The String Edit Distance Matching Problem with Moves

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The edit distance between two strings S and R is defined to be the minimum number of character inserts, deletes and changes needed to convert R to S. Given a text string t of length n, and a pattern string p of length m, informally, the string edit distance matching problem is to compute the smallest edit distance between p and substrings of t.

We relax the problem so that (a) we allow an additional operation, namely, *substring moves*, and (b) we allow approximation of this string edit distance. Our result is a near linear time deterministic algorithm to produce a factor of $O(\log n \log^* n)$ approximation to the string edit distance with moves. This is the first known significantly subquadratic algorithm for a string edit distance problem in which the distance involves nontrivial alignments. Our results are obtained by embedding strings into L_1 vector space using a simplified parsing technique we call *Edit Sensitive Parsing* (ESP).

Categories and Subject Descriptors: F.2.0 [**Analysis of Algorithms and Problem Complexity**]: General

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1. INTRODUCTION

String matching has a long history in computer science, dating back to the first compilers in the sixties and before. Text comparison now appears in all areas of the discipline, from compression and pattern matching to computational biology

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and web searching. The basic notion of string similarity used in such comparisons is that of edit distance between pairs of strings. If R and S are strings then the *edit distance* between R and S, $e(R, S)$, is defined as the minimum number of character insertions, deletions or changes necessary to turn R into S. This distance measure is motivated from a number of scenarios: insertions and deletions on lossy communication channels [Levenshtein 1966]; typing errors in documents [Crochemore and Rytter 1994]; and evolutionary changes in biological sequences [Sankoff and Kruskal 1983].

In the string edit distance matching problem studied in Combinatorial Pattern Matching, we are given a text string t of length n and a pattern string p of length $m < n$. The *string edit distance matching problem* is to compute the minimum string edit distance between p and any prefix of $t[i \dots n]$ for each i; we denote this distance by $D[i]$. It is well known that this problem can be solved in $O(mn)$ time using dynamic programming [Landau and Vishkin 1989; Gusfield 1997]. The open problem is whether this quadratic bound can be improved substantially in the worst case.

There has been some progress on this open problem. Masek and Paterson [Masek and Paterson 1980] show how to compute $D[1]$, the distance between the pattern and the text, using the Four Russians method to improve the bound to $O(mn/\log m)$, which remains the best known bound in general to this date. Progress since then has been obtained by relaxing the problem in a number of ways.

—Restrict $D[i]$'s of interest.
Specifically, the restricted goal is to only determine i's for which $D[i] < k$ for a given parameter k. By again adapting the dynamic programming approach a solution can be found in $O(kn)$ time and space in this case [Landau and Vishkin 1986; Myers 1986]. An improvement was presented by Sahinalp and Vishkin [Şahinalp and Vishkin 1996] (since improved by Cole and Hariharan [Cole and Hariharan 1998]) with an $O(n \operatorname{poly}(k/m))$ time algorithm which is significantly better. These algorithms still have running time of $\Omega(nm)$ in the worst case.

—Consider simpler string distances.
If we restrict the string distances to exclude insertions and deletions, we obtain the Hamming distance measure. In a simple yet significant development, Abrahamson [Abrahamson 1987] gave a $\tilde{O}(n\sqrt{m})$ time solution breaking the quadratic bound[1]; since then, it has been improved to $\tilde{O}(n\sqrt{k})$ [Amir et al. 2000]. Karloff improved this to $\tilde{O}(n)$ by approximating the Hamming distances to $1 + \epsilon$ factor [Karloff 1993]. Hamming distance results however sidestep the fundamental difficulty in string edit distance problem, namely, the need to consider nontrivial *alignment* of the pattern against text when characters are inserted or deleted.

In this paper, we present a near linear time algorithm for the string edit distance matching problem. However, there is a caveat: our result relies strongly on relaxing the problem as described below. Since our result is the first to consider nontrivial alignment between the text and the pattern and still obtain significantly subquadratic algorithm in the worst case, we believe it is of interest. Specifically,

[1]The notation $\tilde{O}$ hides polylog factors.

we relax the string edit distance matching problem in two ways:
(1) We allow approximating $D[i]$'s.
(2) We allow an extra operation, *substring move*, which moves a substring from one position in a string to another.

This modified edit distance between strings S and R is called string edit distance with moves and is denoted $d(R, S)$. There are many applications where substring moves are taken as a primitive: in certain computation biology settings a large subsequence being moved is just as likely as an insertion or deletion; in a text processing environment, moving a large block intact may be considered a similar level rearrangement to inserting or deleting characters. Note that string edit distance with moves still faces the challenge of dealing with nontrivial alignments. Formally, then, $d(R, S)$ is length of the shortest sequence of edit operations to transform R into S, where the permitted operations affect a string are defined as follows:

—A character *deletion* at position i transforms S into $S[1] \dots S[i-1], S[i+1] \dots S[n]$.
—An *insertion* of character a at position i results in $S[1] \dots S[i-1], a, S[i] \dots S[n]$.
—A *replacement* at position i with character a gives $S[1] \dots S[i-1], a, S[i+1] \dots S[n]$.
—A *substring move* with parameters $1 \leq i \leq j \leq k \leq n$ transforms $S[1] \dots S[n]$ into $S[1] \dots S[i-1], S[j] \dots S[k-1], S[i] \dots S[j-1], S[k] \dots S[n]$.

This measure of distince is a metric: the distance between two strings is zero if and only if they are identical; every operation has an equal cost inverse, so $d(R, S) = d(S, R)$; and any distance defined by the minimum number of unit cost editing operations to transform one object into another must obey the triangle inequality. Note that there is no restriction on the interaction of edit operations so, for example, it is quite possible for a substring move to take a substring to a new location and then for a subsequent move to operate on a substring which overlaps the moved substring and its surrounding characters.

Our main result is a deterministic algorithm for the string edit distance matching problem *with moves* which runs in time $O(n \log n)$. The output is the matching array D where each $D[i]$ is approximated to within a $O(\log n \log^* n)$ factor. Our approach relies on an embedding of strings into a space of vectors under the L_1 metric. The L_1 distance between two such vectors is an $O(\log n \log^* n)$ approximation of the string edit distance with moves between the two original strings. This is a general approach, and can be used to solve many other questions of interest beyond the core string edit distance matching problem. These include string similarity search problems such as indexing for string nearest neighbors, outliers, clustering and so on under the metric of edit distance with moves. Recently, this distance was studied by Shapira and Storer [Shapira and Storer 2002], where finding the distance between two strings was shown to be an NP-Complete problem. The authors gave an $O(\log n)$ approximation algorithm for finding the distance between pairs of strings. However, the approximation does not solve the string edit distance matching problem. The nature of the solution presented here is more general and allows other problems to be solved directly using the results presented.

