The steady two-dimensional radial flow of viscous fluid between two inclined plane walls

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This paper considers the steady two-dimensional radial flow of viscous fluid between plane walls which either converge or diverge. A general solution is obtained in terms of elliptic functions and the various mathematically possible types of flow are discussed.

1. Introduction and summary

In this paper we consider the steady radial two-dimensional flow of viscous fluid between plane walls which either converge or diverge. The investigations on this subject were started by Jeffery (1915) and they have proved to be the starting-point for a number of researches by other authors, notably Hamel (1916), Harrison (1919), Karman (1922), Tollmien (1931), Noether (1931) and Dean (1934). These investigations deal with specialized aspects of the question and do not contain a systematic treatment of the general problem. It has been known since 1915 that a general solution is possible in terms of elliptic functions. In spite of this, however, the only case which is quoted in text-books and manuals is that which is, in effect, the degenerate case of the elliptic functions (see, for example, Goldstein 1938, p. 109).

The following investigation was started in the hope that it would lead to a clear picture of the change in flow associated with increasing Reynolds number, R. It was stimulated by the statement which is given by Goldstein (1938, p. 106). The following is a summary of the statement: For inflow the magnitude of the velocity is a maximum in the centre of the channel and diminishes uniformly to zero at the walls. With increasing Reynolds number the velocity profile always remains of this same type but becomes flatter and flatter in the middle dropping very rapidly to zero at the walls. For divergent flow with increasing Reynolds number the state of affairs is quite different. At small Reynolds numbers the magnitude of the velocity is a maximum in the middle of the channel and it decreases uniformly towards the walls. With increasing Reynolds number the flow becomes more and more concentrated in the centre of the channel until finally, at some critical value of R, regions of inflow appear on either side of the region of outflow. With a larger outward flux than that associated with this critical value of R three solutions are possible, with the backward flow near one wall or the other, or near both walls. It seems possible that at still larger values of R the number of possible solutions also increases.

The above statement must, however, be modified in the light of the following investigation, the principal results of which are given below. At this stage it becomes necessary to define more closely what is meant by "outflow" and "inflow". In most previous investigations the Reynolds number R was defined to be of the form $l\,|\,u_{\text{max.}}\,|/\nu$, where l was some representative length, ν the coefficient of kinematic viscosity, and $|\,u_{\text{max.}}\,|$ was the maximum magnitude of the velocity in the flow. As we are now dealing with velocity profiles which contain both outflow and inflow the above definition of R cannot be used, for it makes no distinction between the Reynolds number in the case $|\,u_{\text{max. outwards}}\,| = u_0$, $|\,u_{\text{max. inwards}}\,| = u_1$, and in the case $|\,u_{\text{max. outwards}}\,| = u_1$, $|\,u_{\text{max. inwards}}\,| = u_0$. We therefore define the Reynolds number of the flow as $Q/2\nu$, where Q is the volume of fluid passing from the narrower end of the channel to the wider end, in unit time, between two planes which are at unit distance apart and are perpendicular to both the channel walls. A positive value of R will correspond to average outflow and a negative value of R to average inflow.

In the following investigation it was found advantageous, for purposes of mathematical simplicity, to use the Weierstrassian $\wp$ function as much as possible but to transform to the Jacobian elliptic function for purposes of numerical calculation.

The principal results are as follows: For every pair of values of α and R the number of mathematically possible velocity profiles of radial motion is infinite (see § 4). The profiles may or may not be symmetrical with respect to the central line of the channel. If $\pi > \alpha > \frac{1}{2}\pi$ *pure* outflow is impossible, and there is a range of values of small Reynolds number in which *pure* inflow is impossible. The effect of increasing R in outflow is to exclude, progressively, more and more of the simpler types of flow. No such exclusion is introduced when $|\,R\,|$ is increased in inflow. With increasing Reynolds number in pure inflow, and with small values of α, the velocity profile exhibits all the well-known characteristics of boundary layers near the walls, and an approximately constant velocity across the rest of the channel. In pure outflow the flow becomes more and more concentrated in the centre of the channel as R is increased, until finally regions of inflow occur near the walls.

The analysis described below gives the mathematically possible types of flow but does not indicate which type of flow is actually assumed by the fluid. Considerations of stability would probably show that many of the types of flow are unstable, but the present state of the theory of the stability of fluid motion does not seem to offer much hope that information will be obtained along these lines in the near future. It should be noted that in an actual experiment boundary conditions are imposed not only by the friction at the walls but also by the pressure conditions at the inlet and outlet ends of the channel. The following investigation may be considered as one which determines the proper pressure distributions over the ends which allow of steady laminar radial motion. If in an experiment the pressure distribution is not one of those obtained below, the flow may be neither steady, laminar nor radial. In normal experiments the pressure distribution over the ends approximates most closely to those required for pure inflow and outflow, so that these types are more likely to have been observed than are the other ones.

It is possible, however, to obtain a fairly plausible picture of the change of flow by the help of speculative assumptions. If we make the assumption that "If the pressure conditions over the inlet and outlet ends are not rigidly applied, that is, if the pressure profile can be assumed to be loosely self-adjusting, then the velocity profile will be that one which has the smallest number of crests and troughs", it is possible to deduce a simple systematic sequence of change. In inflow the velocity profile would always be of the symmetrical pure inflow type, and in outflow the sequence of change with increasing R is as shown in figure 4. It is assumed, also, that the non-symmetrical types, and also those which, even though they have inflow along the central line, give a positive value for Q, would not manifest themselves in experiment. This speculation can be pressed still further to provide information as to the stability of the various types of flow (see p. 467).

2. List of symbols and elementary properties of some of the principal functions

2α is the angle between the walls.

OV, OW are the lines of section of the walls with a representative plane which is at right angles to the line of intersection of the walls.

OX is the bisector of OV and OW.

r, θ, h are the cylindrical coordinates, where r is measured from O, θ is measured from the central line OX, and h is measured in a direction perpendicular to the representative plane.

r_0 is the distance along OX of a fixed point O'.

u, v, w are the velocity components in the directions of r, θ, h. (Since the flow is radial $v = w = 0$.)

ρ, μ, ν are the density, coefficient of viscosity, coefficient of kinematic viscosity, of the fluid.

Q is the volume of fluid passing from the narrower end of the channel to the wider end, in unit time, between two planes which are at unit distance apart and are perpendicular to both walls. (Q may be either positive or negative.)

$F(\theta) = ru/2\nu$,

$f(\theta) = -\frac{1}{3}\{F(\theta)+1\} = \wp(\theta-\theta_0)$.

R is the Reynolds number $= Q/2\nu = \int_{-\alpha}^{\alpha} F(\theta)\,d\theta$. (N.B. A positive value of R denotes average outflow and a negative one denotes average inflow.)

a, b, α_0, θ_0 are constants of integration (a, b, α_0 are real and θ_0 complex).

p is the pressure in the fluid at r, θ, h. This is equal to

$$\frac{4\mu\nu}{r^2}F(\theta)+\frac{a\mu\nu}{2r^2}+\text{constant}.$$

$\wp(\theta)$ is the Weierstrassian elliptic function defined by the equation

$$\wp'(\theta)^2 = 4\wp^3-g_2\wp-g_3,$$

where the invariants g_2 and g_3 are respectively equal to $\frac{1}{3}(4-a)$ and $\frac{1}{27}(8-3a-3b)$.

e_1, e_2, e_3 are the roots of the cubic $4s^3-g_2s-g_3 = 0$. If the roots are all real, they are chosen in such a way that $e_1 \geqslant e_2 \geqslant e_3$. If the roots are not all real, then since g_2 and g_3 are real, two of the roots must be complex, and in this case they are chosen so that

$$e_1 = A+iB, \quad e_2 = -2A, \quad e_3 = A-iB,$$

where A and B are real.

$g_2 = -4(e_2e_3+e_3e_1+e_1e_2) = 2(e_1^2+e_2^2+e_3^2)$.

$g_3 = 4e_1e_2e_3$.

$H^2 = (e_2-e_1)(e_2-e_3) = 9A^2+B^2$ when the roots are complex.

Δ is the discriminant of the cubic $4s^3-g_2s-g_3 = 0$. It is equal to

$$16(e_2-e_3)^2(e_3-e_1)^2(e_1-e_2)^2 = g_2^3-27g_3^2.$$

$\omega_1, \omega_2, \omega_3$ are the half-periods of the $\wp$ function. If Δ is positive, that is, if all the roots of the equation $4s^3-g_2s-g_3 = 0$ are real, then the real period is defined to be $2\omega_1$ and the imaginary period is defined to be $2\omega_3$. If Δ is negative, that is, if only one root is real, then the real period is defined to be $2\omega_2$, the other two periods being conjugate complex numbers:

$$2\omega_1 = 2(C-iD), \quad 2\omega_2 = -4C, \quad 2\omega_3 = 2(C+iD),$$

where C and D are real.

