The statistical distribution of the maxima of a random function

By D. E. Cartwright and M. S. Longuet-Higgins
National Institute of Oceanography, Wormley

(*Communicated by G. E. R. Deacon, F.R.S.—Received* 14 *April* 1956)

This paper studies the statistical distribution of the maximum values of a random function which is the sum of an infinite number of sine waves in random phase. The results are applied to sea waves and to the pitching and rolling motion of a ship.

Introduction

Let $f(t)$ denote a continuous, random function of the time t, representing, for example, the height of the sea surface above a fixed point. It is interesting to inquire into the statistical distribution of the heights of the *maxima* of $f(t)$.

There are two distinct problems. On the one hand we may consider the total wave height $2a$, being defined as the difference in level between a crest (maximum) and the preceding trough (minimum). The statistical distribution of a is difficult to determine in the general case, but when $f(t)$ has a narrow frequency spectrum it may be shown that a is distributed according to a Rayleigh distribution

$$p(a) = \frac{2a}{m_0} e^{-a^2/m_0},$$

where $m_0^{\frac{1}{2}}$ is the root-mean-square value of $f(t)$ (see Rayleigh 1880). This distribution has been compared with the observed distribution of the heights of sea waves and it has been shown that many theoretical relations, for example the ratios of the mean wave height to the mean of the highest one-third waves or to the mean of the highest of N consecutive waves, are in close agreement with observation (Longuet-Higgins 1952). Application of the χ^2-test to some histograms of wave heights has also indicated, apparently, no significant departure from the Rayleigh distribution (Watters 1953). It is certain, however, that for functions $f(t)$ having a broad frequency spectrum, the theoretical distribution of a must be different from the Rayleigh distribution.

Alternatively, we may consider the difference in height ξ between a crest and the mean level of the function $f(t)$. Although in practice ξ may be less convenient to measure than a (since the appropriate mean value is sometimes difficult to determine) the theoretical distribution of ξ is easier to obtain, and has been found for a wide class of random functions by Rice (1944, 1945) in connexion with the analysis of electrical noise signals. Rice's solution, which is only one out of many results in a long paper, has not been fully discussed, and it is the purpose of the present paper to examine the solution and to calculate some of the statistical parameters associated with it. We shall also apply the results to ocean waves and to the motion of ships at sea.

In §1 we outline briefly Rice's derivation of the statistical distribution of the maxima ξ. The discussion shows that the distribution depends, surprisingly, on only two parameters: the root-mean-square value of $f(t)$, which we denote by $m_0^{\frac{1}{2}}$, and a parameter ϵ which, as we show in §2, represents the relative width of the frequency spectrum of $f(t)$. When ϵ is small, the distribution of ξ tends to a Rayleigh distribution, as we should expect, and when ϵ approaches its maximum value 1 the distribution of ξ tends to a Gaussian distribution.

One of the main differences between the two variables ξ and a is that ξ may take negative values (since some maxima may lie below the mean level) whereas a is always positive. The proportion r of maxima that are negative can be readily determined in practice, and in §3 we show that this proportion depends only upon ϵ. Hence if r is measured, ϵ can be estimated.

In §§4–6 we calculate the moments of the distribution, the mean values of the highest $1/n$th of all the crest heights and the expectation of the highest in a sample of N crest heights, and we show how these quantities depend upon ϵ.

The distribution of crest heights, as measured from records of ocean wave phenomena, is compared with the theoretical distribution in §7. No significant difference is found. On the other hand, the crest-to-trough heights, examined in §8, are found to depart significantly from the Rayleigh distribution.

1. The distribution of maxima

The random function $f(t)$ is represented as the sum of an infinite number of sine-waves

$$f(t) = \sum_n c_n \cos(\sigma_n t + \epsilon_n), \tag{1·1}$$

where the frequencies σ_n are distributed densely in the interval $(0, \infty)$, the phases ϵ_n are random and distributed uniformly between 0 and 2π, and the amplitudes c_n are such that in any small interval of frequency $\mathrm{d}\sigma$

$$\sum_{\sigma_n=\sigma}^{\sigma+\mathrm{d}\sigma} \tfrac{1}{2}c_n^2 = E(\sigma)\,\mathrm{d}\sigma, \tag{1·2}$$

where $E(\sigma)$ is a continuous function of σ which will be called the energy spectrum of $f(t)$. The total energy per unit length of record is

$$m_0 = \int_0^\infty E(\sigma)\,\mathrm{d}\sigma. \tag{1·3}$$

More generally we shall find it convenient to write

$$m_n = \int_0^\infty E(\sigma)\,\sigma^n\,\mathrm{d}\sigma \tag{1·4}$$

for the nth moment of $E(\sigma)$ about the origin.

To find the distribution of maxima of $f(t)$ we note that, if $f(t)$ has a maximum in the interval $(t, t+\mathrm{d}t)$, then in this interval $f'(t)$ must take values in a range of width $|f''(t)|\,\mathrm{d}t$ very nearly; and the probability of this occurrence, and of f simultaneously lying in the range $(\xi_1, \xi_1+\mathrm{d}\xi_1)$, is

$$\int_{-\infty}^0 [p(\xi_1, 0, \xi_3)\,\mathrm{d}\xi_1\,|\xi_3|\,\mathrm{d}t]\,\mathrm{d}\xi_3, \tag{1·5}$$

where $p(\xi_1, \xi_2, \xi_3)$ is the joint probability distribution of

$$(\xi_1, \xi_2, \xi_3) = (f, f', f''). \tag{1·6}$$

The mean frequency of maxima in the range $\xi_1 < f < \xi_1 + d\xi_1$ is therefore

$$F(\xi_1)\, d\xi_1 = \int_{-\infty}^{0} [p(\xi_1, 0, \xi_3)\, |\, \xi_3\, |\, d\xi_1]\, d\xi_3, \tag{1·7}$$

and the probability distribution of maxima is found by dividing this distribution by the total mean frequency of maxima, which is

$$N_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{0} p(\xi_1, 0, \xi_3)\, |\, \xi_3\, |\, d\xi_1\, d\xi_3. \tag{1·8}$$

Now from (1·6) we have

$$\left.\begin{aligned} \xi_1 &= f(t) = \sum_n c_n \cos(\sigma_n t + \epsilon_n), \\ \xi_2 &= f'(t) = -\sum_n c_n \sigma_n \sin(\sigma_n t + \epsilon_n), \\ \xi_3 &= f''(t) = -\sum_n c_n \sigma_n^2 \cos(\sigma_n t + \epsilon_n). \end{aligned}\right\} \tag{1·9}$$

ξ_1, ξ_2, ξ_3 are therefore each the sum of an infinite number of variables of zero expectation and random phase. Therefore, by the central limit theorem in three dimensions, the joint probability distribution of (ξ_1, ξ_2, ξ_3) is normal (under general conditions assumed to be satisfied by the amplitudes c_n; see Rice 1944, 1945). The matrix of correlations or statistical averages $\Xi_{ij} = \overline{\xi_i \xi_j}$ is seen to be

$$(\Xi_{ij}) = \begin{pmatrix} m_0 & 0 & -m_2 \\ 0 & m_2 & 0 \\ -m_2 & 0 & m_4 \end{pmatrix}. \tag{1·10}$$

Hence

$$p(\xi_1, \xi_2, \xi_3) = \frac{1}{(2\pi)^{\frac{3}{2}} (\Delta m_2)^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}[\xi_2^2/m_2 + (m_4 \xi_1^2 + 2m_2 \xi_1 \xi_3 + m_0 \xi_3^2)/\Delta]\}, \tag{1·11}$$

where

$$\Delta = m_0 m_4 - m_2^2. \tag{1·12}$$

Substituting in (1·7) we have

$$F(\xi_1) = \frac{1}{(2\pi)^{\frac{3}{2}} (\Delta m_2)^{\frac{1}{2}}} \int_{-\infty}^{0} \exp\{-\tfrac{1}{2}(m_4 \xi_1^2 + 2m_2 \xi_1 \xi_3 + m_0 \xi_3^2)/\Delta\}\, |\, \xi_3\, |\, d\xi_3. \tag{1·13}$$

On evaluating the integral and writing

$$\xi_1 / m_0^{\frac{1}{2}} = \eta, \quad \Delta^{\frac{1}{2}} / m_2 = \delta \tag{1·14}$$

we obtain

$$F(\xi_1) = \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{\Delta^{\frac{1}{2}}}{m_0 m_4^{\frac{1}{2}}} e^{-\frac{1}{2}\eta^2} \left[e^{-\frac{1}{2}\eta^2/\delta^2} + (\eta/\delta) \int_{-\eta/\delta}^{\infty} e^{-\frac{1}{2}x^2}\, dx \right]. \tag{1·15}$$

