mu\nu}^+ W^{-\mu\nu} \right\}, \\
\mathcal{L}_{WWZ}^\theta &= \mathcal{L}_{WW\gamma}^\theta (A_\mu \rightarrow Z_\mu),
\end{aligned} \tag{30}$$

with

$$\begin{aligned}
K_{WW\gamma} &= -\frac{g}{2g'} \left[g'^2 + g^2 \right] \kappa_2, \\
K_{WWZ} &= -\frac{g'}{g} K_{WW\gamma}.
\end{aligned} \tag{31}$$

It is important to stress that in both the mNCSM and the nmNCSM there are additional θ corrections to these vertices coming from the Higgs part of the action. This will be elaborated in detail in Section 5.1.

The new parameters in the non-minimal NCSM can be restricted by considering GUTs on non-commutative space-time [31].

4 Electroweak Matter Currents

In this section we concentrate on the fermion electroweak sector of the NCSM. Some terms are derivative valued. Nevertheless, the hermiticity of

the Seiberg-Witten maps for the gauge field guarantees the reality of the action. Using the SW maps of the non-commutative fermion field $\widehat{\psi}$ with corresponding function $\mathcal{R}_\psi(V_\alpha)$

$$\widehat{\psi} = \psi - \frac{1}{2} \theta^{\alpha\beta} \mathcal{R}_\psi(V_\alpha) \partial_\beta \psi + \frac{i}{8} \theta^{\alpha\beta} [\mathcal{R}_\psi(V_\alpha), \mathcal{R}_\psi(V_\beta)] \psi + \mathcal{O}(\theta^2), \quad (32)$$

and it's covariant derivative

$$\begin{aligned} \widehat{D}_\mu \widehat{\psi} &= \partial_\mu \widehat{\psi} - i \mathcal{R}_\psi(\widehat{V}_\mu) * \widehat{\psi} \\ &= D_\mu \left[\psi - \frac{1}{2} \theta^{\alpha\beta} \mathcal{R}_\psi(V_\alpha) \partial_\beta \psi + \frac{i}{8} \theta^{\alpha\beta} [\mathcal{R}_\psi(V_\alpha), \mathcal{R}_\psi(V_\beta)] \psi \right] \\ &\quad - i \mathcal{R}_\psi \left(\frac{1}{4} \theta^{\alpha\beta} \{ \partial_\alpha V_\mu + F_{\alpha\mu}, V_\beta \} \right) \psi + \frac{1}{2} \theta^{\alpha\beta} (\partial_\alpha \mathcal{R}_\psi(V_\mu)) \partial_\beta \psi + \mathcal{O}(\theta^2), \end{aligned} \quad (33)$$

it is straightforward to derive the general expression

$$\begin{aligned} S_\psi &= \int d^4x \overline{\widehat{\psi}} * i \widehat{\mathcal{D}} \widehat{\psi} \\ &= \int d^4x \left(i \overline{\psi} \mathcal{D} \psi - \frac{i}{4} \overline{\psi} \theta^{\mu\nu\rho} \mathcal{R}_\psi(F_{\mu\nu}) D_\rho \psi + \mathcal{O}(\theta^2) \right), \end{aligned} \quad (34)$$

where $\theta^{\mu\nu\rho}$ is a totally antisymmetric quantity:

$$\theta^{\mu\nu\rho} = \theta^{\mu\nu} \gamma^\rho + \theta^{\nu\rho} \gamma^\mu + \theta^{\rho\mu} \gamma^\nu. \quad (35)$$

The terms of the form given in Eq. (34) appear in S_{fermions} (6). One can easily show that $S_{\text{fermions}}^\dagger = S_{\text{fermions}}$, to order $\mathcal{O}(\theta^2)$. From Eq. (34) we have

$$S_\psi^\dagger = S_\psi - \frac{i}{4} \int d^4x (\overline{\psi} \theta^{\mu\nu\rho} \mathcal{R}_\psi(D_\rho F_{\mu\nu}) \psi) + \mathcal{O}(\theta^2).$$

Since $\mathcal{R}_\psi(\theta^{\mu\nu\rho} D_\rho F_{\mu\nu}) = \theta^{\mu\nu\rho} \mathcal{R}_\psi(D_\rho F_{\mu\nu})$ for constant θ , and

$$\theta^{\mu\nu\rho} (D_\rho F_{\mu\nu}) = \theta^{\mu\nu} \gamma^\rho (D_\rho F_{\mu\nu} + D_\nu F_{\rho\mu} + D_\mu F_{\nu\rho}),$$

the θ -dependent term vanishes due to the Bianchi identity

$$D_\rho F_{\mu\nu} + D_\nu F_{\rho\mu} + D_\mu F_{\nu\rho} = 0,$$

thereby proving the reality of the action S_ψ and, hence, the reality of the action S_{fermions} to $\mathcal{O}(\theta^2)$. However, note that the reality of the action is not essential, but is very desirable. ²

²Weinberg writes in his book: “The action is supposed to be real. This is because we want just as many field equations as there are fields. [...] The reality also ensures that the generators of various symmetry transformations are Hermitian operators.” [32]

Next, we express the NCSM results for the electroweak currents in terms of physical fields starting with the left-handed electroweak sector. In the following Ψ_L represents $\Psi_L \in \{L_L^{(i)}, Q_L^{(i)}\}$ and has the general form

$$\Psi_L = \begin{pmatrix} \psi_{\text{up},L} \\ \psi_{\text{down},L} \end{pmatrix}. \quad (36)$$

In this case, according to the Table 2 the representation $\mathcal{R}_{\Psi_L}(V_\mu)$ without $SU(3)$ fields takes the form

$$\mathcal{R}_{\Psi_L}(V_\mu) = g' \mathcal{A}_\mu Y_{\Psi_L} + g B_\mu^a T_L^a. \quad (37)$$

The hypercharge generator Y_{Ψ_L} (see Table 1) can be rewritten as

$$Y_{\Psi_L} = Q_{\psi_{\text{up}}} - T_{3,\psi_{\text{up},L}} = Q_{\psi_{\text{down}}} - T_{3,\psi_{\text{down},L}}, \quad (38)$$

and we make use of Eqs. (26). The left-handed electroweak part of the action S_ψ can be cast in the form

$$\begin{aligned} S_{\psi,\text{ew},L} &= \int d^4x \left(\bar{\Psi}_L i \not{\partial} \Psi_L + \bar{\Psi}_L \mathbf{J}^{(L)} \Psi_L \right) \\ &= \int d^4x \left(\bar{\Psi}_L i \not{\partial} \Psi_L + \bar{\psi}_{\text{up},L} J_{12}^{(L)} \psi_{\text{down},L} + \bar{\psi}_{\text{down},L} J_{21}^{(L)} \psi_{\text{up},L} \right. \\ &\quad \left. + \bar{\psi}_{\text{up},L} J_{11}^{(L)} \psi_{\text{up},L} + \bar{\psi}_{\text{down},L} J_{22}^{(L)} \psi_{\text{down},L} \right), \end{aligned} \quad (39)$$

where $\mathbf{J}^{(L)}$ is a 2×2 matrix whose off-diagonal elements ($J_{12}^{(L)}, J_{21}^{(L)}$) denote the charged currents and diagonal elements ($J_{11}^{(L)}, J_{22}^{(L)}$) the neutral currents. After some algebra we obtain

$$J_{12}^{(L)} = \frac{g}{\sqrt{2}} W^+ + J_{12}^{(L,\theta)} + \mathcal{O}(\theta^2), \quad (40a)$$

$$J_{21}^{(L)} = \frac{g}{\sqrt{2}} W^- + J_{21}^{(L,\theta)} + \mathcal{O}(\theta^2), \quad (40b)$$

$$\begin{aligned} J_{11}^{(L)} &= \left[e Q_{\psi_{\text{up}}} \mathcal{A} + \frac{g}{\cos \theta_W} (T_{3,\psi_{\text{up},L}} - Q_{\psi_{\text{up}}} \sin^2 \theta_W) \mathcal{Z} \right] \\ &\quad + J_{11}^{(L,\theta)} + \mathcal{O}(\theta^2), \end{aligned} \quad (40c)$$

$$\begin{aligned} J_{22}^{(L)} &= \left[e Q_{\psi_{\text{down}}} \mathcal{A} + \frac{g}{\cos \theta_W} (T_{3,\psi_{\text{down},L}} - Q_{\psi_{\text{down}}} \sin^2 \theta_W) \mathcal{Z} \right] \\ &\quad + J_{22}^{(L,\theta)} + \mathcal{O}(\theta^2), \end{aligned} \quad (40d)$$

