

THE STEADY STATE BEHAVIOUR OF MULTISTATE MONOTONE SYSTEMS

BENT NATVIG,^{*} University of Oslo

ARNFRIED STRELLER,^{**} Humboldt-Universität, Berlin

Abstract

In this paper the steady state behaviour of multistate monotone systems of multistate components is considered by applying the theory for stationary and synchronous processes with an embedded point process. After reviewing some general results on stationary availability, stationary interval availability and on stationary mean interval performance probabilities, one is concentrating on systems with independently working and separately maintained components. For this case an explicit formula is given for the mean time which the system in steady state sojourns in states not below a fixed critical level.

MULTISTATE MONOTONE SYSTEMS; STATIONARY AND SYNCHRONOUS PROCESSES WITH AN EMBEDDED POINT PROCESS; STATIONARY AVAILABILITY; STATIONARY INTERVAL AVAILABILITY; STATIONARY MEAN INTERVAL PERFORMANCE PROBABILITIES; REDUNDANCY.

* Postal address: Institute of Mathematics, University of Oslo, P.O.Box 1053, Blindern, Oslo 3, Norway.

**Postal address: Sektion Mathematik, Humboldt-Universität, 1086 Berlin PSF 1297, German Democratic Republic.

1. Introduction

One inherent weakness of traditional reliability theory is that the system and the components are always described just as functioning or failed. Fortunately, by now this theory is being replaced by a theory for multistate systems of multistate components. This enables one for instance in a power generation system to let the system state be the amount of power generated, or in a pipeline system the amount of oil running through a crucial point. In both cases the system state is possibly measured on a discrete scale. Two recent papers in this area are Block and Savits (1982) and Natvig (1982). A summary of the present state of the art of multistate theory is given in Natvig (1984).

Let the set of states of the system be $S = \{0, 1, \dots, M\}$, $M < \infty$. The $M+1$ states represent successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0 . Furthermore, let the set of components be $C = \{1, 2, \dots, n\}$, $n < \infty$, and the set of states of the i th component S_i ($i=1, \dots, n$) where $\{0, M\} \subseteq S_i \subseteq S$. Hence the states 0 and M are chosen to represent the endpoints of a performance scale which might be used for both the system and its components. A slightly more general model, suggested by Reinschke and Klingner (1981), allows $S \subseteq S_i$. The results derived in the present paper will obviously hold for the latter model. Since in most applications there is no need for a more refined description of the performance of any component than of the system, and since it is practical to have a common state, M , describing perfect functioning both for the system and the components, we will in the following concentrate on the former model.

If x_i ($i=1, \dots, n$) denotes the state or performance level of the i th component and $\underline{x} = (x_1, \dots, x_n)$, it is furthermore assumed that the state ϕ of the system is given by the structure function $\phi = \phi(\underline{x})$. Here \underline{x} takes values in $S_1 \times S_2 \times \dots \times S_n$ and ϕ

takes values in S . In this paper we will restrict to the following systems:

Definition 1.1. A system is a multistate monotone system (MMS) iff its structure function ϕ satisfies:

- (i) $\phi(\underline{x})$ is non-decreasing in each argument,
- (ii) $\phi(\underline{0}) = 0$ and $\phi(\underline{M}) = M$ ($\underline{0} = (0, \dots, 0)$, $\underline{M} = (M, \dots, M)$).

Now let X_i denote the random state of the i th component and let $(i = 1, \dots, n, j = 1, \dots, M)$

$$P[X_i > j] = P_{ij}$$

The random system state is then $\phi(\underline{X})$. Let $(j = 1, \dots, M)$

$$P[\phi(\underline{X}) > j] = h_j .$$

In Theorem 3.1 of Natvig (1982) an exact formula for h_j is arrived at. For the special case where X_1, \dots, X_n are independent h_j reduces to a function of merely $\underline{P} = (P_{11}, \dots, P_{1M}, P_{21}, \dots, P_{nM})$ and we write $h_j = h_j(\underline{P})$.