The last integral can be expressed in terms of the known function

$$\operatorname{erf} x = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^x e^{-\frac{1}{2}x^2}\, dx. \tag{1·16}$$

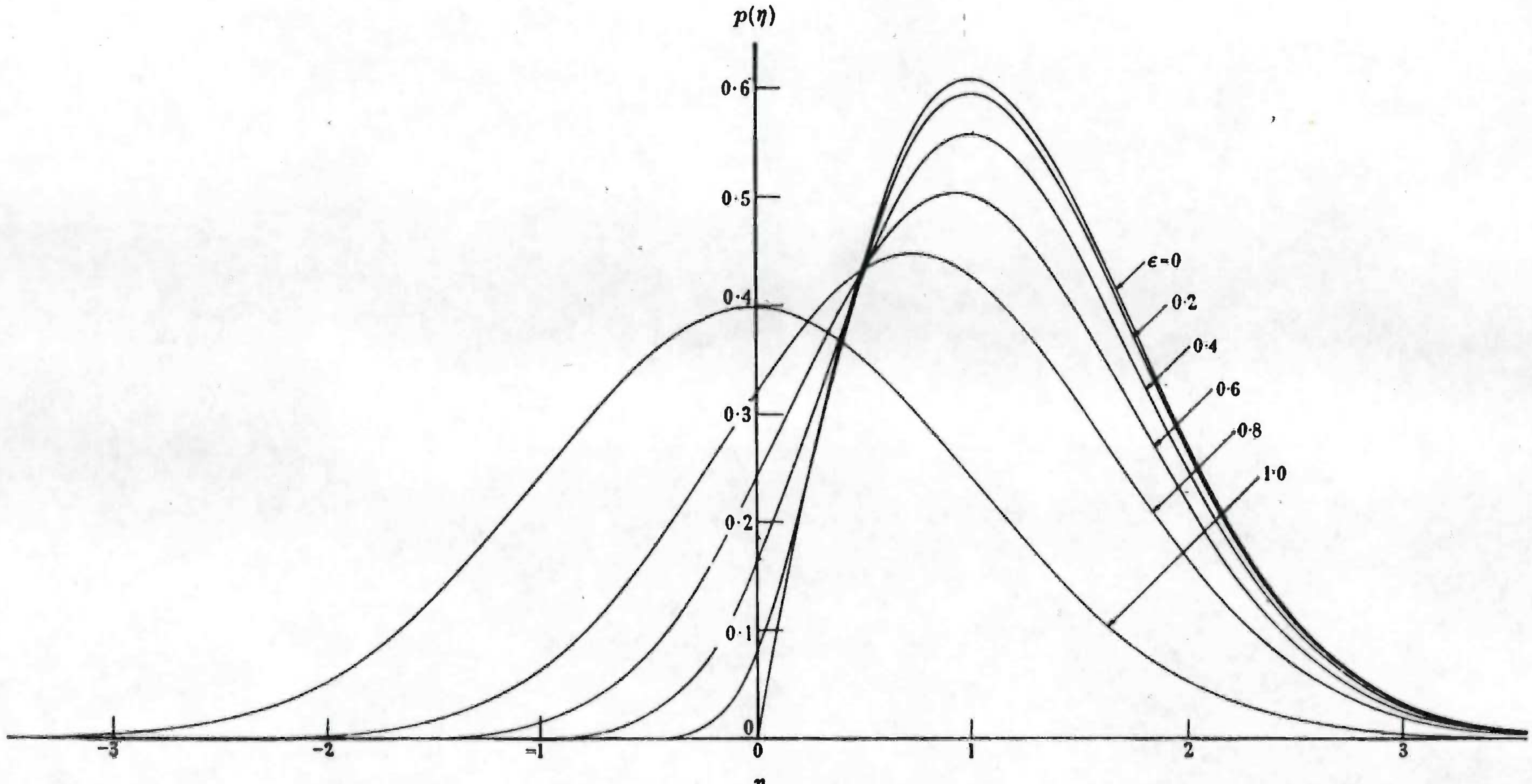

FIGURE 1. Graphs of $p(\eta)$, the probability distribution of the heights of maxima ($\eta = \xi/m_0^{\frac{1}{2}}$) for different values of the width ϵ of the energy spectrum.

The probability distribution of η is $m_0^{\frac{1}{2}}$ times the distribution of ξ_1:

$$p(\eta) = m_0^{\frac{1}{2}} p(\xi_1) = m_0^{\frac{1}{2}} F(\xi_1)/N_1. \tag{1·17}$$

From (1·8) we find

$$N_1 = \frac{1}{2\pi}\left(\frac{m_4}{m_2}\right)^{\frac{1}{2}}, \tag{1·18}$$

and so finally

$$p(\eta) = \frac{1}{(2\pi)^{\frac{1}{2}}}\left[\epsilon\, \mathrm{e}^{-\frac{1}{2}\eta^2/\epsilon^2} + (1-\epsilon^2)^{\frac{1}{2}}\, \eta\, \mathrm{e}^{-\frac{1}{2}\eta^2}\int_{-\infty}^{\eta(1-\epsilon^2)^{\frac{1}{2}}/\epsilon} \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x\right] / \sqrt{m_0} \tag{1·19}$$

where

$$\epsilon^2 = \frac{\delta^2}{1+\delta^2} = \frac{\Delta}{m_0 m_4} = \frac{m_0 m_4 - m_2^2}{m_0 m_4}. \tag{1·20}$$

The function $f(t)$ is statistically symmetrical about the mean level $t = 0$. For, in equation (1·1) each phase angle ϵ_n might be increased or diminished by π without affecting the random character of the phases; and this would merely reverse the sign of $f(t)$. It follows that the statistical distribution of the minima is simply the reflexion of (1·20) in the mean level $\eta = 0$.

2. Discussion

In equation (1·19) η denotes the ratio of the surface height to the r.m.s. height $m_0^{\frac{1}{2}}$. We see that the distribution of η depends only on the single parameter ϵ. A simple interpretation of ϵ is as follows. From (1·12) we have

$$\Delta = m_0 m_4 - m_2^2 = \int_0^\infty \int_0^\infty E(\sigma_1)\, E(\sigma_2)\, (\sigma_2^4 - \sigma_1^2\sigma_2^2)\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2. \tag{2·1}$$

On interchanging σ_1 and σ_2, and adding, we have

$$2\Delta = \int_0^\infty \int_0^\infty E(\sigma_1)\, E(\sigma_2)\, (\sigma_1^2 - \sigma_2^2)^2\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2. \tag{2·2}$$

Since $E(\sigma)$ is essentially positive, it follows that $\Delta \geqslant 0$ and so

$$0 < \epsilon < 1. \tag{2·3}$$

For a very narrow spectrum, with the energy grouped around $\sigma = \sigma_0$, say, $E(\sigma_1)$ and $E(\sigma_2)$ are small except when σ_1 and σ_2 are both near to σ_0; but then the factor $(\sigma_1^2 - \sigma_2^2)^2$ in (2·2) is small and so

$$\epsilon \ll 1. \tag{2·4}$$

In general ϵ is a measure of the r.m.s. width of the energy spectrum E.

Clearly ϵ may take values indefinitely near 0. For a low-pass filter ($E = E_0$ when $\sigma < \sigma_0$, and $E = 0$ when $\sigma > \sigma_0$) we find

$$\epsilon = \tfrac{2}{3}. \tag{2·5}$$

ϵ may also take values indefinitely near 1. For suppose a proportion ϖ of the energy is at frequency $\sigma = \sigma_1$, and $(1-\varpi)$ at $\sigma = \sigma_2$; we have

$$\left.\begin{aligned} m_2 &= m_0\{\varpi\sigma_1^2 + (1-\varpi)\,\sigma_2^2\},\\ m_4 &= m_0\{\varpi\sigma_1^4 + (1-\varpi)\,\sigma_2^4\}.\end{aligned}\right\} \tag{2·6}$$

When $\sigma_2/\sigma_1 \to \infty$ we see that $m_2^2/m_0 m_4 \to 1 - \varpi$ and so

$$\epsilon^2 \to \varpi, \qquad \epsilon^2 = 1 - \frac{m_2^2}{m_0 m_4} \tag{2·7}$$

which can be as near to unity as we please.

The first limiting case ($\epsilon \to 0$) gives the distribution for an infinitely narrow spectrum. From equation (1·19) we have then

$$p(\eta) = \begin{Bmatrix} \eta\, \mathrm{e}^{-\frac{1}{2}\eta^2} & (\eta \geqslant 0), \\ 0 & (\eta \leqslant 0), \end{Bmatrix} \tag{2·8}$$

which is the Rayleigh distribution, or the distribution of the envelope of the waves (see Rice 1944, 1945; Barber 1950; Longuet-Higgins 1952).

