CC

**Preprint n. 1032**
June 28, 1994

Dipartimento di Fisica
Università di Roma "La Sapienza"
I.N.F.N. - Sezione di Roma

SW 9434



*Lorenzo Bertini and Nicoletta Cancrini*

# THE STOCHASTIC HEAT EQUATION: FEYNMAN-KAC FORMULA AND INTERMITTENCE

# The stochastic heat equation: Feynman-Kac formula and intermittence

Lorenzo Bertini[1,*] Nicoletta Cancrini[2]

1 *Courant Institute of Mathematical Sciences, New York University*
New York, NY 10012
E-mail: bertinil@acf4.nyu.edu

2 *Dipartimento di Fisica, Università di Roma 'La Sapienza'*
P.le Aldo Moro 2 - 00185 Roma, Italy.
E-mail: cancrini@sci.uniroma1.it

June 27, 1994

**Abstract**

We study, in one space dimension, the heat equation with a random potential that is a white noise in space and time. This equation is a linearized model for the evolution of a scalar field in a space-time dependent random medium. It has also been related to the distribution of two dimensional directed polymers in a random environment, to the KPZ model of growing interfaces and to the Burgers equation with a conservative noise. We show how the solution can be expressed via a generalized Feynman-Kac formula. We then investigate the statistical properties: the two-point correlation function is explicitly computed and the intermittence of the solution is proven. This analysis is carried out showing how the statistical moments can be expressed through local times of independent brownian motions.

**Keywords**: Stochastic partial differential equations, Feynman-Kac formula, Random media, Moment Lyapunov exponents, Intermittence, Local times.

---

* Permanent address: *Dipartimento di Matematica, Università di Roma 'Tor Vergata'*, Via della Ricerca Scientifica - 00133 Roma, Italy. E-mail: bertini@mat.utovrm.it

# 1 Introduction

We consider the linear stochastic partial differential equation (SPDE)

$$\partial_t \psi_t(x) = \frac{\nu}{2} \Delta \psi_t(x) + \psi_t(x) \eta_t(x) \tag{1.1}$$

where $\psi_t = \psi_t(x)$, $t \geq 0$, is a scalar field in $\mathbf{R}^1$, $\Delta$ the Laplacian, $\nu$ a positive constant and $\eta_t = \eta_t(x)$ is a 2-parameter white noise, i.e.

$$\mathbf{E}\left(\eta_t(x)\, \eta_{t'}(x')\right) = \delta(t-t')\delta(x-x') \tag{1.2}$$

Equation (1.1), often called the stochastic heat equation, is a linearized model for the evolution of a scalar field in a space-time dependent random medium [13]. The parameter $\nu$ has then the interpretation of the viscosity coefficient. The choice of the white noise as random potential corresponds to consider those regimes with very rapid variations, the type of turbulent flows.

A traditional way to investigate the evolution of the field $\psi_t$ is to study its moments. This method is important not only because is constructive but also because the moments themselves have a physical meaning, which is often more important than that of the individual solution. Molchanov [13] studying the moments of the solution of equation (1.1) on a lattice, $(t, x) \in \mathbf{R}^+ \times \mathbf{Z}^d$, shows that the field $\psi_t$ has an intermittent behavior. From a qualitative point of view intermittent random fields are characterized by the appearance of sharp peaks which give the main contribution to the statistical moments.

Our analysis extends the general picture in [13] to the equation (1.1) and improve some quantitative results. In the continuum case, the random potential is singular and a rigorous analysis of (1.1) is not completely trivial. In particular white noise gives gives the same weight to all scales, without introducing any characteristic length or time. The physical requirement behind this choice is that the solution of (1.1), which is supposed to describe macroscopic phenomena, should not be too sensitive with respect to fluctuations occurring at arbitrary small scales. Due to the singularity of white noise, our results are however restricted to one space dimension.

In this paper we construct the solution of the Cauchy problem associated to equation (1.1) via a generalized Feynman-Kac formula. The initial data are in a set in which also distributions are enclosed. In particular we are interested to initial functions which are either localized ($\delta$ type initial conditions) or spatially homogenous (constant initial conditions). For the former case the stochastic evolution has the effect of smoothing the singularity: we prove that for any $t > 0$ the process $\psi_t(x)$ is continuous in the space variable regardless of the initial data.

The Feynman-Kac expression allows us a rather complete analysis of the statistical properties of the solution. The 2 points correlation function is explicitly computed. We then study asymptotic (in time) properties of the solution. In particular we focus on translation invariant initial data, i.e. we assume $\psi_0(x) = \text{cost.}$, and evaluate the moments of the process $\psi_t(x)$. This result establishes the solution of equation (1.1), following the definition given by Molchanov [13], has an intermittent behavior. The key point is a representation of the statistical moments in terms of local times for independent brownian motions. This representation permits to carry out the computations and obtain explicit formulae.

Equation (1.1) arises in several other physical problems: it is satisfied by the partition function of a directed polymer in a two dimensional random medium described by the random potential $\eta$ [11]. It is furthermore related to the random growth of interfaces and to the Burgers equation with noise: introducing (Cole-Hopf transformation) $h_t(x) := \nu \log \psi_t(x)$ it satisfies

$$\partial_t h_t(x) = \frac{\nu}{2}\Delta h_t(x) + \frac{1}{2}\left(\partial_x h_t(x)\right)^2 + \nu\eta_t(x) \tag{1.3}$$

which is the so called KPZ equation [10] proposed as a (non linear) random model of growing interfaces. Here $h_t(x)$ is the height of the interface and $\nu$ the surface tension. By differentiating (1.3) and defining $u_t(x) := -\partial_x h_t(x)$ we get Burgers equation with conservative noise

$$\partial_t u_t(x) = \frac{\nu}{2}\Delta u_t(x) - \partial_x\left(u_t(x)^2 + \nu\eta_t(x)\right) \tag{1.4}$$

which has been largely studied in the physical literature as a simplified model in complex phenomena such as turbulence, intermittence, and large scale structure. A satisfactory mathematical theory of equations (1.4) and (1.3) is however still lacking. See Remark 3 after Theorem 2.2 for a further discussion. The relationship of (1.4) to (1.1) is also exploited in [8], where white noise analysis techniques are used. The less singular problem of Burgers equation with non conservative space-time white noise is studied in [2, 6].

To study rigorously the stochastic heat equation, we realize the white noise as the (generalized) derivative of a Wiener process: $\eta_t = \partial_t B_t$. We can thus rewrite equation (1.1) as

$$d\psi_t = \frac{\nu}{2}\Delta\psi_t dt + \psi_t dB_t \tag{1.5}$$

Since it contains a non trivial diffusion, the stochastic differential $\psi_t dB_t$ presents the well known ambiguities. The correct choice is not a trivial point. In [2], for example, a similar equation, where the random potential is the space integral of white noise, has been studied and it is shown how, in order to obtain that the Cole-Hopf transform of $\psi_t$ gives a solution of Burgers equation, the stochastic differential has to be interpreted in the Stratonovich sense. In the present case, as the random potential is more singular, the situation is more complicated. The Feynman-Kac formula for the linear equation (1.5) when the stochastic differential is interpreted in the Stratonovich sense is not well defined. However after a simple renormalization - the Wick exponential - a meaningful expression is obtained. This renormalized Feynman-Kac formula solves equation (1.5) when the stochastic differential is interpreted in the Ito sense. When the Cole-Hopf transformation is performed this implies a Wick renormalization of the non linear term in equations (1.4) and (1.3), see [3, 5].

We note equation (1.5) in any dimension, with a noise regular in the space variable, has been studied in [16], where the stochastic differential is interpreted in the Stratonovich sense and, more recently, in [15] with both interpretations of the stochastic differential. In the latter paper also the white noise case in one space dimension is discussed.

The paper is structured as follows. In the next section we introduce the mathematical apparatus and state precisely our results. In particular we review in some detail what is meant by intermittence and recall the basic definitions and properties of local times.

In section 3 we prove the Feynman-Kac formula; this allows us to establish an existence and uniqueness theorem for the Cauchy problem for the stochastic heat equation. We also prove some smoothness result for the realizations of the process. The representation in term of local times is introduced and used in a technical point.

Section 4 is devoted to the proof of the statistical properties; here the representation in terms of local times plays a more fundamental role: using known results on their distribution, the proofs are reduced to straightforward computations.