K, E are the complete elliptic integrals of the first and second kinds.
$k = \sqrt{\{(e_2 - e_3)/(e_1 - e_3)\}}$ is the modulus of the elliptic integrals.
$k_1 = \sqrt{\{(e_1 - e_2)/(e_1 - e_3)\}}$ is the complementary modulus $= \sqrt{(1-k^2)}$.
$E(u)$ is the incomplete elliptic integral of the second kind $= \int_0^u \mathrm{dn}^2 u\, du$.
$Z(u)$ is the Jacobian Zeta function. It is defined by the equation

$$Z(u) = E(u) - uE/K.$$

$X = e_1 - e_3$. When e_1, e_2, e_3 are real, $X \geqslant 0$.
$\eta_1, \eta_2, \eta_3, \ldots$ are the positive roots, in ascending order of magnitude, of the equation

$$\mathrm{sn}^2(\eta\sqrt{X}) = \{(1+k^2)X - 1\}/3k^2X.$$

They are connected by the relations

$$\eta_2 = 2\omega_1 - \eta_1, \quad \eta_3 = 2\omega_1 + \eta_1,$$
$$\eta_4 = 4\omega_1 - \eta_1, \quad \eta_5 = 4\omega_1 + \eta_1, \text{ etc.}$$

When $\theta_0 = \omega_3$ and e_1, e_2, e_3 are real, then

$$S = 2\int_0^{\eta_1} F(\eta)\, d\eta > 0,$$
$$T = 2\int_{\eta_1}^{\omega_1} F(\eta)\, d\eta < 0.$$

3. The equations of motion and continuity

We are considering the steady radial motion of an incompressible viscous fluid in the region between inclined plane walls in the absence of external forces. The flow is two-dimensional, the representative plane being perpendicular to the channel walls. In this plane the co-ordinates are r, θ, where r is the distance from O, the intersection of the walls, and θ is measured from the central line OX (cf. figure 1). The channel walls cut the representative plane in the lines OV, OW ($\theta = \mp\alpha$). The co-ordinate perpendicular to this plane is h. The flow diverges from, or converges to, the line represented by $r = 0$. The components of velocity in the directions of r, θ, h are u, v, w. With the assumptions made above it can be seen that $v = w = 0$. The equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} = 0, \tag{3·1}$$

where u is a function of r and θ only. This equation can only be satisfied if

$$u = 2\nu F(\theta)/r, \tag{3·2}$$

where $F(\theta)$ is a function of θ only. The total flux of fluid, per unit length of z axis, is Q where

$$Q = \int_{-\alpha}^{\alpha} ru\,d\theta = 2\nu\int_{-\alpha}^{\alpha} F(\theta)\,d\theta. \tag{3·3}$$

The Reynolds number of the flow, R, can be defined by the equation

$$R = Q/2\nu = \int_{-\alpha}^{\alpha} F(\theta)\,d\theta. \tag{3·4}$$

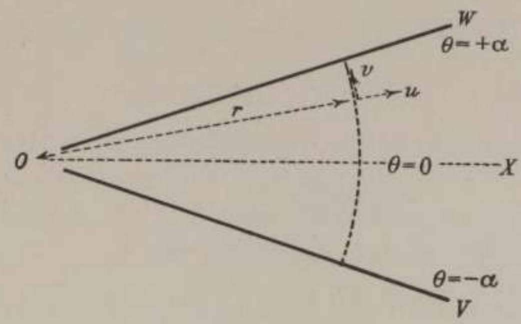

FIGURE 1

The equations of motion are

$$\rho u\frac{\partial u}{\partial r} = -\frac{\partial p}{\partial r}+\mu\left\{\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right\}, \tag{3·5}$$

$$0 = -\frac{\partial p}{\partial \theta}+\frac{2\mu}{r}\frac{\partial u}{\partial \theta}. \tag{3·6}$$

Using the form of u suggested by (3·2), these become

$$-\frac{4\nu^2F^2}{r^3} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{2\nu^2F''}{r^3}, \tag{3·7}$$

$$0 = -\frac{\partial p}{\partial \theta}+\frac{4\mu\nu F'}{r^2}, \tag{3·8}$$

where the accents denote differentiation with respect to θ. From (3·8) we see that

$$p = \frac{4\mu\nu}{r^2}F(\theta)+p_1(r), \tag{3·9}$$

where $p_1(r)$ is a function of r only. If this form is inserted in (3·7), we get

$$\frac{r^3}{\rho}\frac{\partial p_1}{\partial r} = \nu^2(2F''+4F^2+8F). \tag{3·10}$$

The left-hand side of (3·10) is a function of r and the right-hand side is a function of θ. They are both therefore equal to a constant which, from the nature of the expressions, must be real. Putting this constant equal to $-a\nu^2$, we find

$$p_1 = \frac{a\mu\nu}{2r^2} + \text{const.}, \tag{3·11}$$

and

$$2F'' + 4F^2 + 8F + a = 0. \tag{3·12}$$

Multiplying (3·12) by F' and integrating, we obtain

$$F'^2 + \tfrac{4}{3}F^3 + 4F^2 + aF = \text{constant} = b, \text{ say.} \tag{3·13}$$

In this equation put

$$F(\theta) = -\{3f(\theta)+1\}, \tag{3·14}$$

and (3·13) becomes

$$f'(\theta)^2 = 4f^3 - g_2 f - g_3, \tag{3·15}$$

where

$$\left.\begin{aligned} g_2 &= \tfrac{1}{3}(4-a), \\ g_3 &= \tfrac{1}{27}(8-3a-3b). \end{aligned}\right\} \tag{3·16}$$

From the definition of $F(\theta)$ and fróm (3·12) and (3·13) it should be noted that a and b, and hence also g_2 and g_3, are real. Further, a may be either positive or negative, but b must be positive, for u, and hence $F(\theta)$ is zero at the walls and b is therefore equal to the value of $F'(\theta)^2$ when $\theta = \pm\alpha$. The solution of (3·15) is

$$f(\theta) = \wp(\theta - \theta_0), \tag{3·17}$$

where θ_0 is a constant of integration which may, possibly, be complex. The solution of the problem under consideration is therefore

$$\left.\begin{aligned} u &= -\frac{2\nu}{r}\{3\wp(\theta-\theta_0)+1\}, \\ p &= -\frac{4\mu\nu}{r^2}\{3\wp(\theta-\theta_0)+1\} + \frac{a\mu\nu}{2r^2} + \text{constant}, \end{aligned}\right\} \tag{3·18}$$

where the invariants of the $\wp$ function are given by (3·16). This solution is subject to the following limitations:

(i) θ_0 must be a constant which makes u real and non-infinite for all values of θ in the range $\alpha \geqslant \theta \geqslant -\alpha$.

(ii) u must be zero at the walls, that is at $\theta = \pm\alpha$. Hence $F(\theta) = 0$, or $f(\theta) = -\frac{1}{3}$ when $\theta = \pm\alpha$.

In the following investigation we assume that the walls do not actually reach the point $r = 0$, but fall short of it, so that the singularities which occur there are avoided.

4. The velocity profile

Since $f(\alpha) = f(-\alpha) = -\frac{1}{3}$, we have

$$\wp(\alpha-\theta_0) = \wp(\alpha+\theta_0).$$

This equation can only be satisfied if

$$\alpha-\theta_0 = \pm(\alpha+\theta_0)+2m\omega_1+2n\omega_3 \quad (m, n \text{ integers}),$$

and this gives

$$\left.\begin{matrix} -\theta_0 \\ \text{or} \quad \alpha \end{matrix}\right\} = m\omega_1+n\omega_3.$$

In special cases both these possibilities may be satisfied at the same time. Since the flow depends upon $\wp(\theta-\theta_0)$ and not upon θ_0, there is no loss of generality in making θ_0 lie within the fundamental period parallelogram. This is equivalent to restricting θ_0 to one or other of the values $0, \omega_1, \omega_2, \omega_3$. θ_0 cannot be zero, for then u would be infinite at $\theta = 0$. Later in this section it is shown that two cases arise. In the first case, ω_1 and ω_3 are complex conjugate numbers and α, being real, can only have the values $m\omega_2$. In the second case ω_1 is real and ω_3 is imaginary, so that here α can only be put equal to $m\omega_1$.

These possibilities must be considered in conjunction with the fact that $\wp(\theta-\theta_0)$ must be real for all values of θ. The investigation must here be separated into two sections:

(I) The roots of the cubic

$$4s^3-g_2s-g_3 = 0$$

are not all real. This corresponds to the case when the discriminant Δ is negative. Since g_2 and g_3 are real, one of the roots must be real, e_2 say, and the others, e_1 and e_3, must be conjugate complex numbers. We put

$$e_1 = A+iB, \quad e_2 = -2A, \quad e_3 = A-iB,$$

where A and B are real. The discriminant Δ is equal to $-64B^2H^4$, where

$$H = +\sqrt{(9A^2+B^2)}.$$

ω_1 and ω_3 are the roots of the equation

$$\wp(\omega_1) = e_1 \quad \text{and} \quad \wp(\omega_3) = e_3.$$

Since e_1 and e_3 are conjugate complex numbers and since g_2 and g_3 are real, one solution of these equations can be put into the form

$$\omega_1 = C - iD, \quad \omega_2 = -2C, \quad \omega_3 = C + iD,$$

where C and D are real. These values are not unique, since, for example,

$$\omega_1' = -C - iD, \quad \omega_2' = 2iD, \quad \omega_3' = C - iD$$

is another set of values for which e_1 and e_3 are conjugate complex numbers. This second form, and other possible ones, are just alternative ways of describing the fundamental period parallelogram shown in figure 2. They introduce no feature that is not contained by the chosen values of ω_1, ω_2 and ω_3, and they can, therefore, be neglected. If $0 \leqslant \lambda \leqslant 1$, then along the lines ATC $[z = \lambda . 4C]$, BTD $[z = 2\omega_1 + i\lambda . 4D]$, s, which is equal to $\wp(z)$,

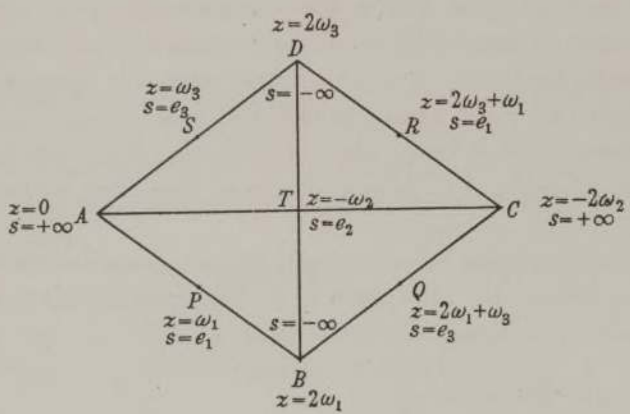

FIGURE 2. $s = \wp(z)$; $\Delta < 0$, g_2 and g_3 real.

assumes all real values from $-\infty$ to $+\infty$ twice over. These then are the only lines within the period parallelogram on which s assumes real values. Along ATC the variable is real so that we must here identify θ_0 with ω_2, which is the only real half-period. Further, the range of θ is $\alpha \geqslant \theta \geqslant -\alpha$, and if $2\alpha > 4C$, i.e. $2\alpha > |2\omega_2|$, the range will contain at least one point congruent with the point of zero argument, where $\wp(z)$ is infinite. This must be avoided and hence the solution here is

$$f(\theta) = \wp(\theta - \omega_2), \tag{4·1}$$

where $|\omega_2| > \alpha$ and where, in order to satisfy the boundary condition, $e_2 \leqslant -\frac{1}{3}$. It should be noted also that this is the only type of solution under (I), for if α were to be equal to an integral multiple of ω_2 the range $\alpha \geqslant \theta \geqslant -\alpha$

would contain at least one point at which the velocity would be infinite. The solution under (I) is therefore

$$f(\theta) = \wp(\theta-\omega_2) = e_2 + \frac{(e_2-e_1)(e_2-e_3)}{\wp(\theta)-e_2} = -2A + \frac{H^2}{\wp(\theta)+2A}$$
$$= -2A + H\frac{1-\operatorname{cn}(2\theta\sqrt{H})}{1+\operatorname{cn}(2\theta\sqrt{H})} = -2A + H\frac{\operatorname{sn}^2(\theta\sqrt{H})\operatorname{dn}^2(\theta\sqrt{H})}{\operatorname{cn}^2(\theta\sqrt{H})}, \tag{4·2}$$

where $k^2 = \dfrac{1}{2}\left(1+\dfrac{3A}{H}\right)$ and $\omega_2 = K/\sqrt{H}$ (see Milne-Thomson 1931, pp. xi and xiv). The boundary condition is

$$\operatorname{cn}(2\alpha\sqrt{H}) = \frac{3H-(6A-1)}{3H+(6A-1)} \tag{4·3}$$

or

$$\frac{\operatorname{sn}^2(\alpha\sqrt{H})\operatorname{dn}^2(\alpha\sqrt{H})}{\operatorname{cn}^2(\alpha\sqrt{H})} = \frac{6A-1}{3H}. \tag{4·3a}$$

In order that this should have a solution $6A \geqslant 1$, which is the condition $e_2 \leqslant -\frac{1}{3}$.