The second limiting case ($\epsilon \to 1$) can occur, as we have shown, when one wave of high frequency and small amplitude is superposed on another disturbance of lower frequency. The high-frequency wave forms a 'ripple' on the remaining waves, and the distribution of maxima tends to the distribution of the surface elevation $(\xi_1/m_0^{\frac{1}{2}})$ itself. On letting ϵ tend to 1 in (1·19) we obtain

$$p(\eta) = \frac{1}{(2\pi)^{\frac{1}{2}}}\mathrm{e}^{-\frac{1}{2}\eta^2}, \tag{2·9}$$

which, as we should expect, is a Gaussian distribution.

The distribution $p(\eta)$ has been plotted in figure 1 for $\epsilon = 0{\cdot}0, 0{\cdot}2, \ldots, 1{\cdot}0$. The transition from the Rayleigh distribution to the Gaussian distribution can be clearly seen.

3. The proportion of negative maxima

This may be found by a simple geometrical argument as follows. Suppose that in a certain interval of time, say $(0, t)$, there are n_0^+ zero up-crossings, at which f passes from negative to positive values, and similarly suppose that there are n_0^- zero down-crossings. Also let there be n_1^+ positive maxima, n_1^- negative maxima, n_2^+ positive minima and n_2^- negative minima. Between a zero up-crossing and the next zero down-crossing the function is always positive, and so the number of maxima exceeds the number of minima by one. In other words, when n_0^- increases by 1, so also does $(n_1^+ - n_2^+)$. Similarly, when n_0^- increases by 1, so does $(n_2^- - n_1^-)$. Therefore, if N_0^+, N_0^-, N_1^+, N_1^-, N_2^+, N_2^- denote the average densities of zero up-crossings, etc., over a long interval we have

$$\begin{rcases} N_0^+ = N_1^+ - N_2^+, \\ N_0^- = N_2^- - N_1^-. \end{rcases} \tag{3·1}$$

Now since $f(t)$ is statistically symmetrical about the mean level it follows that

$$\begin{rcases} N_2^+ = N_1^- = rN_1, \\ N_2^- = N_1^+ = (1-r)\,N_1, \end{rcases} \tag{3·2}$$

where N_1 denotes the total density of maxima, and r denotes the proportion of negative maxima. So from (3·1)

$$N_0^+ = N_1(1-2r), \tag{3·3}$$

or

$$r = \tfrac{1}{2}(1 - N_0^+/N_1). \tag{3·4}$$

But from Rice (1944, 1945) and equation (1·18) we have

$$N_0^+ = \frac{1}{2\pi}\left(\frac{m_2}{m_0}\right)^{\frac{1}{2}}, \quad N_1 = \frac{1}{2\pi}\left(\frac{m_4}{m_2}\right)^{\frac{1}{2}}. \tag{3·5}$$

So equation (3·4) can be written

$$r = \frac{1}{2}\left[1 - \frac{m_2}{(m_0 m_4)^{\frac{1}{2}}}\right] = \tfrac{1}{2}[1-(1-\epsilon^2)^{\frac{1}{2}}]. \tag{3·6}$$

Hence the proportion of negative maxima increases steadily with the relative width of the spectrum. Conversely, we have

$$\epsilon^2 = 1-(1-2r)^2. \tag{3·7}$$

This relation provides us with a ready means of estimating ϵ by simply counting the numbers of positive and negative maxima in a length of record.

4. The moments of $p(\eta)$

The nth moment μ_n' of the probability distribution $p(\eta)$ taken about the origin, is defined by

$$\mu_n' = \int_{-\infty}^{\infty} p(\eta)\,\eta^n\,\mathrm{d}\eta. \tag{4·1}$$

The even moments ($n = 2r$) may be calculated by means of the moment-generating function

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}(\eta t)^2} p(\eta)\,\mathrm{d}\eta \equiv \mu_0' - \frac{t^2}{2.1!}\mu_2' + \frac{t^4}{2^2.2!}\mu_4' \ldots. \tag{4·2}$$

On substituting from (1·19) and evaluating the integral we find

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}(\eta t)^2} p(\eta)\,\mathrm{d}\eta = (1+\epsilon^2t^2)^{\frac{1}{2}}(1+t^2)^{-1}, \tag{4·3}$$

and so on, comparing coefficients of t^{2r} in these two equations, we have

$$\mu_{2r}' = 2^r r!\left[1 - \tfrac{1}{2}\epsilon^2 - \frac{1.1}{2^2.2!}\epsilon^4 - \ldots - \frac{1.1.3\ldots(2r-3)}{2^r.r!}\epsilon^{2r}\right]. \tag{4·4}$$

The odd moments ($n = 2r+1$) may be found in a similar way by means of the moment-generating function

$$\int_{-\infty}^{\infty} \eta t\,\mathrm{e}^{-\frac{1}{2}(\eta t)^2} p(\eta)\,\mathrm{d}\eta \equiv t\mu_1' - \frac{t^3}{2.1!}\mu_3' + \frac{t^5}{2^2.2!}\mu_5' \ldots. \tag{4·5}$$

From (1·18) we have

$$\int_{-\infty}^{\infty} \eta t\,\mathrm{e}^{-\frac{1}{2}(\eta t)^2} p(\eta)\,\mathrm{d}\eta = (\tfrac{1}{2}\pi)^{\frac{1}{2}}(1-\epsilon^2)^{\frac{1}{2}}\,t(1+t^2)^{-\frac{3}{2}}, \tag{4·6}$$

and hence

$$\mu_{2r+1}' = (\tfrac{1}{2}\pi)^{\frac{1}{2}}(1-\epsilon^2)^{\frac{1}{2}}\frac{1.3.5\ldots\ldots(2r+1)}{(r!)^2}. \tag{4·7}$$

In particular we have

$$\left.\begin{aligned} \mu_0' &= 1, \\ \mu_1' &= (\tfrac{1}{2}\pi)^{\frac{1}{2}}(1-\epsilon^2)^{\frac{1}{2}}, \\ \mu_2' &= 2-\epsilon^2, \\ \mu_3' &= (\tfrac{1}{2}\pi)^{\frac{1}{2}}(1-\epsilon^2)^{\frac{1}{2}}.3. \end{aligned}\right\} \tag{4·8}$$

We see that the mean μ_1' is a steadily decreasing function of ϵ, the width of the spectrum. A non-dimensional quantity depending on ϵ is the ratio

$$\rho = \frac{\mu_1'^2}{\mu_0'\mu_2'} = (\tfrac{1}{4}\pi)\frac{1-\epsilon^2}{2-\epsilon^2}. \tag{4·9}$$

The width of the spectrum is given in terms of ρ by the relation

$$\epsilon^2 = \frac{\pi-4\rho}{\pi-2\rho}. \tag{4·10}$$

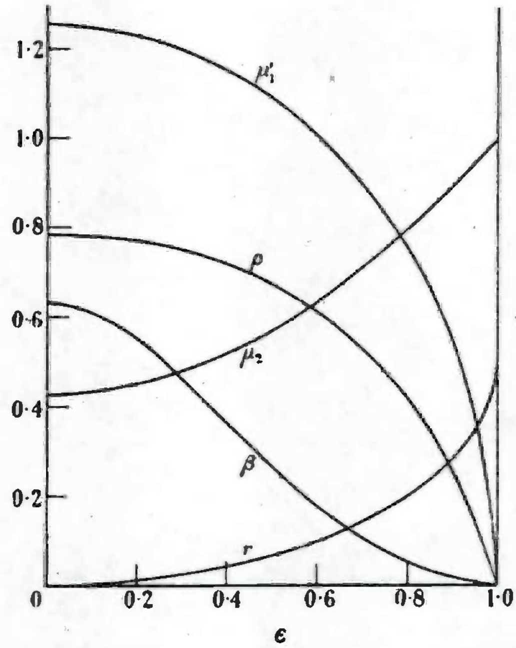

FIGURE 2. Graphs of the mean μ_1', variance μ_2, skewness β, proportion r of negative maxima, and $\rho(=\mu_1'^2/\mu_0'\mu_2')$ as functions of ϵ.