# 2 Preliminaries and Results

*Wiener process and stochastic integrals*

Let $B_t$, $t \geq 0$, be the cylindrical Wiener process on $L^2(\mathbf{R}, dx)$. It is realized as a distribution valued continuous process, i.e. the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is given by $\Omega = C(\mathbf{R}^+; \mathcal{S}')$, here $\mathcal{S}'$ is the Schwartz space of distribution on $\mathbf{R}$, $\mathcal{F}$ is the $\sigma$-algebra generated by the cylindrical sets and $\mathcal{P}$ is the gaussian measure with correlation function

$$\mathbf{E}\left(B_t(f)B_{t'}(g)\right) = t \wedge t' \,(f, g) \tag{2.1}$$

where $f, g \in \mathcal{S}$ are test functions, $a \wedge b = \min\{a, b\}$ and $(\cdot, \cdot)$ is the scalar product in $L^2(\mathbf{R}, dx)$. We denote by $\mathcal{F}_t$ the natural filtration of $B_t$, i.e. the minimal $\sigma$-algebra such that $s \mapsto B_s$ is $\mathcal{F}_t$ measurable for all $s \in [0, t]$.

Let $\lambda_t$, $t > 0$ a $L^2(\mathbf{R}, dx)$ valued, $\mathcal{F}_t$ adapted continuous process such that for any $t > 0$

$$\mathbf{E}\int_0^t ds\; (\lambda_s, \lambda_s) < \infty \tag{2.2}$$

we can then define the Ito integral of $\lambda_t$ with respect to the Wiener process as

$$\int_0^t (\lambda_s, dB_s) := \sum_{i=1}^{\infty} \int_0^t (\lambda_s, e_i)\, dB_s(e_i) \tag{2.3}$$

where $\{e_i\}$ is an orthonormal basis in $L^2(\mathbf{R}, dx)$ and thus $\{B_t(e_i)\}$ are independent one-dimensional Wiener processes; the series is convergent, in $L^2(\mathcal{P})$, by (2.2).

We need a regularized version of $B_t$, which is defined as follows. Let $h \in C_0^\infty(\mathbf{R})$ an even positive function such that $\int dx\, h(x) = 1$. Introduce, for $\kappa > 0$, the mollifier $\delta_\kappa(x) := \kappa\, h(\kappa x)$ and define $B_t^\kappa(x) := B_t(\delta_k(x - \cdot))$. Its correlation function is

$$\mathbf{E}\left(B_t^\kappa(x)B_{t'}^{\kappa'}(x')\right) = t \wedge t'\; C_{\kappa,\kappa'}(x - x') \qquad C_{\kappa,\kappa'} := \delta_\kappa \star \delta_{\kappa'} \tag{2.4}$$

where $\star$ denotes convolution in space; if $\kappa = \kappa'$ we use the notation $C_\kappa := C_{\kappa,\kappa}$

For $\kappa$ finite $B^\kappa_t$ is a nice (i.e. $C^\infty(\mathbf{R})$ valued) process, our results will be obtained letting $\kappa \to \infty$ and showing we have meaningful expressions also in the limit.

In the Feynman-Kac formula, as we will see, it appears a stochastic curvilinear integral that we now define for the regularized process $B^\kappa_t$. Let $s \mapsto \varphi_s$ be a Hölder continuous function from $[0,\infty)$ to $\mathbf{R}$ and $s_i = 2^{-n}\, i\, t$ , $i = 0, \cdots, 2^n$ be a partition of $[0,t]$, introduce

$$M^{\kappa,n}_\varphi(t) := \sum_i \left(B^\kappa_{s_{i+1}}(\varphi_{s_i}) - B^\kappa_{s_i}(\varphi_{s_i})\right) \tag{2.5}$$

It is not difficult to verify that $M^{\kappa,n}_\varphi(t)$ is a Cauchy sequence in $L^2(\mathcal{P})$; its limit defines

$$M^\kappa_\varphi(t) := \lim_{n\to\infty} M^{\kappa,n}_\varphi(t) \quad , \quad t \in [0,\infty) \tag{2.6}$$

which is, under $\mathcal{P}$, a continuous gaussian process and a $\mathcal{F}_t$ martingale.

If $s \mapsto \gamma_s$ is another function the cross variation of $M^\kappa_\varphi(t)$ and $M^{\kappa'}_\gamma(t)$ is

$$\langle M^\kappa_\varphi, M^{\kappa'}_\gamma\rangle_t = \int_0^t ds\, C_{\kappa,\kappa'}(\varphi_s - \gamma_s) \tag{2.7}$$

We note that, since $B^\kappa_t(x)$ is Lipschitz in $x$, this construction is a particular case of the general theory developed in [7].

*Remark.* In [2] an analogous stochastic curvilinear integral was defined for the Brownian sheet, in that case it was proven to be meaningful for the non regularized process. This does not hold with this more singular noise: the variance of $M^\kappa_\varphi(t)$ is $tC_\kappa(0)$ which diverges when $\kappa \to \infty$. As we shall see, this is the reason why the Feynman-Kac formula for the linear equation (1.5) with the Stratonovich stochastic integral needs a renormalization.

*Formulation of the Cauchy problem and Feynman-Kac formula*

Let us introduce the heat kernel

$$G_t(x) := \frac{1}{\sqrt{2\nu\pi t}} \exp\left\{-\frac{x^2}{2\nu t}\right\} \tag{2.8}$$

On the initial datum $\psi_0$ we assume it is a positive Borel measure on $\mathbf{R}$ such that, defining

$$G_t \star \psi_0\,(x) := \int d\psi_0(y)\, G_t(x-y) \tag{2.9}$$

it satisfies

$$\sup_{t\in(0,T]} \sup_{x\in\mathbf{R}} \sqrt{t}\, G_t \star \psi_0(x) < \infty \tag{2.10}$$

for any $T > 0$. This allows a singularity of order $t^{-1/2}$ as $t \to 0^+$ and permits a delta type initial condition. However the application $(t,x) \mapsto G_t \star \psi_0(x)$ is smooth, e.g. differentiable, for any $t > 0$, $x \in \mathbf{R}$.

In the study of the statistical properties we will focus on the cases of the Lebesgue and Dirac measures.

We now formulate the Cauchy problem for the stochastic heat equation (1.5), as an Ito equation, in a convenient mild form.

**Definition 2.1** *Let $\psi_t = \psi_t(x)$, $t > 0$ a continuous, $\mathcal{F}_t$ adapted process such that for any $T > 0$*

$$\sup_{t\in(0,T]} \sup_{x\in\mathbf{R}} \int_0^t ds \int_0^s ds' \int dy\, dy'\, G_{t-s}(x-y)^2\, G_{s-s'}(y-y')^2\, \mathbf{E}\left(\psi_{s'}(y')^2\right) < \infty \tag{2.11}$$

*it is a solution of the stochastic heat equation if for any $t > 0$*

$$\psi_t = G_t \star \psi_0 + \int_0^t G_{t-s} \star \psi_s dB_s \qquad \mathcal{P} - a.s. \tag{2.12}$$

*where*

$$\int_0^t G_{t-s} \star \psi_s dB_s\,(x) := \int_0^t \left(G_{t-s}(x-\cdot)\psi_s\,,\, dB_s\right) \tag{2.13}$$

*is the Ito integral defined in (2.3).*

We remark that, even if the initial datum $\psi_0$ is a measure, we have formulated the stochastic heat equation for processes which are, for any $t > 0$, function valued and satisfy (2.11). We will actually prove that the solution is $C^0(R)$ valued as $t > 0$.

The initial datum $\psi_0$ is satisfied in distribution sense. In fact, using (2.11) and (2.12), it can be verified that if $\psi_t$ is a solution of the stochastic heat equation, for any $f \in C^0(\mathbf{R})$ and uniformly bounded

$$\lim_{t\to 0^+} \int dx\, f(x)\psi_t(x) = \int d\psi_0(x)\, f(x) \tag{2.14}$$

where the limit is $\mathcal{P} - a.s.$

We first define precisely the Feynman-Kac formula at the level of the regularized Wiener process $B_t^\kappa$. Let $b_s$, $s \in [0,t]$ the brownian bridge, with diffusion coefficient $\nu$, between $y$ and $x$; i.e. the gaussian process with mean $y + (x-y)\,s\,t^{-1}$ and correlation function $\Gamma(s', s) = \nu\, t^{-1}\, s'\,(t-s)$ for $s' \le s$. In particular $b_0 = y$ , $b_t = x$. We denote by $P^b_{y,x;t}$ the law of $b$; we write $P^{b,\nu}_{y,x;t}$ when we want to indicate explicitly the dependence on $\nu$. We stress $b$ is independent on the cylindrical Wiener process $B$.

Let finally

$$dP^b_{x,t} := \int d\psi_0(y)\, G_t(x-y)\, dP^b_{y,x;t} \tag{2.15}$$

The expectations with respect to $dP^b_{x,y;t}$ and $dP^b_{x,t}$ are denoted by $\mathbf{E}^b_{x,y;t}$ and $\mathbf{E}^b_{x,t}$ respectively. They are not to be confused with the expectation with respect to $\mathcal{P}$, denoted by $\mathbf{E}$.