where

$$\begin{aligned}
J_{12}^{(L,\theta)} &= \frac{g}{2\sqrt{2}} \theta^{\mu\nu\rho} W_\mu^+ \left\{ -i \overleftarrow{\partial}_\nu \overrightarrow{\partial}_\rho \right. \\
&\quad \left. + e \left[Q_{\psi_{\text{up}}} A_\nu \overrightarrow{\partial}_\rho + Q_{\psi_{\text{down}}} A_\nu \overleftarrow{\partial}_\rho + (Q_{\psi_{\text{up}}} + Q_{\psi_{\text{down}}}) (\partial_\rho A_\nu) \right] \right. \\
&\quad \left. + \frac{g}{\cos \theta_W} \left[\left(T_{3,\psi_{\text{up},L}} - Q_{\psi_{\text{up}}} \sin^2 \theta_W \right) Z_\nu \overrightarrow{\partial}_\rho \right. \right. \\
&\quad \left. \left. + (T_{3,\psi_{\text{down},L}} - Q_{\psi_{\text{down}}} \sin^2 \theta_W) Z_\nu \overleftarrow{\partial}_\rho \right. \right. \\
&\quad \left. \left. + \left((T_{3,\psi_{\text{up},L}} + T_{3,\psi_{\text{down},L}}) - (Q_{\psi_{\text{up}}} + Q_{\psi_{\text{down}}}) \sin^2 \theta_W \right) (\partial_\rho Z_\nu) \right] \right. \\
&\quad \left. - \frac{i e g}{\cos \theta_W} (Q_{\psi_{\text{up}}} T_{3,\psi_{\text{down},L}} - Q_{\psi_{\text{down}}} T_{3,\psi_{\text{up},L}}) A_\nu Z_\rho \right\} \quad (41)
\end{aligned}$$

and

$$\begin{aligned}
J_{11}^{(L,\theta)} &= \frac{1}{2} \theta^{\mu\nu\rho} \left\{ i e Q_{\psi_{\text{up}}} (\partial_\nu A_\mu) \overrightarrow{\partial}_\rho \right. \\
&\quad \left. + \frac{i g}{\cos \theta_W} (T_{3,\psi_{\text{up},L}} - Q_{\psi_{\text{up}}} \sin^2 \theta_W) (\partial_\nu Z_\mu) \overrightarrow{\partial}_\rho \right. \\
&\quad \left. - e^2 Q_{\psi_{\text{up}}}^2 (\partial_\rho A_\mu) A_\nu \right. \\
&\quad \left. - \frac{g^2}{\cos^2 \theta_W} (T_{3,\psi_{\text{up},L}} - Q_{\psi_{\text{up}}} \sin^2 \theta_W)^2 (\partial_\rho Z_\mu) Z_\nu \right. \\
&\quad \left. - \frac{e g}{\cos \theta_W} Q_{\psi_{\text{up}}} (T_{3,\psi_{\text{up},L}} - Q_{\psi_{\text{up}}} \sin^2 \theta_W) [(\partial_\rho A_\mu) Z_\nu - A_\mu (\partial_\rho Z_\nu)] \right. \\
&\quad \left. - \frac{g^2}{2} \left[W_\mu^+ W_\nu^- \overrightarrow{\partial}_\rho + (\partial_\rho W_\mu^+) W_\nu^- \right] \right. \\
&\quad \left. + \frac{i e g^2}{2} (2 Q_{\psi_{\text{up}}} - Q_{\psi_{\text{down}}}) W_\mu^+ W_\nu^- A_\rho \right. \\
&\quad \left. + \frac{i g^3}{2 \cos \theta_W} \left[(2 T_{3,\psi_{\text{up},L}} - T_{3,\psi_{\text{down},L}}) - (2 Q_{\psi_{\text{up}}} - Q_{\psi_{\text{down}}}) \sin^2 \theta_W \right] \right. \\
&\quad \left. \times W_\mu^+ W_\nu^- Z_\rho \right\}, \quad (42)
\end{aligned}$$

while

$$\left. \begin{array}{l} J_{21}^{(L,\theta)} \\ J_{22}^{(L,\theta)} \end{array} \right\} = \left\{ \begin{array}{l} J_{12}^{(L,\theta)} \\ J_{11}^{(L,\theta)} \end{array} (W^+ \leftrightarrow W^-, Q_{\psi_{\text{up}}} \leftrightarrow Q_{\psi_{\text{down}}}, T_{3,\psi_{\text{up},L}} \leftrightarrow T_{3,\psi_{\text{down},L}}) \right\}. \quad (43)$$

Here and in the following we use the notation in which $\vec{\partial}_\rho$ denotes the partial derivative which acts *only* on the fermion field on the right side, while $\overleftarrow{\partial}_\rho$ denotes the partial derivative which acts *only* on the fermion field on the left side, i.e.

$$\partial_\rho \psi \equiv \vec{\partial}_\rho \psi \quad \partial_\rho \bar{\psi} \equiv \bar{\psi} \overleftarrow{\partial}_\rho . \quad (44)$$

We note that in contrast to the SM case, although

$$\left(\int d^4x \bar{\psi}_{\text{up},L} J_{12} \psi_{\text{down},L} \right)^\dagger = \int d^4x \bar{\psi}_{\text{down},L} J_{21} \psi_{\text{up},L} ,$$

we have

$$J_{21}^{(L)} \neq \gamma^0 \left(J_{12}^{(L)} \right)^\dagger \gamma_0 .$$

The reason is the specific form of the interaction term (see Eq. (34)) which contains derivatives, whose presence produce

$$J_{21}^{(L)} = \gamma^0 \left(J_{12}^{(L)} (\vec{\partial} \leftrightarrow \overleftarrow{\partial}) \right)^\dagger \gamma_0 .$$

Now, we turn to the results for the right-handed electroweak sector. Here ψ_R represents $\psi_R \in \{e_R^{(i)}, u_R^{(i)}, d_R^{(i)}\}$, and the representation $\mathcal{R}_{\psi_R}(V_\mu)$ from Table 2 without $SU(3)$ fields is given by

$$\mathcal{R}_{\psi_R}(V_\mu) = g' \mathcal{A}_\mu Y_{\psi_R} = e Q_\psi A_\mu - \frac{g}{\cos \theta_W} Q_\psi \sin^2 \theta_W Z_\mu . \quad (45)$$

For the right-handed fermions, $T_{3,\psi_R} = 0$ and $Y_{\psi_R} = Q_\psi$. The right-handed electroweak part of the action S_ψ is of the form

$$S_{\psi,\text{ew,R}} = \int d^4x \left(\bar{\psi}_R i \not{\partial} \psi_R + \bar{\psi}_R J^{(R)} \psi_R \right) , \quad (46)$$

$$J^{(R)} = \left[e Q_\psi A - \frac{g}{\cos \theta_W} Q_\psi \sin^2 \theta_W Z \right] + J^{(R,\theta)} + \mathcal{O}(\theta^2) , \quad (47)$$

$$\begin{aligned} J^{(R,\theta)} = \frac{1}{2} \theta^{\mu\nu\rho} \left\{ i e Q_\psi (\partial_\nu A_\mu) \vec{\partial}_\rho - \frac{i g}{\cos \theta_W} Q_\psi \sin^2 \theta_W (\partial_\nu Z_\mu) \vec{\partial}_\rho \right. \\ \left. - e^2 Q_\psi^2 (\partial_\rho A_\mu) A_\nu - \frac{g^2}{\cos^2 \theta_W} Q_\psi^2 \sin^4 \theta_W (\partial_\rho Z_\mu) Z_\nu \right. \\ \left. + \frac{e g}{\cos \theta_W} Q_\psi^2 \sin^2 \theta_W [(\partial_\rho A_\mu) Z_\nu - A_\mu (\partial_\rho Z_\nu)] \right\} . \quad (48) \end{aligned}$$

Let us now present our results ³ in a form suitable for further calculations, derivation of Feynman rules and phenomenological applications, i.e. in terms

³We note that in the preceding considerations we have corrected the electroweak currents presented in the appendix of Ref. [6] and expressed them using more compact and transparent notation.

of $\Psi \in \{L^{(i)}, Q^{(i)}\}$, and thus $\psi_{\text{up}} \in \{\nu^{(i)}, u^{(i)}\}$, and $\psi_{\text{down}} \in \{e^{(i)}, d^{(i)}\}$. The electroweak part of the action S_ψ then takes the form

$$\begin{aligned}
S_{\psi,\text{ew}} = & \int d^4x \left\{ \bar{\Psi} i \not{\partial} \Psi \right. \\
& + \bar{\psi}_{\text{up}} J_{12}^{(L)} \frac{1}{2} (1 - \gamma_5) \psi_{\text{down}} + \bar{\psi}_{\text{down}} J_{21}^{(L)} \frac{1}{2} (1 - \gamma_5) \psi_{\text{up}} \\
& + \bar{\psi}_{\text{up}} \frac{1}{2} \left[(J_{11}^{(L)} + J^{(R)}) - (J_{11}^{(L)} - J^{(R)}) \gamma_5 \right] \psi_{\text{up}} \\
& \left. + \bar{\psi}_{\text{down}} \frac{1}{2} \left[(J_{22}^{(L)} + J^{(R)}) - (J_{22}^{(L)} - J^{(R)}) \gamma_5 \right] \psi_{\text{down}} \right\}, \quad (49)
\end{aligned}$$

and the currents $J_{ij}^{(L)}$ can be read from Eqs. (40-43), while $J^{(R)}$ is given by Eqs. (47-48) (with Q_ψ substituted by the corresponding $Q_{\psi_{\text{up}}}$ or $Q_{\psi_{\text{down}}}$).