On the other hand, we have the following two quantities which are independent of ϵ:

$\mu_2'\mu_0' - \frac{\mu_1'^2}{\frac{1}{2}\pi} = 1$

$$\cancel{\mu_2'\mu_0' - \tfrac{1}{2}\pi\mu_1'^2 = 1}, \quad \mu_3'/\mu_1' = 3. \tag{4·11}$$

The moments μ_n about the mean, which are defined by

$$\mu_n = \int_{-\infty}^{\infty} p(\eta)(\eta-\mu_1')^n\, d\eta, \tag{4·12}$$

may be deduced immediately from the moments about the origin. In particular we have from (4·8)

$$\left.\begin{aligned} \mu_0 &= 1, \\ \mu_1 &= 0, \\ \mu_2 &= 1-(\tfrac{1}{2}\pi-1)(1-\epsilon^2), \\ \mu_3 &= (\tfrac{1}{2}\pi)^{\frac{1}{2}}(\pi-3)(1-\epsilon^2)^{\frac{3}{2}}. \end{aligned}\right\} \tag{4·13}$$

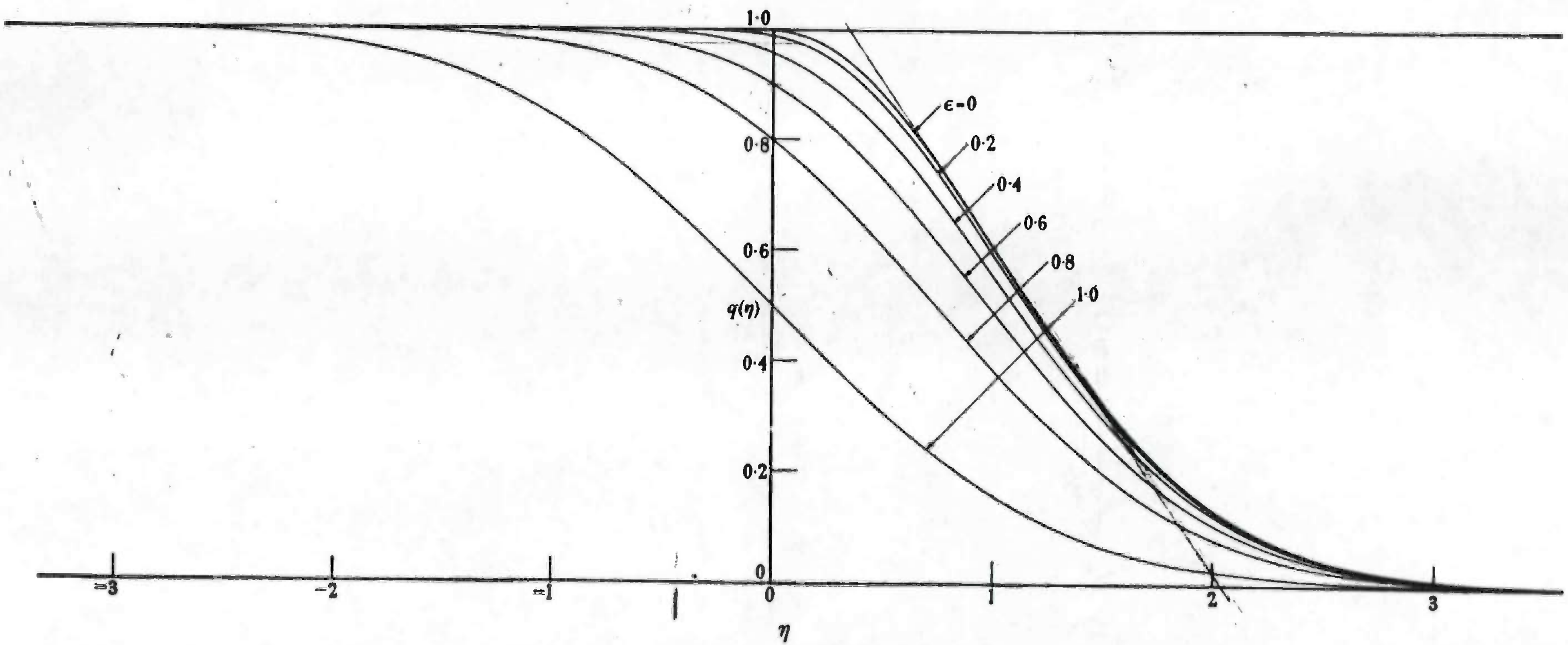

Figure 3. Graphs of the cumulative probability $q(\eta)$, for different values of ϵ.

The coefficient of skewness is given by

$$\beta = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = (\tfrac{1}{2}\pi)^{\frac{1}{2}}(\pi-3)\left[\frac{1-\epsilon^2}{1-(\frac{1}{2}\pi-1)(1-\epsilon^2)}\right]^{\frac{3}{2}}. \tag{4·14}$$

We see that the standard deviation $\mu_2^{\frac{1}{2}}$ steadily increases as ϵ increases. β, on the other hand, steadily decreases.

The mean μ_1', the variance μ_2, the skewness β and the ratios r and ρ are shown graphically as functions of ϵ in figure 2.

In some practical cases we may know the distribution of the maxima $\xi_1 \,(= m_0^{\frac{1}{2}}\eta)$ experimentally and wish to make an estimate of the mean energy $m_0^{\frac{1}{2}}$. Let ν_n' and ν_n denote the nth moments, about the origin and about the mean, of the variate ξ_1. Then

$$\nu_n' = m_0^{\frac{1}{2}(n\cancel{+1})}\mu_n', \quad \nu_n = m_0^{\frac{1}{2}(n\cancel{+1})}\mu_n, \tag{4·15}$$

and so from (4·11)

$$\nu_2'\nu_0' \cancel{- \tfrac{1}{2}\pi\nu_1'^2 = m_0^2}, \quad \nu_3'/\nu_1' = 3m_0. \tag{4·16}$$

$V_2' V_0' - \frac{\nu_1'^2}{\frac{1}{2}\pi} = m_0$

By forming either of these quantities, therefore, we may estimate m_0.

5. The cumulative probability

The cumulative probability $q(\eta)$ may be defined as the probability of η exceeding a given value:

$$q(\eta) = \int_\eta^\infty p(\eta)\,\mathrm{d}\eta. \tag{5·1}$$

Substituting from (1·19) we find

$$q(\eta) = \frac{1}{(2\pi)^{\frac{1}{2}}}\left[\int_{\eta/\epsilon}^\infty \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x + (1-\epsilon^2)^{\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}\eta^2}\int_{-\infty}^{\eta(1-\epsilon^2)^{\frac{1}{2}}/\epsilon} \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x\right]. \tag{5·2}$$

When $\epsilon \to 0$,

$$q(\eta) \to \begin{cases} 1 & (\eta \leqslant 0), \\ \mathrm{e}^{-\frac{1}{2}\eta^2} & (\eta \geqslant 0), \end{cases} \tag{5·3}$$

and when $\epsilon \to 1$,

$$q(\eta) \to \frac{1}{(2\pi)^{\frac{1}{2}}}\int_\eta^\infty \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x. \tag{5·4}$$

Graphs of $q(\eta)$ for these and intermediate values of ϵ are shown in figure 3. The proportion r of negative maxima is given by

$$r = \int_{-\infty}^0 p(\eta)\,\mathrm{d}\eta = 1 - q(0), \tag{5·5}$$

which from (5·2) is

$$r = \tfrac{1}{2}[1-(1-\epsilon^2)^{\frac{1}{2}}], \tag{5·6}$$

in agreement with (3·6).

In some geophysical applications it is found convenient to consider only the higher waves, say the highest $1/n$th of the total number in a sample. The $1/n$th highest maxima correspond to those values of η greater than η', say, where

$$q(\eta') = \int_{\eta'}^\infty p(\eta)\,\mathrm{d}\eta = 1/n. \tag{5·7}$$

The average value of η for these maxima will be denoted by $\eta^{(1/n)}$, so that

$$\eta^{(1/n)} = n\int_{\eta'}^{\infty} p(\eta)\,\eta\,\mathrm{d}\eta. \tag{5·8}$$

Clearly $\eta^{(1)}$ is the same as the mean μ_1'. $\eta^{(1/n)}$ has been computed numerically for $n = 1, 2, 3, 5$ and 10, and for different values of ϵ. The results are shown in figure 4. $\eta^{(1/n)}$ is apparently a decreasing function of ϵ. For small values of ϵ, say $\epsilon < 0{\cdot}5$, the dependence of $\eta^{(1/n)}$ on ϵ is slight, but each curve gradually steepens, and it can be shown that as ϵ approaches 1 the gradient $\partial\eta^{(1/n)}/\partial\epsilon$ tends to $-\infty$. Near $\epsilon = 1$ the curves are all exactly similar in shape, being independent of n.