All of our results rely at the core on a few components. First, we parse strings into a hierarchy of substrings. This relies on deterministic coin tossing (aka local symmetry breaking) that is a well known technique in parallel algorithms [Anderson

and Miller 1991; Cole and Vishkin 1986; Goldberg et al. 1987] with applications to string algorithms [Alstrup et al. 2000; Cormode et al. 2000; Mehlhorn et al. 1997; Muthukrishnan and Ṣahinalp 2000; Ṣahinalp and Vishkin 1994; 1996]. In its application to string matching, precise methods for obtaining hierarchical substrings differ from one application instance to another, and are fairly sophisticated: in some cases, they produce non-trees, in other cases trees of high degree. Inspired by these techniques we present a simple hierarchical parsing procedure called *Edit Sensitive Parsing* (ESP) that produces a tree of degree 3. ESP should not be perceived to be a novel parsing technique; however, it is an attempt to simplify the technical description of applying deterministic coin tossing to obtain hierarchical decomposition of strings. We hope that the simplicity of ESP helps reveal further applications of hierarchical string decompositions.

The second component of our work is the approximately distance preserving embedding of strings into vector spaces based on the hierarchical parsing. This general style of solution has been taken earlier [Cormode et al. 2000; Muthukrishnan and Ṣahinalp 2000]. However, the key difference between our work and previous work is that we embed into L_1 space (in contrast, these prior works embedded into the Hamming space), allowing us to approximate edit distance with moves and obtain a variety of string proximity results based on this distance for the first time (the Hamming embeddings cannot be used to obtain our result).

Previous attempts to relax string edit distance involved additional substring operations such as copying and deleting substrings at unit cost. Although this appears to be a further relaxation of our distance measure, a crucial lemma we use to solve the string edit distance matching problem with substring moves (Lemma 14) no longer holds under this distance, and so the string edit distance matching problem under this metric is open. Our algorithm for the string edit distance problem with moves remains the only significantly subquadratic algorithm known for non-trivial alignments.

Layout. We present our Edit Sensitive Parsing (ESP) and show embedding of strings into the L_1 space in Section 2. We go on to solve problems of approximate string edit distance matching in Section 3. In Section 4 we give our results for other related problems based around approximating the string distance, and concluding remarks are in Section 5.

2. STRING EMBEDDING

In this section, we describe how to embed strings into a vector space so that $d()$, the string edit distance with substring moves, will be approximated by vector distances. Consider any string S over an alphabet set σ. We will embed it as $V(S)$, a vector with an exponential number of dimensions, $O(|\sigma|^{|S|})$; however, the number of dimensions in which the vector is nonzero will be quite small, in fact, $O(|S|)$. This embedding V will be computable in near linear time, and it will have the approximation preserving property we seek.

At the high level, our approach will *parse* S into special substrings, and consider the multiset $T(S)$ of all such substrings. We will ensure that the size of $T(S)$ is at most $2|S|$. Then, $V(S)$ will be the "characteristic vector" for the multiset $T(S)$. These will be defined precisely later.

The technical crux is the parsing of S into its special substrings to generate $T(S)$. We call this procedure as *Edit Sensitive Parsing*, or ESP for short. In what follows, we will first describe ESP, and then describe our vector embedding and prove its approximation preserving properties.

2.1 Edit Sensitive Parsing

We will build a parse tree, called the *ESP tree* (denoted $ET(S)$), for string S: S will be parsed into hierarchical substrings corresponding to the nodes of $ET(S)$. The goal is that string edit operations only have a localized effect on the ET. An obvious parse tree will have all dyadic strings of length 2^i, that is, $S[k2^i \ldots ((k+1)2^i - 1)]$ for all integers k and i; this will yield a fully binary parse tree. But if S is edited by a single character insertion or deletion to obtain S', S and S' will get parsed by this approach into two very different multisets of hierarchical substrings so the resulting embedding will not be approximation preserving.

Given a string S, we now show how to hierarchically build its ESP tree in $O(\log |S|)$ iterations. Each iteration generates a new level of the tree, where each level contains between a half and a third of the number of nodes in the level from which it is derived. At each iteration i, we start with a string S_i and *partition* it into *blocks* of length 2 or 3. We replace each such block corresponding to $S[j \ldots k]$ by its *name*, which is the pair $(i, h(S[j \ldots k]))$. Here h is the one-to-one hash function on substrings of S. Concretely, this can be performed with the Karp-Miller-Rosenberg labelling scheme [Karp et al. 1972]. Then S_{i+1} consists of the names for the blocks in the order in which they appear in S_i. So $|S_{i+1}| \leq |S_i|/2$. We assume $S_0 = S$, and the iterations continue until we are left with a string of length 1. The ESP tree of S consists of levels such that there is a node at level i for each of the blocks of S_{i-1}; their children are the nodes in level $i-1$ that correspond to the symbols in the block. Each character of $S_0 = S$ is a leaf node. We also denote by σ_0 the alphabet σ itself, and the set of names in S_i as σ_i, the alphabet at level-i.

It remains for us to specify how to partition the string S_i at iteration i. This will be based on designating some local features as "landmarks" of the string. A landmark (say $S_i[j]$) has the property that if S_i is transformed into S_i' by an edit operation (say character insertion at k) far away from j i.e., $|k - j| >> 1$), our partitioning strategy will ensure that $S_i'[j]$ will still be designated a landmark. In other words, an edit operation on $S_i[k]$ will only affect j being a landmark if j were close to k. This will have the effect that each edit operation will only change $O(\max_j |k_j - j|)$ nodes of the ESP tree at every level, where k_j is the closest unaffected landmark to j. In order to inflict the minimal number of changes to the ESP tree, we would like this amount to be as small as possible, but still require S_i's to be geometrically decreasing in length.

In what follows, we will describe our method for marking landmarks and partitioning S_i into blocks more precisely. We canonically parse any string into maximal non-overlapping substrings of three types:

(1) Maximal contiguous substrings of S_i that consist of a repeated symbol (so they are of the form a^l for $a \in \sigma_i$ where $l > 1$),

(2) "Long" substrings of length at least $\log^* |\sigma_{i-1}|$ not of type 1 above.

(3) "Short" substrings of length less than $\log^* |\sigma_{i-1}|$ not of type 1.

Each such substring is called a *metablock*. We process each metablock as described below to generate the next level in the parsing.

