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# The Steiner tree polytope and related polyhedra

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We consider the vertex-weighted version of the undirected Steiner tree problem. In this problem, a cost is incurred both for the vertices and the edges present in the Steiner tree. We completely describe the associated polytope by linear inequalities when the underlying graph is series–parallel. For general graphs, this formulation can be interpreted as a (partial) extended formulation for the Steiner tree problem. By projecting this formulation, we obtain some very large classes of facet-defining valid inequalities for the Steiner tree polytope.

*Key words:* Steiner tree, series–parallel graphs, polyhedral characterization, projection, facets, formulations.

## 1. Introduction

The vertex weighted Steiner tree problem in an undirected graph is the problem of finding a tree spanning a prespecified set of vertices at minimum cost where the cost of a tree is equal to the sum of the cost of its edges and the cost of the vertices spanned. In this paper, we consider this problem from a polyhedral point of view.

In Section 2, we formalize the problem and one of its variant, and present an integer programming formulation. Necessary and sufficient conditions for the inequalities in our formulation to be facet-defining are presented in Section 3. In Section 4, we establish that the linear inequalities in our formulation are sufficient to completely characterize the vertex weighted Steiner tree polytope if the underlying graph is series-parallel. Our proof does not explicitly construct a dual optimal solution as is typically done. Instead, we obtain some strong conditions on when an inequality is facet-defining and we use the decomposability of series-parallel graphs to reduce the graph to a cycle. In the last section, we investigate the projection of this extended formulation for the Steiner tree problem for a general graph. From this projection, we obtain some large classes of facet-defining valid inequalities for the undirected Steiner tree polytope. Some of these inequalities are fairly complicated. In particular, we present inequalities whose coefficients take all values between 1 and any odd integer.

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## 2. Problem statement and formulation

### 2.1. Definitions

A *tree* of an undirected graph $G=(V,E)$ is a subgraph $(U,F)$ that is connected and acyclic. Given a set $T\subseteq V$ of *terminals*, a *Steiner tree* is a tree $(U,F)$ spanning $T$, i.e. $T\subseteq U$. The vertices not in $T$ are called *Steiner vertices*. Given a root vertex $r\in V$, $(U,F)$ is an *r-tree* if either $U=\emptyset$ and $F=\emptyset$ or, $r\in U$ and $(U,F)$ is a tree. We shall often describe an $r$-tree by its edge set $F$. When $F$ is non-empty, $U$ is therefore the vertex set $V(F)=\{i\in V: (i,j)\in F$ for some $j\in V\}$ spanned by $F$. However, this might lead to some confusion when $F=\emptyset$ since in this case $U$ can either be $\{r\}$ or the empty set. We shall specify $U$ in this case.

If $w$ is a function defined on $E$ (resp. $V$) then, for any subset $F\subseteq E$ (resp. $U\subseteq V$), we define $w(F)=\sum_{e\in F}w_e$ (resp. $w(U)=\sum_{i\in U}w_i$). For a subset $S$ of $V$, let $E(S)$ denote the set of edges with both endpoints in $S$, let $\delta(S)$ be the set of edges with exactly one endpoint in $S$ and let $G[S]$ denote the subgraph induced by $S$, i.e. the graph $(S,E(S))$. For $e\in E$, let $G-e$ denote the graph $(V,E\backslash\{e\})$ and, for $v\in V$, let $G-v$ denote the graph $G[V\backslash\{v\}]$. More generally, let $G-S$ denote $G[V\backslash S]$.

In this paper, we consider three optimization problems:

**Steiner tree problem.** Given an undirected graph $G=(V,E)$, a set $T\subseteq V$ and a cost function $c$ defined on $E$, find a Steiner tree $(U,F)$ minimizing the cost $c(F)$.

**Vertex-weighted Steiner tree problem.** Given an undirected graph $G=(V,E)$, a set $T\subseteq V$, a cost function $c$ defined on $E$ and a cost function $f$ defined on $V$, find a Steiner tree $(U,F)$ minimizing the total cost $c(F)+f(U)$.

***r*-tree problem.** Given an undirected graph $G=(V,E)$, a root $r\in V$, a cost function $c$ defined on $E$ and a cost function $f$ defined on $V$, find an $r$-tree $(U,F)$ minimizing the total cost $c(F)+f(U)$.

Clearly, the Steiner tree problem is a special case of the vertex-weighted Steiner tree problem. Simply take $f_i=0$ for all $i\in V$. Moreover, the vertex-weighted Steiner tree problem can be seen to be a special case of the $r$-tree problem by selecting some $r\in T$ and replacing $f_i$ for $i\in T$ by some large negative number.

Some formulations, reduction methods, lower bounding techniques, heuristics and polyhedral approaches have been proposed for variations of the vertex-weighted Steiner tree problem (see Segev [27], Duin and Volgenant [12] and Chopra and Gorres [6]).

### 2.2. Formulation

To every $r$-tree or Steiner tree $(U,F)$, we associate an incidence vector $(x,y)$ defined by $x_e=1$ if $e\in F$ and 0 otherwise, and $y_i=1$ if $i\in U$ and 0 otherwise. For Steiner trees, we assume that $y_i$ is defined only for the Steiner vertices $i$ (the terminals are always present, by definition). Let $S_{rT} \subset \{0, 1\}^{|E|+|V|}$ denote the set of incidence vectors of $r$-trees, $S_E \subset \{0, 1\}^{|E|+|V\setminus T|}$ denote the set of incidence vectors of Steiner trees and $S_{ST} = \{x\colon (x, y) \in S_E \text{ for some } y\}$.

The $r$-tree problem can be formulated as the following integer program:

$$\text{Minimize} \quad \sum_{e\in E} c_e x_e + \sum_{i\in V} f_i y_i$$

$$\text{subject to} \quad x(E) = y(V - r), \tag{1}$$

$$x(E(S)) \leqslant y(S-k), \quad k \in S \subseteq V, \ |S| \geqslant 2, \tag{2}$$

$$y_r \leqslant 1, \tag{3}$$

$$x_e \geqslant 0, \quad e \in E, \tag{4}$$

$$y_i \in \mathbb{Z}, \quad i \in V, \tag{5}$$

where $S - k = S\setminus\{k\}$. The constraints (2) are called *generalized subtour elimination* constraints. For a particular value of $S$ and $k$, we shall refer to (2) as the $(S, k)$-inequality. When dealing with $(S, k)$-inequalities, we assume that $(S, k) \neq (V, r)$. In the special case in which $S = \{i, k\}$, the constraint (2) says that $x_{ik} \leqslant y_i$ and, as a result, if $y_i = 0$ we have $x_{ik} = 0$. The constraints $y_i \leqslant y_r$ (and, hence, $y_i \leqslant 1$) can be obtained by subtracting (1) from the $(V, i)$-inequality. When $r \neq k$, $r \in S$ and $S \neq V$, the $(S, k)$, inequality can be obtained by adding $y_k \leqslant y_r$ to the $(S, r)$-inequality. Therefore, in what follows, we shall restrict our attention in (2) to pairs $(S, k)$ such that $k = r$ if $r \in S \neq V$.

The validity of this formulation comes from Edmonds' characterization of the spanning tree polytope [13]. Indeed, given a 0-1 vector $y$ defined by $y_i = 1$ iff $i \in T$, the constraints (2) imply that $x_e = 0$ if $e \notin E(T)$ and that $x(E)S)) \leqslant |S| - 1$ for $\emptyset \neq S \subseteq T$. Moreover, (1) implies that $x(E(T)) = |T| - y_r$ and, hence, either both $y$ and $x$ have all components equal to 0 or $y_r = 1$. As a result, $x$ is a convex combination of incidence vectors of trees spanning $T$.

From the above formulation, we can readily obtain a formulation for the vertex-weighted Steiner tree problem by selecting $r \in T$ and imposing $y_r = 1$ and $y_i = y_r$ for all $i \in T - r$. Eliminating some of the redundant constraints, we obtain the following formulation for the vertex-weighted Steiner tree problem:

$$\text{minimize} \quad \sum_{e\in E} c_e x_e + \sum_{i\in V} f_i y_i$$

$$\text{subject to} \quad x(E) = y(N) + |T| - 1, \tag{6}$$

$$x(E(S)) \leqslant y(S\cap N) + |S \cap T| - 1, \quad S \subseteq V, \ S \cap T \neq \emptyset, \tag{7}$$

$$x(E(S)) \leqslant y(S-k), \quad k \in S \subseteq N, \tag{8}$$

$$y_i \leqslant 1, \quad i \in N, \tag{9}$$

$$x_e \geqslant 0, \quad e \in E, \tag{10}$$

$$y_i \in \mathbb{Z}, \quad i \in N, \tag{11}$$

where $N$ denotes the set of Steiner vertices. This formulation can be considered an extended formulation for the Steiner tree problem (the $E$ in $S_E$ in fact stands for *extended*). A similar extended formulation has been proposed and is being investigated by Lucena.[1]

In this paper, we focus our attention on the linear programming relaxations of the above integer programs obtained by relaxing the constraints (5) or (11). Let $P_{rT}=\{(x, y) \in \mathbb{R}^{|E|+|V|}$ satisfying (1)–(4)$\}$, let $P_E=\{(x, y)\in \mathbb{R}^{|E|+|V\setminus T|}$ satisfying (6)–(10)$\}$ and let $P_{ST}=\{x\in \mathbb{R}^E\colon (x, y)\in P_E$ for some $y\}$.

Given a vector $(x, y)$, for fixed $k$, finding the most violated $(S, k)$-inequality can be reduced to a minimum cut problem (see Rhys [24] or Section III.3.7. in Nemhauser and Wolsey [21]) and, hence, is polynomially solvable. The separation problem for the polytope $P_{rT}$ (or $P_E$) can thus be solved by a sequence of $|V|$ minimum cut problems. Therefore, using the ellipsoid algorithm, we can optimize in polynomial time over $P_{rT}$ (or $P_E$). This implies that we can also optimize in polynomial time over $P_E$. We are not aware of any combinatorial algorithm to optimize in polynomial time over any of these polytopes.

## 3. Facets of conv($S_{rT}$)

In this section, we study some basic properties of the convex hull of $S_{rT}$, denoted by conv($S_{rT}$), and we investigate which inequalities among (2)–(4) define facets of conv($S_{rT}$). For background material on polyhedral theory, we refer the reader to Nemhauser and Wolsey [21].