Let us consider the regularized form of equation (2.12)

$$\psi_t^\kappa = G_t \star \psi_0 + \int_0^t G_{t-s} \star \psi_s^\kappa dB_s^\kappa \tag{2.16}$$

its solution can be expressed, as it is shown in the next section, by the following generalized Feynman-Kac formula

$$\psi_t^\kappa(x) := \mathbf{E}^b_{x,t}\mathcal{E}xp\{M_b^\kappa(t)\} \tag{2.17}$$

where $M_b^\kappa(t)$ is defined pathwise $dP^b_{x,t} - a.s.$ by (2.6) and

$$\mathcal{E}xp\{M_b^\kappa(t)\} := \exp\left\{M_b^\kappa(t) - \frac{1}{2}\langle M_b^\kappa, M_b^\kappa\rangle_t\right\} = \exp\left\{M_b^\kappa(t) - \frac{1}{2}\, t\, C_\kappa(0)\right\} \tag{2.18}$$

in the martingale terminology is the Girsanov exponential of $M_b^\kappa$ or the Wick exponential in the language of quantum field theory. In our contest both of these representations are useful. The diverging term $C_\kappa(0)$ provides the aforementioned renormalization on the Feynman-Kac formula and a meaningful expression is obtained in the limit $\kappa \to \infty$.

**Theorem 2.2** *For any $t > 0$, $x \in \mathbf{R}$, $\psi_t^\kappa(x)$, defined in (2.17), is a Cauchy sequence in $L^2(\mathcal{P})$, denoting by $\psi_t = \psi_t(x)$ its limit we have*

*(i) For all $p \geq 1$, $\psi_t^\kappa(x) \to \psi_t(x)$ in $L^p(\mathcal{P})$ and $\mathcal{P} - a.s.$ The convergence is uniform for $x \in \mathbf{R}$ and for $t$ on compact subsets of $(0, \infty)$.*

*(ii) $\psi_t$ is the unique solution of the stochastic heat equation as formulated in Definition 2.1.*

*(iii) For $(t, x) \in (0, \infty) \times \mathbf{R}$, $(t, x) \mapsto \psi_t(x)$ is $\mathcal{P} - a.s.$ Hölder continuous. The Hölder exponent is $\alpha < 1/2$ in space and $\alpha < 1/4$ in time.*

*(iv) For any $(t, x) \in (0, \infty) \times \mathbf{R}$, $\psi_t(x) > 0$ $\mathcal{P} - a.s.$*

The key point in the proof of the Theorem is to establish that $\psi_t^\kappa(x)$ is a Cauchy sequence. The important statement (*iv*) is essentially contained in Mueller [14] to which we will refer.

*Remark 1.* We have considered, for notational simplicity, deterministic initial data, however our results are easily extended to random initial data.

*Remark 2.* We have considered positive initial data because in the physical problems we have outlined one is mostly interested in positive solutions of (2.12); however, as the equation is linear, the solution with signed initial datum general can be constructed by superposition.

*Remark 3.* By the results in Theorem 2.2, we can construct, as a $C^0(\mathbf{R})$ valued process, $h_t(x) := \nu \log \psi_t(x)$ which describes the interfaces growth in the KPZ model (1.3). Analogously the random field for the Burgers equation with conservative noise can be rigorously defined, as a distribution valued process, by

$$u_t(f) := \nu \int dx\, f'(x) \log \psi_t(x) \tag{2.19}$$

where $f$ is a test function. However, as the non linear terms in the equations (1.3) and (1.4) involve ill defined operations between distributions, those equations have not a rigorous meaning when the cutoff is removed.

*Local times and statistical properties*

We recall the basic definitions and properties of the local times; for a comprehensive discussion see e.g. [17], a book that will be quoted when we need specific results. Given a continuous semimartingale $X$ and $a \in \mathbf{R}$ there exists an increasing process $L_t^a(X)$, called the local time of $X$ in $a$, such that

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)\, dX_s + L_t^a(X) \tag{2.20}$$

where $\operatorname{sgn}(x) = 1$ if $x > 0$ and $\operatorname{sgn}(x) = -1$ if $x \leq 0$.

The process $L_t^a(X)$ can be described informally as $\int_0^t \delta(X_s - a) d\langle X, X\rangle_s$, where $\delta(\cdot)$ is the Dirac's delta function. The following approximation result is instead rigorously proven

$$L_t^a(X) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{(a,a+\varepsilon)}(X_s)\, d\langle X, X\rangle_s \tag{2.21}$$

where $\mathbf{1}_A$ is the characteristic function of the set $A$ and the limit is almost surely.

We will consider only the local times of the brownian bridge and of the brownian motion; in these cases $d\langle X, X\rangle_t = \nu dt$. For notation convenience we define the local times using the measure $dt$ in (2.21), so that our local times are $\nu^{-1}$ the usual ones; thus the local times $L_t^a(X)$ measures (with respect to Lebesgue) the time that $X$ has spent in $a$. Finally we use the notation $L_t(X) := L_t^0(X)$.

The motivation for introducing the local times in our contest lies in the following Proposition which gives a useful representation for the statistical moments of the field $\psi_t$. It is the key point in our analysis on the statistical properties.

**Proposition 2.3** *Let $\vec{b}_s = (b_s^1, \cdots, b_s^n)$, $s \in [0, t]$, $n$ independent brownian bridges between $\vec{y} = (y_1, \cdots, y_n)$ and $x$. Then*

$$\mathbf{E}\left(\psi_t(x)^n\right) = \int \prod_{i=1}^n d\psi_0(y_i)\, G_t(x - y_i) \cdot \mathbf{E}_{\vec{y},x;t}^{\vec{b},\nu}\left(e^{\sum_{i<j} L_t(b^i - b^j)}\right) \tag{2.22}$$

*The 2 point correlation function is given, for $t \leq t'$, by*

$$\mathbf{E}\left(\psi_t(x)\psi_{t'}(x')\right) = \int d\psi_0(y)\, d\psi_0(y')\, dz\; G_t(x - y)\, G_{t'}(x' - y')\, G_{t'-t}(z)\, \mathbf{E}_{y'-y,x'-x+z;t}^{b,2\nu}\left(e^{L_t(b)}\right) \tag{2.23}$$

Let us introduce the notation

$$\Phi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} dy\, e^{-\frac{y^2}{2}} \tag{2.24}$$

for the gaussian distribution.

In the interesting cases in which the initial data is either the Lebesgue or Dirac measure, Proposition 2.3 has the following corollaries.

**Corollary 2.4** *If $\psi_0$ is the Lebesgue measure, we have*

$$\mathbf{E}\left(\psi_t(x)\psi_t(x')\right) = \int_0^t ds\, \frac{|x - x'|}{\sqrt{\pi\nu s^3}}\, e^{-\frac{(x-x')^2}{4\nu s} + \frac{t-s}{4\nu}}\, \Phi\left(\sqrt{\tfrac{t-s}{2\nu}}\right) \tag{2.25}$$

**Corollary 2.5** *If $\psi_0$ is the Dirac measure in* 0, *we have*

$$\begin{aligned}\mathbf{E}\left(\psi_t(x)\psi_t(x')\right) &= \frac{1}{2\pi\nu t}\, e^{-\frac{x^2+(x')^2}{2\nu t}} \int_0^1 ds\, \frac{|x-x'|}{\sqrt{4\pi\nu t}} \frac{1}{\sqrt{s^3(1-s)}} \\ &\quad e^{-\frac{(x-x')^2}{4\nu t}\frac{1-s}{s}} \left(1+\sqrt{\frac{\pi t(1-s)}{\nu}}\, e^{\frac{t}{2\nu}\frac{1-s}{2}}\, \Phi\left(\sqrt{\tfrac{t(1-s)}{2\nu}}\right)\right) \end{aligned} \tag{2.26}$$

*In particular*

$$\mathbf{E}\left(\psi_t(x)^2\right) = \frac{1}{2\pi\nu t}\, e^{-\frac{x^2}{\nu t}} \left(1+\sqrt{\frac{\pi t}{\nu}}\, e^{\frac{t}{4\nu}}\, \Phi\left(\sqrt{\tfrac{t}{2\nu}}\right)\right) \tag{2.27}$$

*Moment Lyapunov exponents and Intermittence*

Before stating our result we briefly review the moment approach to intermittence in space-time dependent random media [1, 4, 13].

