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THE STRICT DETERMINATENESS OF CERTAIN INFINITE GAME Philip Wolfe

# THE STRICT DETERMINATENESS OF CERTAIN INFINITE GAMES 

Philip Wolfe

1. Introduction. Gale and Stewart [1] have discussed an infinite two-person game in extensive form which is the generalization of a game as defined by Kuhn [3] obtained by deleting the requirement of finiteness of the game tree and regarding as plays all unicursal paths of maximal length originating in the distinguished vertex $x_{0}$. In a winlose game the set $S$ of all plays is divided into two sets $S_{I}$ and $S_{I I}$ such that player $I$ wins the play $s$ if $s \in S_{I}$ and player $I I$ wins it if $s \in S_{I I}$. Gale and Stewart have shown that a two-person infinite win-lose game of perfect information with no chance moves (called a GS game here) is strictly determined if $S_{I}$ belongs to the smallest Boolean algebra containing the open sets of a certain topology for $S$. Here we answer affirmatively the question posed by them: Is a GS game strictly determined if $S_{I}$ is a $G_{\delta}$ (or, equivalently, an $F_{\sigma}$ ) ? The notation and results of [1] are used throughout, as well as the partial ordering of $X$ given by : $x>y$ if $f^{n}(x)=y$ for some $n \geqslant 1$.
2. Alternative description of $S_{I}$. Let $\Gamma$ be the game ( $x_{0}, X_{I}, X_{I I}$ $X, f, S, S_{l}, S_{I I}$ ), where

$$
S_{I}=\bigcap_{n=1}^{\infty} E_{n},
$$

$E_{1} \supseteq E_{2} \supseteq \cdots$, and $E_{n}$ is open. Following [3], let the rank $r k(x)$, for $x \in X$, be the unique $k$ such that $f^{k}(x)=x_{0}$. As in [1], $\mathfrak{l}(x)$ is the set of all plays passing through $x$ (the topology for $S$ is that in which $\mathfrak{U}(x)$ is a neighborhood of each play in it). Then for each $n$,

$$
E_{n}=\bigcup\left\{\mathfrak{U}(y): \mathfrak{l}(y) \subseteq E_{n}\right\} ;
$$

and since for any $y \in X$ we have

$$
\mathfrak{U}(y)=\bigcup\{\mathfrak{U}(z): f(z)=y\},
$$

with

$$
r k(z)=1+r k(y),
$$

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there exists for each $n$ a subset $Y_{n}$ of $X$ such that $r k(y)>n$ for all $y \in Y_{n}$ and

$$
E_{n}=\bigcup\left\{\mathfrak{U}(y): y \in Y_{n}\right\} .
$$

Furthermore, since of any two neighborhoods having a non-void intersection, one is contained in the other, each $Y_{n}$ may be chosen so that $\mathfrak{u}(y), \mathfrak{u}\left(y^{\prime}\right)$ are disjoint for different $y, y^{\prime}$ in $Y_{n}$.

Since $s \in S_{I}$ if and only if $s \in E_{n}$ for an infinite number of values of $n$, we have: $s \in S_{I}$ if and only if for infinitely many $n$ there exists $i$ (dependent on $n$ ) such that $s(i) \in Y_{n}$. Thus, since on the one hand $i=r k(s(i))>n$, and on the other for any $n$ there is at most one $i$ such that $s(i) \in Y_{n}$, letting

$$
Y=\bigcup_{n=1}^{\infty} Y_{n}
$$

we have: $s \in S_{I}$ if and only if $s(i) \in Y$ for infinitely many $i$.

## 3. Lemmas.

Lemma 1. If $\Gamma$ is a GS game with

$$
\sum_{I I}^{W}(\Gamma)=\Lambda
$$

and

$$
T=S-\bigcup\left\{\mathfrak{U}(x): \sum_{I I}^{W}\left(\Gamma_{x}\right) \neq A\right\},
$$

then

$$
I_{T}=\left(x_{0}, X_{I}^{T}, X_{I I}^{T}, X^{T}, f^{T}, T, S_{I}^{T}, S_{I I}^{T}\right)
$$

is a subgame of $\Gamma$,

$$
\sum_{1}^{W}\left(\Gamma_{T}\right) \neq \Lambda
$$

implies
and

$$
\begin{gathered}
\sum_{I}^{W}\left(I^{\prime}\right)=\Lambda \\
\sum_{I I}^{W}\left(\left(I_{T}\right)_{x}\right)=\Lambda
\end{gathered}
$$

for all $x \in X^{T}$.
Proof. Since $T$ is a closed nonempty subset of $S, \Gamma_{T}$ is a subgame of $\Gamma$ by Theorem 5 of [1]. The second statement follows from assertion B [1, p. 260]. Finally suppose that

