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The Steady-State Response of a Class of Dynamical Systems to Stochastic Excitation

In this paper a class of coupled nonlinear dynamical systems subjected to stochastic excitation is considered. It is shown how the exact steady-state probability density function for this class of systems can be constructed. The result is then applied to some classical oscillator problems.

1 Introduction

In the last 20 years the response of nonlinear dynamical systems to stochastic excitation has been extensively studied. The diffusion processes approach to this problem leads to the Kolmogorov equations, which have, until now, been explicitly solved only in a few simple cases. For linear systems the transition probability density function can be obtained by a variety of methods [1, 2], whereas in the nonlinear case only some specific one-dimensional systems have been exactly solved so far [3]. An honest survey of the developments in this area can be found in [3, 4]. In recent years the use of approximate techniques in the treatment of random vibrations has become increasingly popular [5–7]. It is expected that in the next decade this trend will continue as computing costs decrease.

Our present knowledge of the steady-state response of nonlinear systems to white noise excitation is also far from a state of maturity [3]. The exact steady-state probability density for any one-dimensional nonlinear system, if it exists, has been found. Some specific nonlinear dynamical systems of higher dimensions have been considered [3], but in general very little is known. If the steady-state probability density of a dynamical system exists and can be found, then it may be possible to obtain the approximate nonstationary response by perturbation analysis [8]; the exact procedures to be used are dependent on the system under consideration. The purpose of this paper is to construct the exact steady-state probability density of a class of nonlinear dynamical systems subjected to stochastic excitation. It will also be shown that some previously published results [3] are particular cases of our present investigation.

2 Construction of Steady-State Solution

Consider the autonomous dynamical system in R^{2n} whose behavior when subjected to white noise excitation is described by the following equation

$$\begin{Bmatrix} \dot{x}_{2i-1} \\ \dot{x}_{2i} \end{Bmatrix} = \begin{pmatrix} h_i(x_{2i}) \\ -f(H)h_i(x_{2i}) - g_i(x_1, x_3, \ldots, x_{2n-1}) \end{pmatrix} + \begin{pmatrix} 0 \\ w_i(t) \end{pmatrix}, \quad x_i(0) = y_i. \tag{1}$$

for $1 = 1, 2, \ldots, n$, where $w_i(t)$ are independent Wiener processes with zero means and $E(dw_i(t)dw_j(t)) = 2D\delta_{ij}dt$. The functions $g_i(x_1, x_3, \ldots, x_{2n-1})$ assumed to arise from a potential function $V(x_1, x_3, \ldots, x_{2n-1})$:

$$g_i(x_1, x_3, \ldots, x_{2n-1}) = \frac{\partial}{\partial x_{2i-1}} V(x_1, x_3, \ldots, x_{2n-1}) \quad i = 1, 2, \ldots, n \tag{2}$$

and H *is* defined by

$$H(\mathbf{x}) = \sum_{i=1}^{n} \int_0^{x_{2i}} h_i(\zeta)d\zeta + V(x_1, x_3, \ldots, x_{2n-1}) \tag{3}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{2n} \end{Bmatrix}$$

At the present stage we further assume that

(*i*) f, H, V have continuous second-order derivatives, $H \geq 0$ and there exists an $H_0 > 0$ such that $f(H) \geq 0$ if $H > H_0$. In addition,

$$f^{-2}\frac{df}{dH} \to 0 \text{ as } H \to \infty$$

(*ii*) There exists a constant L such that

$$\sum_{i=1}^{n} |h_i|^2 + \sum_{i=1}^{n} |f(H)h_i + g_i|^2 \leq L(1 + |\mathbf{x}|^2) \tag{5}$$

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where the vector $\mathbf{x}$ is defined by (4) and the arguments of the functions on the left-hand side are those components of $\mathbf{x}$ previously indicated.