2.1.1 *Type 2: Long strings without repeats.* The discussion here is similar to those in [Goldberg et al. 1987] and [Mehlhorn et al. 1997]. Suppose we are given a string A in which no two adjacent symbols are identical and is counted as a metablock of type 2. We will carry out a procedure on it which will enable it to be parsed into nodes of two or three symbols.

Given a sequence S with no repeats (i.e., $S[i] \neq S[i+1]$ for $i = 1 \ldots |S| - 1$), we will designate at most $|S|/2$ and at least $|S|/3$ substrings of S as *nodes*. The concatenation of these nodes gives S. The first stage consists of iterating an alphabet reduction technique. This is effectively the same procedure as the Deterministic Coin Tossing in [Cole and Vishkin 1986], but applied to strings.

Alphabet reduction For each symbol $A[i]$ compute a new label, as follows. $A[i-1]$ is the left neighbor of $A[i]$, and consider $A[i]$ and $A[i-1]$ represented as binary integers. Denote by l the index of the least significant bit in which $A[i]$ differs from $A[i-1]$, and let $bit(l, A[i])$ be the value of $A[i]$ at the lth bit location. Form $label(A[i])$ as $2l + bit(l, A[i])$ — in other words, as the index l followed by the value at that index.

LEMMA 1. *For any i, if $A[i] \neq A[i+1]$ then $label(A[i]) \neq label(A[i+1])$.*

PROOF. Suppose that the least significant bit position at which $A[i]$ differs from $A[i+1]$ is the same as that at which $A[i]$ differs from $A[i-1]$ (otherwise, $label(A[i]) \neq label(A[i+1])$). But the bit values at this location in each character must differ, and hence $label(A[i]) \neq label(A[i+1])$. □

Following this procedure, we generate a new sequence. If the original alphabet was size τ, then the new alphabet is sized $2 \log |\tau|$. We now iterate (note this iteration is orthogonal to the iteration that constructs the ESP tree of S; we are iterating on A which is a subsequence with no identical adjacent symbols) and perform the alphabet reduction until the size of the alphabet no longer shrinks. This takes $\log^* |\tau|$ iterations. Note that there will be no labels for the first $\log^* |\tau|$ characters.

LEMMA 2. *After the final iteration of alphabet reduction, the alphabet size is 6.*

PROOF. At each iteration of tagging, the alphabet size goes from $|\sigma|$ to $2\lceil \log |\sigma| \rceil$. If $|\sigma| > 6$, then $2\lceil \log |\sigma| \rceil$ is strictly less than this quantity. □

Since A did not have identical adjacent symbols, neither does the final sequence of labels on A using Lemma 1 repeatedly.

Finally, we perform three passes over the sequence of symbols to reduce the alphabet from $\{0 \ldots 5\}$ to $\{0, 1, 2\}$: first we replace each 3 with the least element from $\{0, 1, 2\}$ that does not neighbor the 3, then do the same for each 4 and 5. This generates a sequence of labels drawn from the alphabet {0,1,2} where no adjacent characters are identical. Denote this sequence as A'.

Finding landmarks. We can now pick out special locations, known as *landmarks*, from this sequence that are sufficiently close together. We first select any position

i which is a *local maximum*, that is, $A'[i-1] < A'[i] > A'[i+1]$, as a landmark. Two maxima could still have four intervening labels, so in addition we select as a landmark any i which is a local minimum that is, $A'[i-1] > A'[i] < A'[i+1]$, and is not adjacent to an already chosen landmark. An example of the whole process is given in Figure 1.

LEMMA 3. *For any two successive landmark positions i and j, we have $2 \leq |i-j| \leq 3$.*

PROOF. By our marking procedure, we insist that no adjacent pair of tags are marked — since we cannot have two adjacent maxima, and we specifically forbid marking local minima which are next to minima. Simple case analysis shows that the separation of landmark positions is at most two intervening symbols. □

LEMMA 4. *Determining the closest landmark to position i depends on only $\log^* |\tau|+5$ contiguous positions to the left and 5 to the right.*

PROOF. After one iteration of alphabet reduction, each label depends only on the symbol to its left. We repeat this $\log^* |\tau|$ times, hence the label at position i depends on $\log^* |\tau|$ symbols to its left. When we perform the final step of alphabet reduction from an alphabet of size six to one of size three, the final symbol at position i depends on at most three additional symbols to its left and to its right. We must mark any position that is a local maximum, and then any that is a local minimum not adjacent to a local maximum; hence we must examine at most two labels to the left of i and two labels to the right, which in turn each depend on $\log^* |\tau|+3$ symbols to the left and 3 to the right. The total dependency is therefore as stated. □

Now we show how to partition A into blocks of length 2 or 3 around the landmarks. We treat the leftmost $\log^* |\sigma_{i-1}|$ symbols of the substring as if they were a short metablock (type 3, the procedure for which is described below). The other positions are treated as follows. We make each position part of the block generated by its closest landmark, breaking ties to the right (see Figure 2). Consequent of Lemma 3 each block is now of length two or three.

2.1.2 *Type 1 (Repeating metablocks) and Type 3 (Short metablocks).* Recall that we seek "landmarks" which can be identified easily based only on a local neighborhood. Then we can treat blocks consisting of a single repeated character as large landmarks. Type 1 and Type 3 blocks can each be parsed in a regular fashion, the details we give for completeness. Metablocks of length one would be attached to the repeating metablock to the left or the right, with preference to the left when both are possible, and parsed as described below. Metablocks of length two or three are retained as blocks without further partitioning, while a metablock of length four is divided into two blocks of length two. In any metablock of length five or more, we parse the leftmost three symbols as a block and iterate on the remainder.

2.1.3 *Constructing $ET(S)$.* Having partitioned S_i into blocks of 2 or 3 symbols, we construct S_{i+1} by replacing each block b by $h(b)$ where h is the one-to-one naming (hash) function. Note that blocks of different levels can use hash functions onto distinct domains for computing names, so we focus on any given level i. If we

i	text	c	a	b	a	g	e	h	e	a	d	b	a	g
ii	in binary	010	000	001	000	110	100	111	100	000	011	001	000	110
iii	labels	-	010	001	000	011	010	001	000	100	001	010	000	011
iv	labels as integers	-	2	1	0	3	2	1	0	4	1	2	0	3
v	final labels	-	**2**	1	**0**	1	**2**	1	0	**2**	1	**2**	0	**1**

The original text, drawn from an alphabet of size 8 (i), is written out as binary integers (ii). Following one round of alphabet reduction, the new alphabet is size 6 (iii), and the new text is rewritten as integers (iv). A final stage of alphabet reduction brings the alphabet size to 3 (v) and local maxima and some local minima are used as landmarks (denoted by boxes)

Fig. 1. The process of alphabet reduction and landmark finding

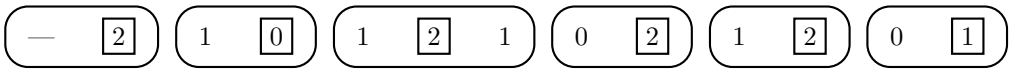


Fig. 2. Given the landmark characters, the nodes are formed.

use randomization, $h()$ can be computed for any block (recall they are of length 2 or 3) in $O(1)$ time using Karp-Rabin fingerprints [Karp and Rabin 1987]; they are one-to-one with high probability. For deterministic solutions, we can use the Karp-Miller-Rosenberge labelling algorithm in [Karp et al. 1972]. Using bucket sorting, hashing can be implemented in $O(1)$ time and linear space. Each block is a node in the parse tree, and its children are the 2 or 3 nodes from which it was formed.