$$
\sum_{I I}^{W}\left(\left(\Gamma_{T}\right)_{x}\right) \neq \Lambda
$$

for some $x \in X^{T}$. Letting, in assertion A [1, p. 260],

$$
F=\mathfrak{u}(x) \cap T,
$$

and noting that $F$ is closed and nonempty and that

$$
\left(I_{T}^{\prime}\right)_{r}=\left(I_{x}\right)_{F},
$$

we have

$$
\sum_{I I}^{W}\left(I_{x}^{\prime}\right) \rightleftharpoons 1,
$$

which is impossible in view of the construction of $T$.
We assume hereafter that $\Gamma$ is a GS game with $S_{I}$ described in terms of $Y \subseteq X$ as in $\S 2$, and that

$$
\sum_{I I}^{W}\left(\Gamma^{\prime}\right)=A,
$$

whence

$$
\sum_{I I}^{W}\left(I_{T}\right)=A
$$

by Lemma 1. The strict determinateness of $\Gamma$ will follow from Lemma 1 and the fact that

$$
\sum_{I}^{W}\left(\Gamma_{T}\right) \neq A,
$$

proved in $\S 4$.
Lemma 2. For $x \in X^{r}$, we have

$$
s \in S_{I}^{T x}
$$

if and only if

$$
s \in S^{T \cdot x} \quad \text { and } \quad s(i) \in Y
$$

for infinitely many $i$.
Lemma 3. For $x \in X^{T}$ there exists

$$
\sigma_{x} \in \sum_{l}\left(\left(\Gamma_{T}\right)_{x}\right)
$$

such that for any

$$
\tau \in \sum_{11}\left(\left(I_{T}\right)_{r}\right)
$$

we have

$$
\left\langle\sigma_{x}, \tau\right\rangle(i) \in Y
$$

for some $i>r k(x)$.
Proof. Let $Y_{x}$ be the set of all

$$
y \in Y \cap X^{p}
$$

such that $y>x$ and no members of $Y$ fall between $x$ and $y$. Let $\Gamma^{\prime}$ be the game

$$
\left(x_{0}, X_{I}^{T x}, X_{I I}^{T x}, X^{T x}, f^{T x}, S^{T x}, S_{I}^{\prime}, S_{I I}^{\prime}\right),
$$

where

$$
S_{1}^{\prime}=S^{T r} \cap \bigcup\left\{\mathfrak{l l}(y): y \in Y_{x}\right\}
$$

and

$$
S_{I I}^{\prime}=S^{T x}-S_{I}^{\prime}
$$

(that is, the game in which $I$ wins if the play passes through any member of $Y$ following $x$ ). Noting that

$$
S_{I}^{T x} \subseteq S_{1}^{\prime},
$$

we have

$$
S_{I I}^{\prime} \subseteq S_{I I}^{7 n}
$$

and hence

$$
\sum_{I I}^{W}\left(I^{\prime}\right)=A .
$$

But $S_{I}^{\prime}$ is open in $S^{T x}$ and so $I^{\prime \prime}$ is strictly determined by Corollary 10 of [1], whence there exists

$$
\sigma_{x} \in \sum_{I}^{W}\left(\Gamma^{\prime}\right),
$$

which satisfies the conclusion of the lemma.
4. Winning $I_{r}^{\prime}$. Let

$$
Y^{\prime}=\left(Y \cap X^{T}\right) \cup\left\{x_{0}\right\} .
$$

For each $x \in Y^{\prime}$ let $\sigma_{x}$ be as given by Lemma 3, and let $\sigma_{x}^{\prime}$ be the restriction of $\sigma_{x}$ to the set of all $z$ in $X^{T}$ such that $x \leqslant z$ and that there exists no $y$ in $Y^{\prime}$ with $x<y \leqslant z$. We show that the domains of the $\sigma_{x}^{\prime}$ cover $X^{r}$ and are disjoint: First, if $x_{0} \in X_{I}^{T}$, then $x_{0}$ belongs to the domain of $\sigma_{x_{0}}$. For

$$
z \in X_{I}^{T}-\left\{x_{0}\right\},
$$

let

$$
x=\max \left\{z^{\prime}: z^{\prime} \in Y^{\prime} \& z^{\prime}<z\right\} .
$$

Then $x \in Y^{\prime}$ and $z$ belongs to the domain of $\sigma_{x}^{\prime}$; thus the domains of the $\sigma_{x}^{\prime}$ cover $X_{I}^{T}$. Now suppose that $x_{1}, x_{2} \in Y^{\prime}, x_{1} \neq x_{2}$, and that there exists $x_{3}$ common to the domains of $\sigma_{x_{1}}^{\prime}$ and $\sigma_{x_{2}}^{\prime}$; then $x_{1} \leqslant x_{3}$ and $x_{2} \leqslant x_{3}$, so that either $x_{1}<x_{2} \leqslant x_{3}$ or $x_{2}<x_{1} \leqslant x_{3}$, which is impossible in view of the restriction imposed upon $\sigma_{x}$ in obtaining $\sigma_{x}^{\prime}$.