These restrictions will be relaxed later on. The preceding assumptions are such that the Lipschitz conditions for system (1) are satisfied. A little manipulation with L and the Lipschitz constants reveals that assumptions (*i*) and (*ii*) are sufficient to guarantee the following result of Ito [3, 9]: there exists an almost everywhere continuous solution of system (1) which is a homogeneous Markov process, the solution being unique up to a stochastic equivalence. Moreover, when an invariant distribution exists, the unique steady-state probability density $p(\mathbf{x})$ may be obtained from the stationary form of the Fokker-Planck equation where $\partial p/\partial t=0$. The previous statement expresses the equivalence under very mild restrictions of the stochastic differential equations approach and the diffusion processes approach [10], a topic that has been rigorously examined by mathematicians. Hence we will have constructed the only steady-state probability density from the stationary Fokker-Planck equation subsequently, under assumptions (*i*) and (*ii*).

We have been interpreting the dynamical system (1) using Ito calculus [10]. It is immaterial whether system (1) is regarded in the sense of Ito or in the sense of Stratonovich [11, 12] since in this particular case the so-called Wong and Zakai [13] corrections terms to the drift vector are identically zero. It is for this reason that assumption (1) need only hold on every finite domain. (Suppose S is a system where conditions (*i*) only hold on every finite region. Define a sequence of systems S_n in the following way: S_n is the restriction of S on the closed ball $B(0, n)$, and f, H, V are assigned suitable constant values outside $B(0, n)$. As $n\to\infty$, S_n tends to S and, for the type of systems considered in this paper [13, 14], the solution of S_n converges to the solution of S. (We have not discussed condition (*ii*) because it will later be removed.) Let $p(\mathbf{x}, t|\mathbf{y})d\mathbf{x}$ be the probability of the system (1) in the range $(x, x+\mathbf{dx})$ at time t given that it is initially at $\mathbf{y}$. The associated system of Fokker-Planck equations has the form

$$\left.\begin{aligned}\frac{\partial p}{\partial t}=-\frac{\partial}{\partial x_{2i-1}}\left[h_i(x_{2i})p\right]+\frac{\partial}{\partial x_{2i}}\Big[(f(H)h_i(x_{2i})+\\ g_i(x_1,x_3,\ldots,x_{2n-1}))p\Big]+D\frac{\partial^2 p}{\partial x_{2i}^2}\\ i=1,2,\ldots,n\end{aligned}\right\}\tag{6}$$

As previously explained, the steady-state density is governed by the following system of linear partial differential equations

$$g_i\frac{\partial p}{\partial x_{2i}}-h_i(x_{2i})\frac{\partial p}{\partial x_{2i-1}}+\frac{\partial}{\partial x_{2i}}\left[f(H)h_i(x_{2i})p+D\frac{\partial p}{\partial x_{2i}}\right]=0\tag{7}$$

$$i=1,2,\ldots,n$$

First we observe that if $p(\mathbf{x})$ satisfies the following conditions it will certainly be a solution of (7)

$$g_i\frac{\partial p}{\partial x_{2i}}-h_i(x_{2i})\frac{\partial p}{\partial x_{2i-1}}=0\tag{8}$$

$$\frac{\partial}{\partial x_{2i}}\left[f(H)h_i(x_{2i})p+D\frac{\partial p}{\partial x_{2i}}\right]=0\tag{9}$$

$$i=1,2,\ldots,n$$

Since (8) constitutes a linear first-order system of partial differential equations, we may solve them by the method of characteristics [15]. The subsidiary equations are

$$-\frac{dx_{2i-1}}{h_i(x_{2i})}=\frac{dx_{2i}}{g_i}=\frac{dp}{0}\quad i=1,2,\ldots,n$$

for which two independent integrals are

$$p=\text{constant}\tag{10}$$

and

$$V(x_1,x_3,\ldots,x_{2n-1})+\int_0^{x_{2i}}h_i(\zeta)d\zeta=k_i\tag{11}$$

$$i=1,2,\ldots,n$$

where k_i is a constant depending on x_{2j}, $j=1, 2, \ldots, n$, $j\neq i$. The system of equations (11) is equivalent to

$$H=V(x_1,x_3,\ldots,x_{2n-1})+\sum_{i=1}^{n}\int_0^{x_{2i}}h(\zeta)d\zeta=\text{constant}\tag{12}$$