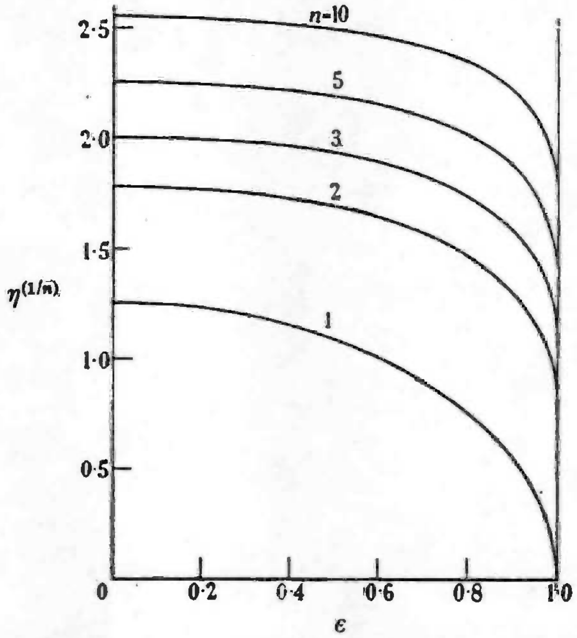

FIGURE 4. Graphs of $\eta^{(1/n)}$, the mean height of the $1/n$th highest maxima, as a function of ϵ, for $n = 1, 2, 3, 5$ and 10.

6. The highest maximum in a sample of N

Suppose that a sample of N maxima is chosen at random; we wish to know the average value of the highest of these, $\eta_{\text{max.}}$. The problem has been considered in the case $\epsilon = 0$ (Longuet-Higgins 1952) and the expectation $\overline{\eta_{\text{max.}}}$ has been computed for values of N up to 20. For values of N greater than 50 (in which we are usually interested) it has been shown that the asymptotic formula

$$\eta_{\text{max.}}/(\mu_2')^{\frac{1}{2}} \doteqdot (\ln N)^{\frac{1}{2}} + \tfrac{1}{2}\gamma(\ln N)^{-\frac{1}{2}} \tag{6·1}$$

is accurate to within 3 %. (Here γ denotes Euler's constant, 0·5772 ....)

The formula (6·1) may be generalized to values of ϵ between 0 and 1 as follows. The probability distribution of $\eta_{\text{max.}}$ is given by*

$$p(\eta_{\text{max.}}) = \frac{\mathrm{d}}{\mathrm{d}\eta_{\text{max.}}}[1-q(\eta_{\text{max.}})]^N, \tag{6·2}$$

* We follow here the same method as in the paper just quoted. But a general study of the limiting form of the distribution of the largest member of a sample has been made by Fisher & Tippett (1928). For a more recent discussion see Gumbel (1954).

where $q(\eta)$ is given by (5·1). Therefore we have

$$\overline{\eta_{\text{max.}}} = \int_{-\infty}^{\infty} \eta \frac{d}{d\eta}[1-q(\eta)]^N d\eta. \tag{6·3}$$

On separating the integral into two parts, from $-\infty$ to 0 and from 0 to ∞, and integrating by parts we find

$$\overline{\eta_{\text{max.}}} = \int_{-\infty}^{0} [1-q(\eta)]^N d\eta + \int_{0}^{\infty} \{1-[1-q(\eta)]^N\} d\eta. \tag{6·4}$$

When N is large $[1-q(\eta)]^N$ is very small unless q is of order $1/N$. Now as x tends to infinity we have

$$\int_{x}^{\infty} e^{-\frac{1}{2}x^2} dx = e^{-\frac{1}{2}x^2}\left[\frac{1}{x} + O\left(\frac{1}{x^3}\right)\right], \tag{6·5}$$

and so from (5·2)

$$q(\eta) = (1-\epsilon^2)^{\frac{1}{2}} e^{-\frac{1}{2}\eta^2} + O\left(\frac{1}{\eta^3} e^{-\frac{1}{2}\eta^2/\epsilon^2}\right) \tag{6·6}$$

for large values of η and when $0 \leqslant \epsilon < 1$. If q is of order $1/N$, η is of order $(\ln N)^{\frac{1}{2}}$. Therefore neglecting terms of order $(\ln N)^{-\frac{3}{2}}$ we have

$$q(\eta) = (1-\epsilon^2)^{\frac{1}{2}} e^{-\frac{1}{2}\eta^2} = (1-\epsilon^2)^{\frac{1}{2}} e^{-\theta}, \tag{6·7}$$

where $\theta = \frac{1}{2}\eta^2$. The first integral in (6·4) is negligible, and on substituting in the second we have

$$\overline{\eta_{\text{max.}}} = \frac{1}{2^{\frac{1}{2}}}\int_{0}^{\infty} \{1-[1-(1-\epsilon^2)^{\frac{1}{2}} e^{-\theta}]^N\} \theta^{-\frac{1}{2}} d\theta. \tag{6·8}$$

Writing
$$\theta_0 = \log[(1-\epsilon^2)^{\frac{1}{2}} N], \quad \theta' = \theta - \theta_0, \tag{6·9}$$

and so
$$e^{-\theta} = \frac{e^{-\theta'}}{(1-\epsilon^2)^{\frac{1}{2}} N}, \tag{6·10}$$

we have
$$\overline{\eta_{\text{max.}}} = \frac{1}{2^{\frac{1}{2}}}\int_{-\theta_0}^{\infty} \left\{1-\left[1-\frac{e^{-\theta'}}{N}\right]^N\right\} (\theta_0+\theta')^{-\frac{1}{2}} d\theta \tag{6·11}$$

$$\doteqdot \frac{1}{2^{\frac{1}{2}}}\int_{-\theta_0}^{\infty} (1-\exp[-e^{-\theta'}])(\theta_0+\theta')^{-\frac{1}{2}} d\theta, \tag{6·12}$$

with relative errors of order $1/N$ only. It may be shown (Longuet-Higgins 1952) that when θ_0 is large the above integral equals

$$2^{\frac{1}{2}}[\theta_0^{\frac{1}{2}} + \tfrac{1}{2}\gamma\theta_0^{-\frac{1}{2}} + O(\theta_0^{-\frac{3}{2}})]. \tag{6·13}$$

Hence we have

$$\overline{\eta_{\text{max.}}} \doteqdot 2^{\frac{1}{2}}\{[\ln(1-\epsilon^2)^{\frac{1}{2}} N]^{\frac{1}{2}} + \tfrac{1}{2}\gamma[\ln(1-\epsilon^2)^{\frac{1}{2}} N]^{-\frac{1}{2}}\}, \tag{6·14}$$

which can also be written

$$\overline{\eta_{\text{max.}}}/(\mu_2')^{\frac{1}{2}} \doteqdot \frac{[\ln(1-\epsilon^2)^{\frac{1}{2}} N]^{\frac{1}{2}} + \frac{1}{2}\gamma[\ln(1-\epsilon^2)^{\frac{1}{2}} N]^{-\frac{1}{2}}}{(1-\frac{1}{2}\epsilon^2)^{\frac{1}{2}}}. \tag{6·15}$$

When $\epsilon \to 0$ this equation reduces to (6·1). The expression on the right-hand side of (6·15) is an increasing function of ϵ, when N is large. It follows that as the spectrum broadens, the ratio of the greatest in a sample to the root-mean-square will tend to increase.

When ϵ approaches 1 (so that $\ln(1-\epsilon^2)^{\frac{1}{2}} N$ is not large compared with 1) the above formula is no longer valid. The corresponding expression for the general case is complicated and probably not of practical importance. We shall simply give the limiting form when $\epsilon \to 1$, and $p(\eta)$ is normal (equation (2·9)). Fisher & Tippett (1928) have shown that the average value of $\eta_{\max.}$ in this case is given by

$$\overline{\eta_{\max.}} = m + \frac{\gamma m}{1+m^2} \tag{6·16}$$

approximately, where m is the mode of the distribution of $\eta_{\max.}$, given by

$$(2\pi)^{\frac{1}{2}} m \, e^{\frac{1}{2}m^2} = N. \tag{6·17}$$

From (6·17) we have
$$m^2 = \ln\left(\frac{N^2}{2\pi}\right) - \ln m^2, \tag{6·18}$$

and so
$$m \doteqdot \left[\ln\left(\frac{N^2}{2\pi}\right) - \ln\ln\left(\frac{N^2}{2\pi}\right)\right]^{\frac{1}{2}}. \tag{6·19}$$

The leading term in (6·16) is thus

$$\overline{\eta_{\max.}} \doteqdot 2^{\frac{1}{2}}\left[\ln\frac{N}{(2\pi)^{\frac{1}{2}}}\right]^{\frac{1}{2}}. \tag{6·20}$$

However, Fisher & Tippett have shown (1928) that for the normal distribution the limiting forms are approached exceptionally slowly. A table of the exact values of $\overline{\eta_{\max.}}$ computed for values of N up to 1000 is given by Tippett (1925).

7. Applications

It is interesting to verify that the distribution just discussed is applicable to records of sea waves and of associated phenomena. In this section we shall consider five such examples: a record of wave pressure at a fixed point on the sea bed; two continuous records of wave height made at sea by a shipborne instrument; one record of the angle of pitch of the ship, and one of the angle of roll. The widths of the corresponding Fourier spectra are fairly representative of the possible range $0 < \epsilon < 1$.