In Section 2 we sketch the main ideas relevant for describing the behaviour of multistate monotone systems in steady state by means of processes with an embedded point process. We then review in Section 3 some general results on stationary availability, stationary interval availability and on stationary mean interval performance probabilities given in Steller (1980). At last in Section 4 we consider multistate monotone systems with independently working and separately maintained components. For this case an explicit formula is given for the mean time which the system in steady state sojourns in states not below a fixed critical level $j \in S$. This formula is a generalization of a result for binary systems obtained in Ross (1975) and discussed in Franken and Steller (1980). As an example we develop the formula for the special case where the i th component consists of M_i branches in parallel and its state is an increasing function of the number

of branches functioning. We furthermore assume that the branches fail and are repaired/replaced independently of each other, all having the same instantaneous failure rate a_i and repair/replacement rate b_i . In Hjort, Natvig and Funnemark (1982) it is shown that the Markov process describing the state of such a component is associated in time.

Finally, it should be made clear that the present paper is much based on Streller (1982 a) and can be considered as the extension of Franken and Streller (1980) to multistate systems.

2. Stationary and synchronous processes with an embedded point process

In the sequel we shall assume that the system is in steady state, i.e., roughly speaking, that it began to work in $-\infty$. Then the time-dependent behaviour can be described by a stationary or a synchronous process with an embedded point process. We summarize some general properties of such processes on $R_+ = [0, \infty]$. A more detailed treatment can be found in e.g. Franken et. al. (1981) or Streller (1982 b).

Consider a real-valued stochastic process $(X(t), t > 0)$ and a sequence $\Phi = (T_m, m > 0)$ both defined on the same probability space.

Definition 2.1. A couple $\Psi = [(X(t), t > 0), (T_m, m > 0)]$ is called a process with an embedded point process (PEP) if

$$0 < T_0 < T_1 < \dots \quad \text{and} \quad T_m \rightarrow \infty \quad \text{a.s. as} \quad m \rightarrow \infty. \quad (2.1)$$

The epochs $T_m, m > 0$, are called embedded points of $(X(t))$.

The embedded points T_m split the process $(X(t))$ into so-called cycles $\Psi_m = (X_m(u), 0 \leq u < D_m)$, where $D_m = T_{m+1} - T_m$ is the cycle length and $X_m(u) = X(T_m + u)$. In view of (2.1) we have that $D_m > 0$ and $\sum D_m = \infty$ a.s. If the cycles Ψ_m are independent, and for $m > 1$ identically distributed, then Ψ is

a regenerative process with regeneration points T_m . If (Ψ_m) is a Markov chain one can obtain important classes of stochastic processes (e.g. semi-Markov. semi-regenerative and piecewise Markovian processes) by specifying the transition probabilities.

Define the shift operator S_u for $u > 0$ as follows:

$$S_u \Psi = [(X(t+u), t > 0), (T_m^u, m > 0)],$$

where $T_m^u = T_{m+N(u)} - u$ and $N(u) = \min(j: T_j > u)$. (Note that $T_0^u = T_{N(u)} - u$.)

Definition 2.2. A PEP Ψ is called synchronous if the sequence of its cycles is stationary. A PEP $\bar{\Psi}$ is called stationary if for every $u > 0$ the shifted PEP $S_u \bar{\Psi}$ has the same distribution as $\bar{\Psi}$.

The stationary and synchronous PEP are different but equivalent descriptions of the system behaviour in steady state, i.e. there exists a one-to-one correspondence between the distributions \bar{P} of a stationary and P of a synchronous PEP. In most cases the distribution P of a synchronous PEP is easier to obtain. Then the distribution \bar{P} of the corresponding stationary PEP is given by the inversion formula (2.1) of Franken and Streller (1980).

Often the embedded points T_m are uniquely determined by the system state process $(X(t))$. For instance, T_m may be entrance epochs of the process into a well-defined subset of the state space.

Remark 2.1. Choosing several sequences of embedded points

$\phi_{(1)}, \phi_{(2)}, \dots$ one can describe the steady state behaviour of the system by considering the corresponding synchronous PEP's. If the mentioned inversion formula is applied to each of them, one derives the distributions of different stationary PEP's $[(\bar{X}_{(1)}(t)), \bar{\Phi}_{(1)}], [(\bar{X}_{(2)}(t)), \bar{\Phi}_{(2)}], \dots$. However, the stationary state processes $(\bar{X}_{(1)}(t)), (\bar{X}_{(2)}(t)), \dots$ are all stochastically equivalent.