This generates the sequence S_{i+1}; we then iterate this procedure until the sequence is of length 1: this is then the root of the tree. Let $|S_i|$ be the number of nodes in $ET(S)$ at level i. Since the first (leaf) level is formed from the characters of the original string, $|S_0| = |S|$. We have $|S_i|/3 \leq |S_{i+1}| \leq \lfloor |S_i|/2 \rfloor$. Therefore, $\frac{3}{2}|S| \leq \sum_i |S_i| \leq 2|S|$. Hence for any i, $|\sigma_i| \leq |S|$ (recall that $h()$ is one-to-one) and so $\log^* |\sigma_i| \leq \log^* |S|$. An example of this is shown in Figure 3.

THEOREM 5. *Given a string S, its ESP tree $ET(S)$ can be computed in time $O(|S| \log^* |S|)$.*

2.1.4 *Properties of ESP.* We can compute $ET(S)$ for any string S as described above (see Figure 4). Each node x in $ET(S)$ represents a substring of S given by the concatenation of the leaf nodes in the subtree rooted at x.

Definition 6. Define the multiset $T(S)$ as all substrings of S that are represented by the nodes of $ET(S)$ (over all levels). We define $V(S)$ to be the "characteristic vector" of $T(S)$, that is, $V(S)[x]$ is the number of times a substring x appears in $T(S)$. Finally, we define $V_i(S)$ the characteristic vector restricted to only nodes which occur at level i in $ET(S)$.

Note that $T(S)$ comprises at most $2|S|$ strings of length at most $|S|$. $V(S)$ is a $O(|\sigma|^{|S|})$ dimensional vector since its domain is any string that may be present in $T(S)$; however, it is a (highly) sparse vector since at most $2|S|$ components are nonzero.

We denote the standard L_1 distance between two vectors u and v by $||u - v||_1$. By definition, $||V(S) - V(R)||_1 = \sum_{x \in T(S) \cup T(R)} |V(S)[x] - V(R)[x]|$. Recall that $d(R, S)$ denotes the edit distance with moves between strings R and S. Our main theorem shows that $V()$ is an approximation preserving embedding of string edit distance with moves.

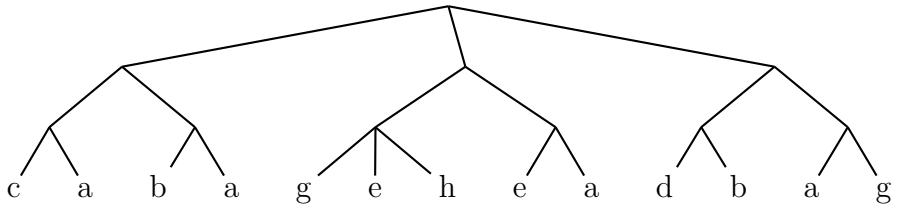
THEOREM 7. *For strings R and S, let n be* $\max(|R|, |S|)$. *Then*

$$d(R, S) \leq 2||V(R) - V(S)||_1 = O(\log n \log^* n) d(R, S)$$

2.2 Upper Bound Proof

$$||V(R) - V(S)||_1 = O(\log n \cdot \log^* n) \cdot d(R, S)$$

PROOF. To show this bound on the L_1 distance, we consider the effect of the editing operations, and demonstrate that each one causes a contribution to the



$T(S) = \{c, a, b, a, g, e, h, e, a, d, b, a, g, ca, ba, geh, ea, db, ag, caba, gehea, dbag, cabageheadbag\}$
Consequently, $V(S)[a] = 4$, $V(S)[caba] = 1$, $V(S)[head] = 0$ and $V(S)[cabageheadbag] = 1$.

Fig. 3. The hierarchical structure of nodes is represented as a parse tree on the string S.

L_1 distance that is bounded by $O(\log n \log^* n)$. Note again that, as mentioned in the introduction, edit operations are allowed to "overlap" on blocks that were previously the subject of moves. We give a Lemma which is similar to Lemma 4 but which applies to any string, not just those with no adjacent repeated characters.

LEMMA 8. *The closest landmark to any symbol of S_i is determined by at most $\log^* |\sigma_i| + 5$ consecutive symbols of S_i to the left, and at most 5 consecutive symbols of S_i to the right.*

PROOF. Given a symbol of S_i, say $S_i[j]$, we show how to find the closest landmark.

Type 1 Repeating metablock Recall that a long repeat of a symbol a is treated as a single, large landmark. $S_i[j]$ is included in such a meta-block if $S_i[j] = S_i[j+1]$ or if $S_i[j] = S_i[j-1]$. We also consider $S_i[j]$ to be part of a repeating substring if $S_i[j-1] = S_i[j-2]$; $S_i[j+1] = S_i[j+2]$; and $S_i[j] \neq S_i[j+1]$ and $S_i[j] \neq S_i[j-1]$ — this is the special case of a metablock of length one. In total, only 2 consecutive symbols to the left and right need to be examined.

Types 2 and 3 Non-repeating metablocks If it is determined that $S_i[j]$ is not part of a repeating metablock, then we have to decide whether it is in a short or long metablock. We examine the substring $S_i[j - \log^* |\sigma_i| - 3 \ldots j - 1]$. If there is any k such that $S_i[k] = S_i[k-1]$ and this is the greatest such k, then there is a repeating metablock terminating at position k. This is a landmark, and so we parse $S_i[j]$ as part of a short metablock, starting from $S[k+1]$ (recall that the first $\log^* |\sigma_i|$ symbols of a long metablock get parsed as if they were in a short metablock). Examining the substring $S[j+1 \ldots j+5]$ allows us to determine if there is another repeating metablock this close to position j, and hence we can determine what node to form containing $S_i[j]$. If there is no repeating metablock evident in $S_i[j - \log^* |\sigma_i| - 3 \ldots j - 1]$ then it is possible to apply the alphabet reduction technique to find a landmark. From Lemma 4, we know that this can be done by examining $\log^* |\sigma_i| + 5$ consecutive symbols to the left and 5 to the right. □

This ability to find the nearest landmark to a symbol by examining only a bounded number of consecutive neighboring symbols means that if an editing operation occurs outside of this region, the same landmark will be found, and so the

same node will be formed containing that symbol. This allows us to prove the following lemma.