On the central line $f(\theta) = e_2 = -2A$ and $F(0)$ is therefore equal to $(6A-1)$ which is positive. On account of the nature of $\wp(\theta-\omega_2)$ we see that $F(\theta)$ has a positive value on the central line and decreases monotonically towards the walls where its value is zero. The case considered under (I) always, therefore, represents outflow which is symmetrical with respect to the central line.

(II) We have discussed the case in which $\Delta < 0$. Let us now discuss that in which $\Delta > 0$. This, since g_2 and g_3 are real, corresponds to the case in which all the roots of the cubic

$$4s^3 - g_2 s - g_3 = 0$$

are real. Let them be denoted by e_1, e_2, e_3 chosen in such a way that $e_1 > e_2 > e_3$. (The degenerate cases, in which two or three of the roots are equal, will be discussed separately. They correspond to $\Delta = 0$.)

In the non-degenerate cases one of the periods, $2\omega_1$ say, is perfectly real, and one, $2\omega_3$ say, is perfectly imaginary (see Whittaker and Watson 1920, p. 444). The period parallelogram is then as shown in figure 3. Along the four lines APB ($z = \lambda . 2\omega_1$), ASD ($z = \lambda . 2\omega_3$), PTR ($z = \omega_1 + \lambda . 2\omega_3$), STQ ($z = \omega_3 + \lambda . 2\omega_1$), where $0 \leqslant \lambda < 1$, the Weierstrassian function $\wp(z)$ takes all real values from $-\infty$ to $+\infty$ twice over, so that these are the only lines within the period parallelogram on which $\wp(z)$ assumes real values.

We cannot identify θ_0 with ω_1, for the argument of $\wp(\theta-\theta_0)$ would then be represented by points on APB. Along this line $\wp(\theta-\theta_0)$ is always positive, and it would therefore be impossible to satisfy the boundary condition $f(\alpha) = \wp(\alpha-\theta_0) = -\frac{1}{3}$. Hence, since α and ω_1 are real, the following possibilities arise:

(i) $\theta_0 = \omega_3$. The boundary condition is

$$\wp(\alpha-\omega_3) = -\tfrac{1}{3}. \tag{4·4}$$

In order that this condition may be satisfied it is essential that

$$e_2 \geqslant -\tfrac{1}{3} \geqslant e_3, \tag{4·5}$$

for the argument of $\wp(\alpha-\omega_3)$ is a point on the line STQ of figure 3, and along this line the value of $\wp(z)$ lies between e_2 and e_3. If (4·5) is satisfied, then (4·4) may be used to determine α. The number of values of α satisfying (4·4) is infinite. This will be referred to later in this section.

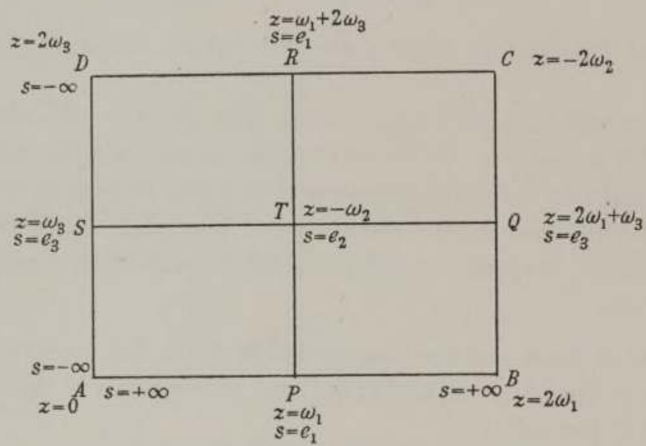

FIGURE 3. $s = \wp(z)$; $\Delta > 0$, g_2 and g_3 real.

(ii) $\theta_0 = \omega_2$. The boundary condition is

$$\wp\{\alpha-\omega_2\} = \wp\{(\alpha-\omega_1)-\omega_3\} = -\tfrac{1}{3}. \tag{4·6}$$

Here again $e_2 \geqslant -\frac{1}{3} \geqslant e_3$ and there is no unique solution of (4·6).

(iii) $\alpha = (2n+1)\,\omega_1$, where n is any positive integer. From a consideration of the values of $\wp(\theta-\theta_0)$ in the range $\alpha \geqslant \theta \geqslant -\alpha$ it is clear that the imaginary part of θ_0 must be equal to ω_3 and that $e_2 \geqslant -\frac{1}{3} \geqslant e_3$. If we put

$$\theta_0 = \alpha_0 + \omega_3, \tag{4·7}$$

where α_0 is real, the equation for α_0 becomes

$$\wp\{(\alpha_0-\omega_1)-\omega_3\} = -\tfrac{1}{3}. \tag{4·8}$$

(iv) $\alpha = 2n\omega_1$, where n is any positive integer. Again we see that θ_0 must be of the form suggested by (4·7) and that $e_2 \geqslant -\frac{1}{3} \geqslant e_3$. The equation for α_0 is here

$$\wp\{\alpha_0 - \omega_3\} = -\tfrac{1}{3}. \tag{4·9}$$

The obvious similarity between the boundary conditions (4·4), (4·6), (4·8) and (4·9) suggests that the four special cases are intimately connected. This is so, as the following method shows. Put

$$X = e_1 - e_3, \quad \text{and} \quad k^2X = e_2 - e_3. \tag{4·10}$$

X and k^2X are always positive. In terms of k^2 and X we have

$$e_1 = \tfrac{1}{3}(2-k^2)\,X;\;\; e_2 = \tfrac{1}{3}(2k^2-1)\,X;\;\; e_3 = -\tfrac{1}{3}(1+k^2)\,X;\;\; \omega_1 = KX^{-\frac{1}{2}}; \tag{4·11}$$

where K is the complete elliptic integral of the first kind, and k is the modulus of the elliptic integral. The condition $e_2 \geqslant -\frac{1}{3} \geqslant e_3$ now becomes

$$(2k^2-1)\,X \geqslant -1 \geqslant -(1+k^2)\,X.$$

This condition assumes different forms according as k^2 is less than or greater than $\frac{1}{2}$. Expressed more precisely the condition (4·5) is:

$$\left.\begin{array}{ll} \text{When } k^2 < \tfrac{1}{2}, & \dfrac{1}{1-2k^2} \geqslant X \geqslant \dfrac{1}{1+k^2}, \\ \text{and when } k^2 \geqslant \tfrac{1}{2}, & \infty \geqslant X \geqslant \dfrac{1}{1+k^2}. \end{array}\right\} \tag{4·12}$$

The fundamental boundary equation, that is, the one from which all the others can be obtained, is taken to be (4·4). Consider the equation

$$\begin{aligned} \wp(\eta - \omega_3) &= e_3 + (e_2 - e_3)\,\mathrm{sn}^2\{\eta\sqrt{(e_1-e_3)}\} \\ &= \tfrac{1}{3}X\{3k^2\,\mathrm{sn}^2(\eta\sqrt{X}) - 1 - k^2\} = -\tfrac{1}{3}. \end{aligned}$$

This can be put into the form

$$\mathrm{sn}^2(\eta\sqrt{X}) = \{(1+k^2)\,X - 1\}/3k^2X. \tag{4·13}$$

The conditions laid down in (4·12) are just those which ensure that (4·13), considered as an equation for η, has real solutions. Let $\eta_1, \eta_2, \eta_3, \ldots$, etc. be the positive roots of (4·13) in ascending order of magnitude. They are connected by the relations

$$\left.\begin{array}{c} \eta_2 = 2\omega_1 - \eta_1, \quad \eta_3 = 2\omega_1 + \eta_1, \\ \eta_4 = 4\omega_1 - \eta_1, \quad \eta_5 = 4\omega_1 + \eta_1, \\ \text{etc.} \end{array}\right\} \tag{4·14}$$

The smallest root, η_1, is less than $\omega_1\,(=KX^{-\frac{1}{2}})$. After these preliminary considerations the subdivisions (i), (ii), (iii) and (iv) made above may be written as

(i′) $\theta_0 = \omega_3$.

In all cases the flow is symmetrical about the central line $\theta = 0$ for $\wp(\theta-\omega_3) = \wp(-\theta-\omega_3)$. In particular the value of $F(\theta)$ when $\theta = 0$ is $-(1+3e_3)$ which is $\{(1+k^2)\,X-1\}$ and is therefore positive. Hence there is always outflow along the central line.

α can have any of the values η_1, η_2, η_3, etc.

If $\alpha = \eta_1$, the flow is pure outflow as shown in figure 4a.