Typical sections of the records are shown in figure 5(a) to (e). Each complete record lasted from 12 to 20 min and contained about 100 maxima and 100 minima. In order to increase the amount of data both maxima and minima were included in the sample. The analysis was carried out as follows. The ordinates A_n of all the stationary points in the record, measured from some common baseline, were numbered consecutively from 1 to N so that the maxima, say, corresponded to even values of n and the minima to odd values of n. The zero of the record was taken to be the mean of A_n:

$$\bar{A} = \frac{1}{N}\sum_{n=1}^{N} A_n. \tag{7·1}$$

The distribution of the variate

$$X_n = (-1)^n (A_n - \bar{A}) \tag{7·2}$$

was then studied. The histograms corresponding to the distribution of X_n are shown in figure 6(a) to (e).

7

To obtain the parameters for the theoretical distribution a harmonic analysis of the original record was made by means of the N.I.O. Fourier analyser (see Darbyshire & Tucker 1953). The range of frequency was divided into a number of equal

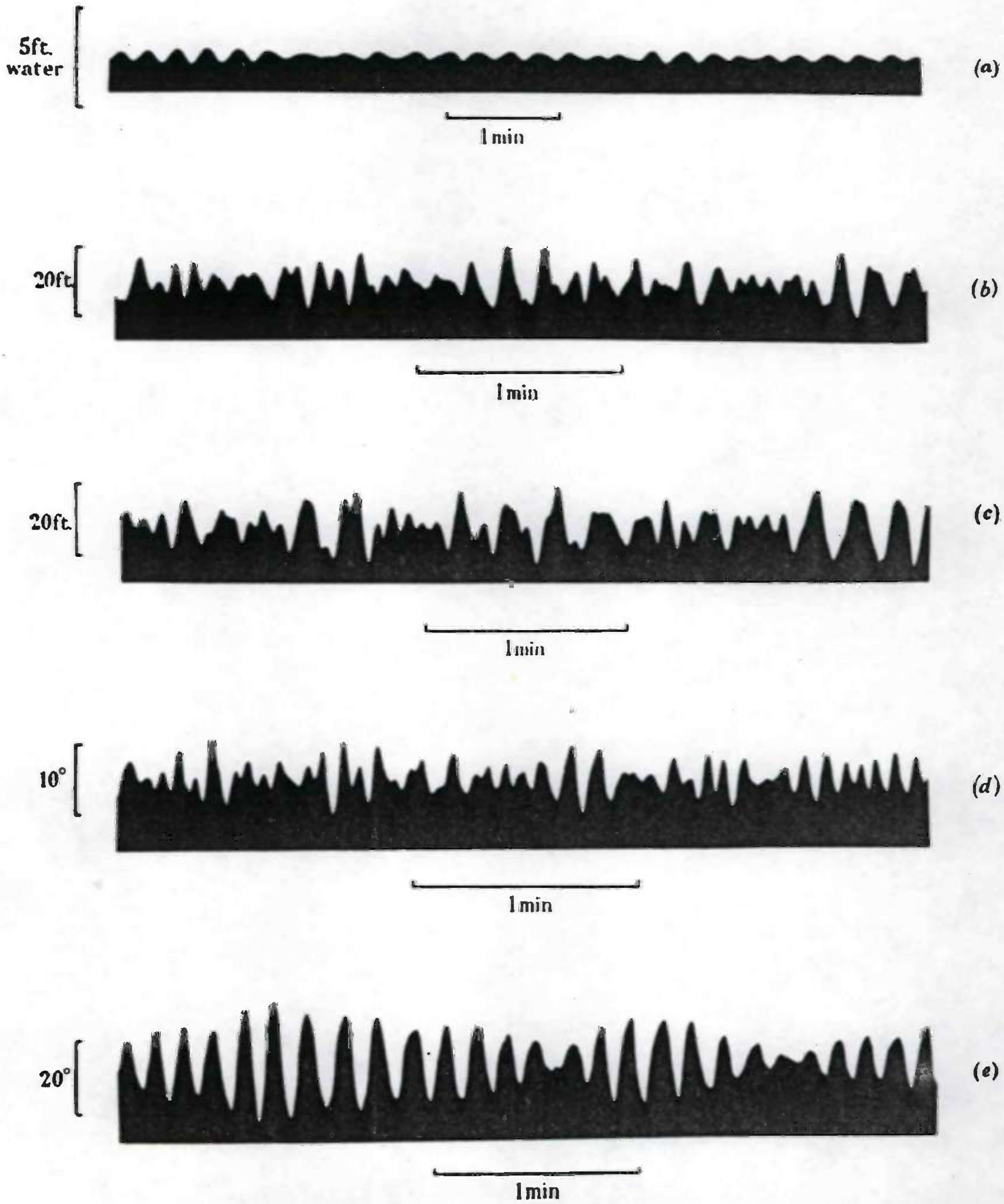

FIGURE 5. Typical short sections of the five records chosen for analysis. (*a*) pressure on the sea bed off Pendeen, Cornwall, 08.00 to 08.20, 15 March 1945; (*b*) wave height in the Bay of Biscay, 19.00 to 19.12, 11 November 1954; (*c*) wave-height in the Bay of Biscay, 02.00 to 02.12, 12 November 1954; (*d*) angle of pitch of R.R.S. *Discovery II*, in North Atlantic, 13.21 to 13.33, 25 May 1954; (*e*) angle of roll of R.R.S. *Discovery II*, in North Atlantic, 14.05 to 14.17, 21 May 1954

narrow ranges each containing about 10 harmonics of the length of the record, and the energy $\Sigma\frac{1}{2}c_n^2$ was summed for each interval. The energy spectra are shown in figure 7 (*a*) to (*e*). The moments m_0, m_2 and m_4 of the distribution were then calculated by multiplying the energy in each small range of frequency by 1, σ^2 and σ^4

6

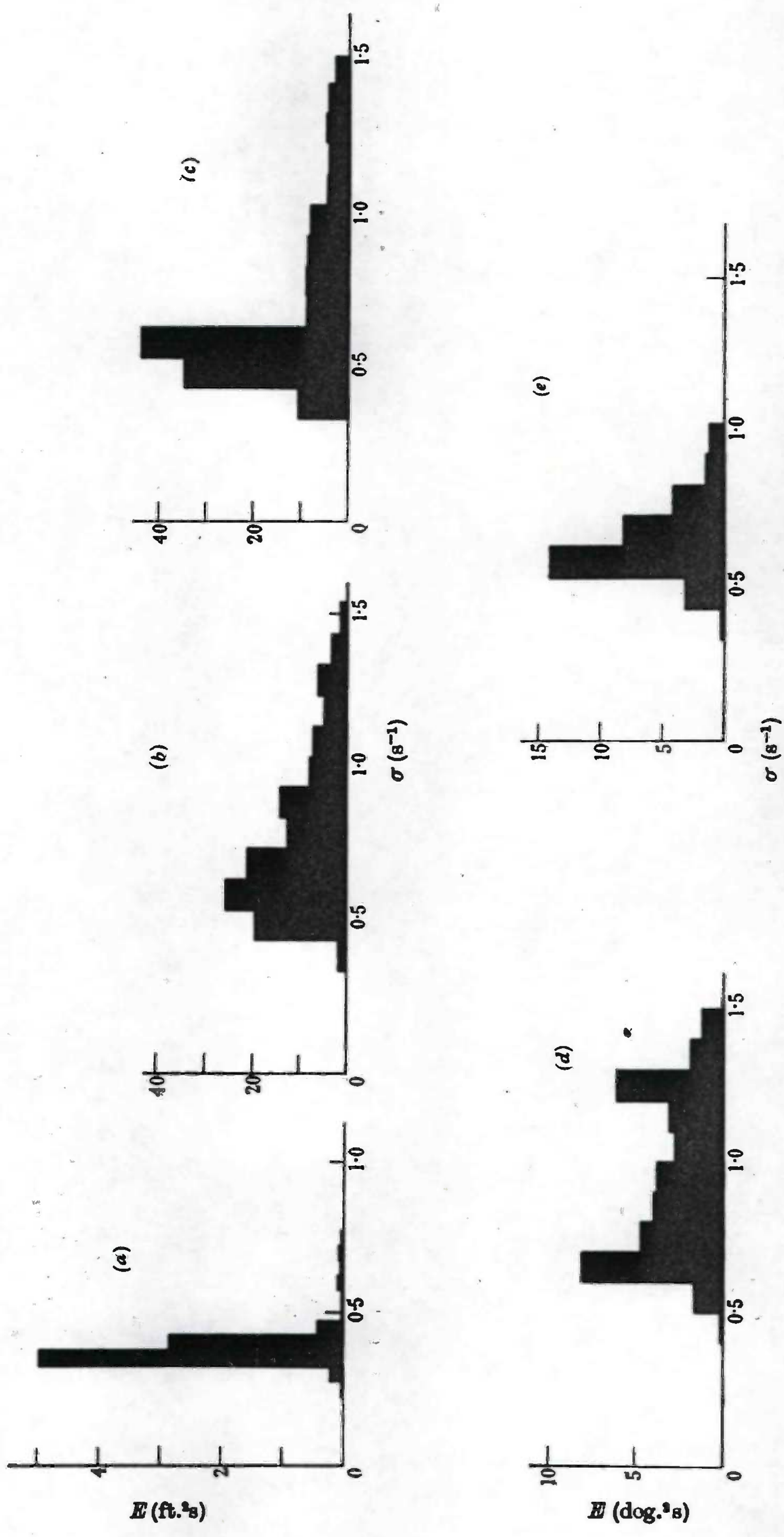

FIGURE 6. Energy spectra of the five records shown in figure 5.