Finally, we note that the fermion fields appearing in this section are not mass but weak-interaction eigenstates. In order to present the results in terms of mass eigenstates, the Cabibbo-Kobayashi-Maskawa matrix (denoted by V_{ij} in the following) enters the quark currents leading to mixing between generations and to the modification of the quark currents by V_{ij} factors:

$$\bar{q}_{\text{up}}^{(i)} V_{ij} J_{12}^{(L)} \frac{1}{2} (1 - \gamma_5) q_{\text{down}}^{(j)}, \quad \bar{q}_{\text{down}}^{(j)} V_{ij}^* J_{21}^{(L)} \frac{1}{2} (1 - \gamma_5) q_{\text{up}}^{(i)},$$

where $q_{\text{up}}^{(i)}$ and $q_{\text{down}}^{(i)}$ represent mass eigenstates. In the NCSM, as in the SM, the neutrino masses are not considered and consequently the leptonic mixing matrix is diagonal in contrast to the neutrino mass extended models. The corresponding non-commutative extensions which include neutrino masses can be made along the lines sketched here (see Section 5.2 for further details on this subject).

In this section, only electroweak interactions were considered. Pure QCD, as well as mixed terms which appear in the NCSM due to the Seiberg-Witten mapping, are left for a future publication [29].

5 Higgs Sector of the NCSM Action

In the preceding section we have expanded the fermionic part of the action and performed a detailed analysis of the electroweak interactions. We devote this section to the analysis of S_{Higgs} and S_{Yukawa} to first order in θ .

5.1 Higgs Kinetic Terms

The expansion of the Higgs part of the action (7) to first order in θ yields ⁴

$$\begin{aligned}
S_{\text{Higgs}} = & \int d^4x \left((\mathbf{D}_\mu \Phi)^\dagger (\mathbf{D}^\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 \right) \\
& + \frac{1}{2} \theta^{\alpha\beta} \int d^4x \Phi^\dagger \left(\mathbf{U}_{\alpha\beta} + \mathbf{U}_{\alpha\beta}^\dagger + \frac{1}{2} \mu^2 \mathbf{F}_{\alpha\beta} - 2i\lambda \Phi (\mathbf{D}_\alpha \Phi)^\dagger \mathbf{D}_\beta \right) \Phi,
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
\mathbf{U}_{\alpha\beta} = & \left(\overleftarrow{\partial}^\mu + i\mathbf{V}^\mu \right) \left(-\partial_\mu \mathbf{V}_\alpha \partial_\beta - \mathbf{V}_\alpha \partial_\mu \partial_\beta + \partial_\alpha \mathbf{V}_\mu \partial_\beta \right. \\
& + i\mathbf{V}_\mu \mathbf{V}_\alpha \partial_\beta + \frac{i}{2} \mathbf{V}_\alpha \mathbf{V}_\beta \partial_\mu + \frac{i}{2} \partial_\mu (\mathbf{V}_\alpha \mathbf{V}_\beta) \\
& \left. + \frac{1}{2} \mathbf{V}_\mu \mathbf{V}_\alpha \mathbf{V}_\beta + \frac{i}{2} \{ \mathbf{V}_\alpha, \partial_\beta \mathbf{V}_\mu + \mathbf{F}_{\beta\mu} \} \right).
\end{aligned} \tag{51}$$

Equation (50) contains the usual covariant derivative of the Higgs boson $\mathbf{D}_\mu = \partial_\mu \mathbf{1} - i\mathbf{V}_\mu$ where $\mathbf{V}_\mu = g' \mathcal{A}_\mu Y_\Phi \mathbf{1} + g B_\mu^a T_L^a$, and $\mathbf{1}$ is a unit matrix suppressed in the following. Also $\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{V}_\nu - \partial_\nu \mathbf{V}_\mu - i[\mathbf{V}_\mu, \mathbf{V}_\nu]$.

Let us construct explicit expressions for the electroweak gauge matrices occurring in (50) and (51). The gauge field \mathbf{V}_μ can be expressed in a matrix form as

$$\mathbf{V}_\mu = \begin{pmatrix} g' \mathcal{A}_\mu Y_\Phi + g T_{3,\phi_{\text{up}}} B_\mu^3 & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & g' \mathcal{A}_\mu Y_\Phi + g T_{3,\phi_{\text{down}}} B_\mu^3 \end{pmatrix}, \tag{52}$$

where from Table 1 one can read⁵: $Y_\Phi = 1/2$, $T_{3,\phi_{\text{up}}} = 1/2$, $T_{3,\phi_{\text{down}}} = -1/2$. The diagonal matrix elements can also be expressed in terms of physical fields using Eqs. (26). Hence, one obtains

$$\begin{aligned}
V_{11,\mu} &= e A_\mu + \frac{g}{2 \cos \theta_W} (1 - 2 \sin^2 \theta_W) Z_\mu, \\
V_{22,\mu} &= -\frac{g}{2 \cos \theta_W} Z_\mu.
\end{aligned} \tag{53}$$

⁴In order to make the presentation more transparent, in this section, we denote the 2×2 matrices appearing in the action by bold letters.

⁵Note $Y_\Phi = Q_{\phi_{\text{up}}} - T_{3,\phi_{\text{up}}} = Q_{\phi_{\text{down}}} - T_{3,\phi_{\text{down}}}$.

The product of two gauge fields is given by

$$\mathbf{V}_\mu \mathbf{V}_\alpha = \begin{pmatrix} V_{11,\mu} V_{11,\alpha} + \frac{g^2}{2} W_\mu^+ W_\alpha^- & \frac{g}{\sqrt{2}} (W_\alpha^+ V_{11,\mu} + W_\mu^+ V_{22,\alpha}) \\ \frac{g}{\sqrt{2}} (W_\alpha^- V_{22,\mu} + W_\mu^- V_{11,\alpha}) & V_{22,\mu} V_{22,\alpha} + \frac{g^2}{2} W_\mu^- W_\alpha^+ \end{pmatrix}, \quad (54)$$

while the product of three gauge fields can be expressed as

$$\mathbf{V}_\mu \mathbf{V}_\alpha \mathbf{V}_\beta = \mathbf{M}_{\mu\alpha\beta}, \quad (55a)$$

with matrix elements

$$\begin{aligned} M_{\mu\alpha\beta,11} &= V_{11,\mu} V_{11,\alpha} V_{11,\beta} \\ &\quad + \frac{g^2}{2} (V_{11,\mu} W_\alpha^+ W_\beta^- + W_\mu^+ W_\alpha^- V_{11,\beta} + W_\mu^+ V_{22,\alpha} W_\beta^-), \\ M_{\mu\alpha\beta,12} &= \frac{g}{\sqrt{2}} (V_{11,\mu} W_\alpha^+ V_{22,\beta} + V_{11,\mu} V_{11,\alpha} W_\beta^+ + W_\mu^+ V_{22,\alpha} V_{22,\beta} \\ &\quad + \frac{g^2}{2} W_\mu^+ W_\alpha^- W_\beta^+), \\ M_{\mu\alpha\beta,21} &= \frac{g}{\sqrt{2}} (V_{22,\mu} W_\alpha^- V_{11,\beta} + V_{22,\mu} V_{22,\alpha} W_\beta^- + W_\mu^- V_{11,\alpha} V_{11,\beta} \\ &\quad + \frac{g^2}{2} W_\mu^- W_\alpha^+ W_\beta^-), \\ M_{\mu\alpha\beta,22} &= V_{22,\mu} V_{22,\alpha} V_{22,\beta} \\ &\quad + \frac{g^2}{2} (V_{22,\mu} W_\alpha^- W_\beta^+ + W_\mu^- W_\alpha^+ V_{22,\beta} + W_\mu^- V_{11,\alpha} W_\beta^+). \end{aligned} \quad (55b)$$

For the field strength one obtains

$$\mathbf{F}_{\mu\nu} = \begin{pmatrix} eA_{\mu\nu} + \frac{g}{2 \cos \theta_W} (1 - 2 \sin^2 \theta_W) Z_{\mu\nu} & \frac{g}{\sqrt{2}} W_{\mu\nu}^+ \\ \frac{g}{\sqrt{2}} W_{\mu\nu}^- & -\frac{g}{2 \cos \theta_W} Z_{\mu\nu} \end{pmatrix} - \frac{ig^2}{2} \begin{pmatrix} W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^- & \sqrt{2} (B_\mu^3 W_\nu^+ - W_\mu^+ B_\nu^3) \\ -\sqrt{2} (B_\mu^3 W_\nu^- - W_\mu^- B_\nu^3) & -W_\mu^+ W_\nu^- + W_\nu^+ W_\mu^- \end{pmatrix}, \quad (56)$$

where $X_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu$ for $X \in \{A, Z, W^+, W^-\}$. By making use of Eq. (26) one can completely express the off-diagonal elements in terms of the physical fields A_μ and Z_μ . The other combinations of fields appearing

in Eqs. (50) and (51) can also be easily obtained. We will not provide the explicit expressions here.