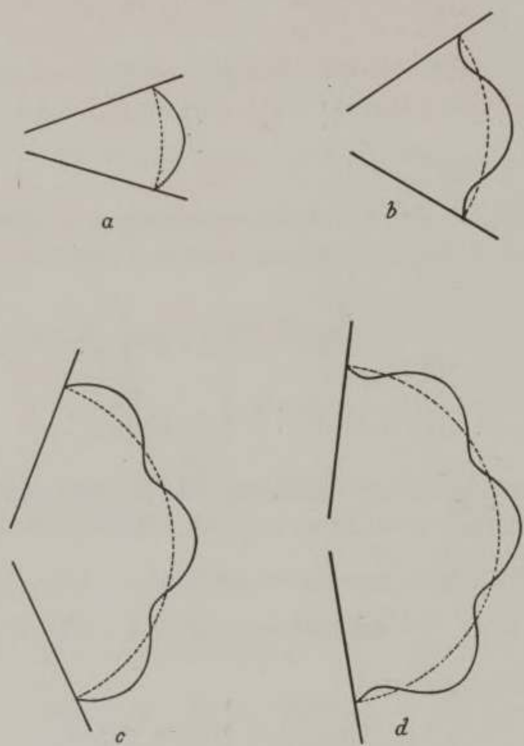

Figure 4. Velocity profiles corresponding to (i′) of § 4.

If $\alpha = \eta_2$, the central outflow is bounded by regions of inflow as in figure 4b.

If $\alpha = \eta_3$, the central outflow has one region of inflow and one of outflow on either side as in figure 4c.

If $\alpha = \eta_4$, the central outflow has two regions of inflow and one of outflow on either side as in figure 4d.

Etc.

(ii′) $\theta_0 = \omega_2$.

In all the cases the flow is symmetrical about the central line for $\wp(\theta-\omega_2) = \wp(-\theta-\omega_2)$. The value of $F(0)$ is $-(1+3e_2)$ which is equal to $\{(1-2k^2)X-1\}$ and is therefore negative. Hence there is always inflow along the central line.

α can have any of the values $\omega_1-\eta_1$, $\omega_1+\eta_1$, $\omega_1+\eta_2$, $\omega_1+\eta_3$, ...

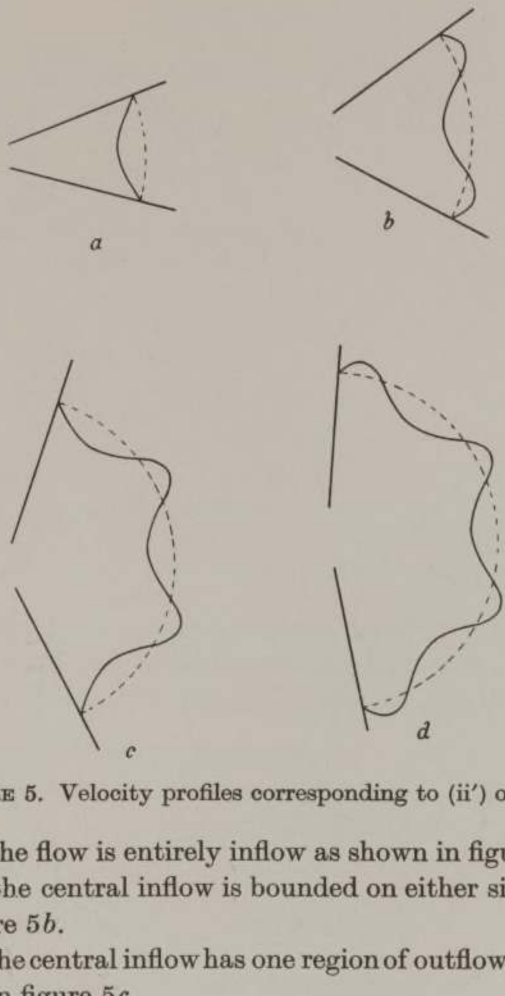

FIGURE 5. Velocity profiles corresponding to (ii′) of § 4.

If $\alpha = \omega_1-\eta_1$, the flow is entirely inflow as shown in figure 5*a*.

If $\alpha = \omega_1+\eta_1$, the central inflow is bounded on either side by a region of outflow as in figure 5*b*.

If $\alpha = \omega_1+\eta_2$, the central inflow has one region of outflow and one of inflow on either side as in figure 5*c*.

If $\alpha = \omega_1+\eta_3$, the central inflow has two regions of outflow and one of inflow on either side as in figure 5*d*.

Etc.

(iii′) and (iv′) $\alpha = (2n+1)\omega_1$ or $\alpha = 2n\omega_1$.

In all cases the flow is unsymmetrical with respect to the central line.

If $\alpha = \omega_1$, then $\theta_0 = \alpha_0 + \omega_3$ where $\alpha_0 = \pm(\omega_1 - \eta_1)$ and the flow is as shown in figure $6a$ or $6a'$. There are no other types.

If $\alpha = 2\omega_1$, then $\theta_0 = \alpha_0 + \omega_3$ where $\alpha_0 = \pm \eta_1$ and the flow is as shown in figure $6b$ or $6b'$. There are no other types.

If $\alpha = 3\omega_1$, then $\theta_0 = \alpha_0 + \omega_3$ where $\alpha_0 = \pm(\omega_1 + \eta_1)$ and the flow is as shown in figure $6c$ or $6c'$. There are no other types.

Etc.

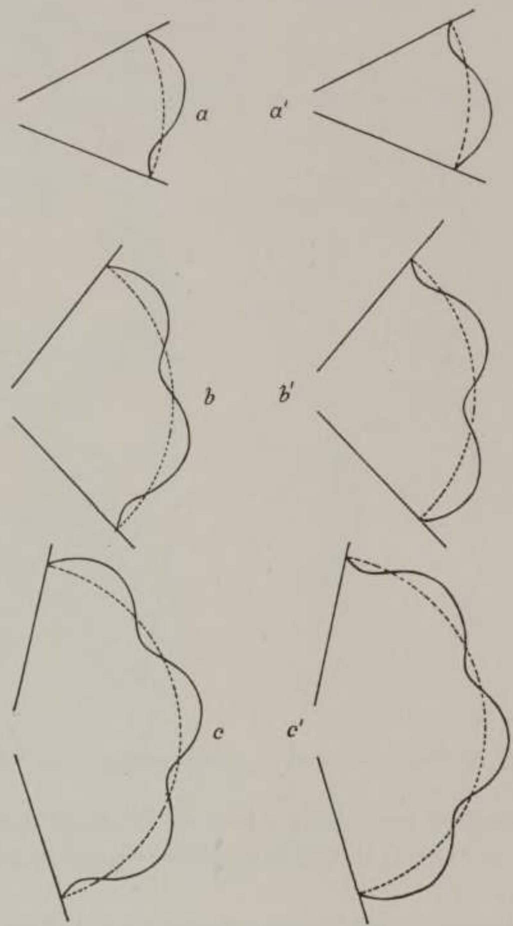

FIGURE 6. Velocity profiles corresponding to (iii′) and (iv′) of § 4.

5. The degenerate cases

The degenerate cases of the Weierstrassian function, when two or three roots of the fundamental cubic are equal, are of interest in that the solutions can be expressed in terms of functions which are more familiar than the elliptic functions. In addition they provide useful pointers to the general theory. These cases correspond to $\Delta = 0$ and can be considered under three heads.

(i) $e_1 = e_2 = e_3$.

Since $e_1 + e_2 + e_3 = 0$, the values of e_1, e_2, e_3 are each zero and the equation for $f(\theta)$ is

$$f'(\theta)^2 = 4f^3. \tag{5·1}$$

Transform this to the form involving $F(\theta)$, by means of the relation $f(\theta) = -\frac{1}{3}\{F(\theta)+1\}$ and it becomes

$$F'(\theta)^2 = -\tfrac{4}{3}\{F+1\}^3. \tag{5·2}$$

This equation cannot have a physical interpretation in connexion with the present problem for it makes $F'(\theta)^2$ negative at the walls, that is where $F(\theta) = 0$.

(ii) $e_2 = e_3 = -\frac{1}{2}e_1$, where e_1 is positive.

The differential equation for $f(\theta)$ is

$$f'(\theta)^2 = 4(f-e_1)(f+\tfrac{1}{2}e_1)^2, \tag{5·3}$$

and the differential equation for $F(\theta)$ is

$$F'(\theta)^2 = -\tfrac{4}{3}(F+1+3e_1)(F+1-\tfrac{3}{2}e_1)^2. \tag{5·4}$$

This too can have no solution capable of physical interpretation, for it makes $F'(\theta)^2$ negative where $F = 0$, except in the one case where $e_1 = \frac{2}{3}$. The only solution then is $F = 0$ which corresponds to no motion between the walls.

(iii) $e_1 = e_2 = -\frac{1}{2}e_3$, where e_1 is positive.

The differential equation for $f(\theta)$ is

$$f'(\theta)^2 = 4(f-e_1)^2(f+2e_1), \tag{5·5}$$

and that for $F(\theta)$ is

$$F'(\theta)^2 = \tfrac{4}{3}(F+1+3e_1)^2(6e_1-1-F). \tag{5·6}$$

As long as $(6e_1-1)>0$ this does not produce an inconsistency at points where $F = 0$. Put $6e_1 - 1 = \lambda$ where λ is positive. Equation (5·6) becomes

$$\frac{dF}{d\theta} = \pm\frac{1}{\sqrt{3}}(2F+\lambda+3)(\lambda-F)^{\frac{1}{2}}. \tag{5·7}$$

If we assume that $dF/d\theta$ is positive at $\theta = \alpha$, we must take the positive sign in equation (5·7). This means that as θ decreases below α, the value of F

decreases below its value at $\theta = \alpha$, which is zero. If we take the positive sign in (5·7) we are considering the case where $F(\alpha-\epsilon)$ is negative, ϵ being a small angle. Since F is zero at $\theta = \pm\alpha$ there must be at least one point in the range, $\theta = \theta'$ say, where $dF/d\theta = 0$. At this position F must be negative, and from (5·7) we see that at $\theta = \theta'$ the value of F must be $-\frac{1}{2}(\lambda+3)$. The solution of (5·7), however, which satisfies the boundary condition at $\theta = \alpha$ is, however,

$$\tanh^{-1}\left(\frac{2\lambda}{3(1+\lambda)}\right)^{\frac{1}{2}} - \left(\frac{1+\lambda}{2}\right)^{\frac{1}{2}}(\theta-\alpha) = \tanh^{-1}\left(\frac{2(\lambda-F)}{3(1+\lambda)}\right)^{\frac{1}{2}}. \tag{5·8}$$