LEMMA 9. *Inserting $k \leq \log^* n + 10$ consecutive characters into S to get S' means $||V_i(S) - V_i(S')||_1 \leq 2(\log^* n + 10)$ for all levels i.*

PROOF. We shall make use of Lemma 8 to show this. We have a contribution to the L_1 distance from the insertion itself, plus its effect on the surrounding locality. Consider the total number of symbols at level i that are parsed into different nodes after the insertion compared to the nodes beforehand. Let the number of symbols at level i which are parsed differently as a consequence of the insertion be M_i. Lemma 8 means that in a non-repeating metablock, any symbol more than 5 positions to the left, or $\log^* |\sigma_i| + 5$ positions to the right of any symbols which have changed, will find the same closest landmark as it did before, and so will be formed into the same node. Therefore it will not contribute to M_i. Similarly, for a repeating metablock, any symbol inside the block will be parsed into the same node (that is, into a triple of that symbol), except for the last 4 symbols, which depend on the length of the block. So for a repeating metablock, $M_i \leq 4$. The number of symbols from the level below which are parsed differently into nodes as a consequence of the insertion is at most $M_{i-1}/2$, and there is a region of at most 5 symbols to the left and $\log^* |\sigma_i| + 5$ symbols to the right which will be parsed differently at level i. Because $|\sigma_i| \leq |S| \leq n$ as previously observed, we can therefore form the recurrence, $M_i \leq M_{i-1}/2 + \log^* n + 10$. If $M_{i-1} \leq 2(\log^* n + 10)$ then $M_i \leq 2(\log^* n + 10)$. From the insertion itself, $M_0 \leq \log^* n + 10$. Finally $||V_i(S) - V_i(S')||_1 \leq 2(M_{i-1}/2)$, since we could lose $M_{i-1}/2$ old nodes, and gain this many new nodes. □

LEMMA 10. *Deleting $k < \log^* n + 10$ consecutive symbols from S to get S' means $||V_i(S) - V_i(S')||_1 \leq 2(\log^* n + 10)$.*

PROOF. Observe that a deletion of a sequence of labels is precisely the dual to an insertion of that sequence at the same location. If we imagine that a sequence of characters is inserted, then deleted, the resultant string is identical to the original string. Therefore, the number of affected nodes must be bounded by the same amount as for an insertion, as described in Lemma 9. □

We combine these two lemmas to show that editing operations have only a bounded effect on the parse tree.

LEMMA 11. *If a single permitted edit operation transforms a string S into S' then $||V(S) - V(S')||_1 \leq 8 \log n(\log^* n + 10)$.*

PROOF. We consider each allowable operation in turn.

Character edit operations The case for insertion follows immediately from Lemma 9 since the effect of the character insertion affects the parsing of at most $2(\log^* n + 10)$ symbols at each level and there are at most $\log_2 n$ levels. In total then $||V(S) - V(S')||_1 \leq 2 \log n(\log^* n + 10)$. Similarly, the case for deletion follows immediately from Lemma 10. Finally, the case for a replacement is shown by noting that a character replacement can be considered to be a deletion immediately adjacent to an insertion.

Substring Moves If the substring being moved is at most $\log^* n + 10$ in length,

then a move can be thought of as a deletion of the substring followed by its re-insertion elsewhere. From Lemma 9 and Lemma 10, then $||V(S) - V(S')||_1 \leq 4 \log n(\log^* n + 10)$. Otherwise, we consider the parsing of the substring using ESP. Consider a character in a non-repeating metablock which is more than $\log^* n + 5$ characters from the start of the substring and more than 5 characters from the end. Then according to Lemma 8, only characters within the substring being moved determine how that character is parsed. Hence the parsing of all such characters, and so the contribution to $V(S)$, is independent of the location of this substring in the string. Only the first $\log^* n + 5$ and last 5 characters of the substring will affect the parsing of the string. We can treat these as the deletion of two substrings of length $k \leq \log^* n + 10$ and their re-insertion elsewhere. For a repeating metablock, if this extends to the boundary of the substring being moved then still only 4 symbols of the block can be parsed into different nodes. So by appealing to Lemmas 9 and 10 then $||V(S) - V(S')|| \leq 8 \log n(\log^* n + 10)$. □

Lemma 11 shows that each allowable operation affects the L_1 distance of a transform by at most $8 \log n(\log^* n + 10)$. Suppose we begin with R, and perform a series of d editing operations, generating $R_1, R_2, \ldots R_d$. At the conclusion, $R_d = S$, so $||V(R_d) - V(S)||_1 = 0$. We begin with a quantity $||V(R) - V(S)||_1$, and we also know that at each step from the above argument that $||V(R_j) - V(R_{j+1})|| \leq 8 \log n(\log^* n + 10)$. Hence, since $d(R, S)$ operations transform R into S, then $||V(R) - V(S)||_1 / 8 \log n(\log^* n + 10) \leq d(R, S)$, giving a bound of $d(R, S) \cdot 8 \log n(\log^* n + 10)$.

2.3 Lower Bound Proof

$$d(R, S) \leq 2||V(R) - V(S)||_1$$

Here, we shall prove a slightly more general statement, since we do not need to take account of any of the special properties of the parsing; instead, we need only assume that the parse structure built on the strings has bounded degree (in this case three), and forms a tree whose leaves are the characters of the string. Our technique is to show a particular way we can use the 'credit' from $||V(R) - V(S)||_1$ to transform R into S. We give a constructive proof, although the computational efficiency of the construction is not important. For the purpose of this proof, we treat the parse trees as if they were static tree structures, so following an editing operation, we do not need to consider the effect this has on the parse structure.

LEMMA 12. *If trees which represent the transforms have degree at most k, then the tree $ET(S)$ can be made from the tree $ET(R)$ using no more than $(k-1)||V(S) - V(R)||_1$ move, insert and delete operations.*

PROOF. We first ensure that any good features of R are preserved. In a top-down, left to right pass over the tree of R, we 'protect' certain nodes — we place a mark on any node x that occurs in the parse tree of both R and S, provided that the total number of nodes marked as protected does not exceed $V_i(S)[x]$. If a node is protected, then all its descendents become protected. The number of marked copies of any node x is $\min(V(R)[x], V(S)[x])$. Once this has been done, the actual editing commences, with the restriction that we do not allow any edit operation to split a protected node.