Thus the general solution for (8) is of the form

$$p=\phi(H)\tag{13}$$

where ϕ is an arbitrary function. Since p and its first partial derivatives vanish as $|\mathbf{x}|\to\infty$, equations (9) imply

$$f(H)h_i(x_{2i})p+D\frac{\partial p}{\partial x_{2i}}=0\quad i=1,2,\ldots,n\tag{14}$$

Substituting (13) into (14), we have

$$h_i(x_{2i})\left[D\frac{d\phi}{dH}+f(H)\phi\right]+0\quad i=1,2,\ldots,n\tag{15}$$

Assuming that none of h_i is identically zero, it follows that

$$\phi=A\exp\left(-\frac{1}{D}\int_0^H f(\zeta)d\zeta\right)$$

where A is a normalizing constant. Hence the steady state density is given by

$$p(\mathbf{x})=\frac{\exp\left(-\frac{1}{D}\int_0^H f(\zeta)d\zeta\right)}{\int_{-\infty}^{\infty}\exp\left(-\frac{1}{D}\int_0^H f(\zeta)d\zeta\right)d\mathbf{x}}\tag{16}$$

where the denominator is a $2n$-fold integral. It can be easily checked that the expression defined in (16) satisfies all the requirements for a probability density function and therefore it represents the unique steady-state density of the coupled nonlinear dynamical system (1), under the assumptions (*i*) and (*ii*).

The assumption (*ii*) is a rather severe growth restriction on the class of systems under consideration. It should be removed if our results are to be of practical use. To this end we recall the concept of well-behaved solutions (see Appendix). Now it can be shown that under assumption (*i*) the solution (16) is a well-behaved solution of the stationary Fokker-Planck equation (7). Since it has been shown that a well-behaved solution of the stationary Fokker-Planck equation is unique [8, 16], we have the following result.

Theorem. The solution (16) is the unique steady-state solution of the dynamical system (1) subjected only to conditions (*i*).

It should be noted that the last theorem can also be established by using only the diffusion processes approach (i.e., a direct interpretation of the solution of the stochastic differential equation (1); using Ito calculus is not required). In this case (see the Appendix) some caution is needed to furnish a rigorous argument because the hard machinery needed comes from the theory of partial differential equations [8]. It is clear that assumption (*i*) is sufficient but not necessary and thus can be relaxed. In order to keep physically interpretable conditions, this will not be done in this paper.

3 Applications and Further Discussion

We shall apply the theorem established in the previous section to some classical oscillator problems, mentioning possible extensions when appropriate.

Example 1. The motion of a Brownian particle in a constant force field with dissipation of Rayleigh type [1, 17] may be described by

$$\ddot{x}+\beta\dot{x}+g=\dot{w}(t) \tag{17}$$

where $E(\dot{w}(S)\dot{w}(t))=2D\delta(t-s)$. The associated Fokker-Planck equation is

$$\frac{\partial p}{\partial t}=-\dot{x}\frac{\partial p}{\partial x}+g\frac{\partial p}{\partial \dot{x}}+\frac{\partial}{\partial \dot{x}}\left[\beta\dot{x}p+D\frac{\partial p}{\partial \dot{x}}\right] \tag{18}$$

We assume that there is a reflecting barrier at $x=0$, so that the particle does not disappear toward $x=-\infty$. We may consider the present system as a particular case of (1) and make the following identifications:

$$n=1$$
$$x_1=x$$
$$x_2=\dot{x}=h_1$$
$$f=\beta$$
$$g_1=g$$

Then $H=1/2\dot{x}^2+gx$ and the unique steady-state density as given by (16) is

$$p(x,\dot{x})=\frac{1}{g}\sqrt{\frac{2\pi D}{\beta}}\exp\left(-\frac{\beta}{D}\left(\frac{1}{2}\dot{x}^2+gx\right)\right) \tag{19}$$

for $x\geq 0$, $-\infty<\dot{x}<\infty$. This is the well-known barometric distribution [1].