First, we show that, without loss of generality, we may restrict our attention to 2-connected graphs. Consider a connected graph $G$ which is not 2-connected. Let $v$ be a cut vertex of $G$ and let $G_1=(V_1, E_1)$, $G_2=(V_2, E_2)$ be such that $V_1\cup V_2=V$, $V_1\cap V_2=\{v\}$, $E_1\cup E_2=E$, $E_1\neq\emptyset$, $E_2\neq\emptyset$, $G_1=G[V_1]$ and $G_2=G[V_2]$. We say that $G_1$ and $G_2$ form a 1-separation of $G$ at vertex $v$. Let $r_1=r$ if $r\in V_1$ and $r_1=v$ otherwise. Similarly, let $r_2=r$ if $r\in V_2$ and $r_2=v$ otherwise. If $(x, y)$ is a vector in $\mathbb{R}^{|E|+|V|}$ then let $(x^i, y^i)$ be the restriction of $(x, y)$ to $G_i$ $(i=1, 2)$. Notice that $(x^1y^1)$ and $(x^2, y^2)$ have only one component in common, namely $y_v$.

**Theorem 1.** *If $v$ is a cut vertex of a graph $G=(V, E)$ then, with the above notation,*

$$\operatorname{conv}(S_{rT})=\{(x, y)\colon (x^1, y^1)\in\operatorname{conv}(S^1_{rT}) \text{ and } (x^2, y^2)\in\operatorname{conv}(S^2_{rT})\}$$

*where $S^i_{rT}$ denotes the set of incidence vectors of $r_i$-trees of $G_i$.*

**Proof.** If $(x, y)$ is the incidence vector of an $r$-tree then $(x^i, y^i)$ is the incidence vector of an $r_i$-tree. Hence,

$$\operatorname{conv}(S_{rT})\subseteq\{(x, y)\colon (x^1y^1)\in\operatorname{conv}(S^1_{rT}) \text{ and } (x^2, y^2)\in\operatorname{conv}(S^2_{rT})\}.$$

Conversely, if $(U_i, F_i)$ is an $r_i$-tree of $G_i$ $(i=1, 2)$ such that $v\in U_1$ iff $v\in U_2$, then

[1]This was communicated to us by L.A. Wolsey.

$(U_1 \cup U_2, F_1 \cup F_2)$ is an $r$-tree of $G$. This follows from the definition of $r_1, r_2$. Now, consider an $(x, y)$ such that $(x^1, y^1) \in \mathrm{conv}(S^1_{rT})$ and $(x^2, y^2) \in \mathrm{conv}(S^2_{rT})$. Then $(x^i, y^i)$ can be seen as a convex combination of incidence vectors of $r_i$-trees of $G_i$ $(i=1, 2)$. Since, for $i=1$ or 2, a fraction of exactly $y_v$ of these $r_i$-trees contain $v$, these $r_i$-trees can be recombined to give $r$-trees of $G$. As a result, $(x, y)$ can be seen as a convex combination of incidence vectors of $r$-trees of $G$. □

As a result, we shall focus our attention on 2-connected graphs. We shall now compute the dimension of $\mathrm{conv}(S_{rT})$. For this purpose, we need the following trivial result.

**Lemma 2.** *Let $G$ be a 2-connected graph. Then, for every distinct $u, v, w \in V$ there exists a path in $G$ from $u$ to $w$ that does not pass through $v$.*

**Proof.** $u$ and $w$ are connected in $G-v$ since this graph is connected. □

**Proposition 3.** *Let $G=(V, E)$ be a 2-connected graph. Then:*

(i) *The affine hull of $S_{rT}$ is $\{(x, y): x(E) = y(V-r)\}$, i.e. the only valid equality for $S_{rT}$ is (1) or a multiple of it.*

(ii) $\dim(\mathrm{conv}(S_{rT})) = |V| + |E| - 1$.

**Proof.** (i) Assume that $\alpha x + \beta y = \gamma$ is satisfied by all $(x, y) \in S_{rT}$. Since $(0, 0), (0, e_r) \in S_{rT}$ where $e_r$ is the $r$th unit vector, we have that $\gamma=0$ and $\beta_r=0$. Consider any edge $e=(i, j) \in E$ with $i \neq r$. By Lemma 2, there exists a path $P$ from $r$ to $j$ that does not go through $i$. This path $P$ is an $r$-tree. Another $r$-tree can be obtained by adding the edge $e$ to $P$. Hence, we must have

$$\beta_i + \alpha_e = 0. \tag{12}$$

Since any two vertices, $i, j \in V-r$ are connected by a path $(i_0=i, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k=j)$ that does not go through $r$, (12) implies that $\beta_{i_0} = -\alpha_{i_0 i_1} = \beta_{i_1} = \cdots = -\alpha_{i_{k-1} i_k} = \beta_{i_k}$. Hence, there exists a scalar $\delta$ such that $\beta_i = -\delta$ for all $i \in V-r$. By the same argument, $\alpha_e = \delta$ for all $e \in E$. Hence, $\alpha x + \beta y = \gamma$ is a multiple of (1).

(ii) follows from (i) by Proposition 2.4 on page 87 of Nemhauser and Wolsey [21]. □

In the following three propositions, we establish which inequalities among (2)–(4) define facets of $\mathrm{conv}(S_{rT})$. Notice that two inequalities of (2)–(4) cannot be obtained from one another by adding a multiple of (1). This together with Proposition 3 implies that they cannot induce the same facet of $\mathrm{conv}(S_{rT})$ and that any other inequality among (2)–(4) implied by these two inequalities cannot be facet-defining. This observation is important in the forthcoming Proposition 6.

**Proposition 4.** *Let $G=(V, E)$ be a 2-connected graph. Then $x_e \geq 0$ defines a facet of* $\mathrm{conv}(S_{rT})$ *iff $G-e$ is 2-connected.*

**Proof.** If $G-e$ is not 2-connected, then it has a cut vertex, say $v$. Let $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ be a 1-separation of $G-e$ at vertex $v$. We assume that $r\in V_1$. Clearly, the $(V_2, v)$-inequality is satisfied at equality by all solutions satisfying $x_e=0$. As a result, $x_e=0$ defines a face of $\text{conv}(S_{rT})$ of dimension strictly less than $\dim(\text{conv}(S_{rT}))-1$ and, hence, not a facet.

Assume now that $G-e$ is 2-connected. By Proposition 3, we know that the only equality satisfied by all $r$-trees of $G-e$ is $x(E-e)=y(V-r)$ (or a multiple of it). In other words, the only valid inequalities that are satisfied at equality by all $r$-trees with $x_e=0$ are equivalent to $x_e\geqslant 0$ (i.e. they can be obtained from a multiple of $x_e\geqslant 0$ by adding a multiple of (1)). Hence, $x_e\geqslant 0$ defines a facet of $\text{conv}(S_{rT})$. □

**Proposition 5.** *Let $G=(V, E)$ be a 2-connected graph. Then $y_r\leqslant 1$ defines a facet of* $\text{conv}(S_{rT})$.

**Proof.** By the same argument as in the proof of Proposition 3, if $\alpha x+\beta y=\gamma$ is satisfied by all $(x, y)\in S_{rT}$ with $y_r=1$ then there exists $\delta$ such that $\beta_i=-\delta$ for all $i\in V-r$, $\alpha_e=\delta$ for all $e\in E$ and $\beta_r=\gamma$. Hence, these equalities can be obtained by combining $y_r=1$ with (1). This proves that $y_r\leqslant 1$ defines a facet of $\text{conv}(S_{rT})$. □

**Proposition 6.** *Let $G=(V, E)$ be a 2-connected graph. Then the $(S, k)$-inequality defines a facet of* $\text{conv}(S_{rT})$ *iff the following conditions are all satisfied*:
- *$G[S]$ is 2-connected if $|S|\geqslant 3$;*
- *$G[S]$ is connected if $|S|=2$;*
- *$G-S$ is connected; and*
- *$k=r$ if $r\in S\neq V$.*

In order to prove this result, we need the following trivial lemma.

**Lemma 7.** *Let $G=(V, E)$ be a 2-connected graph. For any $S\subset V$ and $\mu\notin S$, let $R(u)=\{v\in S$: there exists an $S$-path between $u$ and $v\}$ where an $S$-path is a path with no intermediate vertex in $S$. Then $|R(u)|\geqslant 2$.* □

**Proof of Proposition 6.** If $S=\{i, k\}$ but $G[S]$ is not connected then the $(S, k)$-inequality is $y_i\geqslant 0$ and is implied by the inequalities $y_i\geqslant x_e$ and $x_e\geqslant 0$ for some $e$ incident to $i$.

If $G[S]$ $(|S|\geqslant 3)$ is not 2-connected then it has a cut vertex $v$. Let $G_1=(S_1, E_1)$, $G_2=(S_2, E_2)$ be a 1-separation of $G[S]$ at vertex $v$. Let $k_1=k$ if $k\in S_1$ and $k_1=v$ otherwise. Similarly, let $k_2=k$ if $k\in S_2$ and $k_2=v$ otherwise. The $(S, k)$-inequality can be obtained by summing up the $(S_1, k_1)$-inequality and the $(S_2, k_2)$-inequality. Hence, the $(S, k)$-inequality does not define a facet of $\text{conv}(S_{rT})$.

If $G-S$ is not connected than let $V_1$ be the vertex of a connected component of $G-S$. Without loss of generality, assume that $r\in S\cup V_1$. Let $V_2=V\setminus(S\cup V_1)$. The $(S, k)$-inequality can be obtained by summing up the $(S\cup V_1, r)$-inequality and the $(S\cup V_2, k)$-

inequality and subtracting (1). Hence, the $(S, k)$-inequality does not define a facet of $\mathrm{conv}(S_{rT})$.

If $r \in S \neq V$ and $k \neq r$ then the $(S, k)$-inequality can be obtained by summing up the $(S, r)$-inequality and the inequality $y_k \leqslant y_r$. As previously mentioned, this latter inequality can be obtained by subtracting (1) from the $(V, i)$-inequality. Again, the $(S, k)$-inequality does not define a facet of $\mathrm{conv}(S_{rT})$.

On the other hand, assume that $G[S]$ is 2-connected if $|S| \geqslant 3$ and connected if $|S| = 2$, $G - S$ is connected and $k = r$ if $r \in S \neq V$. We shall prove that the only valid inequalities satisfied at equality by all $(x, y) \in S_{rT}$ with $x(E(S)) = y(S - k)$ are equivalent to the $(S, k)$-inequality. Assume that $\alpha x + \beta y = \gamma$ for all $(x, y) \in S_{rT}$ with $x(E(S)) = y(S - k)$. We break the proof into three parts.