If in this we put $F = -\frac{1}{2}(\lambda+3)$, we find that $\theta' \to -\infty$ which is impossible. We are therefore compelled to take the negative sign in (5·7) and the solution is

$$\tanh^{-1}\left(\frac{2\lambda}{3(1+\lambda)}\right)^{\frac{1}{2}} + \left(\frac{1+\lambda}{2}\right)^{\frac{1}{2}}(\theta-\alpha) = \tanh^{-1}\left(\frac{2(\lambda-F)}{3(1+\lambda)}\right)^{\frac{1}{2}}. \tag{5·9}$$

From equation (5·7) it is clear that $dF/d\theta$ is zero when $F = \lambda$, and in this position

$$\theta' = \alpha - \left(\frac{2}{1+\lambda}\right)^{\frac{1}{2}} \tanh^{-1}\left(\frac{2\lambda}{3(1+\lambda)}\right)^{\frac{1}{2}}.$$

By symmetry θ' must be zero so that

$$\alpha = \left(\frac{2}{1+\lambda}\right)^{\frac{1}{2}} \tanh^{-1}\left(\frac{2\lambda}{3(1+\lambda)}\right)^{\frac{1}{2}}. \tag{5·10}$$

It should be noted also that the maximum value of F is λ, and this occurs at $\theta = 0$. If we put $\lambda = (2H-1)$, where $H > \frac{1}{2}$, equations (5·9) and (5·10) become

$$\left.\begin{aligned} F(\theta) &= (2H-1) - 3H\tanh^2\theta\sqrt{H}, \\ \alpha\sqrt{H} &= \tanh^{-1}\left(\frac{2H-1}{3H}\right). \end{aligned}\right\} \tag{5·11}$$

The equations (5·11) correspond to the case usually described in text-books and manuals (see Goldstein 1938, p. 109, equation (63)), if H is assumed to be so big that $1/H$ can be neglected.

It can therefore be seen that the cases corresponding to positive Δ and negative Δ represent fundamentally different types of flow. We note that

$$27\Delta = \{(4-a)^{\frac{3}{2}} + 8 - 3a - 3b\}\{(4-a)^{\frac{3}{2}} - 8 + 3a + 3b\} = (4-a)^3 - (8-3a-3b)^2 \tag{5·12}$$

in terms of the constants a and b introduced in equations (3·12) and (3·13).

We note also that b must be positive as explained just after equation (3·16). Further

(i) If $a \geqslant 4$, Δ is negative for all values of b.

(ii) If $4 > a > 3$, $\begin{cases} (4-a)^{\frac{3}{2}} + 8 - 3a \text{ is negative,} \\ (4-a)^{\frac{3}{2}} - 8 + 3a \text{ is positive.} \end{cases}$

(iii) If $3 > a$, $\begin{cases} (4-a)^{\frac{3}{2}} + 8 - 3a \text{ is positive.} \\ (4-a)^{\frac{3}{2}} - 8 + 3a \text{ is positive.} \end{cases}$

Hence Δ is negative and the flow is always pure outflow if

$$\left.\begin{array}{lll} \text{(i)} & a \geqslant 3, & \\ \text{or (ii)} & 3 > a \quad \text{and} & 3b > 8 - 3a + (4-a)^{\frac{3}{2}}. \end{array}\right\} \tag{5·13}$$

In addition, Δ is positive and the flow may be either pure inflow, or may contain regions of inflow and outflow, if

$$\text{(iii)} \qquad 3 > a \quad \text{and} \quad 8 - 3a + (4-a)^{\frac{3}{2}} > 3b > 0. \tag{5·14}$$

6. The case $\alpha = 0$

The previous analysis shows that, mathematically at any rate, the velocity profile of the two-dimensional flow between inclined plane walls may have many forms. Many of these are, most probably, unstable to slight disturbances so that they will not have been observed in experiment. It is interesting however to investigate the possibility of the existence of similar velocity profiles in the flow between plane parallel walls. The following analysis shows, however, that as $\alpha \to 0$ the velocity profiles change and, as a limiting form, assume the parabolic shape which emerges from the well-known theoretical investigation of the flow of viscous fluid between plane parallel walls.

Let O' be a point on the line OX of figure 1, distant r_0 from O. Through O' draw a line cutting OV and OW in V' and W'. We put $O'V'$ and $O'W'$ each equal to h and in the subsequent limiting process we keep O', V', W' fixed in space and allow r_0 to tend to infinity. Introduce new variables x and y defined by the equations

$$r = r_0 + x, \quad (r_0 + x)\theta = y. \tag{6·1}$$

x and y will remain finite while r and r_0 tend to infinity and θ tends to zero. Further u, the velocity, must remain finite. From (3·18) we see that

$$p = \frac{2\mu u}{r_0 + x} + \frac{a\mu\nu}{2(r_0 + x)^2} + C. \tag{6·2}$$

The first term tends to zero but the second and third terms are at our disposal since a and C are arbitrary. Hence

$$p \to \left(C + \frac{a\mu\nu}{2r_0^2}\right) - \frac{a\mu\nu}{2r_0^3}x + \frac{3a\mu\nu}{2r_0^4}x^2 - \dots. \tag{6·3}$$

The quantity $\left(C + \frac{a\mu\nu}{2r_0^2}\right)$ is arbitrary and may be put equal to p_0, an arbitrary constant. The expression $(a\mu\nu/2r_0^3)$ is also arbitrary, and for any value of r_0 the magnitude of a can be adjusted to make the ratio finite. Put $(a\mu\nu/2r_0^3)$ equal to P where P is finite. Then, neglecting terms which tend to zero as r_0 tends to infinity, we have

$$p = p_0 - Px. \tag{6·4}$$

If P is positive then $a \to 2Pr_0^3/\mu\nu$, which is very big. Hence from condition (i) of equation (5·13) the flow is pure outflow. Further, after some manipulation, it can be shown that the velocity profile tends to

$$u = u_0\left(1 - \frac{y^2}{h^2}\right), \tag{6·5}$$

where $u_0 = Ph^2/2\mu$. If P is negative, the flow remains of the same type but is reversed in direction.

If the limiting process is carried out in any other manner, the angle α tends to zero in such a way that the two walls collapse and become identical. In such a limiting process all types of velocity profile mentioned previously are possible.

7. The Reynolds number when Δ is negative or zero

The Reynolds number as defined in (3·4) is

$$R = \int_{-\alpha}^{\alpha} F(\theta)\,d\theta = \int_{\alpha}^{-\alpha} \{3f(\theta)+1\}\,d\theta = -2\alpha + 3\int_{\alpha}^{-\alpha} f(\theta)\,d\theta.$$

When Δ is negative the flow is always pure outflow and

$$\left.\begin{aligned} f(\theta) = \wp(\theta-\omega_2) &= -2A + H\frac{\mathrm{sn}^2(\theta\sqrt{H})\,\mathrm{dn}^2(\theta\sqrt{H})}{\mathrm{cn}^2(\theta\sqrt{H})}, \\ \mathrm{cn}\,(2\alpha\sqrt{H}) &= \frac{3H-6A+1}{3H+6A-1}, \\ \omega_2 &= K/\sqrt{H}, \\ k^2 &= \tfrac{1}{2}(H+3A)/H. \end{aligned}\right\} \tag{7·1}$$

It can be shown that

$$\begin{aligned}\int_{\alpha}^{-\alpha} f(\theta)\,d\theta &= 4A\alpha - 2H^{\frac{1}{2}}\int_0^{\alpha\sqrt{H}} \frac{\operatorname{sn}^2\eta\,\operatorname{dn}^2\eta}{\operatorname{cn}^2\eta}\,d\eta \\ &= 4A\alpha - 2H^{\frac{1}{2}}\left[\frac{\operatorname{sn}\eta\,\operatorname{dn}\eta}{\operatorname{cn}\eta} + \eta - 2E(\eta)\right]_0^{\alpha\sqrt{H}} \\ &= 4A\alpha - 2\alpha H + 2H^{\frac{1}{2}}\frac{\operatorname{sn}\alpha\sqrt{H}\,\operatorname{dn}\alpha\sqrt{H}}{\operatorname{cn}\alpha\sqrt{H}} + 4H^{\frac{1}{2}}E(\alpha\sqrt{H}) \\ &= 2\alpha\left[2A - H + 2H\frac{E}{K}\right] - \tfrac{2}{3}(18A-3)^{\frac{1}{2}} + 4H^{\frac{1}{2}}Z(\alpha\sqrt{H}), \qquad (7\cdot2)\end{aligned}$$

where $E(u)$ is the Jacobian elliptic integral of the second kind, $Z(u)$ is the Jacobian zeta function defined by the relation

$$Z(u) = E(u) - \frac{E}{K}u,$$

and E and K are the complete elliptic integrals of the first and second kinds. From these formulae we therefore find

$$R = 2\alpha\left(6A - 3H + 6H\frac{E}{K} - 1\right) - 2(18A-3)^{\frac{1}{2}} + 12H^{\frac{1}{2}}Z(\alpha\sqrt{H}).$$

Using the last of the relations given in (7·1), this becomes

$$R = 2\alpha\left\{H\left(6\frac{E}{K} + 4k^2 - 5\right) - 1\right\} - 2\sqrt{3}\,\{H(4k^2-2)-1\}^{\frac{1}{2}} + 12H^{\frac{1}{2}}\,Z(\alpha\sqrt{H}). \qquad (7\cdot3)$$

It might be noted too that $F(0)$, which gives a measure of the flow on the central line, is given by the relation

$$F(0) = 6A - 1 = (4k^2-2)\,H - 1. \qquad (7\cdot4)$$

The central velocity and the Reynolds number are thus expressed in terms of two parameters, k^2 and H. The evaluation of the Reynolds number corresponding to any pair of values of these parameters, and of the appropriate value of α given by the second of the equations in (7·1), was effected with the aid of two tables of elliptic functions published by Milne-Thomson (1931, 1932). The results of the evaluation are included in table 2.

The type of flow corresponding to the degenerate case described in (5·11) is the limiting case of (7·3) corresponding to $\Delta = 0$. Here $k^2 = 1$ and

$$\left.\begin{aligned} R &= -2\alpha(H+1) + 2\sqrt{(6H-3)}, \\ F(0) &= 2H - 1. \end{aligned}\right\} \qquad (7\cdot5)$$

8. The Reynolds number when Δ is positive

Here again it can be shown that all the characteristics of the fluid motion can be described in terms of two parameters, k^2 and X. It has been shown, however, that many different types of flow are associated with each pair of values of k^2 and X. The type is determined by the value of α which is chosen from the infinite set of values which are available as solutions of the boundary condition. (Physically, however, not all these values are permissible, for α must be less than π.) The exact dependence of the type of flow upon the choice of α can be seen from conditions (i′), (ii′), (iii′), (iv′) of § 4. As can be seen from the diagrams of the types of flow, the two fundamental ones are those which correspond to pure outflow and to pure inflow. The associated values of the Reynolds number can be determined, and it can be seen that the Reynolds numbers of the other types of flow are sums of integral multiples of these fundamental quantities.