Example 2. Consider the following self-excited oscillator corrupted by white noise

$$\ddot{x}-\epsilon(1-\dot{x}^2-x^2)\dot{x}+x=w(t) \tag{20}$$

where $\epsilon>0$. This can be written in the equivalent form

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{bmatrix}x_2\\ \epsilon(1-x_1^2-x_2^2)x_2-x_1\end{bmatrix}+\begin{bmatrix}0\\ \dot{w}(t)\end{bmatrix} \tag{21}$$

Hence by taking $h_1=x_2$, $g_1=x_1$, we have $H=1/2\,x_1^2+1/2x_2^2$, $f=-\epsilon(1-2H)$, and the steady-state density is given by

$$p(x_1,x_2)=C\exp\left(\frac{\epsilon}{D}H(1-H)\right) \tag{22}$$

where $C=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(\epsilon/DH(1-H))dx_1dx_2$. It is easy to check that the function (22) is a well-behaved solution of the associated Fokker-Planck equation. It is also easy to see that all circles on the x_1x_2 plane with centers at the origin are loci of constant probability for the steady-state distribution. Moreover the steady-state density attains a maximum when $H=1/2$ corresponding to $x_1^2+x_2^2=1$, and decreases exponentially on either side of the unit circle. If we now examine the deterministic oscillator obtained by omitting the last term on the right-hand side of (21), we will find by a standard analysis using the Poincaré-Bendixson theorem [18] that the unit circle is the unique limit cycle for the deterministic oscillator.

The information given in the last paragraph suggests that the nonstationary response of the system (21) in the neighborhood of the limit cycle may be obtained by perturbation techniques. This has been done by one of us for the case of weak damping and weak excitation [8], when ϵ, $D<<1$. The approximate spectral density has also been obtained by the same means.

Example 3. A class of generalized Van der Pol-Rayleigh oscillators subjected to white noise excitation is described by the equation [3]

$$\ddot{x}+f(H)\dot{x}+g(x)=\dot{w}(t)\quad x(0)=y,\dot{x}(0)=\dot{y} \tag{23}$$

where $E((dw(t))^2)=2D\,dt$. The equivalent first-order system is

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{bmatrix}x_2\\ -f(H)x_2-g(x_1)\end{bmatrix}+\begin{bmatrix}0\\ \dot{w}(t)\end{bmatrix} \tag{24}$$

where $H=1/2x_2^2+\int_0^{x_1}g(\zeta)d\zeta$ is a measure of the system energy. In this case the associated Fokker-Planck equation is

$$\left.\begin{aligned}&\frac{\partial p}{\partial t}=-x_2\frac{\partial p}{\partial x_1}+\frac{\partial p}{\partial x_2}(f(H)x_2+g(x_1))p+D\frac{\partial^2 p}{\partial x_2^2}\\ &\lim_{t\to 0}p(x_1,x_2,t\,|y,\dot{y})=\delta(x_1-y)\delta(x_2-\dot{y})\end{aligned}\right\} \tag{25}$$

The steady-state solution as given by (16) is

$$p(x_1,x_2)=C\exp\left(-\frac{1}{D}\int_0^H f(\zeta)d\zeta\right) \tag{26}$$

where $C^{-1}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\ (-1/D\int_0^H f(\zeta)d\zeta)dx_1dx_2$. If the system (24) possesses limit cycles, then the steady-state density (26) will have relative peaks on these limit cycles, with exponential decay away from the limit cycles. The nonstationary response in the neighborhood of a limit cycle may be obtained by perturbation techniques, the particular methods used are dependent on the form of $f(H)$. Moreover, asymptotic matching on regions enclosed by two adjacent limit cycles may be used in some cases to determine a uniform approximation. This is the subject of a subsequent paper.

4 Conclusion

In this paper the exact steady-state probability density function of a class of stochastic dynamical systems has been constructed. The construction has been justified by two alternative procedures. The result has been tested in some classical oscillator problems. When the steady-state density is known, the possibility of using perturbation techniques to compute the nonstationary response has been pointed out. In fact, a multiple-scale analysis has been used by one of us in an earlier paper [8] to derive the approximate nonstationary response of a specific oscillator. The conditions (*i*) made in this paper are sufficient but not necessary. Some of the smoothness requirements may be relaxed to handle specific problems.