*Case 1*: $r \notin S$. Since $(0, 0)$ and $(0, e_r)$ must satisfy $\alpha x + \beta y = \gamma$, we have

$$\gamma = 0 \quad \text{and} \quad \beta_r = 0. \tag{13}$$

Let $e = (i, j) \in E(S)$ with $i \neq k$. By Lemma 2, there exists a path $P_1$ in $E(S)$ from $j$ to $k$ that does not pass through $i$. Let $P_2$ be an $S$-path from $r$ to some $l \in S$. By Lemma 7, we can assume that $l \neq i$. Let $P_3$ be a (possibly empty) path connecting $l$ to $P_1$ in $E(S)$ without going through $i$. Clearly, $F = P_1 \cup P_2 \cup P_3$ is an $r$-tree (Figure 1(a)) whose incidence vector satisfies $x(E(S)) = y(S - k)$. Adding $e$ to $F$, we still have such an $r$-tree. Hence, $\alpha_e + \beta_i = 0$. Since $G[S - k]$ is connected, by looking at all edges $e = (i, j) \in E(S)$ with $i \neq k$, we find the existence of a scalar $\alpha_S$ such that

$$\alpha_e = \alpha_S \quad \text{for } e \in E(S) \tag{14}$$

and

$$\beta_i = -\alpha_S \quad \text{for } i \in S - k. \tag{15}$$

Let $e = (i, j)$ with $i \notin S$ and $j \in S$. Let $P_1$ be a path from $i$ to $r$ in $E(V \setminus S)$. The existence of this path follows from the assumption that $G - S$ is connected. Let $P_2$ be a path from $j$ to

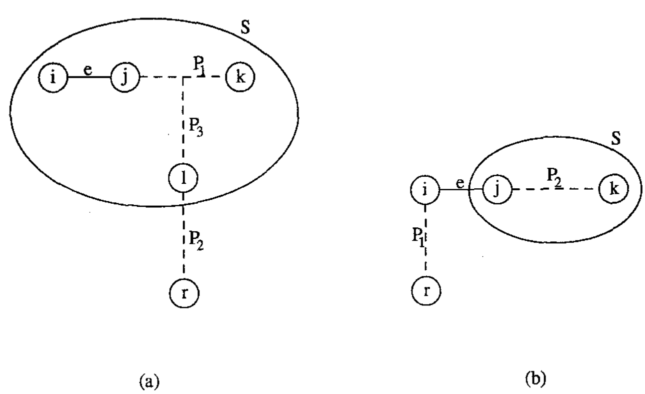


Fig. 1. $r$-trees in the proof of Case 1.

$k$ in $E(S)$. Consider the $r$-trees $F_1 = P_1$ and $F_2 = P_1 \cup P_2 \cup \{e\}$ (Figure 1(b)). Both $r$-trees satisfy the $(S, k)$-inequality at equality and, hence, $\alpha x + \beta y = \gamma$. Using (14) and (15), we get

$$\alpha_e = -\beta_k \quad \text{for } e \in \delta(S). \tag{16}$$

Now let $e = (i, j) \notin E(S)$ with $r \neq i \notin S$ (but $j$ might be in $S$). By Lemma 2, there exists a path $P_1$ from $j$ to $r$ that does not pass through $i$. Since $G[S]$ is connected, we may assume that $|P_1 \cap \delta(S)| \leqslant 2$. If $P_1$ intersects $S$ then connect it to $k$ by edges in $E(S)$ (Figure 2). The resulting $r$-tree $F$ satisfies the $(S, k)$-inequality at equality and so does $F \cup \{e\}$. Hence,

$$\alpha_e + \beta_i = 0 \quad \text{for } e = (i, j) \notin E(S) \text{ with } r \neq i \notin S. \tag{17}$$

Since $G - r$ is connected, any vertex $i \neq r$ is connected to some edge $f$ in $\delta(S)$ by a path in $E(V-r) \setminus E(S)$. Combining (17) with (16), we get

$$\alpha_e = -\beta_k \quad \text{for } e \in E \setminus E(S) \tag{18}$$

and

$$\beta_i = \beta_k \quad \text{for } i \in V \setminus S \setminus \{r\}. \tag{19}$$

Therefore, from (13), (14), (15), (18) and (19), we see that $\alpha x + \beta y = \gamma$ can be derived from $x(E(S)) = y(S-k)$ and (1) by subtracting $\beta_k$ times $x(E) - y(V-r) = 0$ from $\alpha_S + \beta_k$ times $x(E(S)) - y(S-k) = 0$. This means that the $(S, k)$-inequality defines a facet of $\text{conv}(S_{rT})$.

*Case 2*: $r = k$. In this case, $S \neq V$ (otherwise we have (1)). Since $(0, 0)$ and $(0, e_r)$ must satisfy $\alpha x + \beta y = \gamma$, we have $\gamma = 0$ and $\beta_r = 0$.

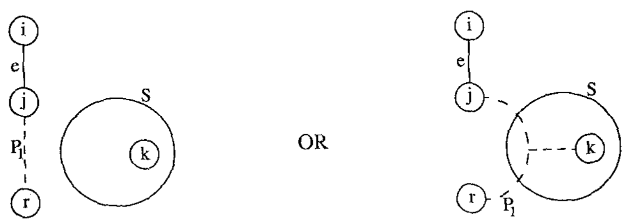


Fig. 2. Another $r$-tree in the proof of Case 1.

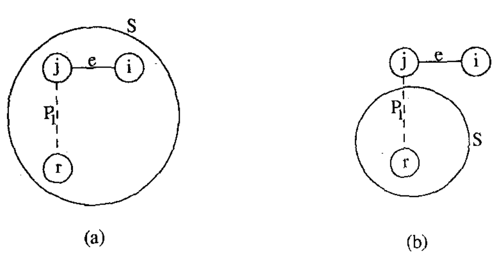


Fig. 3. $r$-trees in the proof of Case 2.

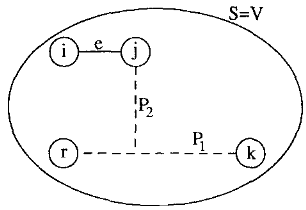


Fig. 4. $r$-tree in the proof of Case 3.

Let $e=(i,j)\in E(S)$ with $i\neq r$. Using Lemma 2, let $P_1$ be a path in $E(S)$ from $j$ to $r$ that does not go through $i$. Since both $P_1$ and $P_1\cup\{e\}$ are $r$-trees (Figure 3(a)) satisfying $x(E(S))=y(S-r)$, we must have $\alpha_e+\beta_i=0$.

Since $G[S-r]$ is connected, this implies the existence of $\alpha_s$ such that $\alpha_e=\alpha_s$ for all $e\in E(S)$ and $\beta_i=-\alpha_s$ for all $i\in S-r$.

Let $e=(i,j)\notin E(S)$ with $i\notin S$. Let $P_1$ be a path from $j$ to $r$ that does not go through $i$. Since $G[S]$ is connected, we may assume that $|P_1\cap\delta(S)|\leqslant 1$. Again, since both $P_1$ and $P_1\cup\{e\}$ are $r$-trees (Figure 3(b)) satisfying $x(E(S))=y(S-r)$, we must have $\alpha_e+\beta_i=0$. Since $G-S$ is connected, there exists $\alpha_{\bar{S}}$ such that $\alpha_e=\alpha_{\bar{S}}$ for all $e\notin E(S)$ and $\beta_i=-\alpha_{\bar{S}}$ for all $i\notin S$.

Therefore, we see that $\alpha x+\beta y=\gamma$ can be derived from $x(E(S))=y(S-r)$ and (1). This means that the $(S,r)$-inequality defines a facet of $\mathrm{conv}(S_{rT})$.

*Case 3*: *$S=V$ and $k\neq r$*. In this case, the $(S,k)$-inequality is equivalent to $y_k\leqslant y_r$.

Since $(0,0)$, must satisfy $\alpha x+\beta y=\gamma$, we have $\gamma=0$.

Let $e=(i,j)\in E$ with $i\notin\{r,k\}$. Let $P_1$ be a path from $r$ to $k$ that does not go through $i$. Let $P_2$ be a path from $j$ to $P_1$ that does not go through $i$. Since both $P_1\cup P_2$ and $P_1\cup P_2\cup\{e\}$ are $r$-trees (Figure 4) satisfying $y_k=y_r$, we must have

$$\alpha_e+\beta_i=0. \tag{20}$$

Let $f$ be any edge. Since $G$ is connected, there exists a path $P_1$ from $r$ to $k$ that uses $f$. Comparing the incidence vector of $P_1$ with $(0,0)$ and using (20), we find that $\alpha_f+\beta_r+\beta_k=0$ for all $f\in E$. Hence, $\beta_i=\beta_r+\beta_k$ for all $i\in V\setminus\{r,k\}$. Therefore, $\alpha x+\beta y=\gamma$ can obtained by subtracting $\beta_r+\beta_k$ times $x(E)-y(V-r)=0$ from $\beta_r$ times $y_r-y_k=0$. This means that the $(V,k)$-inequality defines a facet of $\mathrm{conv}(S_{rT})$. □

## 4. Polyhedral characterization for series-parallel graphs

**Definition 1.** A graph $G$ is *series–parallel* if it does not contain any subgraph homeomorphic[2] to the complete graph $K_4$ on 4 vertices.

Duffin [11] has shown that a 2-connected graph $G=(V,E)$ is series–parallel iff it can

[2] $G_1$ is homeomorphic to $G_2$ if $G_1$ can be obtained from $G_2$ by subdividing edges.

be obtained from the graph consisting of two parallel edges on two vertices by subdividing edges (series operation) and duplicating edges (parallel operation). In fact series-parallel graphs are often defined in this way. Duffin also showed that, given an edge $e \in E$, this construction can be performed in such a way that $e$ does not participate in any series or parallel operation ($e$ is therefore one of the two initial edges). By looking at the last parallel operation in the construction of a 2-connected series–parallel graph, we obtain that these graphs can be decomposed into cycles:

**Lemma 8.** *Let $G$ be a 2-connected series–parallel graph that is not a cycle. Then there exist $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $V_1 \cap V_2 = \{a, b\}$, $V_1 \cup V_2 = V$, $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = E$ and $G_2$ is a cycle on $V_2$ (see Figure 5). Moreover, replacing $G_2$ by a simple path, we obtain a 2-connected series–parallel graph that requires one fewer parallel operation.* □

By the remark preceding the lemma, the decomposition just described can be performed in a way that any prespecified edge $e$ belongs to $E_1$. In particular, this implies that we may assume that $r \in V_1$.