Referring to conditions (i′) and (ii′) mentioned above we see that if $\alpha = \eta_1$ the flow is entirely outflow and that if $\alpha = \omega_1 - \eta_1$ the flow is entirely inflow. The corresponding values of the Reynolds number are S and T where

$$S = 2\int_0^{\eta_1} F(\eta)\,d\eta > 0, \quad T = 2\int_{\eta_1}^{\omega_1} F(\eta)\,d\eta < 0, \tag{8·1}$$

and where η_1 is the smallest positive root of the equation

$$\mathrm{sn}^2(\eta\sqrt{X}) = \{(1+k^2)X - 1\}/3k^2X.$$

The other terms in equations (8·1) are defined by equations which have been derived previously but which, for purposes of convenience, are here repeated:

$$\omega_1 = KX^{-\frac{1}{2}},$$

$$F(\eta) = -3e_3 - 3(e_2 - e_3)\,\mathrm{sn}^2\{\eta(e_1 - e_3)^{\frac{1}{2}}\} - 1$$

$$= 3X\,\mathrm{dn}^2(\eta\sqrt{X}) - (2 - k^2)\,X - 1.$$

From these relations it can be proved that

$$S = 6X^{\frac{1}{2}}\,Z(\eta_1\sqrt{X}) + 2\eta_1\left\{X\left(3\frac{E}{K} - 2 + k^2\right) - 1\right\},$$

$$S + T = 2KX^{-\frac{1}{2}}\left\{X\left(3\frac{E}{K} - 2 + k^2\right) - 1\right\}.$$

The Reynolds numbers of the various types of flow are then:

Type figure $4a$	$R = S,$	Type figure $4c$	$R = 3S+2T,$
$5a$	$R = T,$	$5c$	$R = 2S+3T,$
$4b$	$R = S+2T,$	$4d$	$R = 3S+4T,$
$5b$	$R = T+2S,$	$5d$	$R = 4S+3T,$

etc.

Type figures $6a$ and $6a'$	$R = S+T,$
$6b$ and $6b'$	$R = 2S+2T,$
$6c$ and $6c'$	$R = 3S+3T,$

etc.

Values of S and T are given in table 1 for different values of k^2 and X. The table covers the domain $\Delta \geqslant 0$ and gives the limits of X corresponding to each value of k^2. The cases $k^2 = 1{\cdot}0$ and $k^2 = 0{\cdot}0$ are critical ones and have been considered as limiting cases of $k^2 = 1-\epsilon$ and $k^2 = \epsilon$ as $\epsilon \to 0$.

TABLE 1. VALUES OF k^2 AND X WHEN $\Delta \geqslant 0$

I (a) $k^2 = 1{\cdot}0$; $\infty \geqslant X \geqslant 0{\cdot}5$; $K = \infty$.

X	η_1	ω_1	S	T
0·500	0·000	∞	0·000	$-\infty$
1·000	0·658	∞	0·832	$-\infty$
1·100	0·666	∞	0·997	$-\infty$
1·500	0·657	∞	1·613	$-\infty$
4·000	0·503	∞	4·136	$-\infty$
9·000	0·360	∞	7·082	$-\infty$
16·000	0·277	∞	9·870	$-\infty$
25·000	0·224	∞	12·600	$-\infty$
36·000	0·188	∞	15·278	$-\infty$
$X \geqslant 100$ (say)	$1{\cdot}146X^{-\frac{1}{2}}$	∞	$2{\cdot}607X^{\frac{1}{2}}$	$-\infty$

I (b) $k^2 = 1-\epsilon$, where ϵ is small; $\infty \geqslant X \geqslant 0{\cdot}5(1+\frac{1}{2}\epsilon)$; $K \approx \frac{1}{2}\log_e(16/\epsilon)$.

When X is very big

$S \approx 2{\cdot}607X^{\frac{1}{2}}$, $\qquad \eta_1 \approx 1{\cdot}146X^{-\frac{1}{2}}$,

$T \approx \{3{\cdot}393 - \log_e(16/\epsilon)\}X^{\frac{1}{2}}$, $\qquad \omega_1 \approx \{\frac{1}{2}\log(16/\epsilon)\}X^{-\frac{1}{2}}$.

I (c) $k^2 = 0{\cdot}8$; $\infty \geqslant X \geqslant 0{\cdot}556$; $K = 2{\cdot}2572$.

X	η_1	ω_1	S	T
0·556	0·000	3·030	0·000	$-4{\cdot}826$
1·000	0·648	2·257	0·659	$-3{\cdot}519$
1·440	0·673	1·881	1·323	$-3{\cdot}101$
4·000	0·529	1·129	3·854	$-2{\cdot}804$
9·000	0·383	0·752	6·705	$-2{\cdot}251$
25·000	0·240	0·451	12·010	$-4{\cdot}652$
100·000	0·122	0·226	24·732	$-8{\cdot}627$
$X \geqslant 100$ (say)	$1{\cdot}230X^{-\frac{1}{2}}$	$2{\cdot}257X^{-\frac{1}{2}}$	$2{\cdot}499X^{\frac{1}{2}}$	$-0{\cdot}846X^{\frac{1}{2}}$

TABLE I (*continued*)

I (*d*) $k^2 = 0\cdot5$; $\infty \geqslant X \geqslant 0\cdot667$; $K = 1\cdot8541$.

X	η_1	ω_1	S	T
0·667	0·000	2·271	0·000	−2·466
1·000	0·635	1·854	0·407	−1·573
1·690	0·731	1·426	2·015	−1·563
3·240	0·672	1·030	2·991	−0·476
9·000	0·490	0·618	6·472	−0·082
$X \geqslant 200$ (say)	$1\cdot854X^{-\frac{1}{2}}$	$1\cdot854X^{-\frac{1}{2}}$	$2\cdot542X^{\frac{1}{2}}$	0·000

I (*e*) $k^2 = 0\cdot4$; $5 \geqslant X \geqslant 0\cdot714$; $K = 1\cdot7775$.

X	η_1	ω_1	S	T
0·714	0·000	2·103	0·000	−1·918
1·000	0·631	1·778	0·324	−1·171
3·610	0·758	0·936	3·339	−0·059
5·000	0·795	0·795	4·467	0·000

I (*f*) $k^2 = 0\cdot2$; $1\cdot667 \geqslant X \geqslant 0\cdot833$; $K = 1\cdot6596$.

X	η_1	ω_1	S	T
0·833	0·000	1·818	0·000	−0·910
1·000	0·623	1·660	0·161	−0·521
1·667	1·286	1·286	1·250	0·000

I (*g*) $k^2 = \epsilon$ (small); $1+2\epsilon \geqslant X \geqslant 1-\epsilon$; $K = \frac{1}{2}\pi(1+\frac{1}{4}\epsilon)$.

X	η_1	ω_1	S	T
$1-\epsilon$	0·000	$\frac{1}{2}\pi(1+\frac{3}{4}\epsilon)$	0·000	$-\frac{3}{2}\pi\epsilon$
1·000	0·615 (app.)	$\frac{1}{2}\pi(1+\frac{1}{4}\epsilon)$	$0\cdot800\epsilon$	$-2\cdot371\epsilon$
$1+2\epsilon$	$\frac{1}{2}\pi(1-\frac{3}{4}\epsilon)$	$\frac{1}{2}\pi(1-\frac{3}{4}\epsilon)$	$\frac{3}{2}\pi\epsilon$	0

I (*h*) $k^2 = 0$; $X = 1$; $K = \frac{1}{2}\pi$.

X	η_1	ω_1	S	T
1	$0 \rightarrow \frac{1}{2}\pi$	$\frac{1}{2}\pi$	0·000	0·000

The results of the above table were then analysed in the following way. Tables were made giving corresponding values of α and R for different types of flow. Table 2 gives these values for the symmetrical outflow (cf. figure 4*a*). The values for symmetrical inflow (cf. figure 5*a*) are given in table 3. Corresponding values of α and R were plotted, as shown in detail in figure 7. Each value of k^2 which was considered gave rise to a curve. These curves mapped out regions in which pure outflow and pure inflow were respectively possible. From these it is clear that, corresponding to each value of α, there is a range of values of R in which the type of flow considered is possible. Similar regions were mapped out for different types of flow, and the appropriate ranges in which they are possible are shown in table 4. Some of the limits were read off from graphs and are therefore subject to error—but the general run, and the order of magnitude, of the limits are as tabulated. The limits of the ranges of the symmetrical types of flow are plotted in figure 8. Those for the non-symmetrical types are shown in figure 9. A number of deductions can be made from these figures, and, in addition, the forms of the curves give rise to a number of speculations as to the stability of the various types of flow.

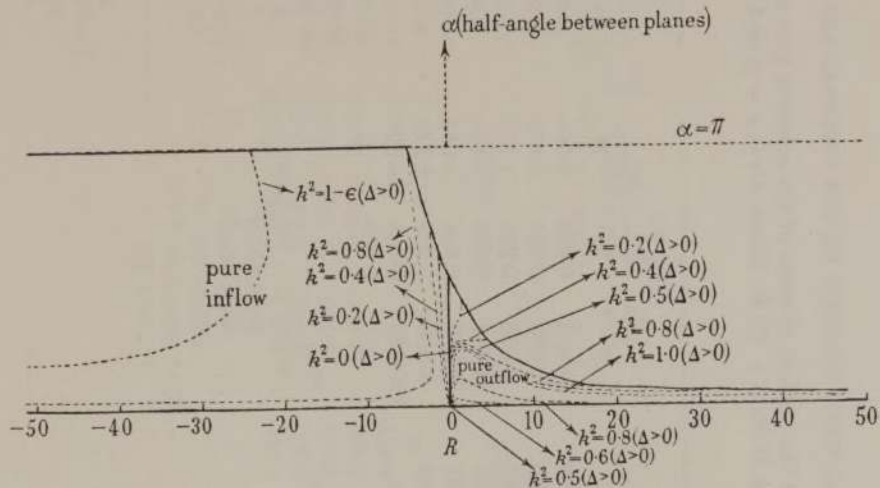

FIGURE 7. Ranges of R for pure inflow and for pure outflow.