It is not difficult to see that the value of the Higgs field that minimizes the (non-commutative) Higgs potential is the same as in the commutative case because of the following: We are looking for the minimum value of the potential attained for *constant* fields and hence can ignore all derivative terms and all star products. This leaves terms like $\theta^{\alpha\beta}V_\alpha V_\beta\Phi$ in the hybrid SW map that could possibly lead to corrections of the vacuum expectation value of the Higgs. Taking into account also the potential of the gauge fields it is, however, clear that we should consider only $V_\alpha = 0$, i.e. $\widehat{\Phi} = \Phi$ when fixing the vacuum expectation value.

The Higgs field is chosen to be in the unitary gauge

$$\Phi(x) \equiv \phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h(x) + v \end{pmatrix}, \quad (57)$$

where $v = \sqrt{-\mu^2/\lambda}$ represents the Higgs vacuum expectation value, while $h(x)$ is the physical Higgs field.

There are several points that need to be mentioned in connection with the NCSM version of the S_{Higgs} part of the action (50). From (57) one trivially obtains

$$\int d^4x \phi^\dagger H \phi = \int d^4x (h(x) + v) H_{22} (h(x) + v),$$

where H stands here for any 2×2 matrix. Taking into account this along with (52) and (54-56), it is easy to see that terms containing one or more Higgs fields $h(x)$ as well as terms containing solely gauge bosons reside in (50).

First, let us examine the contributions of the last two θ -dependent terms in Eq. (50). By making use of (50-56) for the Higgs field in unitary gauge we find

$$\begin{aligned} & \frac{1}{2}\theta^{\alpha\beta} \int d^4x \phi^\dagger \left(\frac{1}{2}\mu^2 \mathbf{F}_{\alpha\beta} - 2i\lambda \phi (\mathbf{D}_\alpha \phi)^\dagger \mathbf{D}_\beta \right) \phi \\ &= \frac{1}{8}\theta^{\alpha\beta} \left\{ ig^2 \int d^4x (h+v)^2 [\mu^2 + \lambda(h+v)^2] W_\alpha^+ W_\beta^- \right. \\ & \quad \left. + \frac{g}{\cos\theta_W} \int d^4x (h+v)^2 [-\mu^2(\partial_\alpha Z_\beta) + 2\lambda(h+v)(\partial_\alpha h)Z_\beta] \right\}. \quad (58) \end{aligned}$$

Owing to the Stokes theorem the term containing only one Z field vanishes. Similarly, by performing partial integration and taking into account $v^2 =$

$-\mu^2/\lambda$, the spuriously looking two-field terms vanish and (58) simplifies to

$$\frac{1}{8}\theta^{\alpha\beta}\lambda\int d^4x h(h+v)(h+2v)\left\{ig^2(h+v)W_\alpha^+W_\beta^-+2\frac{g}{\cos\theta_W}(\partial_\alpha h)Z_\beta\right\}. \quad (59)$$

Second, let us note that, in contrast to the SM case, in the NCSM action S_{Higgs} (50) there are terms proportional to v^2 that cannot be identified as the mass terms of the Higgs and weak gauge bosons fields but represent interaction terms. Hence, after the identification of the mass terms ($-1/2 m_H^2 h^2$), $M_W^2 W_\mu^+ W^{-\mu}$ and $1/2 M_Z^2 Z_\mu Z^\mu$ with Higgs, W and Z boson masses

$$m_H^2 = 2\mu^2 = -2v^2\lambda, \\ M_W^2 = \frac{1}{4}v^2g^2, \quad M_Z^2 = \frac{1}{4}v^2(g^2 + g'^2) = \frac{M_W^2}{\cos^2\theta_W}, \quad (60)$$

respectively, additional terms remain which describe interactions of Higgs and gauge bosons and interactions of solely gauge bosons. The latter behaviour is novel in comparison with the Standard Model and is introduced by the Seiberg-Witten mapping. The analysis of Eq. (50) reveals that, in addition to the interaction terms contained in S_{gauge} (21), the last three terms of the second bracket in $\mathbf{U}_{\alpha\beta}$ (51) give rise to order θ contributions to the three- and four-gauge-boson couplings. Specifically, the three-gauge-boson interaction terms from S_{Higgs} read $(-1/4)v^2\theta^{\alpha\beta}[I_{\alpha\beta} + I_{\alpha\beta}^\dagger]_{22}$, where $I_{\alpha\beta} = \mathbf{V}^\mu[(\partial_\mu \mathbf{V}_\alpha)\mathbf{V}_\beta + \mathbf{V}_\alpha(\partial_\beta \mathbf{V}_\mu) + (\partial_\beta \mathbf{V}_\mu)\mathbf{V}_\alpha]$. By making use of (55) one arrives at explicit expressions for the W^+W^-Z , W^+W^-Z and ZZZ interaction terms:

$$\begin{aligned} & \frac{-1}{4}v^2\theta^{\alpha\beta}\left[I_{\alpha\beta} + I_{\alpha\beta}^\dagger\right]_{22} \\ &= \frac{e}{2}M_W^2\theta^{\alpha\beta}\left[(W^{+\mu}W_\alpha^- + W^{-\mu}W_\alpha^+)A_{\mu\beta} + (\partial_\beta A_\alpha)W^{+\mu}W_\mu^-\right] \\ & \quad - \frac{g}{4\cos\theta_W}M_W^2\theta^{\alpha\beta}\left\{Z^\mu\left[W_\mu^+(\partial_\beta W_\alpha^-) + W_\mu^-(\partial_\beta W_\alpha^+)\right] \right. \\ & \quad \left. + (Z^\mu W_\alpha^+ + Z_\alpha W^{+\mu})W_{\mu\beta}^- + (Z^\mu W_\alpha^- + Z_\alpha W^{-\mu})W_{\mu\beta}^+ \right. \\ & \quad \left. - \cos 2\theta_W\left[(W^{+\mu}W_\alpha^- + W^{-\mu}W_\alpha^+)Z_{\mu\beta} + (\partial_\beta Z_\alpha)W^{+\mu}W_\mu^-\right]\right\} \\ & \quad + \frac{g}{4\cos\theta_W}M_Z^2\theta^{\alpha\beta}Z^\mu Z_\alpha(2\partial_\beta Z_\mu - \partial_\mu Z_\beta). \end{aligned} \quad (61)$$

The four-gauge-boson interaction terms can be analysed analogously.