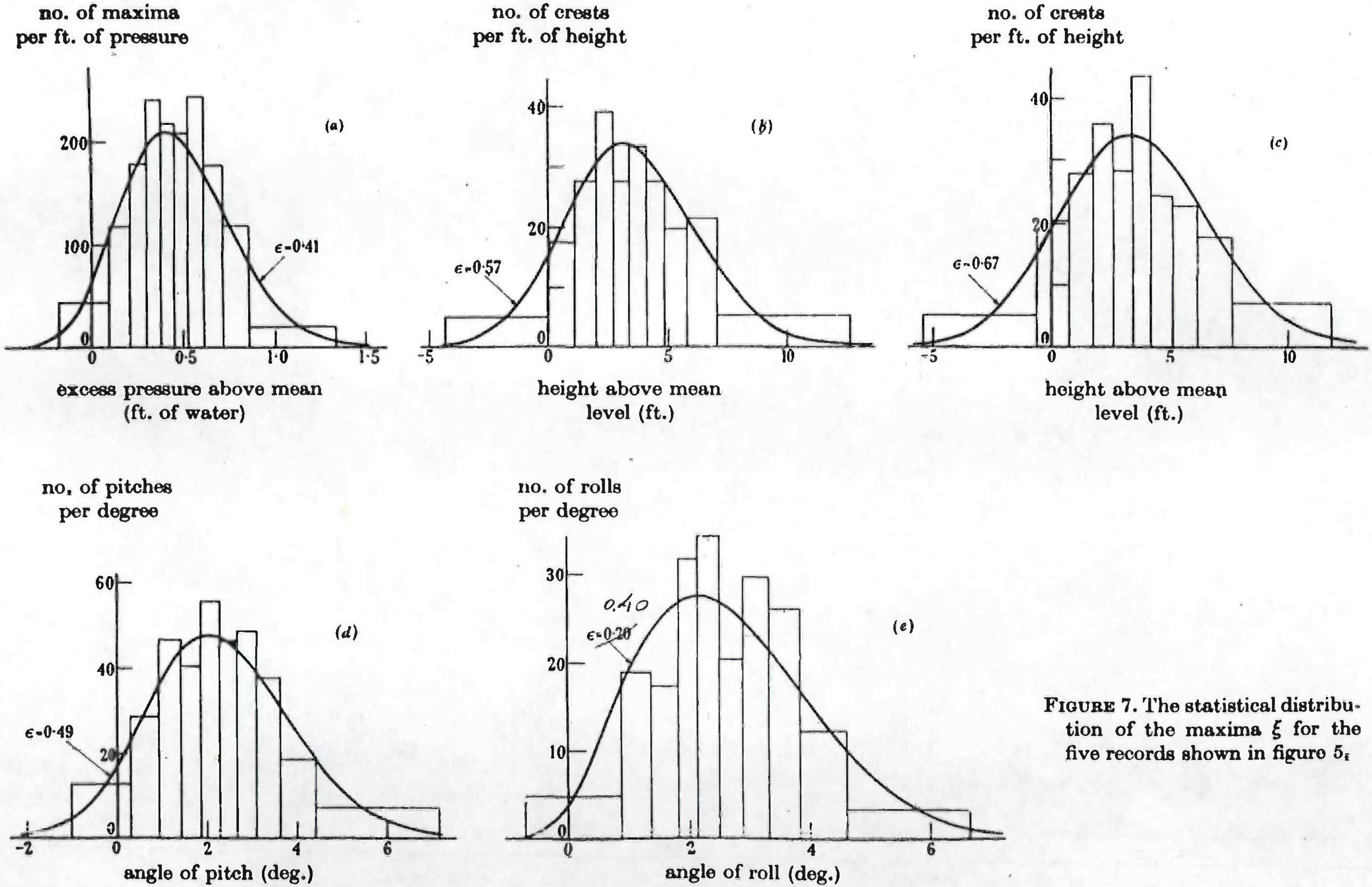

FIGURE 7. The statistical distribution of the maxima ξ for the five records shown in figure 5.

respectively. From these three moments the parameter ϵ defined by equation (1·20) was calculated. The corresponding curves of probability $p(\eta)$, multiplied by the total number N in each sample, are shown in figure 6 (*a*) to (*e*).

In constructing the histograms the horizontal scale has been divided, not into equal intervals, but into intervals such that the expected numbers of maxima in each interval (according to the theoretical distribution) are equal. The purpose is to avoid the small classes that must otherwise occur at the two ends of the distribution, and which make the application of the χ^2 significance test unsatisfactory unless the classes are amalgamated in some arbitrary way. The vertical scale is so chosen that, for each separate subclass, a rectangle whose height indicated the expected frequency of maxima would enclose the same area as is enclosed by the curve of theoretical frequency. The width of the two outermost rectangles is chosen quite arbitrarily, but this does not affect in any way the application of the χ^2 test. Some relevant data concerning the five records are given in table 1. The first record is of wave pressure measured on the sea bed in a depth of 110 ft. of water by a powerphone pressure recorder, in March 1945 (described by Barber & Ursell, 1948). The section of record in figure 5 (*a*) indicates a long, regular swell with a fairly narrow spectrum ($\epsilon = 0{\cdot}41$). However, it contains a certain amount of energy outside the main frequency band.

TABLE 1. DATA FOR THE RECORDS IN FIGURES 5 TO 7

example	N	ϵ (from energy spectrum)	$P(\chi^2)$	ϵ (from r)	ϵ (from ρ)
(*a*)	164	0·41	0·60	0·31	0·37
(*b*)	220	0·57	0·62	0·58	0·66
(*c*)	270	0·67	0·55	0·68	0·69
(*d*)	180	0·48	0·67	0·44	0·45
(*e*)	250	~~0·20~~ 0·40	0·12	0·26	—

The second and third records are of waves in deep water (Bay of Biscay) measured by the shipborne wave recorder installed in R.R.S. *Discovery II*. The instrument has been described by Tucker (1952). The two records are somewhat more irregular than the pressure record and have correspondingly broader spectra ($\epsilon = 0{\cdot}57$ and $\epsilon = 0{\cdot}67$ respectively). This is due partly to the fact that the records of wave height contain more energy of higher frequency than the record of pressure.

The last two records are of the pitching and rolling motion of R.R.S. *Discovery II* in a seaway in the North Atlantic. The angles of pitch and roll were measured in the conventional manner by gyroscopes. The roll, in particular, has a very narrow spectrum ($\epsilon = 0{\cdot}20$) and the record is correspondingly regular. This is as we should expect, since the rolling motion of a ship is only lightly damped, and is tuned sharply to oscillations having a period close to its period of free motion.

For each of the above records the quantity χ^2 was calculated, and also the probability of χ^2 exceeding this value. Since two parameters have been estimated from the sample (the mean height and the total frequency) χ^2 has in each case 8 degrees of freedom. From table 1 it will be seen that for none of the records is the probability of χ^2 significantly small.

For each measured sample of X_n the quantities r (the proportion of negative maxima) and $\rho\,(=\mu_1'^2/\mu_0'\mu_2')$ have been found, and from the relations (3·7) and (4·10) two independent estimates of ϵ have been made. These are also given in table 1. It will be seen that in examples (*b*), (*c*) and (*d*) the values of ϵ are in good agreement with that derived from the moments of the energy function $E(\sigma)$. In examples (*a*) and (*e*) the estimate derived from r is not in such good agreement, but this is hardly surprising, since the number of negative maxima on which the estimate is based is rather small. In example 5, the estimate derived from ρ gives a small negative value for ϵ^2, which is of course impossible. In all the other cases the alternative estimates of ϵ are so close to the original estimate as to make no significant difference to the probability of χ^2.

8. Crest-to-trough wave heights

In view of the agreement of the observed distributions of the heights of crests with the theoretical distribution it is interesting to study also the distribution of the crest-to-trough wave heights in the same records.