Since the domains of the $\sigma_{x}^{\prime}$ cover $X_{1}^{r}$ and are disjoint, they have
a common extension $\sigma^{*}$, which necessarily maps the elements of $X_{I}^{T}$ on their immediate successors, and thus belongs to $\sum_{1}\left(I_{T}\right)$.

We show that $\sigma^{*}$ wins $\Gamma_{T}$. Let

$$
\tau \in \sum_{I I}\left(\Gamma_{r}\right)
$$

For this $\tau$ and any $x$ in $Y^{\prime}$, let $i(x)$ be the least $i$ such that $\left\langle\sigma_{x}, \tau\right\rangle(i) \in Y^{\prime}$, whose existence is given by Lemma 3. Define $\left\{x_{n}\right\}$ inductively by

$$
x_{n+1}=\left\langle\sigma^{*}, \tau\right\rangle\left(i\left(x_{n}\right)\right) \quad n=0,1, \cdots
$$

( $x_{0}$ is the distinguished vertex). Since

$$
r k\left(x_{n+1}\right)=i\left(x_{n}\right)>r k\left(x_{n}\right),
$$

and $x_{n}, x_{n+1}$ are on a common path, we have $x_{n+1}>x_{n}$ for all $n$, and so if $x_{n} \in Y^{\prime}$ then

$$
x_{n+1}=\left\langle\sigma^{*}, \tau\right\rangle\left(i\left(x_{n}\right)\right)=\left\langle\sigma_{x_{n}}, \tau_{x_{n}}\right\rangle\left(i\left(x_{n}\right)\right) \in Y^{\prime},
$$

where

$$
\tau_{x_{n}} \in \sum_{I I}\left(\left(\Gamma_{r}\right)_{x_{n}}\right)
$$

is the restriction of $\tau$ to $X_{I I}^{\tau_{1} x_{n}}$. Thus by induction $x_{n} \in Y^{\prime}$ for all $n$, and hence

$$
\left\langle\sigma^{*}, \tau\right\rangle(i) \in Y
$$

for infinitely many values of $i$, so that

$$
\left\langle\sigma^{*}, \tau\right\rangle \in S_{j}^{T^{\prime}}
$$

Since $\tau$ is arbitrary,

$$
\sigma^{*} \in \sum_{I}^{W}\left(\Gamma_{T}\right),
$$

so that by Lemma 1, we have

$$
\sum_{I}^{W}(\Gamma) \neq \Lambda .
$$

As this is the consequence of the sole fact that

$$
\sum_{I I}^{W}(\Gamma)=\Lambda
$$

$\Gamma$ is strictly determined.
Reversing the roles of the players in the above gives the result that a GS game is strictly determined if $S_{I}$ is an $F_{\sigma}$.

The strict determinateness of a two-person zero-sum game with G payoff having chance moves can be shown. The proof is more complicated, but uses the same ideas [4].
5. An application. Let

$$
I^{\prime}=\left(x_{0}, X_{1}, X_{I I}, X, f, S, \Phi\right)
$$

be a zero-sum two-person infinite game of perfect information with no chance moves having payoff $\phi$ such that there exists a real function $h$ on $X(|h(x)|<K<\infty)$ with

$$
\Phi(s)=\lim _{i \rightarrow \infty} \sup h(s(i)) \text { for all } s \in S .
$$

$\Gamma$ is the result of an attempt to reduce the following situation to a game: The tree $K$ of a GS game and a function $h$ as above are given; the two players make choices in $K$ in the belief that every play will terminate in some unknown, but distant, vertex $x$, at which time player $I$ will receive the amount $h(x)$ from player $I I$. A payoff function $\Phi$ is sought such that $\Phi(s)(-\Phi(s))$ expresses the utility to player $I(I I)$ of a play $s$ in $K$.