Let the process $\varphi_t(x)$ be homogenous and ergodic with respect to translation of the space variable; its moments (one point correlation functions)

$$m_n(t) := \mathbf{E}\left(\varphi_t(x)^n\right) \tag{2.28}$$

does not depend on $x$. The $n$-moment Lyapunov exponent can be defined if the limit

$$\gamma_n := \lim_{t\to\infty} \frac{\log m_n(t)}{t} \tag{2.29}$$

is finite. The process $\varphi_t$ is *intermittent* if the strict inequalities

$$\gamma_1 < \frac{1}{2}\gamma_2 < \cdots < \frac{1}{n}\gamma_n < \cdots \tag{2.30}$$

are satisfied.

To explain the rationale behind this definition, let $\alpha \in (\gamma_1, \gamma_2/2)$ and consider the following random set

$$B_{t,\alpha} := \{x : \varphi_t(x) > e^{\alpha t}\} \tag{2.31}$$

The ergodic theorem ensure that the volume density of this set

$$\rho_{t,\alpha} := \lim_{R\to\infty} \frac{\mathrm{Vol}\left(B_{t,\alpha} \cap \{|x| < R\}\right)}{\mathrm{Vol}\left(\{|x| < R\}\right)} \tag{2.32}$$

exists and it is given by $P\{\varphi_t(x) > \exp\{\alpha t\}\}$. By Chebyshev inequality we then have

$$\rho_{t,\alpha} = P\{\varphi_t(x) > \exp\{\alpha t\}\} \leq e^{-\alpha t}\mathbf{E}\left(\varphi_t(x)\right) \sim e^{-(\alpha-\gamma_1)t} \tag{2.33}$$

The notation $\sim$ denotes logarithmic equivalence, i.e.

$$f(t) \sim g(t) \Longleftrightarrow \lim_{t\to\infty} \frac{\log f(t) - \log g(t)}{t} = 0 \tag{2.34}$$

For large $t$ the density of the set $B_{t,\alpha}$ is thus exponentially small.

The second moment can be written as

$$m_2(t) = \mathbf{E}\left(\varphi_t^2(x)\right) = \mathbf{E}\left(\varphi_t^2(x)\,\mathbf{1}_{B_{t,\alpha}}\right) + \mathbf{E}\left(\varphi_t^2(x)\,\mathbf{1}_{\mathbf{R}\setminus B_{t,\alpha}}\right) \tag{2.35}$$

where $\mathbf{1}_{B_{t,\alpha}}$ is the indicator of the set $B_{t,\alpha}$. The second term in (2.35) does not exceed $\exp\{2\alpha t\}$, furthermore $\exp\{2t\alpha\} \ll \exp\{\gamma_2 t\}$, hence

$$m_2(t) \sim \mathbf{E}\left(\varphi_t^2(x)\,\mathbf{1}_{B_{t,\alpha}}\right) \tag{2.36}$$

Therefore the second moment is generated almost entirely by the sharp fluctuations of the field $\varphi_t(x)$ concentrated in the set $B_{t,\alpha}$, whose density, as we saw above, is exponentially small.

In the same way, choosing a parameter sequence $\{\alpha_n\}$ such that

$$\frac{1}{n}\gamma_n < \alpha_n < \frac{1}{n+1}\gamma_{n+1} \tag{2.37}$$

a hierarchical succession of sets

$$B_{t,\alpha_1} \supset B_{t,\alpha_2} \supset B_{t,\alpha_3} \supset \cdots \tag{2.38}$$

is obtained. Each of these sets is a collection of small islands, the distribution of which is exponentially small. Repeating the same argument for the second moment it is easy to understand how every moment is generated by the values that the process $\varphi_t(x)$ assumes in the corresponding set of the hierarchy. This shows how the strict inequalities (2.30) imply the presence of a peculiar local structure, hence the name *intermittence.*

We now discuss the moment Lyapunov exponents for the stochastic heat equation. We consider deterministic translation invariant initial data, i.e. we assume $\psi_0$ to be the Lebesgue measure. For such initial datum we can state the following theorem.

**Theorem 2.6** *The n-moment of $\psi_t(x)$ is given by*

$$\mathbf{E}\left(\psi_t(x)^n\right) = 2\,\exp\left\{\frac{n(n^2-1)}{4!\nu}\,t\right\}\,\Phi\left(\sqrt{\tfrac{n(n^2-1)}{12\nu}\,t}\right) \tag{2.39}$$

*In particular the n-moment Lyapunov is*

$$\gamma_n = \frac{1}{4!\,\nu}\,n\,(n^2-1) \tag{2.40}$$

*Remark.* In the directed polymer case one is interested in a delta initial condition $\psi_0 = \delta_0$ and in evaluating the moments of $\psi_t := \int dx\,\psi_t(x)$, see [12]. They are still given by formula (2.39).

The cylindrical Wiener process $B_t(x)$ is homogenous and ergodic with respect to translation of the space variable and we are considering homogenous and ergodic initial datum with respect to spatial translations, so the process $\psi_t(x)$, being a functional of $B_t(x)$, is homogenous and ergodic. We can thus conclude, on the base of Theorem 2.6 that, for each (positive) value of the parameter $\nu$, the process $\psi_t(x)$ has an intermittent behavior.

As we remarked in the introduction, the stochastic heat equation has been extensively studied on a lattice [13]. The discrete case has some differences from the continuous one. The renormalization term $C_\kappa(0)$ is finite and no regularization is needed in constructing the solution. All the Lyapunov exponents ($n > 2$) are estimated as functions of $\gamma_2$, of which there is not an explicit expression but only the qualitative behavior as a function of the viscosity $\nu$: it tends to one in the limit $\nu \to 0$ and to zero in the limit $\nu \to \infty$. In the continuum case instead, the moment Lyapunov exponents diverge in the limit $\nu \to 0$.

We note formula (2.40) has been obtained in [9] showing the $n-th$ moment Lyapunov exponent is given by the lowest eigenvalue of a $n$ body Schrödinger operator with a two body delta potential; the result (2.40) is obtained when the self-interactions are ignored. Our approach is instead purely probabilistic: the use of local times permits an exact and rigorous calculus of the statistical moments, from which the Lyapunov exponents are then obtained as leading order. Furthermore the discussion about the stochastic differential and the renormalization on the Feynman-Kac formula gives a clear mathematical meaning to the physical hypotheses of ignoring self-interactions.

In the physical literature [9, 12] the free energy of a directed polymer in a random environment is obtained from (2.40) via the replica method. The exact formula (2.39) shows explicitly the problems connected with the analytic continuation: the argument of $\Phi$ becomes imaginary for $n \in (0,1)$.

# 3 Feynman-Kac formula

This section is devoted to the proof of Theorem 2.2. We first show the Feynman-Kac formula (2.17) is meaningful when $\kappa \to \infty$. We next prove the limiting process is the unique solution of the stochastic heat equation. Finally we establish the Hölder continuity of the trajectories. The representation in term of local times is introduced and used to obtain the necessary bounds on the moments of $\psi_t^\kappa$.

The first step in the proof of Theorem 2.2 consists in verifying that $\psi_t^\kappa(x)$, defined in (2.17), is a Cauchy sequence in $L^2(\mathcal{P})$. Let $dP_{x,t}^{b'}$ be an independent copy of the measure $dP_{x,t}^b$. We have

$$\begin{aligned}
&\mathbf{E}\left(\psi_t^\kappa(x) - \psi_t^{\kappa'}(x)\right)^2 \\
&= \mathbf{E}_{x,t}^b \mathbf{E}_{x,t}^{b'} \mathbf{E}\left[\left(\mathcal{E}xp\{M_b^\kappa(t)\} - \mathcal{E}xp\{M_b^{\kappa'}(t)\}\right)\left(\mathcal{E}xp\{M_{b'}^\kappa(t)\} - \mathcal{E}xp\{M_{b'}^{\kappa'}(t)\}\right)\right] \quad (3.1)
\end{aligned}$$

since $M_b^\kappa(t)$, under $\mathcal{P}$, is a gaussian variable, the expectation can be explicitly computed; recalling (2.7) we get

$$\mathbf{E}^B\left(\psi_t^\kappa(x) - \psi_t^{\kappa'}(x)\right)^2 = \mathbf{E}_{x,t}^b \mathbf{E}_{x,t}^{b'}\left[e^{\int_0^t ds\, C_\kappa(b_s - b'_s)} - 2e^{\int_0^t ds\, C_{\kappa,\kappa'}(b_s - b'_s)} + e^{\int_0^t ds\, C_{\kappa'}(b_s - b'_s)}\right]$$

$$= \int d\psi_0(y)\, d\psi_0(y')\, G_t(x-y)\, G_t(x-y')$$
$$\mathbf{E}^{b,2\nu}_{y-y',0;t}\left[e^{\int_0^t ds\, C_\kappa(b_s)} - 2e^{\int_0^t ds\, C_{\kappa,\kappa'}(b_s)} + e^{\int_0^t ds\, C_{\kappa'}(b_s)}\right] \tag{3.2}$$

as $b_s - b'_s$ is, in law, the brownian bridge from $y - y'$ to 0 with diffusion coefficient $2\nu$.