5.2 Yukawa Terms

Next, we proceed to the θ -expansion of the S_{Yukawa} action (8). Similarly to the analysis of the electroweak currents presented in Section 4, let us first analyse the general form for the Yukawa action,

$$S_{\psi, \text{Yukawa}} = - \int d^4x \sum_{i,j=1}^3 \left[\left(G_{\text{down}}^{(ij)} (\overline{\Psi}_L^{(i)} * h_{\psi_{\text{down}}}(\widehat{\Phi}) * \widehat{\psi}_{\text{down},R}^{(j)} + \text{h.c.}) \right) + \left(G_{\text{up}}^{(ij)} (\overline{\Psi}_L^{(i)} * h_{\psi_{\text{up}}}(\widehat{\Phi}_c) * \widehat{\psi}_{\text{up},R}^{(j)} + \text{h.c.}) \right) \right]. \quad (62)$$

Here G_{down} and G_{up} are general 3×3 matrices which comprise Yukawa couplings while $\widehat{\psi}_{\text{up},R}^{(j)}$ and $\widehat{\psi}_{\text{down},R}^{(j)}$ denote up and down fermion fields of the generation j . As we analyse a simple non-commutative extension of the SM, G_{up}^{ij} vanishes for leptons. Furthermore, as in the SM one can find a biunitary transformation that diagonalizes the G matrices

$$G_{\text{down}} = \frac{\sqrt{2}}{v} S_{\text{down}} M_{\text{down}} T_{\text{down}}^\dagger, \quad G_{\text{up}} = \frac{\sqrt{2}}{v} S_{\text{up}} M_{\text{up}} T_{\text{up}}^\dagger,$$

and obtain the diagonal 3×3 mass matrices M_{down} and M_{up} . Next, one redefines the fermion fields to mass eigenstates

$$\begin{aligned} \overline{\psi}_{\text{down},L}^{(i)} S_{\text{down}}^{(ij)} &\rightarrow \overline{\psi}_{\text{down},L}^{(j)} & T_{\text{down}}^{\dagger(ij)} \widehat{\psi}_{\text{down},R}^{(j)} &\rightarrow \widehat{\psi}_{\text{down},R}^{(i)} \\ \overline{\psi}_{\text{up},L}^{(i)} S_{\text{up}}^{(ij)} &\rightarrow \overline{\psi}_{\text{up},L}^{(j)} & T_{\text{up}}^{\dagger(ij)} \widehat{\psi}_{\text{up},R}^{(j)} &\rightarrow \widehat{\psi}_{\text{up},R}^{(i)}. \end{aligned}$$

This redefinition of the fields introduces the fermion mixing matrix $V = S_{\text{up}}^\dagger S_{\text{down}}$ in the electroweak currents (49), and, owing to the hybrid SW mapping of the Higgs field, in the Yukawa part of the NCSM action as well. We introduce the matrix V_f , which like in the SM, corresponds to

$$V_f = \begin{cases} \mathbf{1} & \text{for } f = \ell \\ V \equiv V_{CKM} & \text{for } f = q \end{cases}, \quad (63)$$

where ℓ and q denote leptons and quarks, respectively. Hence, the quark mixing is described by the CKM matrix, while the mixing in the lepton sector is absent but can be additionally introduced following the commonly accepted modifications of the SM which comprise neutrino masses. Furthermore, as the Higgs part of the NCSM action introduces mass dependent gauge boson couplings (see Eq. (61)), the Yukawa part of the NCSM action introduces fermion mass dependent interactions. In contrast to the NCSM, in the SM

fermion mass dependent interactions always include an interaction with the Higgs field.

Using Eq. (12) we find

$$\begin{aligned}
& \int d^4x \widehat{\Psi}_L^{(i)} * h_{\psi_{\text{down}}}(\widehat{\Phi}) * \widehat{\psi}_{\text{down},R}^{(j)} \\
&= \int d^4x (\overline{\Psi}_L^{(i)} \Phi \psi_{\text{down},R}^{(j)}) + \frac{1}{2} \int d^4x \theta^{\mu\nu} \overline{\Psi}_L^{(i)} \left[-i \overleftarrow{\partial}_\mu \Phi \overrightarrow{\partial}_\nu \right. \\
&\quad - \overleftarrow{\partial}_\nu \mathcal{R}_{\Psi_L}(V_\mu) \Phi - \Phi \mathcal{R}_{\psi_{\text{down},R}}(V_\mu) \overrightarrow{\partial}_\nu \\
&\quad - \mathcal{R}_{\Psi_L}(V_\mu) (\partial_\mu \Phi) - (\partial_\mu \Phi) \mathcal{R}_{\psi_{\text{down},R}}(V_\mu) \\
&\quad + i \mathcal{R}_{\Psi_L}(V_\mu) \mathcal{R}_{\Psi_L}(V_\nu) \Phi + i \Phi \mathcal{R}_{\psi_{\text{down},R}}(V_\mu) \mathcal{R}_{\psi_{\text{down},R}}(V_\nu) \\
&\quad \left. - i \mathcal{R}_{\Psi_L}(V_\mu) \Phi \mathcal{R}_{\psi_{\text{down},R}}(V_\nu) \right] \psi_{\text{down},R}^{(j)}. \tag{64}
\end{aligned}$$

The representations $\mathcal{R}_{\Psi_L}(V_\mu)$ and $\mathcal{R}_{\psi_{\text{down},R}}(V_\mu)$ can be read from Table 2. Expressions valid for both leptons and quarks, with strong interactions omitted, are given in Eqs. (37) and (45). For the Higgs field (57) is used.

Finally, using (64), after some algebra we obtain the following result for (62) expressed in terms of physical fields (and with gluons omitted):

$$\begin{aligned}
S_{\psi, \text{Yukawa}} &= \int d^4x \sum_{i,j=1}^3 \left[\overline{\psi}_{\text{down}}^{(i)} \left(N_{dd}^{V(ij)} + \gamma_5 N_{dd}^{A(ij)} \right) \psi_{\text{down}}^{(j)} \right. \\
&\quad + \overline{\psi}_{\text{up}}^{(i)} \left(N_{uu}^{V(ij)} + \gamma_5 N_{uu}^{A(ij)} \right) \psi_{\text{up}}^{(j)} \\
&\quad + \overline{\psi}_{\text{up}}^{(i)} \left(C_{ud}^{V(ij)} + \gamma_5 C_{ud}^{A(ij)} \right) \psi_{\text{down}}^{(j)} \\
&\quad \left. + \overline{\psi}_{\text{down}}^{(i)} \left(C_{du}^{V(ij)} + \gamma_5 C_{du}^{A(ij)} \right) \psi_{\text{up}}^{(j)} \right]. \tag{65}
\end{aligned}$$

The neutral currents read

$$\begin{aligned}
N_{dd}^{V(ij)} &= -M_{\text{down}}^{(ij)} \left(1 + \frac{h}{v} \right) + N_{dd}^{V,\theta(ij)} + \mathcal{O}(\theta^2), \\
N_{dd}^{A(ij)} &= N_{dd}^{A,\theta(ij)} + \mathcal{O}(\theta^2), \\
N_{uu}^{V(ij)} &= -M_{\text{up}}^{(ij)} \left(1 + \frac{h}{v} \right) + N_{uu}^{V,\theta(ij)} + \mathcal{O}(\theta^2), \\
N_{uu}^{A(ij)} &= N_{uu}^{A,\theta(ij)} + \mathcal{O}(\theta^2), \tag{66}
\end{aligned}$$

where

$$\begin{aligned}
& N_{dd}^{V,\theta(ij)} \\
&= -\frac{1}{2}\theta^{\mu\nu} M_{\text{down}}^{(ij)} \left\{ i\frac{(\partial_\mu h)}{v} \vec{\partial}_\nu \right. \\
&\quad - \left[eQ_{\psi_{\text{down}}} A_\mu + \frac{g}{2\cos\theta_W} (T_{3,\psi_{\text{down},L}} - 2Q_{\psi_{\text{down}}} \sin^2\theta_W) Z_\mu \right] \frac{(\partial_\nu h)}{v} \\
&\quad + \left[eQ_{\psi_{\text{down}}} (\partial_\nu A_\mu) + \frac{g}{2\cos\theta_W} (T_{3,\psi_{\text{down},L}} - 2Q_{\psi_{\text{down}}} \sin^2\theta_W) (\partial_\nu Z_\mu) \right. \\
&\quad \left. \left. - i\frac{g^2}{2} W_\mu^+ W_\nu^- \right] \left(1 + \frac{h}{v} \right) \right\}, \tag{67}
\end{aligned}$$

$$\begin{aligned}
N_{dd}^{A,\theta(ij)} &= \frac{g}{4\cos\theta_W} T_{3,\psi_{\text{down},L}} \theta^{\mu\nu} M_{\text{down}}^{(ij)} \left(1 + \frac{h}{v} \right) Z_\mu \\
&\quad \times \left[\left(\overleftarrow{\partial}_\nu - \overrightarrow{\partial}_\nu \right) + 2ieQ_{\psi_{\text{down}}} A_\nu \right], \tag{68}
\end{aligned}$$

and

$$\left. \begin{array}{l} N_{uu}^{V,\theta(ij)} \\ N_{uu}^{A,\theta(ij)} \end{array} \right\} = \left\{ \begin{array}{l} N_{dd}^{V,\theta(ij)} \\ N_{dd}^{A,\theta(ij)} \end{array} \right. (W^+ \leftrightarrow W^-, \text{down} \rightarrow \text{up}). \tag{69}$$