The payoff $\Phi$ defined above arises from ascription to players $I$ and $I I$ respectively of "optimistic" and " pessimistic" behaviors in this way : Player $I$ assumes that the play $s$ will terminate in some " distant" vertex $s(i)$ at which $h$ assumes nearly its supremum on all "distant" vertices of $s$; he thus makes his choices so as to maximize the expression

$$
\lim _{i \rightarrow \infty} \sup h(s(i))=\Phi(s) ;
$$

and player $I I$ supposes that $s$ will terminate in some "distant" vertex at which his gain $-h(s(i))$ assumes nearly its infimum for all such vertices, and thus seeks to maximize

$$
\liminf _{i \rightarrow \infty}-h(s(i))=-\Phi(s),
$$

that is, to minimize $\Phi$. The derived game is thus zero-sum. Ascription, however, of such "optimistic" or "pessimistic" payoffs to both players yields, in general, a non-zero sum game.

We show now that the game $\Gamma$ of this section is strictly determined, using the method of Theorem 15 of [1] which asserts the strict determinateness of $I^{\prime}$ for the more special case of continuous $\Phi$. (Gillette [2] has shown the strict determinateness of an infinite game of perfect information with chance moves which consists in repeated play from a finite set of finite games and has payoff

$$
\limsup _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} g_{n}(s),
$$

where $g_{n}(s)$ is the gain from the $n$th game played.)
First, as a converse to the equivalence of $\S 2$, let $Y \subseteq X$, and denote by $Y_{n}$ the set of all members of $Y$ having rank greater than $n$. Then

$$
\begin{aligned}
\{s: s(i) \in Y \text { for infinitely many } i\} & =\bigcap_{n}\left\{s: s(i) \in Y_{n} \text { for some } i\right\} \\
& =\bigcap_{n} \cup\left\{\mathfrak{l}(y) ; y \in Y_{n}\right\},
\end{aligned}
$$

which is a $G_{\delta}$.
Now in $I$, for $t$ real, let

$$
S_{I}^{t}=\{s: h(s(i))>t \text { for infinitely many } i\}
$$

and $S_{I I}^{t}=S-S_{I}^{t}$. Then $S_{I}^{t}$ is a $G_{\delta}$, and thus the GS game

$$
\Gamma_{t}=\left(x_{0}, X_{1}, X_{I I}, X, f, S, S_{I}^{t}, S_{I I}^{t}\right)
$$

is strictly determined. Let

$$
v=\sup \left\{t: \sum_{t}^{W}\left(\Gamma_{t}\right) \neq \Lambda\right\} .
$$

Since $S_{I}^{K}=\Lambda, S_{l}^{-K}=S$, and $S_{l}^{t}$ is a decreasing function of $t$, we have

$$
-K \leqslant v \leqslant K, \quad \sum_{I}^{W}\left(\Gamma_{t}\right) \neq 1 \quad \text { if } \quad t<v,
$$

and

$$
\sum_{I I}^{W}\left(\Gamma_{t}\right) \neq 1 \text { if } t>v .
$$

Given $\varepsilon>0$, choose

$$
\sigma_{v} \in \sum_{I}^{W}\left(\Gamma_{v-\varepsilon}\right) \quad \text { and } \quad \tau_{v} \in \sum_{I I}^{W}\left(\Gamma_{v+\varepsilon}\right) .
$$

Then for any

$$
\sigma \in \sum_{l}(\Gamma), \quad \tau \in \sum_{I I}(\Gamma),
$$

we have

$$
h\left(\left\langle\sigma_{0}, \tau\right\rangle(i)\right)>v-\varepsilon \text { for infinitely many } i
$$

and do not have

$$
h\left(\left\langle\sigma, \tau_{0}\right\rangle(i)\right)>v+\varepsilon \text { for infinitely many } i ;
$$

so that

$$
\Phi\left(\left\langle\sigma_{0}, \tau\right\rangle\right) \geqslant v-\varepsilon \quad \text { and } \quad \Phi\left(\left\langle\sigma, \tau_{0}\right\rangle\right)<v+2 \varepsilon .
$$

Hence

$$
v-\varepsilon \leqslant \sup _{\sigma} \inf _{\tau} \Phi(\langle\sigma, \tau\rangle) \leqslant \inf _{\tau} \sup _{\sigma} \Phi(\langle\sigma, \tau\rangle) \leqslant v+2 \varepsilon ;
$$

thus $\Gamma^{\prime}$ is strictly determined, and has value $v$.

## References

1. David Gale and F. M. Stewart, Infinite Games with Perfect Information. Ann. of Math. Studies 28 (Contributions to the Theory of Games II), 245-266. Princeton, 1953.
2. Dean Gillette, Representable Infinite Games. Thesis, University of California, Berkeley, June 1953.
3. H. W. Kuhn, Extensive Games and the Problem of Information. Ann. of Math. Studies 28, 193-216.
4. Philip Wolfe, Games of Infinite Length. Thesis, University of California (1954).

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