3. Stationary characteristics

We now assume that there exists a state $j \in \{1, \dots, M\}$ which can be interpreted as the minimal level ensuring a sufficiently good system performance. Then the results of Section 3 in Franken and Steller (1980) can formally be carried over to multistate systems by defining $G = \{\underline{x}: \phi(\underline{x}) > j\}$ as the set of "good" states and $B = \{\underline{x}: \phi(\underline{x}) < j\}$ as the set of "bad" states. We shall consider the stationary availability to level j and the stationary interval availability to level j defined respectively by

$$A_j = \bar{P}(\phi(\bar{X}(0)) > j) \quad \text{and} \quad A_j(s) = \bar{P}(\phi(\bar{X}(u)) > j, 0 < u < s).$$

The embedded points $T_m, m > 0$ are now chosen in such a way that

$$\begin{aligned} \text{"All points } T \text{ satisfying } \phi(\underline{X}(T-0)) < j-1 \text{ and} \\ \phi(\underline{X}(T+0)) > j \text{ are embedded points"} \end{aligned} \quad (3.1)$$

and

$$\phi(\underline{X}(T_m+0)) > j \quad \text{for all } m = 0, 1, \dots \quad (3.2)$$

This means that we may also choose some epochs of transitions within G as embedded points. The properties (3.1) and (3.2) ensure that a generic cycle of the synchronous PEP starts with an "up-time" of length U , satisfying $P(U > 0) = 1$, where the system is in G , eventually followed by a "down-time" of length D , satisfying $P(D = 0) > 0$, where the system is in B . Thus in view of Theorem 3.1 in Franken and Steller (1980) we have, as shown in Steller (1980)

$$A_j = E_P(U)/\Delta, \quad \Delta = E_P(U+D), \quad (3.3)$$

and

$$A_j(s) = \Delta^{-1} \int_s^\infty P(U > t) dt. \quad (3.4)$$

Consider now the cumulative system performance

$$\bar{Z}(s) = \int_0^s \phi(\bar{X}(u)) du$$

generated by the stationary PEP $[(\bar{X}(t)), (\bar{T}_m)]$ and ϕ . If the system performance over a time interval of length s is considered to be sufficiently good if $\bar{Z}(s)/s > j$ (eventually

"lost" performance is recuperated within the interval), then it is natural, instead of $A_j(s)$, to consider the characteristic

$$P_j(s) = \bar{P}(\bar{Z}(s)/s > j) ,$$

which we call the stationary mean interval performance probability to level j. (Obviously, we are here free to let j take any value in $(0, M]$.)

To date we we are not aware of an explicit formula for $P_j(s)$. However, an approximation formula for large s is given in Streller (1980). Consider the sequence

$$Z_m = \int_{T_m}^{T_{m+1}} \phi(\underline{X}(u)) du , \quad m = 0, 1, \dots ,$$

generated by the m th cycle of the synchronous PEP $[(\underline{X}(t)), (T_m)]$ and ϕ . Then, as a consequence of the Central Limit Theorem for generalized cumulative processes, given in the mentioned paper, we obtain the following proposition.

Proposition 3.1. Assume that

- (i) $K = E_P(Z_0) < \infty$, $\Delta = E_P(D_0) < \infty$,
- (ii) $\sigma_0^2 = E_P(Z_0 - \frac{K}{\Delta} D_0)^2 < \infty$ and that
- (iii) the sum

$$\sigma^2 = \sigma_0^2 + 2 \sum_{n=1}^{\infty} E_P((Z_0 - \frac{K}{\Delta} D_0)(Z_n - \frac{K}{\Delta} D_n))$$

converges absolutely.

If $\sigma^2 > 0$, then

$$P_j(s) \approx \frac{1}{\sqrt{2\pi} u} \int_0^{\infty} \exp(-t^2/2) dt ,$$

where

$$u = \frac{\sqrt{\Delta s}}{\sigma} (j - \frac{K}{\Delta}) .$$

A sufficient condition for (iii) to hold is also given in Streller (1980).