We shall proceed bottom-up in $\log n$ rounds ensuring that after round i when we have created R_i that $||V_i(S) - V_i(R_i)||_1 = 0$. The base case to create R_0 deals with individual symbols, and is trivial: for any symbol a, if $V_0(R)[a] > V_0(S)[a]$ then we delete the $(V_0(R)[a] - V_0(S)[a])$ unmarked copies of a from R; else if $V_0(R)[a] < V_0(S)[a]$ then at the end of R we insert $(V_0(S)[a] - V_0(R)[a])$ copies of a. In each case we perform exactly $|V_0(R)[a] - V_0(S)[a]|$ operations, which is the contribution to $||V_0(R) - V_0(S)||_1$ from symbol a. R_0 then has the property that $||V_0(R_0) - V_0(S)||_1 = 0$.

Each subsequent case follows an inductive argument: assuming we have enough nodes of level $i-1$ (so $||V_{i-1}(S)-V_{i-1}(R_{i-1})||_1 = 0$), we show how to make R_i using just $(k-1)||V_i(S)-V_i(R)||_1$ move operations. Consider each node x at level i in the tree $ET(S)$. If $V_i(R)[x] \geq V_i(S)[x]$, then we would have protected $V_i(S)[x]$ copies of x and not altered these. The remaining copies of x will be split to form other nodes. Else $V_i(S)[x] > V_i(R)[x]$ and we would have protected $V_i(R)[x]$ copies of x. Hence we need to build $V_i(S)[x] - V_i(R)[x]$ new copies of x, and the contribution from x to $||V_i(S) - V_i(R)||_1$ is exactly $V_i(S)[x] - V_i(R)[x]$: this gives us the credit to build each copy of x. To make each of the copies of x, we need to bring together at most k nodes from level $i-1$. So pick one of these, and move the other $k-1$ into place around it (note that we can move any node from level $i-1$ so long as its *parent* is not protected). We do not care *where* the node is made — this will be taken care of at higher levels. Because $||V_{i-1}(S) - V_{i-1}(R_{i-1})||_1 = 0$ we know that there are enough nodes from level $i-1$ to build every level i node in S. We then require at most $k-1$ move operations to form each copy of x by moving unprotected nodes. □

Since this inductive argument holds, and we use at most $k - 1 = 2$ moves for each contribution to the L_1 distance, the claim follows.

3. SOLVING THE STRING EDIT DISTANCE MATCHING PROBLEM

In this section, we present an algorithm to solve the string edit distance problem with moves. For any string S, we will assume that $V(S)$ can be stored in $O(|S|)$ space by listing only the nonzero components of $|S|$. More formally, we store $V(S)[x]$ if it is nonzero in a table indexed by $h(x)$, and we store x as a pointer into S together with $|x|$.

The result below on pairwise string comparison follows immediately from Theorems 5 and 7 together with the observation that given $V(R)$ and $V(S)$, $||V(R) - V(S)||_1$ can be found in $O(|R| + |S|)$ time.

THEOREM 13. *Given strings S and R with $n = \max(|S|, |R|)$, there exists a deterministic algorithm to approximate $d(R, S)$ accurate up to an $O(\log n \log^* n)$ factor in $O(n \log^* n)$ time with $O(n)$ space.*

3.1 Pruning Lemma

In order to go on to solve the string edit distance problem, we need to "compare" pattern p of length m against $t[i \ldots n]$ for each i, and there are $O(n)$ such "comparisons" to be made. Further, we need to compute the distance between p and $t[i \ldots k]$ for all possible $k \geq i$ in order to compute the best alignment starting at position i, which presents $O(mn)$ subproblems in general. The classical dynamic

programming algorithm performs all these comparisons in a total of $O(mn)$ time in the worst case by using the dependence amongst the subproblems. Our algorithm will take a different approach. First, we make the following crucial observation:

LEMMA 14. (Pruning Lemma) *Given a pattern p and text t, $\forall l, r : 1 \leq l \leq r \leq n$,*

$$d(p, t[l \dots l+m-1]) \leq 2\, d(p, t[l \dots r]).$$

PROOF. Observe that for all r in the lemma, $d(p, t[l \dots r]) \geq |(r-l+1) - m|$ since this many characters must be inserted or deleted. Using triangle inequality of edit distance with moves, we have for all r, $d(p, t[l \dots l+m-1])$

$$\begin{aligned} &\leq d(p, t[l \dots r]) + d(t[l \dots r],\ t[l \dots l+m-1) \\ &= d(p, t[l \dots r]) + |(r-l+1)\ -\ m| \\ &\leq 2d(p, t[l \dots r]) \end{aligned}$$

which follows by considering the longest common prefix of $t[l \dots r]$ and $t[l \dots l+m-1]$. □

The significance of the Pruning Lemma is that it suffices to approximate only $O(n)$ distances, namely, $d(p, t[l \dots l+m-1)$ for all l, in order to solve the string edit distance problem with moves, correct up to a factor 2 approximation.[2] Hence, it prunes candidates away from the "quadratic" number of distance computations that a straightforward procedure would entail.

Still, we cannot directly apply Theorem 13 to compute $d(p, t[l \dots l+m-1])$ for all l, because that will be expensive. It will be desirable to use the answer for $d(p, t[l \dots l+m-1])$ to compute $d(p, t[l+1 \dots l+m])$ more efficiently. In what follows, we will give a more general procedure that will help compute $d(p, t[l \dots l+m-1])$ very fast for every l, by using further properties of ESP.

3.2 ESP subtrees

We know that we can approximate the distance between the pattern p and any substring of t by comparing the ESP parsings of the two. However, to compute the parsing of many substrings of t will be expensive and wasteful. In this section we show that given the ESP tree for a string, then taking the subtree induced by any substring means that this subtree will have the same edit-sensitive properties as the whole tree.

Definition 15. Let $ET_i(S)_j$ be the jth node in level i of the parsing of S.
Define $range(ET_i(S)_j)$ as the set of values $[a \dots b]$ so that the leaf labels of the subtree rooted at $ET_i(S)_j$ correspond to the substring $S[a \dots b]$.
We define an *ESP Subtree* of S, $EST(S, l, r)$ as the subtree of $ET(S)$ which contains all nodes $S[i]$ for $l \leq i \leq r$, and all ancestors of these nodes. Formally, we find all nodes of $ET_i(S)_j$ where $[l \dots r] \cap range(ET_i(S)_j) \neq \emptyset$. The name of a node derived from $ET_i(S)_j$ is $(i, h(S[range(ET_i(S)_j) \cap [a \dots b]]))$.