The charged currents are given by

$$\begin{aligned}
C_{ud}^{V(ij)} &= C_{ud}^{V,\theta(ij)} + \mathcal{O}(\theta^2), \\
C_{ud}^{A(ij)} &= C_{ud}^{A,\theta(ij)} + \mathcal{O}(\theta^2), \tag{70}
\end{aligned}$$

where

$$\begin{aligned}
& C_{ud}^{V,\theta(ij)} \\
&= -\frac{g}{4\sqrt{2}}\theta^{\mu\nu} \left(1 + \frac{h}{v} \right) \left\{ \left[(V_f M_{\text{down}})^{(ij)} + (M_{\text{up}} V_f)^{(ij)} \right] (\partial_\nu W_\mu^+) \right. \\
&\quad + \left[(V_f M_{\text{down}})^{(ij)} \overrightarrow{\partial}_\nu + (M_{\text{up}} V_f)^{(ij)} \overleftarrow{\partial}_\nu \right] W_\mu^+ \Big] \\
&\quad + ie \left[(V_f M_{\text{down}})^{(ij)} Q_{\psi_{\text{up}}} - (M_{\text{up}} V_f)^{(ij)} Q_{\psi_{\text{down}}} \right] A_\mu W_\nu^+ \\
&\quad + i\frac{g}{\cos\theta_W} \left[(V_f M_{\text{down}})^{(ij)} (2T_{3,\psi_{\text{up},L}} - Q_{\psi_{\text{up}}} \sin^2\theta_W) \right. \\
&\quad \left. - (M_{\text{up}} V_f)^{(ij)} (2T_{3,\psi_{\text{down},L}} - Q_{\psi_{\text{down}}} \sin^2\theta_W) \right] Z_\mu W_\nu^+ \Big\}, \tag{71}
\end{aligned}$$

and

$$C_{ud}^{A,\theta(ij)} = C_{ud}^{V,\theta(ij)} (M_{\text{up}} \rightarrow -M_{\text{up}}), \tag{72}$$

while

$$\begin{aligned} C_{du}^{V(ij)} &= \left(C_{ud}^{V(ij)} (\vec{\partial} \leftrightarrow \overleftarrow{\partial}) \right)^\dagger, \\ C_{du}^{A(ij)} &= - \left(C_{ud}^{A(ij)} (\vec{\partial} \leftrightarrow \overleftarrow{\partial}) \right)^\dagger. \end{aligned} \quad (73)$$

Note that $\vec{\partial}$ and $\overleftarrow{\partial}$ are defined in (44).

At the end, observe that the simplified introduction of the fermion mass and the use of the relation

$$\begin{aligned} S_{\psi,m} &= \int d^4x \overline{\widehat{\psi}} * (i\widehat{\mathcal{D}} - m) \widehat{\psi} \\ &= \int d^4x \left[\overline{\psi} (i\mathcal{D} - m) \psi - \frac{1}{4} \overline{\psi} \mathcal{R}_\psi(F_{\mu\nu}) (i\theta^{\mu\nu\rho} D_\rho - m\theta^{\mu\nu}) \psi \right. \\ &\quad \left. + \mathcal{O}(\theta^2) \right]. \end{aligned} \quad (74)$$

is valid only in the case of pure QED and pure QCD.

6 Feynman Rules

On the basis of the results presented in Sections 4 and 5, it is now straightforward to derive the Feynman rules needed for phenomenological applications of the NCSM, i.e. for the calculation of physical processes. In this section, we list a number of selected Feynman rules for the NCSM pure electroweak interactions up to order θ . We omit interactions with the Higgs particle, boson interactions with four and more gauge fields, and fermion interactions with more than two gauge bosons.

The following notation for vertices has been adopted: *all gauge boson momenta are taken to be incoming; following the flow of the fermion line, the momenta of the incoming and outgoing fermions are given by p_{in} and p_{out} , respectively.* In the following we denote fermions by f , and the generation indices by i and j . Furthermore, $f_u^{(i)} \in \{\nu^{(i)}, u^{(i)}\}$ and $f_d^{(i)} \in \{e^{(i)}, d^{(i)}\}$.

For the Feynman rules we use the following definitions:

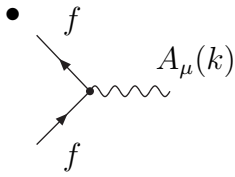
$$\begin{aligned} c_{V,f} &= T_{3,f_L} - 2Q_f \sin^2 \theta_W, \\ c_{A,f} &= T_{3,f_L}. \end{aligned} \quad (75)$$

The charge Q and the weak isospin T_3 can be read from Table 1. The notation V_f is introduced in (63), while $\theta^{\mu\nu\rho}$ is defined in (35). We also make use of $(\theta k)^\mu \equiv \theta^{\mu\nu} k_\nu = -k_\nu \theta^{\nu\mu} \equiv -(k\theta)^\mu$ and $(k\theta p) \equiv k_\mu \theta^{\mu\nu} p_\nu$.

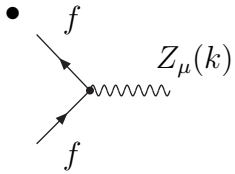
6.1 Minimal NCSM

In this subsection we present selected Feynman rules for the mNCSM containing SM contributions and θ corrections. The θ corrections to vertices containing fermions are obtained using Eq. (49) and the Yukawa part of the action (65) has to be taken into account as well, because it generates additional mass dependent terms which modify some interaction vertices. In comparison with the SM, this is a novel feature. Similarly, the gauge boson couplings present in (24) receive additional θ dependent corrections from the Higgs part of the action (50) and even new three- and four-gauge boson couplings appear, see (61).

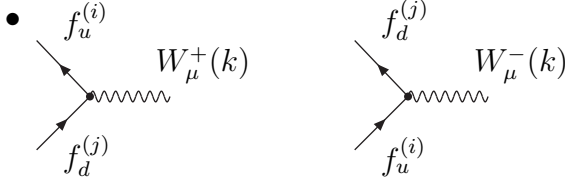
First, we list three-vertices that appear in the SM as well.



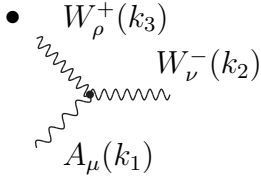
$$\begin{aligned}
 & i e Q_f \left[\gamma_\mu - \frac{i}{2} k^\nu (\theta_{\mu\nu\rho} p_{\text{in}}^\rho - \theta_{\mu\nu} m_f) \right] \\
 &= i e Q_f \gamma_\mu \\
 & \quad + \frac{1}{2} e Q_f [(p_{\text{out}} \theta p_{\text{in}}) \gamma_\mu - (p_{\text{out}} \theta)_\mu (\not{p}_{\text{in}} - m_f) - (\not{p}_{\text{out}} - m_f) (\theta p_{\text{in}})_\mu] ,
 \end{aligned} \tag{76}$$



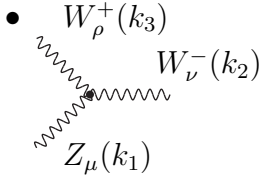
$$\begin{aligned}
 & \frac{i e}{\sin 2\theta_W} \left\{ \left(\gamma_\mu - \frac{i}{2} k^\nu \theta_{\mu\nu\rho} p_{\text{in}}^\rho \right) (c_{V,f} - c_{A,f} \gamma_5) \right. \\
 & \quad \left. - \frac{i}{2} \theta_{\mu\nu} m_f \left[p_{\text{in}}^\nu (c_{V,f} - c_{A,f} \gamma_5) - p_{\text{out}}^\nu (c_{V,f} + c_{A,f} \gamma_5) \right] \right\} ,
 \end{aligned} \tag{77}$$



$$\frac{ie}{2\sqrt{2}\sin\theta_W} \begin{pmatrix} V_f^{(ij)} \\ V_f^{*(ij)} \end{pmatrix} \left\{ \left[\gamma_\mu - \frac{i}{2} \theta_{\mu\nu\rho} k^\nu p_{\text{in}}^\rho \right] (1 - \gamma_5) - \frac{i}{2} \theta_{\mu\nu} \left[\begin{pmatrix} m_{f_u^{(i)}} \\ m_{f_d^{(j)}} \end{pmatrix} p_{\text{in}}^\nu (1 - \gamma_5) - \begin{pmatrix} m_{f_d^{(j)}} \\ m_{f_u^{(i)}} \end{pmatrix} p_{\text{out}}^\nu (1 + \gamma_5) \right] \right\}, \quad (78)$$



$$ie \left\{ g^{\mu\nu} (k_1 - k_2)^\rho + g^{\nu\rho} (k_2 - k_3)^\mu + g^{\rho\mu} (k_3 - k_1)^\nu + \frac{i}{2} M_W^2 \left[\theta^{\mu\nu} k_1^\rho + \theta^{\mu\rho} k_1^\nu + g^{\mu\nu} (\theta k_1)^\rho - g^{\nu\rho} (\theta k_1)^\mu + g^{\rho\mu} (\theta k_1)^\nu \right] \right\}, \quad (79)$$



$$ie \cot\theta_W \left\{ g^{\mu\nu} (k_1 - k_2)^\rho + g^{\nu\rho} (k_2 - k_3)^\mu + g^{\rho\mu} (k_3 - k_1)^\nu + \frac{i}{2} M_W^2 \left[\theta^{\mu\nu} k_1^\rho + \theta^{\mu\rho} k_1^\nu + g^{\mu\nu} (\theta k_1)^\rho - g^{\nu\rho} (\theta k_1)^\mu + g^{\rho\mu} (\theta k_1)^\nu \right] - \frac{i}{4} M_Z^2 \left[\theta^{\mu\nu} (k_1 - k_2)^\rho + \theta^{\nu\rho} (k_2 - k_3)^\mu + \theta^{\rho\mu} (k_3 - k_1)^\nu - 2g^{\mu\nu} (\theta k_3)^\rho - 2g^{\nu\rho} (\theta k_1)^\mu - 2g^{\rho\mu} (\theta k_2)^\nu \right] \right\}. \quad (80)$$