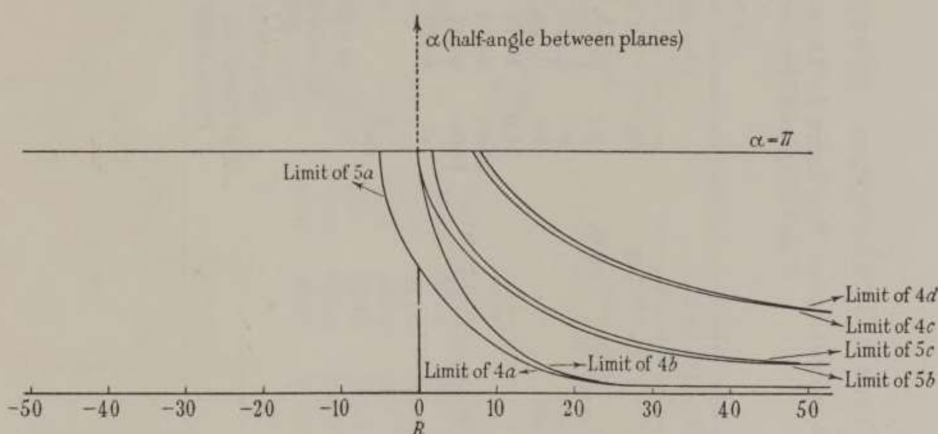

FIGURE 8. Ranges of R for different types of symmetrical flow. [*Note*. "Limit of 4*a*" should be interpreted as the "Upper limit of the Reynolds number for which flow of the type shown in figure 4*a* is possible." Similar interpretations should be made of the other inscriptions. All the curves tend asymptotically to zero according to the law (constant)/R.]

TABLE 2. PURE OUTFLOW. SYMMETRICAL WITH RESPECT TO THE CENTRAL LINE. See figure 4*a*

α is plotted against R in figure 7. The notes at the bottom of some of the columns give approximate values of α, R and $F(0)$, valid for very large H (or X). [N.B. When $\Delta > 0$, $\alpha = \eta_1$ and $R = S$.]

$\Delta < 0$									$\Delta = 0$		
$k^2 = 0{\cdot}5$			$k^2 = 0{\cdot}6$			$k^2 = 0{\cdot}8$			$k^2 = 1{\cdot}0$		
H	α	R	H	α	R	H	α	R	H (or X)	α	R
∞	0·000	0·000	2·5	0·000	0·000	0·833	0·000	0·000	0·500	0·000	0·000
			6·0	0·114	0·205	1·000	0·261	0·079	1·000	0·658	0·832
			9·0	0·104	0·361	3·000	0·328	1·112	1·100	0·666	0·997
			25·0	0·070	0·858	4·000	0·299	1·477	1·500	0·657	1·613
			600·0	0·015	4·891	6·000	0·256	2·062	4·000	0·503	4·136
			900·0	0·012	5·910	8·000	0·226	2·519	9·000	0·360	7·082
			∞	0·000	∞	200·000	0·048	14·990	16·000	0·277	9·870
						∞	0·000	∞	25·000	0·224	12·600
									36·000	0·188	15·278
									100·000	0·115	26·07
									∞	0·000	∞

$k^2 = 0{\cdot}5$: *Note*
$\alpha R \to 0$
$R/\alpha F(0) \to 1{\cdot}33$

$k^2 = 0{\cdot}6$: *Note*
When $H > 600$ (say)
$\alpha \approx 0{\cdot}368 H^{-\frac{1}{2}}$
$R \approx 0{\cdot}197 H^{\frac{1}{2}}$
$\alpha R \approx 0{\cdot}072$
$R/\alpha F(0) \approx 1{\cdot}33$
$F(0) \approx 0{\cdot}4H$

$k^2 = 0{\cdot}8$: *Note*
When $H > 200$ (say)
$\alpha \approx 0{\cdot}681 H^{-\frac{1}{2}}$
$R \approx 1{\cdot}060 H^{\frac{1}{2}}$
$\alpha R \approx 0{\cdot}722$
$R/\alpha F(0) \approx 1{\cdot}30$
$F(0) \approx 1{\cdot}2H$

$k^2 = 1{\cdot}0$: *Note*
When $H > 100$ (say)
$\alpha \approx 1{\cdot}146 H^{-\frac{1}{2}}$
$R \approx 2{\cdot}607 H^{\frac{1}{2}}$
$\alpha R \approx 2{\cdot}988$
$R/\alpha F(0) \approx 1{\cdot}14$
$F(0) \approx 2H$

$\Delta > 0$

$k^2 = 0\cdot8$			$k^2 = 0\cdot5$			$k^2 = 0\cdot4$			$k^2 = 0\cdot2$		
X	α	R	X	α	R	X	α	R	X	α	R
0·556	0·000	0·000	0·667	0·000	0·000	0·714	0·000	0·000	0·833	0·000	0·000
1·000	0·648	0·659	1·000	0·635	0·407	1·000	0·631	0·324	1·000	0·623	0·161
1·440	0·673	1·323	1·690	0·731	2·015	3·610	0·758	3·339	1·667	1·286	1·250
4·000	0·529	3·854	3·240	0·672	2·991	5·000	0·795	4·467			
9·000	0·383	6·705	9·000	0·490	6·472						
25·000	0·240	12·01	∞	0·000	∞						
100·000	0·122	24·73									
∞	0·000	∞									

Note ($k^2 = 0\cdot8$)

When $X > 100$ (say)

$\alpha \approx 1\cdot230 X^{-\frac{1}{2}}$

$R \approx 2\cdot499 X^{\frac{1}{2}}$

$\alpha R \approx 3\cdot014$

$R/\alpha F(0) \approx 1\cdot13$

$F(0) \approx 1\cdot8X$

Note ($k^2 = 0\cdot5$)

When $X > 200$ (say)

$\alpha \approx 1\cdot854 X^{-\frac{1}{2}}$

$R \approx 2\cdot542 X^{\frac{1}{2}}$

$\alpha R \approx 4\cdot713$

$R/\alpha F(0) \approx 0\cdot91$

$F(0) \approx 1\cdot5X$

$k^2 = \epsilon$ (small)		
X	α	R
$1-\epsilon$	0·000	0·000
1·000	0·615 (app.)	$\frac{4}{5}\epsilon$
$1+2\epsilon$	$\frac{1}{2}\pi(1-\frac{3}{4}\epsilon)$	$\frac{3}{2}\pi\epsilon$

$k^2 = 0$		
X	α	R
1	$0 \to \frac{1}{2}\pi$	0·000

TABLE 3. PURE INFLOW. SYMMETRICAL WITH RESPECT TO THE CENTRAL LINE. See figure 5a

α is plotted against R in figure 7. The notes at the bottom of the first two columns give approximate values of α, R and $F(0)$ valid for very large X. [N.B. $\Lambda > 0$, $\alpha = \omega_1 - \eta_1$, $R = T$.]

$k^2 = 1 - \epsilon$	$k^2 = 0{\cdot}8$			$k^2 = 0{\cdot}5$			$k^2 = 0{\cdot}4$		
	X	α	$-R$	X	α	$-R$	X	α	$-R$
Note	0·556	3·030	4·826	0·667	2·271	2·466	0·714	2·103	1·918
When X is very big	1·000	1·609	3·519	1·000	1·219	1·573	1·000	1·147	1·171
$\alpha \approx (\frac{1}{2}\log 16/\epsilon - 1{\cdot}146)\, X^{-\frac{1}{2}}$	1·440	1·208	3·101	1·690	0·695	1·563	3·610	0·174	0·059
$R \approx (3{\cdot}393 - \log 16/\epsilon)\, X^{\frac{1}{2}}$	4·000	0·600	2·804	3·240	0·358	0·476	5·000	0·000	0·000
$F(0) \approx -(1-2\epsilon)\, X$	9·000	0·369	2·251	9·000	0·128	0·082			
$R/\alpha F(0) \to 2$	25·000	0·211	4·652	$X > 200$	0·000	0·000			
	100·000	0·104	8·627						
	∞	0·000	∞						

	$k^2 = 0{\cdot}2$			$k^2 = \epsilon$ (small)			$k^2 = 0{\cdot}0$		
Note	X	α	$-R$	X	α	$-R$	X	α	$-R$
When $X > 100$ (say)	0·833	1·818	0·910	$1-\epsilon$	$\frac{1}{2}\pi(1+\frac{3}{4}\epsilon)$	$\frac{3}{8}\pi\epsilon$	1·000	$\frac{1}{2}\pi \to 0$	0
$\alpha \approx 1{\cdot}027 X^{-\frac{1}{2}}$	1·000	1·037	0·521	1·000	$0{\cdot}956 + \frac{1}{8}\pi\epsilon$	$2{\cdot}371\epsilon$			
$-R \approx 0{\cdot}846 X^{\frac{1}{2}}$	1·667	0·000	0·000	$1+2\epsilon$	0·000	0·000			
$-\alpha R \approx 0{\cdot}869$									
$R/\alpha F(0) \approx 1{\cdot}37$									
$F(0) \approx -0{\cdot}6X$									