4. The case of independent components

In this section we shall assume that all the components are independently working and separately maintained. In particular, this means that the rule for improving the performance of a component and the corresponding random time for doing so do not depend on the behaviour of the other components. (Obviously, this rule must be specified for each concrete model.)

Consider now the i th component. As embedded points $(T_{mi}, m > 0)$ we choose the epochs where a transition of the state of this component occurs; i.e. $X_i(T_{mi}-0) \neq X_i(T_{mi}+0)$. Furthermore, let $(\bar{X}(t), t > 0)$ be the stationary state vector process and $[\underline{X}_{(i)}(t), \Phi_{(i)}]$, where $\underline{X}_{(i)}(t) = (X_{1i}(t), \dots, X_{ni}(t))$, the corresponding synchronous PEP with embedded points T_{mi} .

Remark 4.1. In view of the independence of the components the processes $(\bar{X}(t))$ and $(\underline{X}_{(i)}(t))$ differ only in their i th component where they take the values $(\bar{X}_i(t))$ and $(X_{ii}(t))$ respectively. For $k \neq i$ the processes $(\bar{X}_k(t))$ and $(X_{ki}(t))$ are statistically equivalent.

In addition to (T_{mi}) we consider other embedded points (T'_m) and (T_m^*) ; in T'_m the system crosses level j downwards while in T_m^* it crosses level $j-1$ upwards. More precisely,

$$\begin{aligned} \phi(\underline{X}(T'_m-0)) &\geq j & \text{and} & \phi(\underline{X}(T'_m+0)) < j-1; \\ \phi(\underline{X}(T_m^*-0)) &\leq j-1 & \text{and} & \phi(\underline{X}(T_m^*+0)) \geq j. \end{aligned}$$

Recall that the intensity of a stationary point process is defined as the mean number of points within the interval $[0,1)$. In view of the intensity conservation law the stationary point processes $\bar{\Phi}' = (\bar{T}'_m)$ and $\bar{\Phi}^* = (T_m^*)$ have the same intensity λ , say.

Furthermore, it is obvious that each epoch \bar{T}'_m of $\bar{\Phi}'$ is contained in the union $\bigcup_{i=1}^n \bar{\Phi}_{(i)}$ of all stationary epochs where a

Finally, consider the synchronous PEP $[(\underline{X}^*(t)), \Phi^*]$. A generic cycle starts with a working phase of length U , with $P(U>0) = 1$, where the system state is in $\{j, \dots, M\}$, followed by a failure phase of length D , with $P(D>0) = 1$, where the system state is in $\{0, \dots, j-1\}$. Thus, we have in view of (3.3)

$$\begin{aligned} h_j(\bar{P}) &= A_j = E_P(U)/E_P(U+D) \\ \lambda^{-1} &= E_P(U+D) \end{aligned} \tag{4.4}$$

Consequently, from (4.1), (4.3) and (4.4) we prove the following theorem

Theorem 4.1. The mean steady state working phase $E_P(U)$ is given by

$$E_P(U) = \lambda^{-1} h_j(\bar{P}) \tag{4.5}$$

where

$$\lambda = \sum_{i=1}^n \sum_{(k,l \in S_i: k>l)} \lambda_i^{k,l} [h_j((e^k)_i, \bar{P}) - h_j((e^l)_i, \bar{P})]. \tag{4.6}$$

As mentioned in Section 1 an exact formula for h_j is given in Theorem 3.1 of Natvig (1982).

Obviously, for each specific model, it remains to arrive at expressions for the intensities $\lambda_i^{k,l}$ corresponding to the transition $k \rightarrow l$, $k > l$. This depends on the performance process $(X_{ii}(t))$ of the i th component.

As an example we develop the formula (4.6) for the special case where the i th component consists of M_i branches in parallel and its state is an increasing function, f_i , of the number of branches functioning, satisfying $f_i(0) = 0$ and $f_i(M_i) = M$. However, we have an arbitrary multistate monotone structure function ϕ organizing the states of the components. This system is just a multistate version of the one arrived at when introducing redundancy at the component level in a traditional binary system. Remember according to Theorem 1.3 of Barlow and Proschan ((1975), p.23) this is really worthwhile.