[2]The Pruning Lemma also holds for the classical edit distance where substring moves are not allowed since we have only used the triangle inequality and unit cost to insert or delete characters in its proof; hence, it may be of independent interest. However, it does not hold when substrings may be copied or deleted, such as the "LZ distance" [Cormode et al. 2000].

This yields a proper subtree of $ET(S)$, since a node is included in the subtree if and only if at least one of its children is included (as the ranges of the children partition the range of the parent). As before, we can define a vector representation of this tree.

Definition 16. Define $VS(S, l, r)$ as the characteristic vector of $EST(S)$ by analogy with $V(S)$, that is, $VS(S, l, r)[x]$ is the number of times the substring x is represented as a node in $EST(S, l, r)$.

Note that $EST(S, 1, |S|) = ET(S)$, but in general it is not the case that $EST(S, l, r) = ET(S[l \dots r])$. However, $EST(S, l, r)$ shares the properties of the edit sensitive parsing. We can now state a theorem that is analogous to Theorem 7.

THEOREM 17. *Let d be $d(R[l_p \dots r_p], S[l_q \dots r_q])$. Then*

$$d \leq 2||VS(R, l_p, r_p]) - VS(S, l_q, r_q])||_1 = O(\log n \log^* n)d$$

PROOF. Certainly, since Lemma 12 makes no assumptions about the structure of the tree, then the lower bound holds, that the editing distance is no more than twice the size of the difference between the ESP subtrees.

For the upper bound, consider applying the necessary editing operations to the substrings of R and S. We study the effect on the original ESP trees, $ET(R)$ and $ET(S)$. Theorem 7 proved that each editing operation can cause a difference of at most $O(\log n \log^* n)$ between $V(S)$ and $V(S')$. It follows that the difference in $VS(S, l, r)$ must be bounded by the same amount: it is not possible that any more nodes are deleted or removed, since the nodes of $EST(S, l, r)$ are a subset of the nodes of $ET(S)$. Therefore, by the same reasoning as in Theorem 7, the total difference $||VS(R, l_p, r_p) - VS(S, l_q, r_q)||_1 = d \cdot O(\log n \log^* n)$. □

We need one final lemma before proceeding to build an algorithm to solve the String Edit Distance problem with moves.

LEMMA 18. *$VS(S, l{+}1, r{+}1)$ can be computed from $VS(S, l, r))$ in time $O(\log |S|)$.*

PROOF. Recall that a node is included in $EST(S, l, r)$ if and only if one of its children is. A leaf node corresponding to $S[i]$ is included if and only if $i \in [l \dots r]$. This gives a simple procedure for finding $EST(S, l + 1, r + 1)$ from $EST(S, l, r)$, and so for finding $VS(S, l + 1, r + 1)$: (1) At the left hand end, let x be the node corresponding to $S[l]$ in $EST(S, l, r)$. We must remove x from $EST(S, l, r)$, and also remove any ancestors which do not contain $S[l]$, to produce $EST(S, l + 1, r)$. We must also adjust every ancestor of x to ensure that their name is correct. A node at level i corresponding to the substring $S[j \dots k]$ which is an ancestor of x was previously represented in the subtree by the name $(i, h(S[l \dots k]))$; this needs to be replaced by the name $(i, h(S[l + 1 \dots k]))$. (2) At the right hand end let y be the node corresponding to $S[r + 1]$ in $ET(S)$. We must add y to $EST(S, l + 1, r)$ to produce $EST(S, l + 1, r + 1)$, and set the parent of y to be its parent in $ET(S)$, adding any ancestor if it is not present. We then adjust every ancestor of y to ensure that their name is correct: an ancestor of y corresponding to the string $S[j \dots k]$ should be given the name $(i, h(S[j \dots r + 1]))$. Since in both cases we only consider ancestors of one leaf node, and the depth of the tree is $O(\log |S|)$, it follows that this procedure takes time $O(\log |S|)$. □

3.3 String Edit Distance Matching Algorithm

Combining these results allows us to solve the main problem we study in this paper.

THEOREM 19. *Given text t and pattern p, we can solve the string edit distance matching problem with moves by computing an $O(\log n \log^* n)$ approximation to $D[i] = \min_{i \leq k \leq n} d(p, t[i \ldots k])$ for each i, in time $O(n \log n)$.*

PROOF. Our algorithm is as follows: given pattern p of length m and text t of length n, we compute $ET(p)$ and $ET(t)$ in time $O(n \log^* n)$ as per Theorem 5. We then compute $EST(t, 1, m)$. This can be carried out in time at worst $O(n)$ since we have to perform a pre-order traversal of $ET(t)$ to discover which nodes are in $EST(t, 1, m)$. From this we can compute $\hat{D}[1] = ||VS(t, 1, m) - VS(p, 1, m)||_1$. We then iteratively compute $||VS(t, i+1, i+m) - VS(p, 1, m)||_1$ from $||VS(t, i, i+m-1) - VS(p, 1, m)||_1$ by using Lemma 18 to find which nodes to add or remove to $EST(t, i, i+m-1)$ and adjusting the count of the difference appropriately. This takes n comparisons, each of which takes $O(\log n)$ time. By Theorem 17 and Lemma 14, $D[i] \leq \hat{D}[i] \leq O(\log n \log^* n) D[i]$. □

If $\log^* n$ is $O(\log m)$ as one would expect for any reasonable sized pattern and text, then a tighter analysis can show the running time to be $O(n \log m)$. This is because we only need to consider the lower $\log m$ levels of the parse trees; above this $EST(t, i, i+m-1)$ has only a single node in each level.

4. APPLICATIONS AND EXTENSIONS OF OUR TECHNIQUES

Our embedding of strings into vector spaces in an approximately distance preserving manner has many other applications as such, and with extensions. In this section, we describe some further results. In contrast to the previous results, which have all been deterministic, many of these applications make use of randomized techniques.

4.1 Sketches in the Streaming model

We consider the embedding of a string S into a vector space as before, but now suppose S is truly massive, too large to be contained in main memory. Instead, the string arrives as a stream of characters in order: $(s_1, s_2 \ldots s_n)$. This is the *data streams* model [Henzinger et al. 1998], and we must perform computations with $o(n)$ space, preferably polylogarithmic space, without backtracking on the data. We are able to perform our embedding in this restricted model, the first such result known in string matching, as shown below. The result of our computations is a sketch vector for the string S. The idea of sketches is that they can be used as much smaller surrogates for the actual objects in distance computations.