Let
$$L^\kappa_t(b) := \int_0^t ds\, C_\kappa(b_s) \tag{3.3}$$

The proof that the right hand side of (3.2) converges to 0 as $\kappa, \kappa' \to \infty$ will be completed after the next two Lemmata which provide the necessary bounds on $L^\kappa_t(b)$. They are based on elementary properties of the local times.

**Lemma 3.1** *Let $L_t(b)$ as defined in (2.21). For any $p \in [1, \infty)$, there exists a constant $c > 0$ such that for all $t \in [0, \infty)$*
$$\sup_{z\in\mathbf{R}} \|L^\kappa_t(b) - L_t(b)\|_{L^p(dP^b_{z,0;t})} \le c\, t^{1/4}\, k^{-1/2} \tag{3.4}$$

*Proof.*

By scaling we have, in law, $L_t(b) = t^{1/2} L_1(\tilde{b})$ where $\tilde{b}_s$, $s \in [0,1]$ is the brownian bridge between $z\,t^{-1/2}$ and 0. On the other hand, introducing $\tilde{h} := h \star h$ and recalling the definition of $C_\kappa$, we have
$$L^\kappa_t(b) = \kappa \int_0^t ds\, \tilde{h}(\kappa b_s) \overset{\text{Law}}{=} t^{1/2}\, t^{1/2}\kappa \int_0^1 ds\, \tilde{h}(t^{1/2}\kappa \tilde{b}_s) \tag{3.5}$$

Let us recall the occupation time formula, [17], ch. VI, 1.6, which states that for any positive Borel measurable function $f$, the following identity holds $dP^b_{x,0;t} - a.s.$
$$\int_0^t ds\, f(b_s) = \int da\, L^a_t(b)\, f(a) \tag{3.6}$$

Using the identity (3.5), the occupation time formula (3.6) and recalling the normalization $\int da\, \tilde{h}(a) = 1$ , we have
$$\begin{aligned}
\|L^\kappa_t(b) - L_t(b)\|_{L^p(dP^b_{z,0;t})} &= t^{\frac{1}{2}} \left\| t^{1/2}\kappa \int da\, \tilde{h}(t^{1/2}\kappa a)\, L^a_1(b) \; - \; L_1(b) \right\|_{L^p(dP^b_{zt^{-1/2},0;1})} \\
&\le\; t^{\frac{1}{2}} \int da\, \tilde{h}(a) \left\| L_1^{a(t^{1/2}\kappa)^{-1}}(b) - L_1(b) \right\|_{L^p(dP^b_{zt^{-1/2},0;1})} \\
&\le\; c_1\, t^{\frac{1}{2}} \int da\, \tilde{h}(a)\, (a(t^{1/2}\kappa)^{-1})^{\frac{1}{2}} = c\, t^{1/4}\, \kappa^{-1/2}
\end{aligned} \tag{3.7}$$

where, in the last inequality, we used the $L^p$ Hölder continuity of exponent 1/2 of the local time of the brownian bridge, [17] ch. VI, 1.8. □

**Lemma 3.2** *For any $p > 0$, $T > 0$ there exists a constant $c > 0$ such that, for any $\kappa > 0$*
$$\sup_{z\in\mathbf{R},\, t\in[0,T]} \mathbf{E}^b_{z,0;t}\left(e^{pL^\kappa_t(b)}\right) \le c \tag{3.8}$$

*Proof.*

Retaining the notations introduced in the previous Lemma, by scaling and the occupation time formula (3.6), we have

$$\begin{aligned} \mathbf{E}^b_{z,0;t}\left(e^{pL^\kappa_t(b)}\right) &= \mathbf{E}^b_{zt^{-1/2},0;1}\left(\exp\left\{p\,t^{\frac{1}{2}}\int da\,\tilde{h}(a)\,L_1^{a(t^{1/2}\kappa)^{-1}}(b)\right\}\right) \\ &\leq \mathbf{E}^b_{zt^{-1/2},0;1}\int da\,\tilde{h}(a)\,e^{p\,t^{\frac{1}{2}}L_1^{a(t^{1/2}\kappa)^{-1}}(b)} \end{aligned} \tag{3.9}$$

where we used Jensen inequality.

The bound (3.8) is then proven once we show there exists $c_1 > 0$ such that for each $\lambda \in [0, \Lambda]$

$$\sup_{a,z\in\mathbf{R}} \mathbf{E}^b_{z,0;1}\left(e^{\lambda L_1^a(b)}\right) \leq c_1 \tag{3.10}$$

The intuition about the local times suggests that the supremum in (3.10) is attained for $a = z = 0$, a computation shows that then the expectation is finite. We prove below this fact introducing appropriate stopping times and reducing to the local time in 0 for the brownian bridge from 0 to 0.

Let us introduce, for the brownian bridge $b$ starting from $z$ and arriving in 0 at time 1, the stopping time $T_a := \{\inf t : b_t = a\}$ and denote by $P_{a,z}$ its distribution. Using the strong Markov property and the additivity of the local time we have

$$\mathbf{E}^b_{z,0;1}\left(e^{\lambda L_1^a(b)}\right) = \int_0^1 P_{a,z}(dt)\,\mathbf{E}^b_{a,0;1-t}\left(e^{\lambda L^a_{1-t}(b)}\right) = \int_0^1 P_{a,z}(dt)\,\mathbf{E}^{\tilde{b}}_{0,a;1-t}\left(e^{\lambda L^a_{1-t}(\tilde{b})}\right) \tag{3.11}$$

in the last identity we used the time reversal property of the brownian bridge, i.e. if $b_s$, $s \in [0,\tau]$ is a brownian bridge from $a$ to 0 then $\tilde{b}_s := b_{\tau-s}$, is, in law, a brownian bridge from 0 to $a$.

We now introduce, for the brownian bridge $\tilde{b}$ starting from 0 and arriving in $a$ at time $1-t$, the stopping time $\tilde{T}_a := \{\inf s : \tilde{b}_s = a\}$ and denote by $\tilde{P}_{a,t}$ its distribution.

We can then write (3.11) as

$$\begin{aligned} \mathbf{E}^b_{z,0;1}\left(e^{\lambda L_1^a(b)}\right) &= \int_0^1 P_{a,z}(dt)\int_0^{1-t}\tilde{P}_{a,t}(ds)\,\mathbf{E}^{\tilde{b}}_{a,a;1-t-s}\left(e^{\lambda L^a_{1-t-s}(\tilde{b})}\right) \\ &= \int_0^1 P_{a,z}(dt)\int_0^{1-t}\tilde{P}_{a,t}(ds)\,\mathbf{E}^b_{0,0;1}\left(e^{\lambda\sqrt{1-t-s}L_1(b)}\right) \end{aligned} \tag{3.12}$$

the last identity being obtained by translation and scaling.

The right hand side of (3.12) can now be bounded using the following result, see [17] ch. XII, 3.8. If $b_s$, $s \in [0,1]$ is a brownian bridge (with diffusion coefficient $\nu$) from 0 to 0, then, in law, $L_1(b) = \sqrt{2\gamma}$, where $\gamma$ is an exponential random variable with mean $\nu^{-1}$. □

*Proof of Theorem 2.2.*

We conclude the proof that $\psi_t^\kappa(x)$ in (2.17) is a Cauchy sequence in $L^2(\mathcal{P})$. By Lemmata 3.1 and 3.2, we have

$$\begin{aligned}
&\mathbf{E}^{b,2\nu}_{y-y',0;t}\left|e^{\int_0^t ds\, C_\kappa(b_s)} - e^{\int_0^t ds\, C_{\kappa,\kappa'}(b_s)}\right| \\
\le\; & \left\|e^{\int_0^t ds\, C_\kappa(b_s)} + e^{\int_0^t ds\, C_{\kappa,\kappa'}(b_s)}\right\|_{L^2(dP^{b,2\nu}_{y-y',0;t})} \left\|\int_0^t ds\, C_\kappa(b_s) - \int_0^t ds\, C_{\kappa,\kappa'}(b_s)\right\|_{L^2(dP^{b,2\nu}_{y-y',0;t})} \\
\le\; & c_1\,(\kappa\wedge\kappa')^{-\frac{1}{2}} \qquad (3.13)
\end{aligned}$$

where $c_1$ is independent on $y, y' \in \mathbf{R}$, $t \in [0,T]$.