Many combinatorial optimization problems that are NP-hard on general graphs are polynomially solvable on series–parallel graphs because of the decomposability of these graphs (see e.g. [2, 3, 5, 9, 10, 25, 28, 30, 31]). This is formalized in various ways by Arnborg and Proskurowski [2], Arnborg, Lagergren and Seese [3] and Takamizawa, Nishizeki and Saito [28]. Not surprisingly, the Steiner tree problem (Wald and Colbourn [30] and Rardin et al. [23]), its vertex-weighted version and the $r$-tree problem are all polynomially solvable on series–parallel graphs. Simple decomposition-based algorithms similar to those in Takamizawa et al. [28] can indeed be developed.[3] These algorithms can even be implemented in linear time (series–parallel graphs can be decomposed in linear time, see e.g. Wagner [29]).

The fact that most combinatorial optimization problems are polynomially solvable on series–parallel graphs suggests that it might be possible to obtain simple explicit descriptions of the corresponding polytopes by linear inequalities. Results of this kind were obtained for

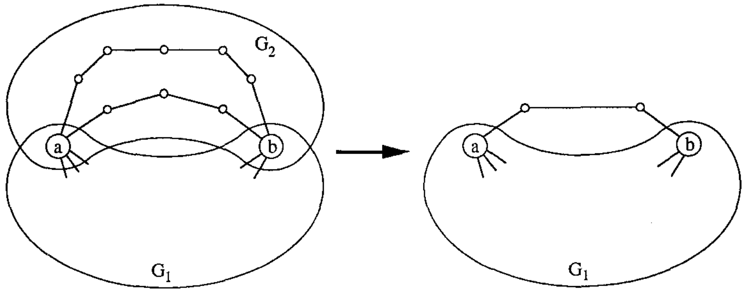


Fig. 5. Decomposition property of series-parallel graphs.

[3]Such an algorithm for the $r$-tree problem is in fact hidden behind the proof of the forthcoming Theorem 9.

various combinatorial optimization problems defined on series–parallel graphs: stable set problem (Boulala and Uhry [5], Mahjoub [18]), directed Steiner tree problems (Prodon et al. [22], Goemans [14], Schaffers [26]), 2-connected and 2-edge-connected subgraph problems (Cornuéjols et al. [9], Coullard et al. [10], Mahjoub [19]) and the traveling salesman problem (Coullard et al. [10]). Martin et al. [20] propose a general technique to derive from dynamic programming algorithms complete characterizations of most combinatorial optimization polytopes (including the Steiner tree polytope) on series–parallel graphs. However, their characterizations use auxiliary variables and are therefore not in the space of the natural variables. In this section, we obtain a complete polyhedral characterization of both the $r$-tree polytope and the vertex-weighted Steiner tree polytope on series–parallel graphs.

**Theorem 9.** *If $G$ is series–parallel then $P_{rT} = \text{conv}(S_{rT})$.*

By Theorem 1, we know that it is sufficient to prove Theorem 9 for 2-connected graphs. For that purpose, we use a non-algorithmic proof technique communicated to us by Pochet and Wolsey. This technique is essentially equivalent to showing that any facet is defined by one of the inequalities in the proposed linear inequality system (see Lovász [17] for an illustration for the matching polytope).

**Theorem 10.** *Let $P = \{x \in \mathbb{R}^n\colon A^{=}x = b^{=}, A^{\leqslant}x \leqslant b^{\leqslant}\}$ and let $S = P \cap \mathbb{Z}^n$. Assume that for each inequality $ax \leqslant b$ in $A^{\leqslant}x \leqslant b^{\leqslant}$, there exists $\tilde{x} \in S$ such that $a\tilde{x} < b$. For a cost function $c$, let $\mathcal{O}$ be the set of optimal solutions to $\min\{cx\colon x \in S\}$.*

*Then $P = \text{conv}(S)$ iff, for any cost function $c$, either $c = uA^{=}$ for some row vector $u$ or there exists an inequality $ax \leqslant b$ among $\{A^{\leqslant}x \leqslant b^{\leqslant}\}$ such that $ax^* = b$ for all $x^* \in \mathcal{O}$.* □

From now on, given a cost function, $(c, f)$, $\mathcal{O}$ denotes the set of incidence vectors of minimum cost $r$-trees of $G$.

The fact that all inequalities in $P_{rT}$ satisfy the technical condition of Theorem 10, namely that they are satisfied strictly by the incidence vector of some $r$-tree, follows from Proposition 3.

In order to apply Theorem 10, we prove the following result:

**Lemma 11.** *Let $i$ be a degree 2 vertex of a graph $G$ with $i \neq r$. Let $e = (i, j)$ and $e' = (i, j')$ be the two edges incident to $i$. If, for some cost function $(c, f)$, no $(S, k)$-inequality with $|S| = 2$ is satisfied at equality by all solutions in $\mathcal{O}$, then $c_e = c_{e'}$ and $f_i + c_e \leqslant 0$.*

**Proof.** By assumption, there exists an $(x, y) \in \mathcal{O}$ such that $y_i > x_{e'}$. Hence, $y_i = 1$ and $x_{e'} = 0$. Since $i$ is connected to $r$, we must have $x_e = 1$. Replacing $y_i$ by 0 and $x_e$ by 0, we obtain the incidence vector $(x', y')$ of another $r$-tree whose cost $cx' + fy'$ is at least as much as the cost of $(x, y)$ by optimality of $(x, y)$. Hence, $f_i + c_e \leqslant 0$. Similarly,

$$f_i + c_{e'} \leqslant 0. \tag{21}$$

By assumption, there also exists an $(x, y) \in \mathscr{O}$ such that $y_j > x_e$. Hence, $y_j = 1$ and $x_e = 0$. Two cases can happen. If $y_i = 0$ then replacing $y_i$ by 1 and $x_e$ by 1, we obtain another $r$-tree $(x', y')$ such that $cx' + fy' \geq cx + fy$. Hence, $f_i + c_e \geq 0$. Together with (21), this implies that $c_{e'} \leq c_e$. If $y_i = 1$ then the fact that $i$ is connected to $r$ implies that $x_{e'} = 1$. Replacing $x_{e'}$ by 0 and $x_e$ by 1, we obtain another $r$-tree. Comparing the cost of these $r$-trees, we find that $c_{e'} \leq c_e$. So, in any case, we have $c_{e'} \leq c_e$. By symmetry, we also have that $c_e \leq c_{e'}$. Hence, $c_e = c_{e'}$. □

Our proof of Theorem 9 proceeds by induction on the number of parallel operations needed in the construction of $G$. For the base case, we have:

**Proposition 12.** *If $G = (V, E)$ is a cycle then $P_{rT} = \text{conv}(S_{rT})$.*

**Proof.** The proof is based on Theorem 10. Consider any cost function $(c, f)$. If some $(S, k)$-inequality with $|S| = 2$ is satisfied at equality by all solutions in $\mathscr{O}$ then we are done. Otherwise, by Lemma 11, we know that there exists $\gamma$ such that $c_e = \gamma$ for all $e \in E$ and $f_i + \gamma \leq 0$ for all $i \in V - r$. In this case, the cost of any $r$-tree $(U, F) \neq (\emptyset, \emptyset)$ is given by $f_r + \sum_{i \in U - r}(f_i + \gamma)$. Since $f_i + \gamma \leq 0$ for all $i \in V - r$, the cost of an optimal $r$-tree is given by $\min\{0, f_r + \sum_{i \in V - r}(f_i + \gamma)\}$. If $f_i + \gamma < 0$ for some $i \in V - r$ then $y_i = y_r$ for every $(x, y) \in \mathscr{O}$, i.e. the $(V, i)$-inequality is satisfied at equality for every $(x, y) \in \mathscr{O}$. If $f_i + \gamma = 0$ for all $i \in V - r$ then the cost of an optimal $r$-tree is given by $\min\{0, f_r\}$. We have three cases:

1. If $f_r > 0$ then every optimal solution satisfies $x_e = 0$ for any $e \in E$.
2. If $f_r < 0$ then every optimal solution satisfies $y_r = 1$.
3. If $f_r = 0$ then the cost function $(c, f)$ is a multiple of the equality constraint (1).

Therefore the result follows from Theorem 10. □

We are now ready to prove Theorem 9.

**Proof of Theorem 9.** By Theorem 1, we may restrict our attention to 2-connected series–parallel graphs $G = (V, E)$. As previously mentioned, the proof is based on Theorem 10 and proceeds by induction on $p$, the number of parallel operations needed in the construction of $G$. The case $p = 0$ was treated in Proposition 12.

Suppose we have proved the theorem for some $p$ and consider a graph requiring $p + 1$ parallel operations. By Lemma 8, there exist $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $V_1 \cap V_2 = \{a, b\}$, $V_1 \cup V_2 = V$, $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = E$ and $G_2$ is a cycle on $V_2$. $G_2$ can be seen as the union of two paths between $a$ and $b$. Let $V_{21}$ and $V_{22}$ be the intermediate vertices of these two paths and let $E_{21}$ and $E_{22}$ be the edges of these two paths. Moreover, we may assume that $r \in V_1$, i.e. $r \notin (V_{21} \cup V_{22})$.