THEOREM 20. *A sketch $sk(V(S))$ can be computed in the streaming model to allow approximation of the string edit distance with moves using $O(\log n \log^* n)$ space. For a combining function f, then $d(R, S) \leq f(sk(V(R)), sk(V(S))) \leq O(\log n \log^* n) d(R, S)$ with probability $1 - \delta$. Each sketch is a vector of length $O(\log 1/\delta)$ that can be manipulated in time linear in its size. Sketch creation takes total time $O(n \log^* n \log 1/\delta)$.*

PROOF. It is precisely the properties of ESP that ensures edit operations have local effect on the parse structure that also allow us to process the stream with very little space requirements. Since Lemma 8 tells us that the parsing of any symbol at any level S_j depends only on at most $O(\log^* n)$ other symbols, we only need to have these symbols in memory to make the parsing. This is true at every level in the parsing: only $O(\log^* n)$ nodes at each level need to be held in memory to make the parsing. When we group nodes of level i together to make a node of level $i+1$ we can conceptually "pass up" this new node to the next level in the process. Each node corresponds to addition of one to an entry in $V(S)$. We cannot store all the entries of the vector $V(S)$ without using linear space. Instead, we can store a short summary of $V(S)$ which can be used as a surrogate for distance computations. Existing techniques can be adapted to achieve this, in particular the streaming techniques described in Theorem 2 of [Indyk 2000]. For vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we can compute *sketches*, $sk(\boldsymbol{x})$ and $sk(\boldsymbol{y})$ so with probability $1-\delta$ for a combining function f:

$$(1-\epsilon)||\boldsymbol{x}-\boldsymbol{y}||_1 \leq |f(sk(\boldsymbol{x}), sk(\boldsymbol{y})) \leq (1+\epsilon)||\boldsymbol{x}-\boldsymbol{y}||_1$$

Such methods are necessarily approximate and randomized. The requirement for a naming function $h()$ is solved by using Karp-Rabin signatures for the substrings [Karp and Rabin 1987]. These have the useful property that the signature for a long substring can be computed from the signatures of its two or three component substrings. Thus, we can compute entries of $V(S)$ and so build a sketch of $V(S)$ using poly-logarithmic space. Since $V(S)$ can be used to approximate the string edit distance with moves up to a $O(\log n \log^* n)$ factor, it follows that these sketches achieve the same order of approximation. Overall, the total working space needed to create the parsing is $\log n$ levels each keeping $\log^* n$ nodes, totalling $O(\log n \log^* n)$ space. □

This type of computation on the data stream is useful in the case where the string is too large to be stored in memory, and so is held on secondary storage, or is communicated over a network. Sketches allow rapid comparison of strings: hence they can be used in many situations to allow approximate comparisons to be carried out probabilistically in time $O(\log 1/\delta)$ instead of the $O(n)$ time necessary to even inspect both strings.

4.2 Tighter Bounds

With further analysis, we can tighten the bounds of approximation slightly. This is useful for applications where the distances being approximated are large.

LEMMA 21. *For strings R and S, let $n = \max(|R|, |S|)$. Then*

$$d(R,S) \leq 2||V(R) - V(S)||_1 = O(\log(n/d(R,S)) \log^* n) d(R,S)$$

PROOF. We improve the upper bound by observing that in the top levels of the tree, there are only a limited number of nodes, so although these might change many times during a sequence of editing operations we have the bound $||V_i(S) - V_i(R)||_1 \leq |S_i| + |R_i|$. A worst case argument says that the greatest number of nodes that could be affected is when one level in the tree is completely altered. The size of this level is $|S_i| + |R_i| = 8d(\log^* n + 10)$, and the number of nodes above it in the tree (which

may all be affected) is $\sum_{j \geq i} |S_i| + |R_i| \leq 16d(\log^* n + 10)$. Below this, we may assume that the worst case is when each edit operation contributes $8(\log^* n + 10)$ to $||V_j(S) - V_j(R)||$ for $j < i$. Thus $||V(S) - V(R)||_1 \leq d(\log n - \log(d \log^* n))8(\log^* n + 10) + 16d(\log^* n + 10) = O(d \log^* n \log(n/d \log^* n)) = O(d \log(n/d) \log^* n)$. □

We note that since the approximation depends on $\log(n/d)$, the quality of the approximation actually increases the less alike the strings are.

4.3 Other Applications

The embedding into L_1 distance leads to a variety of results as an immediate consequence, using existing algorithms in this vector space. We outline some examples of these.

—Outlier finding: given a set of strings, these can be preprocessed in polynomial time. Then for a query string, we can find a string which is at least $O(\epsilon n)$ away from the query for some constant fraction ϵ, with constant probability. The procedure takes time $O(n \log k + \log^3 k)$ per query, using the results of [Goel et al. 2001] with Lemma 21.

—Approximate Nearest Neighbors: preprocess a set of strings so that given a query sequence we can find an approximate nearest neighbor. Using the embedding to L_1 and the algorithms of [Indyk and Motwani 1998] we can find $O(\log n \log^* n)$ approximations to the nearest neighbor with polynomial time pre-processing and $O(n \log k + k^{1/2} \log n)$ query time.

—Other similar problems can be solved for the string edit distance with moves using the L_1 embedding. For example, we can perform clustering, similarity search, minimum spanning tree and so on by adapting algorithms for these problems that work in L_1 space to use the embedding.

5. CONCLUSION

We have provided a deterministic near linear time algorithm that is an $O(\log n \log^* n)$ approximation to the string edit distance matching problem with moves. This is the first substantially subquadratic algorithm known for any string edit distance matching problem with nontrivial alignment. Our result was obtained by embedding this string distance into the L_1 vector distance; this embedding is of independent interest since it can be used to solve a variety of string proximity problems such as nearest neighbors, outlier detection, and stream-based approximation of string distances. All of these results are the first known for this string edit distance. It is open whether the $O(\log n \log^* n)$ factor in our approximations can be improved.

With only minor modifications, the techniques in this paper can allow the distances being approximated to incorporate additional operations such as linear scalings, substring reversals, copies, deletions and interchanges. However, the outstanding open problem is to understand the standard string edit distance matching problem (or quite simply computing the standard edit distance between two strings) where substring moves are not allowed, and find solutions faster than the essentially quadratic upper bounds. We have yet to make progress on this problem.

Additional Note Subsequent to completing this work, we learned of [Ṣahinalp and Vishkin 1995] where strings are parsed into 2-3 trees similar to our ESP method.

That paper proposes a novel approach for data compression by parsing strings hierarchically online; their methods do not involve embeddings, and do not yield results in this paper.

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