Recalling (3.2) and the hypotheses (2.10) on $\psi_0$, we have thus proven

$$\mathbf{E}\left(\psi_t^\kappa(x) - \psi_t^{\kappa'}(x)\right)^2 \le c_1\,[G_t \star \psi_0\,(x)]^2\,(\kappa\wedge\kappa')^{-\frac{1}{2}} \le c(\kappa\wedge\kappa')^{-\frac{1}{2}} \qquad (3.14)$$

uniformly for $x \in \mathbf{R}$ and $t$ in compact subsets of $(0,\infty)$.

From the $L^2(\mathcal{P})$ convergence and Lemma 3.2 we get also

$$\|\psi_t(x)\|_{L^2(\mathcal{P})} \le c\,G_t \star \psi_0\,(x) \qquad (3.15)$$

where $c$ is independent on $t \in [0,T]$, $x \in \mathbf{R}$.

From the above estimate we have

$$\begin{aligned}
&\int_0^t ds \int_0^s ds' \int dy\,dy'\, G_{t-s}(x-y)^2\, G_{s-s'}(y-y')^2\, \mathbf{E}\left(\psi_{s'}(y')^2\right) \\
&\quad\le\; c\int_0^t ds \int_0^s ds' \int dy\,dy'\, G_{t-s}(x-y)^2\, G_{s-s'}(y-y')^2\, \frac{1}{\sqrt{s'}}\, G_{s'} \star \psi_0(y') \qquad (3.16)
\end{aligned}$$

where we used assumption (2.10) and the positivity of $\psi_0$.

The bound (2.11) follows then from (3.16) noting that $G_t(x)^2 = (4\pi t)^{-1/2} G_{t/2}(x)$ and using the semigroup property of the heat kernel.

We prove the other statements of the Theorem.

*(i)* Let $n$ an even integer; the $L^n(\mathcal{P})$ norm of $\psi_t^\kappa(x)$ can be computed analogously to (3.1). Let $\vec{b} = (b^1, \cdots b^n)$ $n$ independent brownian bridges between $y_i$ and $x$ in time $t$. Let $L_t^\kappa(b^i - b^j) := \int_0^t ds C_\kappa(b^i_s - b^j_s)$, we have

$$\mathbf{E}\left(\psi_t^\kappa(x)^n\right) = \int \prod_{i=1}^n d\psi_0(y_i)\, G_t(x-y_i)\;\cdot\;\mathbf{E}^{\vec{b}}_{\vec{y},x;t}\left(e^{\sum_{i<j} L_t^\kappa(b^i-b^j)}\right) \le c \qquad (3.17)$$

where, by Lemma 3.2 and condition (2.10), $c$ is independent on $\kappa > 0$, $x \in \mathbf{R}$ and $t$ in compact subsets of $(0,\infty)$.

Let $p > 1$, by the Cauchy-Schwartz inequality

$$\left\|\psi_t^\kappa(x) - \psi_t^{\kappa'}(x)\right\|_{L^p(\mathcal{P})} \le \left\|\psi_t^\kappa(x) - \psi_t^{\kappa'}(x)\right\|^{\frac{1}{p}}_{L^2(\mathcal{P})} \cdot \left\|\psi_t^\kappa(x) - \psi_t^{\kappa'}(x)\right\|^{\frac{p}{p-1}}_{L^{2(p-1)}(\mathcal{P})} \qquad (3.18)$$

which converges to 0 by the $L^2(\mathcal{P})$ convergence and the uniform bound (3.17). From $L^p(\mathcal{P})$ convergence of $\psi_t^\kappa$ and Lemma 3.2 it follows the bound (3.15) also in $L^p(dP^B)$:

$$\|\psi_t(x)\|_{L^p(dP^B)} \leq c\, G_t \star \psi_0\,(x) \tag{3.19}$$

To prove the $\mathcal{P} - a.s.$ convergence, we note that, by the Borel-Cantelli lemma, is enough to show that, for some $p > 1$, $\alpha > 0$, there exists $c > 0$ such that for any $\kappa > 0$, $x \in \mathbf{R}$ and $t$ in compact subsets of $(0, \infty)$

$$\mathbf{E}\,|\psi_t^\kappa(x) - \psi_t(x)|^p \leq c\,\frac{1}{\kappa^{1+\alpha}} \tag{3.20}$$

The bound (3.20) holds for $p = 6$ with $\alpha = 1/2$. This can be proven computing, as in (3.1) and (3.2), the $L^6(\mathcal{P})$ norm of $\psi_t^\kappa(x) - \psi_t(x)$. It can be written as a sum of many (i.e. $2^6$) terms; they can be associated in such a way that each of them contains the product of at least three factor of the form $[\exp\{L_t^\kappa(b^i - b^j)\} - \exp\{L_t(b^i - b^j)\}]$. Proceeding as in (3.13) and using again Lemmata 3.1 and 3.2 the bound (3.20) is then obtained. We omit the tedious algebraic details.

*(ii)* The bound (2.11) has already been proven. To conclude $\psi_t$ is the solution of the stochastic heat equation, we first verify that $\psi_t^\kappa$ in (2.17) is the solution of (2.16).

From definitions (2.17) and (2.15), using the Markov property for the brownian bridge, we have

$$\begin{aligned}
&\int_0^t G_{t-s} \star \psi_s^\kappa dB_s^\kappa\,(x) \\
&= \int_0^t \int d\psi_0(z)\, dy\; G_s(z-y)\, G_{t-s}(y-x)\, \mathbf{E}^b_{z,y;s}\left(\mathcal{E}xp\{M_b^\kappa(s)\}\right)\, dB_s^\kappa(y) \\
&= \int_0^t \int d\psi_0(z)\, G_t(z-x) \mathbf{E}^b_{z,x;t}\left(\mathcal{E}xp\{M_b^\kappa(s)\}\, dB_s^\kappa(b_s)\right)
\end{aligned} \tag{3.21}$$

where we used the expression for the transition probability of the brownian bridge.

By the definition of the stochastic curvilinear integral (2.5), $dB_s^\kappa(b_s)$ equals $dM_b^\kappa(s)$. Recalling that $\mathcal{E}xp\{M_b^\kappa(s)\}$ is the Girsanov exponential (2.18), the stochastic integral in (3.21) is easily computed obtaining

$$\int d\psi_0(z)\, G_t(z-x)\, \mathbf{E}^b_{z,x;t}\left(\mathcal{E}xp\{M_b^\kappa(t)\} - 1\right) = \psi_t^\kappa(x) - G_t \star \psi_0\,(x) \tag{3.22}$$

which proves the claim.

As $\psi_t^\kappa(x)$ converges to $\psi_t(x)$ uniformly for $t$ in compact subsets of $(0, \infty)$, $x \in \mathbf{R}$, $(t, x) \mapsto \psi_t(x)$ is $\mathcal{P} - a.s.$ continuous. Thus for $t > 0$ the process $\psi_t$ is, by construction, $\mathcal{F}_t$ adapted, continuous and $C^0(\mathbf{R})$ valued. Since $\psi_t^\kappa$ satisfies (2.16) and converges to $\psi_t$, once we show

$$\lim_{\kappa\to\infty} \left\| \int_0^t G_{t-s} \star \psi_s dB_s\,(x) - \int_0^t G_{t-s} \star \psi_s^\kappa dB_s^\kappa\,(x) \right\|_{L^2(\mathcal{P})} = 0 \tag{3.23}$$

we can conclude $\psi_t$ satisfies, $\mathcal{P}-a.s.$, equation (2.12).

To prove (3.23) let us consider first

$$\mathbf{E}\left(\int_0^t G_{t-s}\star(\psi_s-\psi_s^\kappa)dB_s\,(x)\right)^2=\int_0^t ds\int dy\,G_{t-s}(x-y)^2\,\mathbf{E}\,(\psi_s(y)-\psi_s^\kappa(y))^2 \quad (3.24)$$

using (3.2), Lemmata 3.1, 3.2 and (3.15), it can be bounded by

$$c_1\,\kappa^{-\frac{1}{2}}\int_0^t ds\int dy\,G_{t-s}(x-y)^2[G_s\star\psi_0\,(y)]^2\le c_2\,\kappa^{-\frac{1}{2}}\int_0^t ds[s(t-s)]^{-\frac{1}{2}}G_t\star\psi_0\,(x)\le c_3\,\kappa^{-\frac{1}{2}} \quad (3.25)$$

where we used the hypothesis (2.10) on $\psi_0$.