Let $(c, f)$ be any cost function and let $\mathscr{O}$ be the corresponding set of optimal solutions. We would like to show that either there exists an inequality among (2)–(4) which is satisfied at equality by all solutions in $\mathscr{O}$ or $(c, f)$ is a multiple of (1), i.e. there exists $\gamma$ such that $c_e = \gamma$ for all $e \in E$, $f_r = 0$ and $f_i = -\gamma$ for all $i \in V - r$. By Lemma 11, we know that we may assume that $(c, f)$ is such that there exist $\gamma_1$ and $\gamma_2$ with $c_e = \gamma_l$ for all $e \in E_{2l}$ and $f_i + \gamma_l \leq 0$ for all $i \in V_{2l}$ $(l = 1, 2)$. This latter fact means that, whenever $y_a = 1$ or $y_b = 1$

for some $(x, y) \in \mathscr{O}$, we incur no increase in cost by linking any vertex in $V_{21} \cup V_{22}$ to either $a$ or $b$ using edges in $E_2$. Therefore, for any solution $(x, y) \in \mathscr{O}$, the *contribution* of $G_2$ to the cost of $(x, y)$, defined by

$$C = \sum_{i \in V_2:\ y_i = 1} f_i + \sum_{e \in E_2:\ x_e = 1} c_e,$$

is equal to:

1. $C_{00} = 0$ if $y_a = y_b = 0$;
2. $C_{10} = f_a + \sum_{i \in V_{21}} (f_i + \gamma_1) + \sum_{i \in V_{22}} (f_i + \gamma_2)$ if $y_a = 1$, $y_b = 0$;
3. $C_{01} = f_b + \sum_{i \in V_{21}} (f_i + \gamma_1) + \sum_{i \in V_{22}} (f_i + \gamma_2)$ if $y_a = 0$, $y_b = 1$;
4. $C_{11}^1 = f_a + f_b + \sum_{i \in V_{21}} (f_i + \gamma_1) + \sum_{i \in V_{22}} (f_i + \gamma_2)$ if $y_a = 1$, $y_b = 1$ and $a$ and $b$ are connected through $G_1$;
5. $C_{11}^2 = f_a + f_b + \min(\gamma_1, \gamma_2) + \sum_{i \in V_{21}} (f_i + \gamma_1) + \sum_{i \in V_{22}} (f_i + \gamma_2)$ if $y_a = 1$, $y_b = 1$ and $a$ and $b$ are connected through $G_2$.

Letting $\delta = \sum_{i \in V_{21}} (f_i + \gamma_1) + \sum_{i \in V_{22}} (f_i + \gamma_2)$ and assuming without loss of generality that $\gamma_1 \leqslant \gamma_2$, we obtain that $C_{00} = 0$, $C_{10} = f_a + \delta$, $C_{01} = f_b + \delta$, $C_{11}^1 = f_a + f_b + \delta$, $C_{11}^2 = f_a + f_b + \delta + \gamma_1$. Since $f_i + \gamma_l \leqslant 0$ for all $i \in V_{2l}$ $(l = 1, 2)$, we have that $\delta \leqslant 0$. In order to reduce $G$ to a graph which requires only $p$ parallel operations, we "simulate" $G_2$ by a path between $a$ and $b$. The actual construction depends on whether $\delta = 0$ or not.

*Case 1*: $\delta = 0$. $\delta = 0$ implies that $f_i = -\gamma_l$ for all $i \in V_{2l}$ and $l = 1, 2$. Let $G' = (V', E')$ be the graph obtained from $G$ by replacing $G_2$ by an edge $e'$ between $a$ and $b$ (see Figure 6). We assume that the vertices and edges in $G_1$ are referred in the same way in $G$ and $G'$. The cost of the vertices and edges in $G_1$ are unchanged while $e'$ is assigned a cost of $\gamma_1$. Let $\mathscr{O}'$ be the set of optimal solutions corresponding to this new problem.

Edge $e$ simulates $G_2$ in the following sense. Given any $r$-tree $F$ of $G$, we can obtain an $r$-tree of $G'$ of smaller or equal cost by simply removing $F \cap E_2$ and replacing it by $e'$ if $F$ contains one of the two paths in $G_2$ between $a$ and $b$. Conversely, given any $r$-tree $F'$ of $G'$, we can obtain an $r$-tree $F$ of $G$ of the same cost by letting $F = F'$ if $e' \notin F'$ and $F = F' \setminus \{e\} \cup E_{21}$ if $e' \in F'$. In other words,

$$\{(x^1, y^1): (x, y) \in \mathscr{O}\} = \{(x^1, y^1): (x, y) \in \mathscr{O}'\}, \tag{22}$$

where the superscript 1 indicates the restriction to $G_1$.

By the inductive hypothesis, either there exists an inequality among (2)–(4) for $G'$ which is satisfied at equality by all solutions in $\mathscr{O}'$ or the cost function is a multiple of the

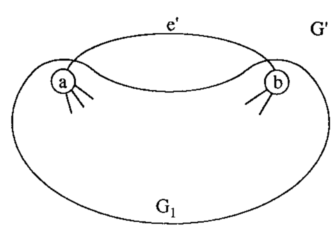


Fig. 6. Graph $G'$ for Case 1.

constraint (1) corresponding to $G'$. In the latter case, we distinguish between two cases. If $\gamma_2 = \gamma_1$ then the original cost function was a multiple of the constraint (1) corresponding to $G$ and so we are done. If $\gamma_2 > \gamma_1$ then the $(V \setminus V_{22}, r)$-inequality is satisfied at equality by all $(x, y) \in \mathscr{O}$.

Therefore, we assume that there exists an inequality among (2)–(4) for $G'$ which is satisfied at equality by all solutions in $\mathscr{O}'$.

- If $x_{e'}$ does not appear in this inequality then the same inequality is satisfied at equality by all solutions in $\mathscr{O}$. This follows from (22).
- If the inequality is $x_{e'} \geqslant 0$ then the $(V_1, r)$-inequality is satisfied at equality by all solutions in $\mathscr{O}$.
- Finally, if the inequality is an $(S, k)$-inequality (with $\{a, b\} \in S$) then the $(S \cup V_2, k)$-inequality is satisfied at equality by all solutions in $\mathscr{O}'$.

Therefore, in any case, we either have an inequality among (2)–(4) for $G$ which is satisfied at equality by all solutions in $\mathscr{O}$ or the cost function is a multiple of the constraint (1) corresponding to $G$.

*Case 2:* $\delta < 0$. $\delta < 0$ implies that $f_s + \gamma_l < 0$ for some $s \in V_{2l}$ and $l = 1$ or 2. As a result, any solution $(x, y) \in \mathscr{O}$ with $y_a = 1$ or $y_b = 1$ (or both) satisfies $y_s = 1$ since otherwise the contribution in $G_2$ of $(x, y)$ can be decreased. In other words, $y_j \leqslant y_s$ for all $(x, y) \in \mathscr{O}$ and all $j \in V_2 \cup \{a, b\}$. This means that $s$ plays a role of root in $G_2$.

Let $G' = (V', E')$ be the graph obtained from $G$ by replacing $G_2$ by an additional vertex $v'$ linked to $a$ and $b$ by the edges $e'$ and $f'$ respectively (see Figure 7). The cost of the vertices and edges in $G_1$ are unchanged while $e'$ and $f'$ are assigned a cost of $\gamma_1$ and $v'$ is assigned a cost of $\delta - \gamma_1$. Let $\mathscr{O}'$ be the set of optimal solutions corresponding to this new problem. Since $f_{v'} + c_{e'} = f_{v'} + c_{f'} = \delta < 0$, whenever $(x, y) \in \mathscr{O}'$ with $y_a = 1$ (or $y_b = 1$), we have that $y_{v'} = 1$.

Again, this path that replaces $G_2$ simulates it in the sense that

$$\{(x^1, y^1): (x, y) \in \mathscr{O}\} = \{(x^1, y^1): (x, y) \in \mathscr{O}'\}. \tag{23}$$

By the inductive hypothesis, either there exists an inequality among (2)–(4) for $G'$ which is satisfied at equality by all solutions in $\mathscr{O}'$ or the cost function is a multiple of the constraint (1) corresponding to $G'$. This latter case is impossible since $f_{v'} + c_{e'} < 0$.

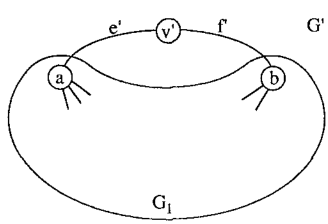


Fig. 7. Graph $G'$ for Case 2.

So assume that there exists an inequality among (2)–(4) for $G'$ which is satisfied at equality by all solutions in $\mathcal{O}'$.

- If none of $x_{e'}$, $x_{f'}$ and $y_{v'}$ appear in this inequality then the same inequality is satisfied at equality by all solutions in $\mathcal{O}$.
- If the inequality is $x_{e'} \geqslant 0$ (resp. $x_{f'} \geqslant 0$) then this means that any $(x, y) \in \mathcal{O}'$ satisfies $y_a = 0$ (resp. $y_b = 0$). Indeed, if $y_a = 1$ and $x_{e'} = 1$ then we must have $y_{v'} = 1$ and $x_{f'} = 1$. We obtain a contradiction by interchanging $x_{e'}$ with $x_f$, since the resulting solution is still optimal and has $x_{e'} = 1$. But (23) now implies that $y_a = 0$ (resp. $y_b = 0$) for all $(x, y) \in \mathcal{O}$. Hence, $x_e = 0$ (resp. $x_f = 0$) for all $(x, y) \in \mathcal{O}$ where $e$ (resp. $f$) is any edge in $G$ incident to $a$ (resp. $b$).
- If the inequality is an $(S, k)$-inequality with $|S| = 2$ and $v' \in S$ (i.e. it is one of the following $x_{e'} \leqslant y_{v'}$, $x_{f'} \leqslant y_{v'}$, $x_{e'} \leqslant y_a$, $x_{e'} \leqslant y_b$) then any $(x, y) \in \mathcal{O}$ with $y_a = y_b = 1$ satisfies $x_{e'} = x_{f'} = 1$. As a result, any $(x, y) \in \mathcal{O}'$ with $y_a = y_b = 1$ has a path between $a$ and $b$ in $G_2$ in which all edges have weight 1. Hence, the $(V_2, s)$-inequality is satisfied at equality by all optimal solutions in $\mathcal{O}$.
- Otherwise, the inequality is an $(S, k)$-inequality with $|S| \geqslant 3$ and $v' \in S$. By Proposition 6, we can assume without loss of generality that $G[S]$ is 2-connected, implying that $a$, $b \in S$. Let $l = k$ if $k \neq v'$ and $l = s$ if $k = v'$. Then the $(S - v' \cup V_2, l)$-inequality is satisfied at equality by all optimal solutions in $\mathcal{O}$.

Therefore, in any case, we either have an inequality among (2)–(4) for $G$ which is satisfied at equality by all solutions in $\mathcal{O}$ or the cost function is a multiple of the constraint (1) corresponding to $G$. This completes the proof of the result. □

By replacing some inequalities by equalities in the linear description of a polyhedron, we do not create new extreme points. As a consequence, such an operation maintains the integrality of a polyhedron. Therefore, by imposing to $P_{rT}$ that $y_r = 1$ and $y_i = y_r$ for all $i \in T - r$, we obtain a complete characterization of vertex-weighted Steiner trees for series–parallel graphs.