The local crest-to-trough wave amplitude a_n may be defined as half the absolute difference in height between a crest and the preceding trough, or between a trough and the preceding crest. Thus

$$a_n = \tfrac{1}{2}(X_n + X_{n-1}). \tag{8·1}$$

The statistical distribution of a_n is more difficult to obtain theoretically than that of X_n for general values of ϵ. However, when $\epsilon \ll 1$ the function $f(t)$ is a regular sine-wave with slowly varying phase and amplitude, so that $a_n = X_n$ very nearly. So we may expect a_n to be distributed according to the Rayleigh distribution (2·8). By considering a disturbance consisting of a small ripple superposed on a long wave ($\epsilon \sim 1$) it can be seen that the distribution of a_n must in general be different from the Rayleigh distribution, though not necessarily by very much. The general distribution no doubt depends on other parameters besides ϵ. Yet it is reasonable to expect that for small values of ϵ the observed distribution of a_n will be in better agreement with the Rayleigh distribution than for larger values of ϵ.

In figure 8 are shown the observed distributions of a_n in the five examples discussed in §7, together with the corresponding Rayleigh distributions

$$p(a) = \frac{a}{\bar{a}^2}\,e^{-a^2/2\bar{a}^2},$$

where $\bar{a}$ is the root-mean-square wave amplitude. The values of χ^2 and $P(\chi^2)$ are given in table 2. (χ^2 again has 8 degrees of freedom, since two parameters—in this case the total number in the sample and the root-mean-square amplitude—have been estimated.)

The table shows that the records with the smallest value of ϵ (examples (*a*), (*d*) and (*e*)) do not give significantly small values of $P(\chi^2)$. On the other hand, those with the two largest values of ϵ give very significant values of $P(\chi^2)$. This verifies our expectation that the observed distribution departs more from the Rayleigh distribution as the width of the energy spectrum increases.

From figure 8 it will be seen that the records with the two broad spectra deviate especially from the Rayleigh distribution for low values of the wave amplitude, having relatively more waves in that range. It appears that the mode of the distribution has a tendency to move to the left in the broader spectra.

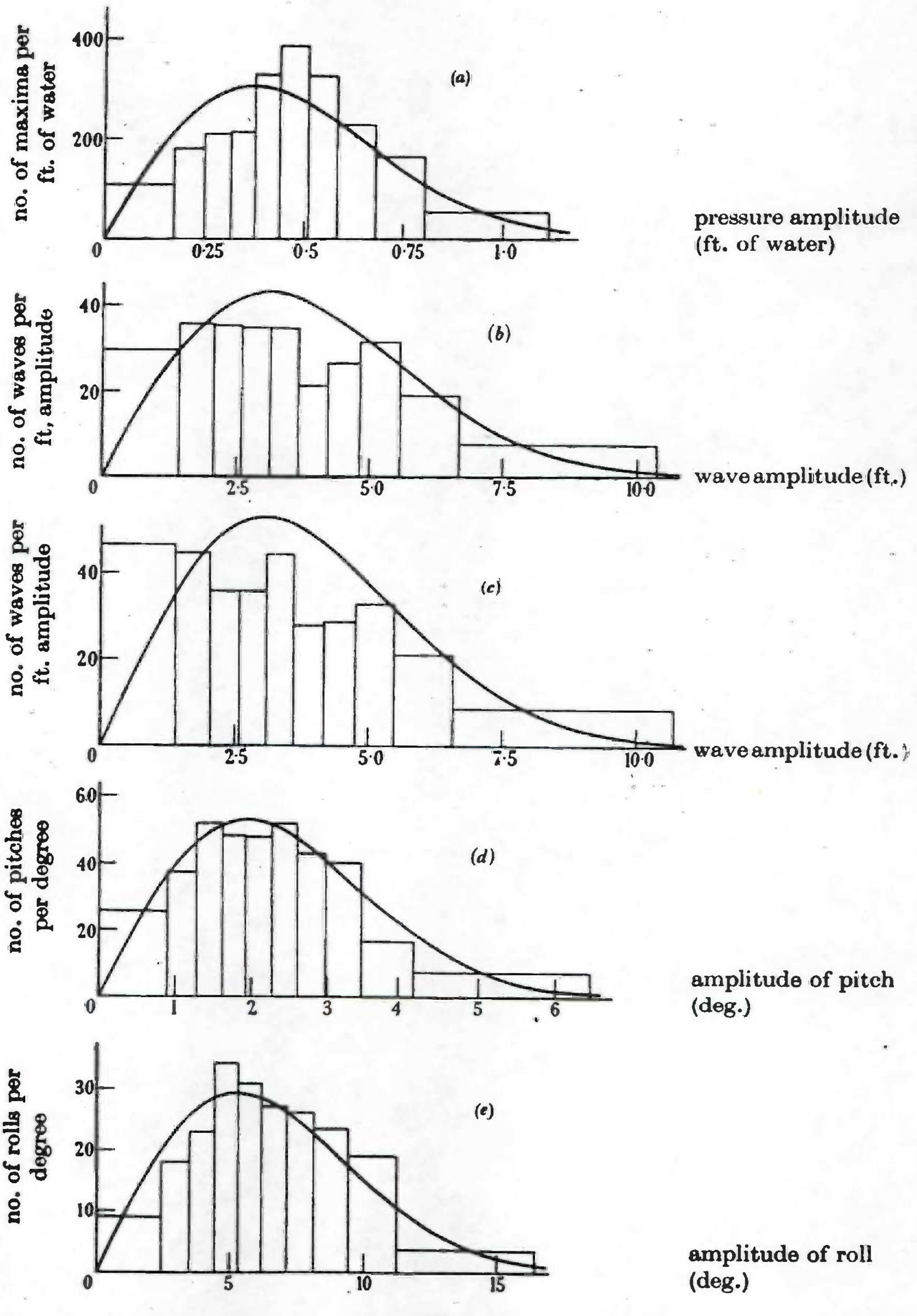

Figure 8. The statistical distribution of the crest-to-trough amplitudes for the five records shown in figure 5.

Our conclusions may be compared with those of Watters (1953) who studied histograms of wave heights of 109 records, and compared 38 of these with the corresponding Rayleigh distributions (with variance chosen so as to give the best fit). Although some of the values of $P(\chi^2)$ were low (as small as 0·05) the values taken as a whole did not show a significant departure from the Rayleigh distributions. There are two possible explanations for this. First, the intervals of wave height were equal, and so there were many classes containing only very few heights. In applying the test these classes were arbitrarily pooled, and it can be shown that in several cases pooling the classes in a different way would have resulted in much lower values of χ^2. (The difficulty is avoided by our present method of making the theoretical classes of uniform size.) Secondly, the widths of the energy spectra of the records studied by Watters were probably less than in examples (*b*) and (*c*) of the present paper, which were in fact chosen on account of their exceptional breadth.

Table 2. Data for the distributions of figure 8

example	ϵ	$P(\chi^2)$
(*a*)	0·41	0·33
(*b*)	0·57	0·001
(*c*)	0·67	0·000
(*d*)	0·48	0·55
(*e*)	0·20	0·51

9. Conclusions

If ξ denotes the height of a maximum of the random function $f(t)$ above the mean level, and if $m_0^{\frac{1}{2}}$ is the r.m.s. value of $f(t)$, then the statistical distribution of η ($= \xi/m_0^{\frac{1}{2}}$) is a function only of η and one other parameter ϵ, which defines the relative width of the energy spectrum of $f(t)$. ϵ lies between 0 and 1. When $\epsilon \to 0$, $p(\eta)$ tends to a Rayleigh distribution; when $\epsilon \to 1$, $p(\eta)$ tends to a Gaussian distribution. As ϵ increases from 0 to 1, the mean of $p(\eta)$ gradually decreases, the variance increases and the shewness decreases. The proportion of maxima that are negative steadily increases. The mean height of the highest $1/\eta$th of the waves varies little for small values of ϵ, but tends always to decrease. The highest maximum in a sample of N maxima tends to decrease relative to m_0 but to increase relative to the r.m.s. height of the maxima.

The records of ocean waves and of ship motion which are discussed in the present paper show good agreement with the theoretical distributions, for various values of ϵ ranging from 0·20 to 0·68.

The theoretical distribution of crest-to-trough heights is known only for a narrow spectrum ($\epsilon = 0$), when it is a Rayleigh distribution. In three of the examples in this paper, for which $\epsilon < 0{\cdot}5$ and the total number in the sample was less than 300, there was no significant departure from the Rayleigh distribution. On the other hand, the examples with the broadest spectra ($\epsilon = 0{\cdot}57$ and $\epsilon = 0{\cdot}67$) did show significant departures.

This indicates the need for a theoretical derivation of the crest-to-trough height distribution when $\epsilon > 0$. Meanwhile, for the purpose of practical prediction, it would