On the other hand

$$\begin{aligned}&\mathbf{E}\left(\int_0^t G_{t-s}\star\psi_s^\kappa(dB_s-dB_s^\kappa)\,(x)\right)^2\\ &=\ \mathbf{E}\int_0^t ds\left(G_{t-s}(x-\cdot)\psi_s^\kappa\,,\,(1-\delta_\kappa+C_\kappa-\delta_\kappa)\,G_{t-s}(x-\cdot)\psi_s^\kappa\right) \quad (3.26)\end{aligned}$$

we consider just the term with $(1-\delta_\kappa)$, the other one is analogous. It can be bounded by

$$\begin{aligned}&\int_0^t ds\int dy\,dy'\,\Big[G_{t-s}(x-y)^2\,\delta_k(y-y')\,\|\psi_s^\kappa(y)\|_{L^2(\mathcal{P})}\|\psi_s^\kappa(y)-\psi_s^\kappa(y')\|_{L^2(\mathcal{P})}\\ &\quad+G_{t-s}(x-y)\,\delta_k(y-y')\,\|\psi_s^\kappa(y)\|^2_{L^2(\mathcal{P})}\,|G_{t-s}(x-y)-G_{t-s}(x-y')|\Big] \quad (3.27)\end{aligned}$$

The first line vanishes in the limit $k\to\infty$ by Lemma 3.2 and the $L^2(\mathcal{P})$ convergence (uniform in $x$) of $\psi_t^\kappa(x)$. For the second line we note that, by dominated convergence theorem, we can pass to the limit inside the time integral and conclude it converges to 0. Together with (3.25) this implies (3.23).

We finally prove uniqueness in the class of processes in Definition 2.1. Since the equation is linear it is enough to show that any solution of (2.12) with 0 initial datum is identically 0. For such a solution we have

$$\mathbf{E}\left(\psi_t(x)^2\right)=\int_0^t ds\int dy\,G_{t-s}(x-y)^2\,\mathbf{E}\left(\psi_s(y)^2\right) \quad (3.28)$$

Iterating (3.28) and using the condition (2.11) we get

$$\begin{aligned}\sup_{x\in\mathbf{R}}\mathbf{E}\,(\psi_t(x))^2 &= \sup_{x\in\mathbf{R}}\int_0^t ds_1\cdots\int_0^{s_{n-1}}ds_n\int dy_1\cdots dy_n G_{t-s_1}(x-y_1)^2\\ &\qquad\cdots G_{s_{n-1}-s_n}(y_{n-1}-y_n)^2\mathbf{E}\left(\psi_{s_n}(y_n)^2\right)\\ &\le c\sup_{x\in\mathbf{R}}\int_0^t ds_1\cdots\int_0^{s_{n-3}}ds_{n-2}\int dy_1\cdots dy_{n-2}\,G_{t-s_1}(x-y_1)^2\\ &\qquad\cdots G_{s_{n-3}-s_{n-2}}(y_{n-3}-y_{n-2})^2 \quad (3.29)\end{aligned}$$

which converges to 0 as $n\to\infty$.

*(iii)*

We first show the Hölder continuity of $t \mapsto \psi_t(x)$. Thanks to the bound (3.15), it can be easily verified that for any $t > 0$, $\delta > 0$ exists $c$ such that for any $h > 0$ , $x \in \mathbf{R}$

$$\left\| \int_0^{t+h} G_{t+h-s} \star \psi_s dB_s (x) - \int_0^t G_{t-s} \star \psi_s dB_s (x) \right\|_{L^2(\mathcal{P})} \leq c\, h^{\frac{1}{4}-\delta} \tag{3.30}$$

By an application of the Burkholder-Davis-Gundy inequality, see e.g. [17], IV 4.2, and using (3.19) instead of (3.15), the same bound can be proven also in $L^p(\mathcal{P})$. We omit the details, see however [2] for an analogous estimate. Using the equation (2.12) and the Kolmogorov theorem the $\mathcal{P} - a.s.$ Hölder continuity follows.

The following bound is proven with the same argument as above. For any $p > 1$, $t > 0$, $\delta > 0$ exists $c$ such that for any $h > 0$ , $x \in \mathbf{R}$

$$\left\| \int_0^t G_{t-s} \star \psi_s dB_s (x+h) - \int_0^t G_{t-s} \star \psi_s dB_s (x) \right\|_{L^p(\mathcal{P})} \leq c\, h^{\frac{1}{2}-\delta} \tag{3.31}$$

This implies that $x \mapsto \psi_t(x)$ is $\mathcal{P} - a.s.$ Hölder continuous with exponent $\alpha < 1/2$.

*(iv)* Let first note that a comparison principle holds. Let $\psi_t^i$, $i = 1, 2$, be the solution with initial datum $\psi_0^i$; from the linearity of the equation and the Feynman-Kac formula we have that $\psi_0^1 \leq \psi_0^2$ (as measures) implies

$$\forall\, (t, x) \in (0, \infty) \times \mathbf{R} \qquad \psi_t^1(x) \leq \psi_t^2(x) \qquad \mathcal{P}\ - a.s. \tag{3.32}$$

For initial conditions which are absolutely continuous with respect to Lebesgue and whose density is continuous and with compact support the strict inequality $\psi_t(x) > 0$ is proven in Muller [14] up to an explosion time which is infinite by our results.

Using Mueller's result and the comparison relation (3.32), statement *(iv)* is proven for initial data with continuous density with respect to Lebesgue. General initial data are then reduced to this case thanks to the Markov property and the fact that they became continuous in space in an arbitrary small time. □

# 4 Statistical Properties

In this section we prove the explicit formulae for the correlation functions and the intermittent behavior of the solution we constructed. Proposition 2.3, which allows to express the moments of $\psi_t$ in terms of local times, is a straightforward consequence of the machinery already developed. Corollaries 2.4 and 2.5 follow then from known results on the distribution of local times. Finally Theorem 2.6 exploits elementary properties of the local times: the $n$-th moment is computed reducing it to the evaluation of an exponential moment for the local time of a single brownian motion.

*Proof of Proposition 2.3*

For any integer $n$, by Theorem 2.2, $(i)$

$$\mathbf{E}\,(\psi_t(x)^n) = \lim_{\kappa \to \infty} \mathbf{E}\,(\psi_t^\kappa(x)^n) \tag{4.1}$$

Introducing $n$ independent copies of $b$ and computing the expectation with respect to $\mathcal{P}$ we have

$$\mathbf{E}\left(\psi_t^\kappa(x)^n\right) = \int \prod_{i=1}^n d\psi_0(y_i)\, G_t(x-y_i) \;\cdot\; \mathbf{E}_{\vec{y},x;t}^{\vec{b}}\left(\exp\left\{\sum_{i<j}\int_0^t ds\, C_\kappa(b_s^i - b_s^j)\right\}\right) \tag{4.2}$$

By Lemmata 3.1 and 3.2, the right hand side of (4.2) converges to

$$\int \prod_{i=1}^n d\psi_0(y_i)\, G_t(x-y_i) \;\cdot\; \mathbf{E}_{\vec{y},x;t}^{\vec{b}}\left(e^{\sum_{i<j} L_t(b^i - b^j)}\right) \tag{4.3}$$

which proves (2.22). The proof of (2.23) is analogous. $\square$

*Proof of Corollaries 2.4 and 2.5*

If $\psi_0$ is the Lebesgue measure we can express the correlation functions in terms of local time of brownian motions. Let $\beta_s$, $s \geq 0$ the brownian motion, with diffusion coefficient $\nu$, starting from $x$; denote by $dP_x^\beta$ its law. Realizing the brownian bridge as conditional brownian motion, we have

$$dP_x^{\beta,\nu} = \int dy\, G_t(x-y)\, dP_{y,x;t}^{b,\nu} \tag{4.4}$$

By (2.23) we can thus express the correlation function as

$$\begin{aligned}
\mathbf{E}\Big(\psi_t(x)\,\psi_t(x')\Big) &= \mathbf{E}_{x-x'}^{\beta,2\nu}\left(e^{L_t(\beta)}\right) = \mathbf{E}_0^{\beta,2\nu}\left(e^{L_t^{x-x'}(\beta)}\right) \\
&= \mathbf{E}_0^{\beta,1}\left(\exp\left\{(2\nu)^{-\frac{1}{2}}\, L_t^{(x-x')/\sqrt{2\nu}}(\beta)\right\}\right)
\end{aligned} \tag{4.5}$$

where the last identity is obtained by scaling.