TABLE 4. RANGES OF THE REYNOLDS NUMBER IN WHICH THE VARIOUS TYPES OF FLOW ARE POSSIBLE

α	Pure outflow, Symmetrical, See figure 4a: R		Pure inflow, Symmetrical, See figure 5a: R		Central outflow with a region of inflow on each side, Symmetrical, See figure 4b: R		Central inflow with a region of outflow on each side, Symmetrical, See figure 5b: R		Central outflow with a region of inflow and one of outflow on each side, Symmetrical, See figure 4c: R	
0·0	0·0 to	$+\infty$	0·0 to	$-\infty$	$-\infty$ to	∞	$-\infty$ to	$+\infty$	$-\infty$ to	$+\infty$
0·1	0·0	47·0	0·0	$-\infty$	$-\infty$	47	$-\infty$	188	$-\infty$	424
0·2	0·0	24·0	0·0	$-\infty$	$-\infty$	24	$-\infty$	94	$-\infty$	212
0·3	0·0	16·5	0·0	$-\infty$	$-\infty$	18·8	$-\infty$	63	$-\infty$	127
0·4	0·0	13·5	0·0	$-\infty$	$-\infty$	16·5	$-\infty$	47	$-\infty$	106
0·5	0·0	11·0	0·0	$-\infty$	$-\infty$	14·5	$-\infty$	33	$-\infty$	85
0·75	0·0	7·5	0·0	$-\infty$	$-\infty$	10·5	$-\infty$	23·5	$-\infty$	66
1·00	0·0	4·0	0·0	$-\infty$	$-\infty$	8·0	$-\infty$	17·0	$-\infty$	53
$\frac{1}{2}\pi$	0·0	0·0	0·0	$-\infty$	$-\infty$	4·0	$-\infty$	9·5	$-\infty$	29
2·00	0·0	0·0	$-2·0$	$-\infty$	$-\infty$	2·0	$-\infty$	5·5	$-\infty$	19
2·50	0·0	0·0	$-4·0$	$-\infty$	$-\infty$	1·0	$-\infty$	2·5	$-\infty$	12
π	0·0	0·0	$-5·0$	$-\infty$	$-\infty$	0·0	$-\infty$	0·0	$-\infty$	7

α	Central inflow with a region of outflow and one of inflow on each side, Symmetrical, See figure 5c: R		Central outflow with two regions of inflow and one of outflow on each side, Symmetrical, See figure 4d: R		Outflow and inflow, Non-symmetrical, See figure 6 (a and a'): R		Outflow and inflow, Non-symmetrical, See figure 6 (b and b'): R	
0·0	$-\infty$ to	$+\infty$	$-\infty$ to	$+\infty$	$-\infty$ to	$+\infty$	$-\infty$ to	$+\infty$
0·1	$-\infty$	188	$-\infty$	424	$-\infty$	47	$-\infty$	188
0·2	$-\infty$	94	$-\infty$	212	$-\infty$	31	$-\infty$	94
0·3	$-\infty$	63	$-\infty$	127	$-\infty$	25	$-\infty$	63
0·4	$-\infty$	47	$-\infty$	106	$-\infty$	20	$-\infty$	47
0·5	$-\infty$	33	$-\infty$	85	$-\infty$	17	$-\infty$	35
0·75	$-\infty$	24·5	$-\infty$	66	$-\infty$	11·5	$-\infty$	27
1·00	$-\infty$	18·5	$-\infty$	53	$-\infty$	7	$-\infty$	22
$\frac{1}{2}\pi$	$-\infty$	11·5	$-\infty$	29	$-\infty$	0·5	$-\infty$	14
2·00	$-\infty$	7·5	$-\infty$	19	$-\infty$	$-1·5$	$-\infty$	8
2·50	$-\infty$	4·5	$-\infty$	12·5	$-\infty$	$-3·5$	$-\infty$	4
π	$-\infty$	2·0	$-\infty$	7·5	$-\infty$	$-4·5$	$-\infty$	0

In order to make a rapid survey of the velocity profiles at large Reynolds numbers the characteristics of the flow can be compared with two hypothetical cases, (i) a flow in which the velocity is constant across a circular arc, centre O, spanning the walls, and (ii) a flow which is "parabolic" in form across the same arc. In this paragraph the main interest centres on "pure outflow" and on "pure inflow". The other types of flow are not without interest but it is doubtful whether they will have been observed in normal

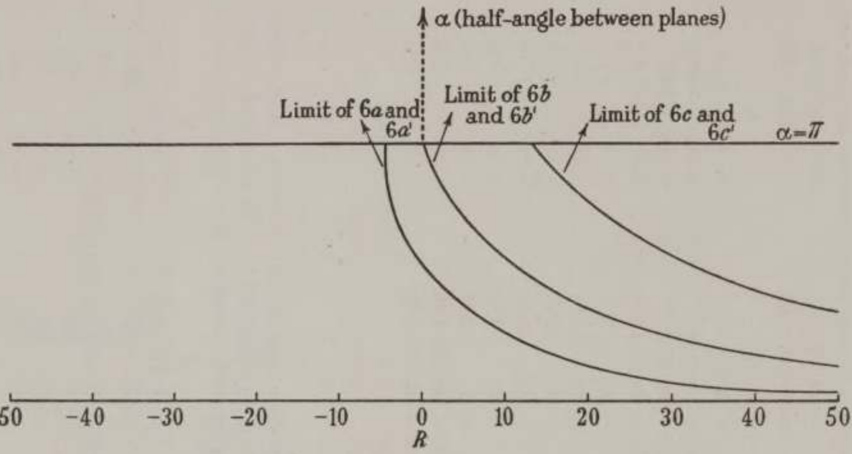

FIGURE 9. Ranges of R for different types of non-symmetrical flow. [*Note*. See legend below figure 8.]

FIGURE 10

experiment. I am, however, indebted to Mr H. B. Squire, M.A., of the Royal Aircraft Establishment, Farnborough, for a suggestion that at low Reynolds numbers, at any rate, it might be possible to reproduce some of the types of flow by means of apparatus represented schematically in figure 10. By means of suitable apparatus it might be possible to obtain slow steady motion in long parallel channels between plane thin walls. At the walls the velocity would be zero. The outer walls of the set of channels could be connected to slowly converging walls and in the ensuing motion, which would be laminar at small Reynolds numbers, there would be vaguely defined layers of zero velocity separating streams of fluid moving in opposite directions. It is probable, however, that many of these composite types of flow are unstable. This will be referred to later. Returning however to the cases of pure inflow and outflow and to the hypothetical cases mentioned above, if the velocity is constant across a span it can be shown that $R/\alpha F(0)$ is equal to 2, and if the velocity profile is of the form

$$u = u_0\left(1-\frac{\theta^2}{\alpha^2}\right),$$

the value of $R/\alpha F(0)$ is $1\frac{1}{3}$. These numbers can be compared with those occurring in the notes at the feet of some of the columns in tables 2 and 3. In table 2, relating to outflow, αR increases uniformly from 0 to 4·713 as $R/\alpha F(0)$ decreases from 1·33 to 0·91, the Reynolds number being large. Hence for a constant small value of α, the flow changes from the parabolic form, and becomes more and more concentrated in the centre of the channel, as R increases. As R increases still further the value of $du/d\theta$ becomes zero at the walls and changes sign. This changes the type of flow from that shown in 4*a* to that in 4*b*. For large Reynolds number in inflow, however, the value of $R/\alpha F(0)$ is approximately 2, suggesting a fairly constant velocity across the span dropping very rapidly to zero at the walls. As the Reynolds number is decreased the region in which the flow is approximately constant becomes smaller and finally disappears. This phenomenon has been commented on, and in greater detail, by other investigators, but not from the above point of view.

When the flow changes from the comparatively simple ones discussed in the preceding paragraph we must turn to figures 8 and 9 for a general survey. A representative example will explain the variation of flow with increasing Reynolds number. Let us consider the case $\alpha = 0·3$ radian. When $R = -\infty$ all types of flow are possible, except, of course, the pure outflow of type 4*a*. [N.B. We shall specify each type by the number of the diagram in which it occurs.] This means that the maximum and minimum velocities can be adjusted at will in any one of the types to make the total flux equal to 2ν times the required Reynolds number. This state of affairs continues right up to and including $R = 0$. In the range $0 < R < 16·5$ all types of flow, except of course the pure inflow of 5*a*, are possible. If R increases beyond this limit, then there must be a change of type of flow *only* if the original type was that of 4*a*. Other types of flow would continue unchanged. Beyond $R = 16·5$ the type 4*a* is not possible. Similarly beyond 18·8 all types of flow except 4*a* and 4*b* are possible. And so on! In addition, non-symmetrical types of flow are possible, and, where they occur, a mirror image of the flow with respect to the central line gives a second possible flow, as, for example, in figures $6a$ and $6a'$.

It can be seen quite readily that the state of affairs is extremely complicated. The above mathematical analysis only yields information as to the types of flow which are possible but gives no help on the question as to which is the type actually assumed by the flow when R and α are given. It might be noted here that in a normal experiment boundary conditions are imposed not only by the friction at the walls but also by the imposed pressure conditions over the inlet and outlet ends of the channel. The above investigation may be considered as one which determines the proper pressure distributions over the ends which allow of steady laminar radial flow. If in an experiment the pressure distribution is not one of those obtained above, the flow may be neither steady, laminar nor radial. It might be noted too that the imposed pressure conditions in experiment are closer to those implied by the pure outflow and inflow than those implied by the other types, so that pure outflow and pure inflow are more likely to occur than are the other types. It is clear also, from figure 8, that if $\alpha > \frac{1}{2}\pi$ pure outflow is quite impossible—whatever the Reynolds number may be—and that there is a range of small Reynolds number within which pure inflow is impossible.

Considerations of stability are undoubtedly the dominant ones in deciding the type of flow assumed by the fluid. On the other hand, the present state of the theory of stability of fluid does not seem to offer much hope that information will be obtained along these lines. It is possible, however, to make a fairly plausible assumption here which will have the effect of systematizing our picture of the way in which the velocity profile changes with increasing Reynolds number. The assumption is that "If the pressure conditions over the inlet and outlet ends are not rigidly applied—that is, if the pressure-profile can be assumed to be loosely self-adjusting—then the velocity-profile will be that one which has the smallest number of crests and troughs". If this assumption is accepted, and if we leave out of consideration the non-symmetrical velocity profiles and those which have inflow along the central axis and yet give a positive total flux outwards, then the sequence of changes is as follows. For inflow the velocity profile is always that of figure $5a$. For outflow the velocity profile passes through the sequence of changes shown in figure 4. If these types of flow are possible for the whole range of Reynolds number, there seems to be no plausible reason why the other velocity profiles, that is, those in figures $5b, c, d$, etc., and in figure 6, should ever manifest themselves, unless they can be produced by rigidly imposed pressure conditions at the ends of the channel.

This sequence of speculations can be pressed still further, and one can surmise as to the stability of the various types of flow. If at any angle α there is a rapid variation of the type of flow for a small change in the Reynolds number, it seems legitimate to suggest that the particular type of flow is unstable in the region of the Reynolds number in question. For example, when α is 0·2, and R is about 24, a small increase in R produces a rapid transition from type $4a$ to type $4b$ and then to type $4c$. This type of instability seems to manifest itself for all types of outflow but not for inflow.

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