Furthermore, we assume that the branches fail and are repaired/replaced independently of each other, all having the same instantaneous failure rate a_i and repair/replacement rate b_i . Hence we have a birth and death process on $\{0, 1, \dots, M_i\}$, registering the number of working branches, with birth and death rates given respectively by

$$\begin{aligned} b_{ik} &= (M_i - k)b_i & k = 0, \dots, M_i \\ a_{ik} &= k a_i & k = 0, \dots, M_i \end{aligned}$$

From Karlin and Taylor ((1975), p.137), we then get

$$\bar{P}_{if_i}(k) = \bar{P}(\bar{X}_i(t) = f_i(k)) = \binom{M_i}{k} (b_i/a_i)^k (1 + b_i/a_i)^{-M_i} \quad k=0, \dots, M_i$$

Hence \bar{P} is established.

Now let $P_{(i)f_i}(k) = P_{(i)}(X_{ii}(t) = f_i(k))$. By an heuristic argument using Bayes' formula we get

$$\begin{aligned} \bar{P}_{(i)f_i}(k) &= (b_{ik} + a_{ik}) \bar{P}_{if_i}(k) / \sum_{j=0}^{M_i} [(b_{ij} + a_{ij}) \bar{P}_{if_i}(j)] \\ &= [(M_i - k)b_i + a_i k] \binom{M_i}{k} (b_i/a_i)^k / [2M_i b_i (1 + b_i/a_i)^{M_i - 1}] \end{aligned} \quad (4.7)$$

That this is in fact the correct distribution can easily be verified by checking that it satisfies the equations

$$P_{(i)f_i}(k) = \sum_{j=0}^{M_i} P_{(i)f_i}(j) \pi_{jk} \quad k = 0, \dots, M_i, \quad \sum_{k=0}^{M_i} P_{(i)f_i}(k) = 1,$$

where the transition probabilities of the embedded Markov chain are given by

$$\pi_{jk} = \begin{cases} ja_i / (ja_i + (M_i - j)b_i) & k = j-1, j = 1, \dots, M_i - 1 \\ (M_i - j)b_i / (ja_i + (M_i - j)b_i) & k = j+1, j = 1, \dots, M_i - 1 \\ 1 & j = 0, k = 1 \quad \text{and} \quad j = M_i, k = M_i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since now λ_i is the inverse of the mean time between two successive transitions in the embedded Markov chain, we have from (4.7)

$$\lambda_i^{-1} = \sum_{k=0}^{M_i} P_{(i)f_i}(k) / [(M_i - k)b_i + a_i k] = (a_i + b_i) / [2a_i b_i M_i] \quad (4.8)$$

Hence we finally get from (4.6), the deduction leading to (4.3), (4.7) and (4.8) that

$$\begin{aligned}
 \lambda &= \sum_{i=1}^n \sum_{k=1}^{M_i} \lambda_i^{f_i(k), f_i(k-1)} [h_j((e^{f_i(k)})_{i, \underline{P}}) - h_j((e^{f_i(k-1)})_{i, \underline{P}})] \\
 &= \sum_{i=1}^n \sum_{k=1}^{M_i} \lambda_i^{P(i)} (E_i^{f_i(k), f_i(k-1)}) [h_j((e^{f_i(k)})_{i, \underline{P}}) - h_j((e^{f_i(k-1)})_{i, \underline{P}})] \\
 &= \sum_{i=1}^n \sum_{k=1}^{M_i} \lambda_i^{P(i)} f_i(k) \frac{k a_i}{(M_i - k) b_i + a_i k} [h_j((e^{f_i(k)})_{i, \underline{P}}) - h_j((e^{f_i(k-1)})_{i, \underline{P}})] \\
 &= \sum_{i=1}^n \sum_{k=1}^{M_i} k a_i \binom{M_i}{k} (b_i/a_i)^{k(1+b_i/a_i)}^{-M_i} \\
 &\quad \times [h_j((e^{f_i(k)})_{i, \underline{P}}) - h_j((e^{f_i(k-1)})_{i, \underline{P}})].
 \end{aligned}$$

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