Here we give the new three-gauge-boson coupling which follows from the Higgs action (50), i.e., Eq. (61)

- $$\frac{e M_Z^2}{2 \sin 2\theta_W} \left[\theta^{\mu\nu} (k_1 - k_2)^\rho + \theta^{\nu\rho} (k_2 - k_3)^\mu + \theta^{\rho\mu} (k_3 - k_1)^\nu - 2g^{\mu\nu} (\theta k_3)^\rho - 2g^{\nu\rho} (\theta k_1)^\mu - 2g^{\rho\mu} (\theta k_2)^\nu \right]. \quad (81)$$

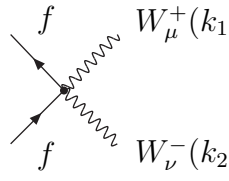
Additionally, from the Higgs action (50) one can derive the θ corrections to the electroweak four-gauge-boson vertices already present in SM (see (24)), as well as, new four-gauge-boson vertices.

Equation (49) also describes the interaction vertices involving fermions and two or three gauge bosons. These do not appear in the SM. In the following we provide all contributions to such vertices with four legs and corresponding mass-dependent contributions from (65).

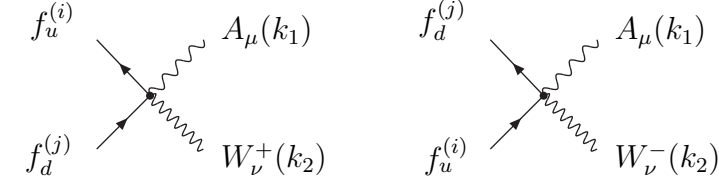
- $$-\frac{e^2 Q_f^2}{2} \theta_{\mu\nu\rho} (k_1^\rho - k_2^\rho), \quad (82)$$

- $$\frac{-e^2 Q_f}{2 \sin 2\theta} \times \left[\theta_{\mu\nu\rho} (k_1^\rho - k_2^\rho) (c_{V,f} - c_{A,f} \gamma_5) - 2\theta_{\mu\nu} m_f c_{A,f} \gamma_5 \right], \quad (83)$$

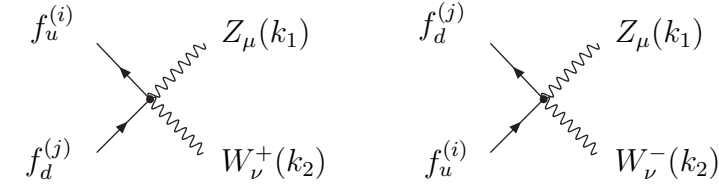
- $$\frac{-e^2}{2 \sin^2 2\theta} \theta_{\mu\nu\rho} (k_1^\rho - k_2^\rho) (c_{V,f} - c_{A,f} \gamma_5)^2, \quad (84)$$

- 

$$\frac{-e^2}{8 \sin^2 \theta_W} \left[\theta_{\mu\nu\rho} (p_{\text{in}}^\rho + k_1^\rho)(1 - \gamma_5) + 2\theta_{\mu\nu} m_f \right], \quad (85)$$

- 

$$\frac{-e^2}{4\sqrt{2} \sin \theta_W} \left\{ \theta_{\mu\nu\rho} \left[\begin{pmatrix} Q_{f_u^{(i)}} \\ Q_{f_d^{(j)}} \end{pmatrix} (p_{\text{in}}^\rho + k_1^\rho) - \begin{pmatrix} Q_{f_d^{(j)}} \\ Q_{f_u^{(i)}} \end{pmatrix} (p_{\text{in}}^\rho + k_2^\rho) \right] (1 - \gamma_5) \right. \\ \left. + \theta_{\mu\nu} \left[\begin{pmatrix} m_{f_u^{(i)}} Q_{f_d^{(j)}} \\ m_{f_d^{(j)}} Q_{f_u^{(i)}} \end{pmatrix} (1 - \gamma_5) - \begin{pmatrix} m_{f_d^{(j)}} Q_{f_u^{(i)}} \\ m_{f_u^{(i)}} Q_{f_d^{(j)}} \end{pmatrix} (1 + \gamma_5) \right] \right\} \begin{pmatrix} V_f^{(ij)} \\ V_f^{*(ij)} \end{pmatrix}, \quad (86)$$

- 

$$\frac{-e^2}{4\sqrt{2} \sin \theta_W \sin 2\theta_W} \begin{pmatrix} V_f^{(ij)} \\ V_f^{*(ij)} \end{pmatrix} \\ \left\{ \theta_{\mu\nu\rho} \left[\begin{pmatrix} c_{V,f_u^{(i)}} + c_{A,f_u^{(i)}} \\ c_{V,f_d^{(j)}} + c_{A,f_d^{(j)}} \end{pmatrix} (p_{\text{in}}^\rho + k_1^\rho) - \begin{pmatrix} c_{V,f_d^{(j)}} + c_{A,f_d^{(j)}} \\ c_{V,f_u^{(i)}} + c_{A,f_u^{(i)}} \end{pmatrix} (p_{\text{in}}^\rho + k_2^\rho) \right] (1 - \gamma_5) \right. \\ \left. + \theta_{\mu\nu} \left[\begin{pmatrix} m_{f_u^{(i)}} [c_{V,f_d^{(j)}} + 3c_{A,f_d^{(j)}}] \\ m_{f_d^{(j)}} [c_{V,f_u^{(i)}} + 3c_{A,f_u^{(i)}}] \end{pmatrix} (1 - \gamma_5) \right. \right. \\ \left. \left. - \begin{pmatrix} m_{f_d^{(j)}} [c_{V,f_u^{(i)}} + 3c_{A,f_u^{(i)}}] \\ m_{f_u^{(i)}} [c_{V,f_d^{(j)}} + 3c_{A,f_d^{(j)}}] \end{pmatrix} (1 + \gamma_5) \right] \right\}. \quad (87)$$

Similarly, $ffWWZ$, $ffWW\gamma$ and $ff\gamma WZ$ can be extracted from Eq. (49) as well. They have no mass-dependent corrections.

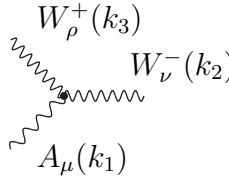
6.2 Non-Minimal NCSM

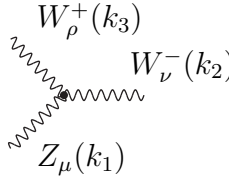
Here we give the selected Feynman rules for the non-minimal NCSM introduced in Subsection 3.2. Observe that the fermion sector is not affected by the change of the representation in the gauge part of the action. Let us define

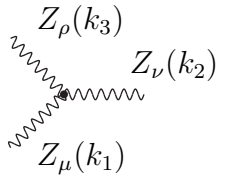
$$\begin{aligned}
\Theta_3((\mu, k_1), (\nu, k_2), (\rho, k_3)) &= -\theta^{\mu\nu}[k_1^\rho(k_2 k_3) - k_2^\rho(k_1 k_3)] \\
&+ (\theta k_1)^\mu [g^{\nu\rho}(k_2 k_3) - k_2^\rho k_3^\nu] - (\theta k_1)^\nu [g^{\rho\mu}(k_2 k_3) - k_2^\rho k_3^\mu] \\
&- (\theta k_1)^\rho [g^{\mu\nu}(k_2 k_3) - k_2^\mu k_3^\nu] + (k_1 \theta k_2) [k_3^\mu g^{\nu\rho} - k_3^\nu g^{\rho\mu}] \\
&+ \text{cyclical permutations of } (\mu_i, k_i).
\end{aligned} \tag{88}$$

We use the simplified notation $\mu_1 \equiv \mu$, $\mu_2 \equiv \nu$ and $\mu_3 \equiv \rho$.