**Corollary 13.** *If $G$ is series-parallel then* $P_E = \text{conv}(S_E)$. □

## 5. Projection of $P_E$

In Section 2.2, we defined $P_{ST}$ as the projection of $P_E$ onto the $x$ variables. In this section, we show that $P_{ST}$ has a very rich polyhedral structure.

To obtain a (partial) description of $P_{ST}$ in terms of linear inequalities, we use the projection method introduced by Balas and Pulleyblank [4] (see also Liu [16]). For this purpose, we consider the polyhedral cone $W$ associated with (6)–(9):

$$W=\left\{(p,q,r): \sum_{S:\, i\in S,\, S\cap T\neq\emptyset} p_S + \sum_{(S,k):\, i\in S-k,\, k\in S\subseteq N} q_{S,k} - r_i = 0, \quad i\in N,\right.$$

$$p_S \geqslant 0, \quad S\neq V,\ S\cap T\neq\emptyset,$$

$$q_{S,k}\geqslant 0, \quad k\in S\subseteq N,$$

$$\left. r_i \geqslant 0, \quad i\in N\right\}$$

where $p_S$ corresponds to (7) (or to (6) if $S=V$), $q_{S,k}$ corresponds to (8) and $r_i$ corresponds to (9). $W$ can easily be seen to be a pointed cone. Given any $(p,q,r)\in W$, a valid inequality for $P_{ST}$ can be constructed by summing up the inequalities (6), (7), (8) and (9) multiplied by the corresponding $p_S$, $q_{S,k}$ or $r_i$. The resulting *projected generalized subtour elimination* inequality is

$$\sum_{e\in E}\alpha_e x_e \geqslant \beta \tag{24}$$

where

$$\alpha_e = -\sum_{S:\, e\in E(S),\, S\cap T\neq\emptyset} p_S - \sum_{(S,k):\, e\in E(S),\, k\in S\subseteq N} q_{S,k}, \tag{25}$$

$$\beta = -\sum_{S:\, S\cap T\neq\emptyset}(|S\cap T|-1)p_S - \sum_{i\in N} r_i. \tag{26}$$

Moreover, all valid inequalities for $P_{ST}$ arise in this way (see Liu [16]). Therefore, it "suffices" to obtain the set of extreme rays for the cone $W$ in order to describe $P_{ST}$ by a system of linear inequalities in the natural set of variables.

A projected generalized subtour elimination inequality is valid for a specific graph $G$. However, Chopra and Rao [7] in their study of the Steiner tree polytope have obtained lifting theorems that allow to derive facet-defining valid inequalities for a graph $G'$ from facet-defining valid inequalities for a minor $G$ of $G'$. We refer the reader to Section 4.1. or [7] for details.

In order to reduce the dimension of the cone $W$, we may restrict our attention to those variables corresponding to the facet-defining valid inequalities among (6)–(9). Another reduction in dimension follows from the following observation. $W$ is symmetric in all variables of the form $p_{N\cup M}$ where $\emptyset\neq M\subset T$, in the sense that all these variables appear in the same constraints with the same coefficients. Similarly, for a fixed $L\subset N$, $W$ is symmetric in all variables of the form either $p_{L\cup M}$ where $\emptyset\neq M\subseteq T$ or $q_{L\cup\{k\},k}$ where $k\in N\setminus L$. As a result, for any extreme ray, at most one variable for each of these equivalent classes is nonzero. This means that we may simply consider the following cone obtained by selecting a representative for each equivalence class:

$$W' = \left\{ (r, s, t): \sum_{L \subseteq N:\ i \in L} s_L = t + r_i, \ i \in N, \right.$$
$$s_L \geqslant 0, \ L \subseteq N,$$
$$\left. r_i \geqslant 0, \ i \in N \right\}.$$

Here $s_L$ denotes any variable of the form $p_{L \cup M}$ ($M \subseteq T$, $M \neq T$ if $L = N$) or $q_{L \cup \{k\}, k}$ ($k \in N \setminus L$) and $t$ denotes $-p_V$. The cone $W'$ has a gigantic number of extreme rays. Although extreme rays of $W'$ do not necessarily define facets of $P_{ST}$ or even of conv$(S_{ST})$, we shall see shortly that many do. In the rest of this section, we shall describe some special subclasses of projected generalized subtour elimination constraints.

### 5.1. Steiner partition facets

The Steiner partition facets for the Steiner tree polytope conv$(S_{ST})$ were introduced by Chopra and Rao in [7]. We show that these inequalities are in fact projected generalized subtour elimination constraints and are therefore valid for $P_{ST}$. Consider a *Steiner partition* of $V$, i.e. a partition $\{V_1, \ldots, V_k\}$ of $V$ where $V_i \cap T \neq \emptyset$ for $i = 1, \ldots, k$. Let $\delta(V_1, \ldots, V_k)$ denote the set of edges having endpoints in two distinct members of the partition. The Steiner partition inequality is

$$x(\delta(V_1, \ldots, V_k)) \geqslant k - 1. \tag{27}$$

This inequality corresponds to the ray of $W$ obtained by letting $q_{S,k} = 0$ for all $(S, k)$, $r_i = 0$ for all $i \in N$ and

$$p_S = \begin{cases} -1, S = V, \\ 1, \quad S \in \{V_1, \ldots, V_k\}, \\ 0, \quad \text{otherwise.} \end{cases}$$

Indeed, $(p, q, r) \in W$ and, using (25) and (26), we verify easily that $\alpha_e = 1$ if $e \in \delta(V_1, \ldots, V_k)$ and 0 otherwise, and that $\beta = k - 1$. Chopra and Rao [7] give necessary and sufficient conditions for (27) to define a facet of conv$(S_{ST})$.

### 5.2. Odd hole facets

The odd hole facets were also introduced by Chopra and Rao [7]. These inequalities are parameterized by an odd integer $k$ and are defined on a graph $G_k = (V_k, E_k)$ with terminal set $T_k = \{u_0, \ldots, u_{k-1}\}$ and Steiner vertex set $V_k \setminus T_k = \{v_0, \ldots, v_{k-1}\}$. The edge set $E_k$ consists of the edges of the form $(u_i, v_i)$, $(u_i, v_{i+1})$, $(v_i, v_{i+1})$ for $i = 0, \ldots, k-1$ with the convention that $v_k = v_0$. The odd hole inequality is

$$x(E_k) \geqslant 2(k-1). \tag{28}$$

This inequality corresponds to the ray of $W$ obtained by letting $q_{S,k}=0$ for all $(S, k)$, $r_i=0$ for all $i \in N$ and

$$p_S = \begin{cases} -2, & S=V, \\ 1, & S=\{u_i, v_i, v_{i+1}\} \text{ for } i=0,\dots, k-1, \\ 0, & \text{otherwise.} \end{cases}$$

If we consider a graph $G=(V_k, E)$ obtained from $G_k$ by adding edges then the above ray corresponds to the inequality

$$x(E_k)+2x(E \setminus E_k) \geqslant 2(k-1).$$

This inequality is shown to be a lifted version of the odd hole inequality in Chopra and Rao [7]. It defines a facet of $\mathrm{conv}(S_{ST})$ (Chopra and Rao [7]).

### 5.3. Combinatorial design facets

We now describe a new class of valid inequalities for $P_{ST}$ (and, hence, for $\mathrm{conv}(S_{ST})$) which, among many others, include the lifted odd hole inequalities. This class of *combinatorial design* valid inequalities is obtained by looking at rays $(p, q, r)$ of $W$ satisfying $q_{S,k}=0$ for all $(S, k)$, $r_i=0$ for all $i \in N$ and $p_S=0$ for all $S \neq V$ with $|S \cap T| \geqslant 2$. In other words, the only nonzero components of $(p, q, r)$ are of the form $p_S$ with $|S \cap T|=1$ or $S=V$.

Consider a graph $G=(V, E)$ with $V=T \cup N$ where $T=\{u_1,\dots, u_l\}$ and $N=\{v_1,\dots, v_k\}$. Let $\{N_j: j=1,\dots, l\}$ be subsets of $N$. Define $T_i=\{u_j \in T: v_i \in N_j\}$ for $i=1,\dots, k$. Define also the $k \times l$ matrix $A=[a_{ij}]$ where $a_{ij}=1$ if $u_j \in T_i$ (or $v_i \in N_j$) and 0 otherwise. We impose two conditions on $A$, i.e. on the collection of subsets $\{N_j: j=1,\dots, l\}$ of $N$.

**Assumption 1.** $\mathrm{rank}(A)=l$.

**Assumption 2.** The vector

$$e=\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^k$$

belongs to the cone generated by the columns of $A$, i.e. there exists $x \in \mathbb{R}^l_+$ such that $Ax=e$.

In other words, there exists $d \in \mathbb{Z}_+$ and a vector $\beta \in \mathbb{Z}^l_+$ with $A\beta=de$ and $\gcd(\beta_1,\dots, \beta_l, d)=1$. $A\beta=de$ can also be written as

$$\sum_{u_j \in T_i} \beta_j = d \tag{29}$$

for all $i=1,\dots, k$.

For $e \in E$, define

$$d_e = \begin{cases} \sum_{j:\ e \in E(N_j)} \beta_j, & e \in E(N), \\ \beta_j, & e = (v_i, u_j) \text{ with } v_i \in N_j, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $d_e = 0$ if $e \in E(T)$. For $e = (v_p, v_q) \in E(N)$, $d_e$ can also be expressed as $\sum_{u_j \in (T_p \cap T_q)} \beta_j$.

**Theorem 14.** *The* combinatorial design *inequality*

$$\sum_{e \in E} (d - d_e) x_e \geqslant d(l-1) \tag{30}$$

*is a valid inequality for* $P_{ST}$.

**proof.** Consider the vector $(p, q, r) \in W$ with $q_{S,k} = 0$ for all $(S, k)$, $r_i = 0$ for all $i \in N$, $p_V = -d$, $p_S = \beta_j$ if $S = \{u_j\} \cup N_j$ for $j = 1, \ldots, l$ and $p_S = 0$ otherwise. By definition of $d_e$, it is easily verified that this vector generates (30). □

In the following theorem, we show that if $E$ contains enough edges then (30) defines a facet of $\text{conv}(S_{ST})$ (and hence also of $P_{ST}$).

**Theorem 15.** *If*

(i) $G[T] = (T, E(T))$ *is 2-connected*;

(ii) $\{(v_i, u_j) : v_i \in N_j$ *for some* $j = 1, \ldots, l\} \subseteq \delta(T)$; *and*

(iii) $G_j = G[N_j]$ *is connected for* $j = 1, \ldots, l$;

*then the inequality* (30) *defines a facet of* $\text{conv}(S_{ST})$.