Let $\xi := (x-x')/\sqrt{2\nu}$, introduce the stopping time $T_\xi := \inf\{s : \beta_s = \xi\}$ and denote by $P_\xi(ds)$ its law. By the strong Markov property and the additivity of the local time we have

$$\begin{aligned}
\mathbf{E}_0^{\beta,1}\left(e^{(2\nu)^{-1/2} L_t^\xi(\beta)}\right) &= \int_0^t P_\xi(ds)\, \mathbf{E}_0^\beta\left(e^{(2\nu)^{-1/2} L_{t-s}(\beta)}\right) \\
&= 2\int_0^t ds\, \frac{|\xi|}{\sqrt{2\pi s^3}}\, e^{-\frac{\xi^2}{2s}} \int_0^\infty dy\, G_{t-s}(y)\, e^{y\,(2\nu)^{-1/2}}
\end{aligned} \tag{4.6}$$

where we used the explicit expression for $P_\xi(ds)$, [17], ch. III, 3.7 and for the law of $L_t(\beta)$, [17], ch. VI, 2.2. Equation (2.25) is just a convenient rewriting of (4.6).

Corollary 2.5 is proven following the same steps. When $\psi_0(dx) = \delta_0(dx)$ from (2.23) we have

$$\mathbf{E}\left(\psi_t(x)\psi_t(x')\right) = \frac{1}{2\pi\nu t}\, e^{-\frac{x^2+(x')^2}{2\nu t}}\, \mathbf{E}_{0,(x-x')/\sqrt{2\nu t};1}^{b,1}\left(e^{\sqrt{\frac{t}{2\nu}}\, L_1(b)}\right) \tag{4.7}$$

On the other hand, by the time reversal property of the brownian bridge

$$\begin{aligned}
\mathbf{E}_{0,a;1}^{b,1}\left(e^{\lambda L_1(b)}\right) &= \mathbf{E}_{a,0;1}^{b,1}\left(e^{\lambda L_1^a(b)}\right) = \int_0^1 \widetilde{P}_a(ds)\, \mathbf{E}_{0,0;1-s}^{b,1}\left(e^{\lambda L_1(b)}\right) \\
&= \int_0^1 \widetilde{P}_a(ds)\, \mathbf{E}_{0,0;1}^{b,1}\left(e^{\lambda\sqrt{1-s}L_1(b)}\right)
\end{aligned} \tag{4.8}$$

where $\tilde{P}_a(ds)$ is the law of the stopping time $T_a := \inf\{t : b_t = a\}$ for the brownian bridge from 0 to $a$ in time 1, realizing it as a conditional brownian motion it can be verified that

$$\tilde{P}_a(ds) = \frac{|a|}{\sqrt{2\pi s^3(1-s)}} e^{-\frac{a^2}{2}\frac{1-s}{s}} ds \tag{4.9}$$

The formula (2.26) follows then from (4.7), (4.8), (4.9) and the following result, see [17] ch. XII, 3.8. If $b$ is a brownian bridge (with diffusion coefficient 1) from 0 to 0 in time 1, the local time $L_1(b)$ has the same law of $\sqrt{2\gamma}$, where $\gamma$ is an exponential random variable of mean 1. □

*Proof of Theorem 2.6*

As in the proof of Corollary 2.4 we express the moments of $\psi_t(x)$ in terms of the local times of independent brownian motions. Let $\vec{\beta} := (\beta^1, \cdots, \beta^n)$ $n$ independent brownian motions with diffusion coefficient $\nu$, from Proposition 2.3 we have

$$\mathbf{E}\left(\psi_t(x)^n\right) = \mathbf{E}_x^{\vec{\beta},\nu}\left(e^{\sum_{i<j} L_t(\beta^i-\beta^j)}\right) \tag{4.10}$$

where we used (4.4).

By Tanaka formula (2.20) we have

$$\sum_{i<j}\left|\beta_t^i - \beta_t^j\right| = 2\nu\, w_t + 2\nu \sum_{i<j} L_t(\beta^i - \beta^j) \tag{4.11}$$

where

$$w_t := \frac{1}{2\nu}\sum_{i<j}\int_0^t \operatorname{sgn}(\beta_s^i - \beta_s^j)\, d(\beta_s^i - \beta_s^j) \tag{4.12}$$

It can be rewritten as

$$w_t = \frac{1}{2\nu}\sum_{i=1}^n \int_0^t \left(\sum_{j\neq i}\operatorname{sgn}(\beta_s^i - \beta_s^j)\right) d\beta_s^i \tag{4.13}$$

from which we get

$$\langle w, w\rangle_t = \frac{t}{4\nu}\sum_{i=1}^n (n-1-2(i-1))^2 = \frac{t}{4\nu}\frac{n(n^2-1)}{3} \tag{4.14}$$

by Levy characterization theorem $w$ is thus, in law, a brownian motion starting from 0 and with diffusion coefficient $n(n^2-1)/12\nu$.

Using a deterministic procedure in (4.11), the Skorohod lemma, see [17] ch. VI, 2.1, we have

$$\sum_{i<j} L_t(\beta^i - \beta^j) = \sup_{s\leq t}(-w_s) \tag{4.15}$$

Recalling (4.10) we have proven

$$\mathbf{E}\left(\psi_t(x)^n\right) = \mathbf{E}_x^{\vec{\beta},\nu}\left(\exp\left\{\sup_{s\leq t}(-w_s)\right\}\right) = \mathbf{E}_0^{\beta,1}\left(\exp\left\{\left[\frac{n(n^2-1)}{12\nu}\right]^{\frac{1}{2}}\sup_{s\leq t}(-\beta_s)\right\}\right) \tag{4.16}$$

from which (2.39) follows by the reflection principle: if $\beta$ is a brownian motion starting from 0, then $\sup_{s\leq t}\beta_s$ has the same law of $|\beta_t|$, see e.g. [17], III 3.7. □

## Acknowledgements

We are grateful to G. Jona-Lasinio for previous collaboration and his helpful and deep comments on this problem. We also thank E. Scoppola, for the interesting discussions, and E. Presutti for useful remarks at an early stage of this work.

## References

[1] Baxendale P.H., Rozowskii B.L. : *Kinematic Dynamo and Intermittence in a Turbulent Flow.* CAMS Preprint # 92-2 (1992).

[2] Bertini L., Cancrini N., Jona-Lasinio G. : *The Stochastic Burgers Equation.* To appear in Comm. Math. Phys. .

[3] Bertini L., Cancrini N., Jona-Lasinio G.: *Burgers equation forced by conservative or noncoservative noise.* To be published in the Proceedings of the NATO-ASI School 'Stochastic Analysis and Applications in Physics', Madeira August 1993.

[4] Bogachev L.V. : *The moment approach to intermittence of random media. The model of a mean-field diffusion.* Preprint Nr. 116 Institut für Mathematik Ruhr-Universität-Bochum (1991).

[5] Cancrini N. : *Soluzione del problema di Cauchy per l'equazione di Burgers stocastica in una dimensione spaziale* Ph.D. Dissertation, Dipartimento di Fisica, Università di Roma 'La Sapienza' [in Italian] (1993).

[6] Da Prato G., Debussche A., Temam R. : *Stochastic Burger Equation* Preprint di Matematica n. 27 'Scuola Normale Superiore' Pisa. (1993).

[7] Gihkman I.I., Skorohod A.V. : *Stochastic Differential Equations.* Berlin Heidelberg New York: Springer-Verlag 1972.

[8] Holden H., Lindstrøm T., Øksendal B., Ubøe J., Zhang T.-S. : *The Burgers equation with a noisy force and the stochastic heat equation* Preprint (1993).

[9] Kardar M. : *Replica Bethe Ansatz Studies of Two-Dimensional Interfaces with Quenched Random Impurities.* Nucl. Phys. B **290**, 582-602 (1987).

[10] Kardar M., Parisi G., Zhang Y.-C. : *Dynamical Scaling of Growing Interfaces* Phys. Rev. Lett. **56**, 889-892 (1986).

[11] Kardar M., Zhang Y.-C. : *Scaling of Directed Polymers in Random Media* Phys. Rev. Lett. **58**, 2087-2090 (1987).

[12] Krug J., Spohn H. : *Kinetic Roughening of growing surfaces* In "Solids far from equilibrium: Growth, Morphology and Defects", C. Godrèche ed. Cambridge University Press 1991.

[13] Molchanov S.A. : *Ideas in Theory of Random Media.* Acta Appl. Math. **22**, 139-282 (1991) and references therein.

[14] Mueller C. : *On the Support of Solutions to the Heat Equation with Noise.* Stochastics and Stochastics Reports, **37**, 225-245 (1991).

[15] Noble J.M. : *Evolution Equation with Gaussian Potential.* Preprint (1993).

[16] Pastur L.: *Large time behaviour of moments of solution of parabolic differential equations with random coefficients.* Séminaire 1991/92 Centre de Mathematiques, Ecole Polytechnique, Palaiseau, France.

[17] Revuz D., Yor M. : *Continuous Martingales and Brownian Motion.* Berlin Heidelberg New York: Springer-Verlag 1991.