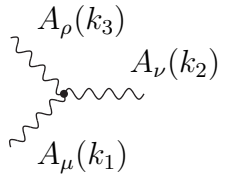
First, we list the Feynman rules for the modified $W^+W^-\gamma$, W^+W^-Z and ZZZ vertices already present in the mNCSM:

-  Eq. (79) + $2 e \sin 2\theta_W K_{WW\gamma} \Theta_3((\mu, k_1), (\nu, k_2), (\rho, k_3)),$ (89)

-  Eq. (80) + $2 e \sin 2\theta_W K_{WWZ} \Theta_3((\mu, k_1), (\nu, k_2), (\rho, k_3)),$ (90)

-  Eq. (81) + $2 e \sin 2\theta_W K_{ZZZ} \Theta_3((\mu, k_1), (\nu, k_2), (\rho, k_3)).$ (91)

Additionally, we give the new gauge boson vertices $\gamma\gamma\gamma$, $Z\gamma\gamma$ and $ZZ\gamma$:

-  $2 e \sin 2\theta_W K_{\gamma\gamma\gamma} \Theta_3((\mu, k_1), (\nu, k_2), (\rho, k_3)),$ (92)

- $$Z_\nu(k_2) \quad -2 e \sin 2\theta_W K_{Z\gamma\gamma} \Theta_3((\mu, k_1), (\nu, k_2), (\rho, k_3)), \quad (93)$$

- $$Z_\nu(k_2) \quad -2 e \sin 2\theta_W K_{ZZ\gamma} \Theta_3((\mu, k_1), (\nu, k_2), (\rho, k_3)). \quad (94)$$

The functions K are not independent and they are defined in Eqs. (29,31).

7 Conclusions

The main purpose of this article is to complete the Non-Commutative Standard Model constructed in [6,9], and thus to make it accessible to phenomenological considerations and further research. The NCSM action are given in terms of physical fields and mass eigenstates. The freedom in the choice of traces in kinetic terms for gauge fields produces two versions of the NCSM, namely the mNCSM and the nmNCSM. However, such freedom does not affect the matter sector of the action and the fermion-gauge boson interactions remain the same in both versions of the NCSM. We have provided an explicit expression for selected vertices of which some already appear in the original SM, but in the NCSM they gain θ -dependent corrections, whereas others appear for the first time in the non-commutative version of the SM. We have presented a careful discussion of electroweak charged and neutral currents as well as a derivation of the Higgs and Yukawa terms of the NCSM action.

Among the novel features in comparison with previous works [6,9] are the appearance of additional gauge boson interaction terms (30) and (61) in the gauge (21) and in the Higgs (50) parts of the action, and the appearance of mass-dependent corrections to the boson-boson and fermion-boson couplings stemming from the Higgs and Yukawa parts of the action, respectively. In Eqs. (76-78) the mass-dependent terms stem from the Yukawa interactions (64-73), while in Eqs. (79,80) the mass corrections arise from the θ -expanded Higgs action (61). To first order in θ , equation (65) contains coupling of fermions to gauge bosons that depend on the mass of the fermion involved. Also the appearance of new terms (61) would certainly produce important contributions in a number of physical processes. All the above features are introduced by the Seiberg-Witten maps.

CP violation induced by space-time non-commutativity has potential to be a particular sensitive probe of non-commutativity [33]. The analysis of C,P,T properties of the NCSM and the NCGUTs [31] shows that θ transforms under C, P, T in such a way that it preserves these discrete symmetries in the action. However, considering θ as a fixed background (or spectator) field, there will be spontaneous breaking of CP (relative to the background), just as one has spontaneous breaking of Lorentz symmetries in non-commutative theories. Consequently, non-commutative effects can also mix with the CKM-matrix CP-violating parameter δ in the spirit of Ref. [33]. Since the fermion sectors of the mNCSM and the nmNCS are equal, the above conclusion is valid for both models. It should be noted that in the present work, the unitary CKM mixing matrix has been considered with matrix elements not as functions of space-time but as constants. Furthermore, the θ -expansions of the SW map and the star product have been worked out only up to first order.

In conclusion, the thorough analysis of the electroweak sector considered in this paper facilitates further research on reliable bounds on non-commutativity from hadronic and leptonic physics.

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References

- [1] M. Kontsevich, *Lett. Math. Phys.* **66** (2003) 157 [q-alg/9709040].
- [2] N. Seiberg and E. Witten, *JHEP* **09** (1999) 032 [hep-th/9908142].
- [3] J. Madore, S. Schraml, P. Schupp and J. Wess, *Eur. Phys. J. C* **16** (2000) 161 [hep-th/0001203].
- [4] B. Jurčo, S. Schraml, P. Schupp and J. Wess, *Eur. Phys. J. C* **17** (2000) 521 [hep-th/0006246].

- [5] B. Jurčo, L. Möller, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C**21** (2001) 383 [hep-th/0104153].
- [6] X. Calmet, B. Jurčo, P. Schupp, J. Wess and M. Wohlgenannt, Eur. Phys. J. C**23** (2002) 363 [hep-ph/0111115].
- [7] M. Chaichian, P. Presnajder and A. Tureanu, hep-th/0409096; to appear in Phys. Rev. Lett.
- [8] F. Koch and E. Tsouchnika, hep-th/0409012.
- [9] W. Behr, N.G. Deshpande, G. Duplančić, P. Schupp, J. Trampetić and J. Wess, Eur. Phys. J. C**29** (2003) 441 [hep-ph/0202121].
- [10] G. Duplančić, P. Schupp and J. Trampetić, Eur. Phys. J. C**32** (2003) 141 [hep-ph/0309138].
- [11] P. Schupp, J. Trampetić, J. Wess and G. Raffelt, Eur. Phys. J. C **36** (2004) 405 [hep-ph/0212292].
- [12] P. Minkowski, P. Schupp and J. Trampetić, Eur. Phys. J. C **37** (2004) 123 [hep-th/0302175].
- [13] J. Trampetić, Acta Phys. Polon. B**33** (2002) 4317 [hep-ph/0212309].
- [14] P. Schupp and J. Trampetić, hep-ph/0405163.
- [15] T. Ohl and J. Reuter, Phys. Rev. D**70** (2004) 076007 [hep-ph/0406098].
- [16] A. Armoni, Nucl. Phys. B**593** (2001) 229 [hep-th/0005208].
- [17] A. Bichl, J. Grimstrup, H. Grosse, L. Popp, M. Schweda and R. Wulkenhaar, JHEP **06** (2001) 013 [hep-th/0104097].
- [18] J. Grimstrup, B. Kloibock, L. Popp, V. Putz, M. Schweda and M. Wickenhauser, hep-th/0210288.
- [19] F. Brandt, C.P. Martín and F. Ruiz Ruiz, JHEP **07** (2003) 068 [hep-th/0307292].
- [20] J. Gomis and Th. Mehen, Nucl. Phys. B**591** (2000) 265 [hep-th/0005129].
- [21] X. Calmet and M. Wohlgenannt, Phys. Rev. D**68** (2003) 025016 [hep-ph/0305027].

- [22] M. Dimitrijević, L. Jonke, L. Möller, E. Tschouchnika, J. Wess and M. Wohlgenannt, *Eur. Phys. J. C* **31** (2003) 129 [hep-th/0307149].
- [23] M. Dimitrijević, F. Meyer, L. Möller and J. Wess, *Eur. Phys. J. C* **36** (2004) 117 [hep-th/0310116].
- [24] W. Behr and A. Sykora, *Nucl. Phys. B* **698** (2004) 473 [hep-th/0309145].
- [25] J.M. Grimstrup, T. Jonsson and L. Thorlacius, *JHEP* **12** (2003) 001 [hep-th/0310179].
- [26] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, *Phys. Lett. B* **604** (2004) 98 [hep-th/0408069].
- [27] M. Chaichian, P. Presnajder, M.M. Sheikh-Jabbari and A. Tureanu, *Eur. Phys. J. C* **29** (2003) 413 [hep-th/0107055].
- [28] M. Chaichian, A. Kobakhidze and A. Tureanu, hep-th/0408065.
- [29] B. Melic, K. Passek-Kumericki, J. Trampetic, P. Schupp and M. Wohlgenannt, hep-ph/0503064.
- [30] S. Denk, V. Putz and M. Wohlgenannt, hep-th/0402229.
- [31] P. Aschieri, B. Jurčo, P. Schupp and J. Wess, *Nucl. Phys. B* **651** (2003) 45 [hep-th/0205214].
- [32] S. Weinberg, “The Quantum theory of fields. Vol. 1: Foundations,” Section 7.2, Cambridge University Press 2000.
- [33] I. Hinchliffe and N. Kersting, *Phys. Rev. D* **64** (2001) 116007 [hep-ph/0104137].