**Proof.** Consider an inequality $ax \geqslant b$ valid for $S_{ST}$ such that

$$\left\{ x \in S_{ST} :\ \sum_{e \in E} (d - d_e) x_e = d(l-1) \right\} \subseteq \{ x \in S_{ST} :\ ax = b \}.$$

We show that this implies that $ax \geqslant b$ is a multiple of (30).

*Claim 1.* There exists $\alpha_T$ such that $a_e = \alpha_T$ for all $e \in E(T)$ and $b = (l-1)\alpha_T$.

*Proof of Claim 1.* Suppose $a_f \neq a_g$ for $f, g \in E(T)$. Consider a cycle $C$ in $G[T]$ that goes through $f$ and $g$. The existence of this cycle follows from condition (i). Augment $C$ to obtain a spanning tree on $G[T]$ plus one additional edge in $C$. By removing $f$ or $g$, we obtain two spanning trees $(T, T_f)$ and $(T, T_g)$ of $G[T]$. Since any spanning tree in $G[T]$ has $l-1$ edges and since $d_e = 0$ for $e \in E(T)$, the incidence vector $x$ of any spanning tree on $G[T]$ satisfies (30) at equality. Hence, it also satisfies $ax = b$. Therefore, $\sum_{e \in T_f} a_e = \sum_{e \in T_g} a_e = b$, implying that $a_f = a_g$. Let $\alpha_T = \alpha_f$. Also, $b = |T_f| \alpha_T = (l-1)\alpha_T$.

*Claim 2.* Let $j \in \{1, \ldots, l\}$. There exists $\alpha_j$ such that $a_e = \alpha_j$ for all $e \in \{(v_i, u_j) : v_i \in N_j\}$.

*Proof of Claim 2.* Let $(v_p, v_q) \in E(N_j)$. Consider a Steiner tree in which $(v_p, v_q)$ is present, the vertices in $T_p$ are linked to $v_p$, the vertices in $T_q \setminus T_p$ are linked to $v_q$ and the vertices in $T \setminus (T_p \cup T_q)$ are connected to vertices in $T_p \cup T_q$ by edges in $E(T)$.

This Steiner tree satisfies (30) at equality since

$$(d-d_{v_pv_q})+\sum_{u_r\in(T_p\cup T_q)}(d-\beta_r)+d|T\setminus(T_p\cup T_q)|$$

$$=d-d_{v_pv_q}+d|T_p\cup T_q|-\sum_{u_r\in T_p}\beta_r-\sum_{u_r\in T_q}\beta_r+\sum_{u_r\in T_p\cap T_q}\beta_r+d|T\setminus(T_p\cup T_q)|$$

$$=d(l+1)-d_{v_pv_q}-d-d+d_{v_pv_q}$$

$$=d(l-1),$$

by (29) and the definition of $d_{v_pv_q}$. Hence, it must also satisfy $ax=b$. This also holds if we replace $(v_p, u_j)$ by $(v_q, u_j)$. Hence, $a_{v_pu_j}=a_{v_qu_j}$.

By choosing $(v_p, v_q)$ from a spanning tree of $G[N_j]$ ($G[N_j]$ is assumed to be connected by condition (iii)), we obtain that $a_e$ is constant for $e\in\{(v_i, u_j): v_i\in N_j\}$.

*Claim 3.* For $j\in\{1,\ldots, l\}$, $d\alpha_j=(d-\beta_j)\alpha_T$.

*Proof of Claim 3.* Fix $i$ in $\{1,\ldots, k\}$. Consider a Steiner tree in which $v_i$ is linked to all vertices in $T_i$ and the vertices in $T\setminus T_i$ are connected to $T_i$ through edges in $E(T)$. This Steiner tree satisfies (30) at equality since

$$\sum_{u_r\in T_i}(d-\beta_r)+d|T\setminus T_i|=d|T_i|-\sum_{u_r\in T_i}\beta_r+d|T\setminus T_i|=d(l-1)$$

by (29). Hence, it also satisfies $ax=b$ implying that

$$\sum_{u_j\in T_i}\alpha_j+(l-|T_i|)\alpha_T=(l-1)\alpha_T \tag{31}$$

where we have used Claims 1 and 2. Rearranging (31), we obtain that, for any $i$ in $\{1,\ldots, k\}$,

$$\sum_{u_j\in T_i}(\alpha_T-\alpha_j)=\alpha_T,$$

or $A\gamma=\alpha_Te$ where $\gamma_j=\alpha_T-\alpha_j$. Since $A$ has full rank, we obtain that

$$\alpha_T-\alpha_j=\beta_j\alpha_T/d$$

i.e. $d\alpha_j=(d-\beta_j)\alpha_T$, proving the claim.

*Claim 4.* For $e\in E(N)$, $da_e=(d-d_e)\alpha_T$.

*Proof of Claim 4.* Let $e=(v_p, v_q)\in N$. Using the same construction as in Claim 2, we obtain a Steiner tree satisfying $ax=b$, i.e.

$$(l-1)\alpha_T = a_e + \sum_{u_j \in T_p \cup T_q} \alpha_j + (l - |T_p \cup T_q|)\alpha_T$$

$$= a_e + \sum_{u_j \in T_p \cup T_q} \frac{d-\beta_j}{d}\alpha_T + (l - |T_p \cup T_q|)\alpha_T$$

$$= a_e + l\alpha_T - \frac{\alpha_T}{d} \sum_{u_j \in T_p \cup T_q} \beta_j$$

$$= a_e + l\alpha_T - (\alpha_T/d)(d+d-d_e)$$

$$= a_e + (l-2+d_e/d)\alpha_T$$

implying that $da_e = (d-d_e)\alpha_T$.

*Claim 5.* Let $e=(v_i, u_j) \in \delta(T)$. If $u_j \notin T_i$ then $a_e = \alpha_T$.

*Proof of Claim 5.* Consider the Steiner tree in which $v_i$ is linked to $T_i \cup \{u_j\}$ and the vertices in $T \setminus (T_i \cup \{u_j\})$ are connected to $T_i \cup \{u_j\}$ by edges in $E(T)$. This Steiner tree satisfies (30) at equality and, hence, satisfies $ax=b$. Comparing it to the Steiner tree constructed in the proof of Claim 3, we see that $a_e = \alpha_T$.

In summary, we have proved that $a_e = (d-d_e)\alpha_T/d$ for all $e \in E$ and $b=(l-1)\alpha_T$. This proves that the inequality (30) defines a full-dimensional face of $\text{conv}(S_{ST})$. □

The conditions in Theorem 15 although sufficient are not necessary. For example, any lifted odd hole inequality is in fact of the form (30) but many of them, although facet-defining, fail the conditions of Theorem 15.

By looking at rays of $W$ with $p_S \neq 0$ for some $S$ with $|S \cap T| \geq 2$, we obtain some even larger classes of valid inequalities. However, we don't have some simple conditions as in Theorem 15 for these inequalities to be facet-defining.

*Block design facets*

We obtain a large subclass of (30) by considering a collection of subsets $\{T_i: i=1,\ldots,k\}$ that constitutes a balanced incomplete block design (BIBD) (see e.g. Anderson [1]). In the notation of design theorists, a BIBD, or a $(v, k, \lambda)$-design, is a collection of subsets (called *blocks*) of cardinality $k$ of a set $S$ of cardinality $v$ such that every pair of elements of $S$ occur in exactly $\lambda$ subsets. In our setting[4], $S$ is $T$, $v$ is $l$, $k$ is going to be $d$, and $\lambda$ is an additional parameter. From now on, we shall consider $(l, d, \lambda)$-designs. In an $(l, d, \lambda)$-design, the number of blocks, represented by $k$ in our case, can be seen to be equal to $\lambda l(l-1)/(d(d-1))$.

If $\{T_i: i=1,\ldots,k\}$ is a $(l, d, \lambda)$-design then clearly Assumption 2 is satisfied since we can choose all $\beta_j$'s to be 1. Moreover, assumption 1 is also satisfied since the incidence matrix $A$ of any BIBD satisfies $\text{rank}(A) \geq \text{rank}(A^TA) = l$ and hence has full rank. This is a basic result in design theory (see e.g. pp. 17–18 in [1]). The valid inequalities of the form

[4] The notation for designs is in fact fairly standard but, unfortunately, does not correspond to what we have been using so far!

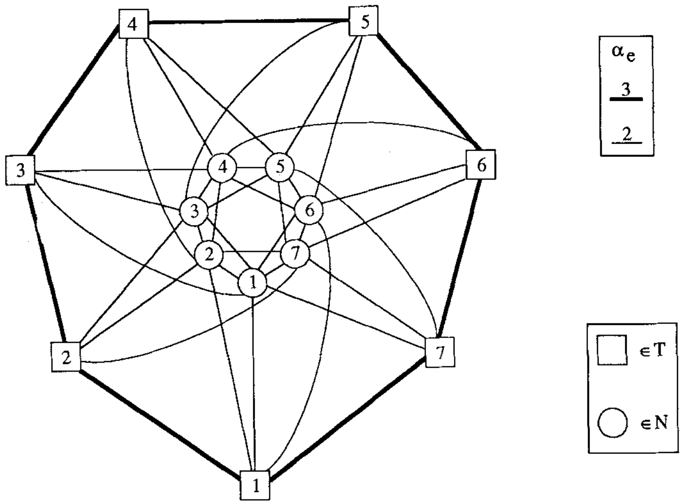


Fig. 8. The inequality $\Sigma\alpha_e x_e \geq 18$ is a facet-defining block design inequality corresponding to a (7, 3, 1)-design.

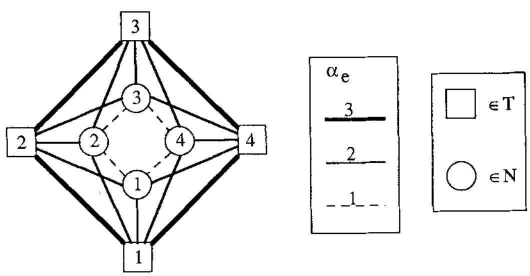


Fig. 9. The inequality $\Sigma\alpha_e x_e \geq 9$ is a facet-defining block design inequality corresponding to a (4, 3, 2)-design.