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APPENDIX

Many problems in mechanics and related fields involving the response of dynamical systems to stochastic excitation can be modeled by stochastic differential equations of the form

$$d\mathbf{x}(t) = \mathbf{a}(t,\mathbf{x}(t))dt + \sum_{k=1}^{m} \sigma_k(t,\mathbf{x}(t))dw_k(t)$$
$$\mathbf{x}(t_0) = \mathbf{y} \qquad (27)$$

where $\mathbf{x}$, $\mathbf{a}$, $\sigma_k \epsilon R^m$ for $k=1, 2, \ldots, m$, and the $w_k(t)$ for $k=1, 2, \ldots, m$ are independent Wiener processes, with $E(dw_i(t)dw_j(t)) = \delta_{ij}dt$. It can be shown that the response in this case is a Markov process. In appropriate circumstances [3, 9], the transition probability density function satisfies the Fokker-Planck equation in a region D

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{m} \frac{\partial}{\partial x_i}[a_i(t,\mathbf{x})]p + \frac{1}{2}\sum_{i=1}^{m} \frac{\partial}{\partial x_i \partial x_j}$$
$$[b_{ij}(t,\mathbf{x})p] = Lp \qquad (28)$$

with initial condition

$$\lim_{t-s} p(\mathbf{x},t\,|\mathbf{y},s) = \delta(\mathbf{x}-\mathbf{y})$$

The coefficients a_i, b_{ij} are derived in the following way: a_i are the components of $\mathbf{a}(t, \mathbf{x})$ and

$$b_{ij} = \sum_{k=1}^{m} \sigma_{ik}(t,\mathbf{x})\sigma_{jk}(t,\mathbf{x}) \qquad (29)$$

where σ_{rs}, $r=1, 2, \ldots, m$, are the components of $\boldsymbol{\sigma}_s(t, \mathbf{x})$ defined in (27). A well-behaved solution of the Fokker-Planck equation (28) is defined in the following way:

(*I*) If p_s is a solution of the stationary Fokker-Planck equation $Lp=0$, then it is well-behaved if

$$\sum_{i=1}^{m}\left[a_i p_s - \frac{1}{2}\sum_{j=1}^{m}\frac{\partial b_{ij}p}{\partial x_j}\right]\cdot n_i = 0 \quad p_s = 0 \qquad (30)$$

on the boundary ∂D of the region D, where n_i are the components of the outward normal to ∂D. If D is an infinite domain, then (30) should be taken in a limiting sense.

(*II*) If p is a solution of the time-dependent Fokker-Planck equation, it is well behaved if equations (30) are satisfied with p_s replaced by p, and for all solutions p_s of the stationary equation $Lp=0$,

$$\left.\begin{aligned} &\int_D P_s^{-1}p^2 d\mathbf{x} < \infty \\ &\int_D p_s^{-1}\left(\frac{\partial p}{\partial t}\right)^2 d\mathbf{x} < \infty \end{aligned}\right\} \qquad (31)$$

for all $t>0$, with the convergence being uniform in t if D is an infinite domain.

The following has been established [8, 16].

Theorem

Well-behaved solutions to the Fokker-Planck equation are unique. Under some mild restrictions [8] the well-behaved solution p of the time-dependent Fokker-Planck equation converges in L^1 to a function of p_s as $t\to\infty$ and p_s is exactly a solution obtained by solving that stationary equation $L_p=0$.

Because of the exponential nature of p defined in (16) and the conditions (*i*), it is easy to check that the solution (16) satisfies (30) and is thus the unique well-behaved steady-state solution. By assuming that the time-dependent solution of (28) is well behaved [8], a self-consistent diffusion processes approach based on the Fokker-Planck equation may be developed to derive the same results as in Section 2. In this case the intermediate use of assumption (*II*) is not needed.