(30) arising from BIBDs will be called *block design inequalities*. As seen in Theorem 15, for any BIBD, these inequalities are facet-defining if certain conditions on the edge set are met.

The existence of many designs has been established (see [1]). One of the most wellknown class of BIBDs consists of $(l, 3, 1)$-designs, which are called *Steiner triple systems*. These systems exist for all $l \equiv 1$ or 3 (mod 6), $l \geq 3$ (see [1]). In Figure 8, we have represented a facet-defining valid inequality corresponding to the Steiner triple system with $l = 7$. The class of *finite projective plans* is another class of BIBDs. A finite projective plane of order $n$ is a $(n^2+n+1, n+1, 1)$-design $(n \geq 2)$ and their existence has been proved whenever $n$ is the power of some prime number. Many more BIBDs exist. In Figures 9 and 10, we have represented some facet-defining valid inequalities that arise from small BIBDs.

*Facets with many different coefficients*

We show now that the Steiner tree polytope $\text{conv}(S_{ST})$ has some very complicated facet-defining valid inequalities. The block design inequalities that we have just described,

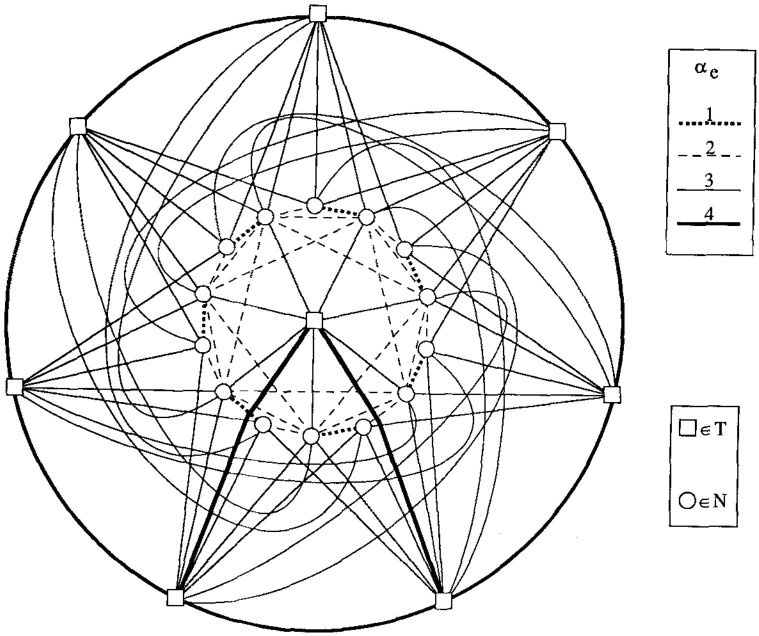


Fig. 10. The inequality $\sum \alpha_e x_e \geqslant 28$ is a facet-defining block design inequality corresponding to a (8, 4, 3)-design.

although fairly complex, have at least a strong combinatorial flavor. We shall describe, for any odd integer $n=2m+1$ $(m \geqslant 3)$, combinatorial design inequalities having coefficients $1, 2, \ldots, n$. A slightly different construction can be done for even $n$. We already know that these inequalities are facet-defining for graphs for which $E$ is suitably chosen (Theorem 15).

Our graphs have $n=2m+1$ Steiner vertices $v_0, \ldots, v_{2m}$ and $n=2m+1$ terminals $u_0, \ldots, u_{2m}$. The collection of subsets $\{T_i\colon i=0, \ldots, 2m\}$ is given by:

$$T_i = \begin{cases} \{0, 2m\}, & i=0, \\ \{i, 2m+1-i\}, & 1 \leqslant i \leqslant m, \\ \{0, 2m-i, i\}, & m+1 \leqslant i \leqslant 2m-1, \\ \{0, 1, m-1, m\}, & i=2m. \end{cases}$$

We first claim that this collection of subsets satisfies Assumption 1, i.e. the incidence matrix $A$ corresponding to this collection is invertible. Indeed, let $E_0, \ldots, E_{2m}$ be the rows by $A$. Subtracting $(2m-2)E_0+3E_1+2\sum_{i=2}^{m-1}E_i+E_m-E_{m+1}-2\sum_{i=m+2}^{2m-1}E_i$ from $E_{2m}$ and subtracting $E_0+E_{2m-1}$ from $E_i$ for $m+1 \leqslant i \leqslant 2m-1$, we obtain an upper-triangular matrix with all diagonal entries being except the last one which is $-(2m+1)$. Moreover, Assumption 2 is also satisfied since we can select $d=n$, $\beta_0=1$ and $\beta_j=j$ for $j=1, \ldots, 2m$.

The combinatorial design inequality associated with $\{T_i\colon i=0, \ldots, 2m\}$ is $\sum_{e \in E}(n-d_e)x_e \geqslant n(n-1)$. If $e=(v_i, u_j)$ with $u_j \in T_i$ and $1 \leqslant j < 2m = n-1$ then $n-d_e=n-j$ which can take all values between 1 and $n-1$. Moreover, for $e \in E(T)$,

$n - d_e = n$. Therefore, for $E$ suitably chosen (e.g. a complete edge set), the coefficients of this facet-defining inequality have all values between 1 and $n$.

*Relationship with bidirected formulation*

So far, we have not discussed any relationship between $P_{ST}$ and other formulations for the Steiner tree problem (except for the fact that the Steiner partition inequalities and odd hole inequalities are valid for $P_{ST}$). For a recent survey of formulations, see Goemans and Myung [15]. Since any Steiner tree can be directed away from a terminal vertex $r$, the undirected Steiner tree problem can be formulated as a bidirected problem. This observation readily gives another extended formulation for the problem (Wong [32]). Let

$$Q_E = \{z: z(\delta^-(S)) \geqslant 1, \quad S: r \notin, S \cap T \neq \emptyset, \tag{32}$$

$$z_e \geqslant 0, \quad e \in E_b\} \tag{33}$$

where $\delta^-(S)$ denotes the set of arcs incoming to $S$ and $E_b$ is the st of bidirected arcs (we disregard the arcs incoming to $r$). Let $Q_{ST} = \{x: x_{ij} = z_{ij} + z_{ji}$ for all $(i, j) \in E$ and some $z \in Q_E\}$. In [15], Goemans and Myung prove that $Q_{ST}$ is the dominant of $P_{ST}$, i.e. $Q_{ST} = \{x'$: $x' \geqslant x$ for some $x \in P_{ST}\}$. (In fact, they also show a stronger result, namely they completely characterize by linear inequalities the set $\{z: x \in P_{ST}$ where $x_{ij} = z_{ij} + z_{ji}$ for all $(i, j) \in E\}$.) Most of the valid inequalities for $P_{ST}$ (or $\text{conv}(S_{ST})$) we have derived in this section have nonnegative coefficients, and are therefore valid for its dominant. In other words, these valid inequalities could have been obtained by projecting $Q_E$ onto the $x$ variables. Although this projection seems in general more complicated, we shall illustrate how obtain the class of combinatorial design inequalities (30) from $Q_E$. Let $r = u_1$. Multiply (32) by

$$\begin{cases} d - \beta_j, & S = \{u_j\}, \quad j = 2, \ldots, l, \\ \beta_j, & S = \{u_j\} \cup N_j, \quad j = 2, \ldots, l, \\ 0, & \text{otherwise}, \end{cases}$$

and (33) by

$$\begin{cases} \beta_1, & e = (u_j, v_i), \quad r \in T_i - u_j, \\ \beta_1, & e = (v_h, v_i), \quad r \in T_i \setminus T_h, \\ 0, & \text{otherwise}, \end{cases}$$

and add up all these inequalities. We claim that the resulting inequality $\sum_{e \in E_b} a_e w_e \geqslant b$ satisfies $a_e = d - d_e$ and $b = d(l-1)$ where, for an arc $e$, $d_e$ represents the value of its undirected counterpart. This immediately establishes the validity of (30) for $Q_{ST}$.

To show that the resulting inequality is of the required form, first notice that $b = \sum_{j=2}^{l} [(d - \beta_j) + \beta_j] = d(l-1)$. To compute $a_e$, we consider several cases.

- If $e = (u_i, u_j) \in E_b(T)$ (hence, $u_j \neq r$) then $a_e = (d - \beta_j) + \beta_j = d = d - d_e$.
- If $e = (v_i, u_j)$ (hence, $u_j \neq r$) and $v_i \in N_j$ then $a_e = d - \beta_j = d - d_e$.
- If $e = (v_i, u_j)$ (hence, $u_j \neq r$) and $v_i \notin N_j$ then $a_e = (d - \beta_j) + \beta_j = d = d - d_e$.
- If $e = (u_j, v_i)$ with $v_i \in N_j$ then, for $r \notin (T_i \setminus \{u_j\})$, we have

$$a_e = \sum_{u_k \in (T_i \setminus \{u_j\})} \beta_k = d - \beta_j = d - d_e$$

and, for $r \in (T_i \setminus \{u_j\})$, we have

$$a_e = \sum_{u_k \in (T_i \setminus \{u_j, r\})} \beta_k + \beta_1 = d - \beta_j = d - d_e.$$

- If $e = (u_j, v_i)$ with $v_i \notin N_j$ then, for $r \notin T_i$, we have

$$a_e = \sum_{u_k \in T_i} \beta_k = d = d - d_e$$

and, for $r \in T_i$, we have

$$a_e = \sum_{u_k \in (T_i \setminus \{r\})} \beta_k + \beta_1 = d = d - d_e.$$

- If $e = (v_h, v_i) \in E_b(N)$ then, for $r \notin (T_i \setminus T_h)$, we have

$$a_e = \sum_{u_k \in (T_i \setminus T_h)} \beta_k = d - \sum_{u_k \in (T_i \cap T_h)} \beta_k = d - d_e$$

and, for $r \in (T_i \setminus T_h)$, we have

$$a_e = \sum_{u_k \in (T_i \setminus (T_h \cup \{r\}))} \beta_k + \beta_1 = d - d_e.$$

## 6. Open problem

We would like to conclude by mentioning that, to this date, a complete characterization of the undirected Steiner tree polytope for series–parallel graphs is still unknown. Chopra and Rao [8] have conjectured that the Steiner partition inequalities and the odd hole inequalities are sufficient to describe the dominant of this polytope. We have given an extended formulation for the undirected Steiner tree problem on series–parallel graphs and have shed some light on the projection. However, in the case of series–parallel graphs, most of the generalized subtour elimination constraints do not define facets. A technique to eliminate such rays of $W$ is therefore needed.

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