Discrete Comput Geom (2012) 48:128–141
DOI 10.1007/s00454-011-9392-2

The Strong Thirteen Spheres Problem

Oleg R. Musin · Alexey S. Tarasov

Received: 21 July 2010 / Revised: 27 July 2011 / Accepted: 16 October 2011 /
Published online: 29 February 2012

Abstract The thirteen spheres problem asks if 13 equal-size non-overlapping spheres in three dimensions can simultaneously touch another sphere of the same size. This problem was the subject of the famous discussion between Isaac Newton and David Gregory in 1694. The problem was solved by Schütte and van der Waerden only in 1953.

A natural extension of this problem is the strong thirteen-sphere problem (or the Tammes problem for 13 points), which calls for finding the maximum radius of and an arrangement for 13 equal-size non-overlapping spheres touching the unit sphere. In this paper, we give a solution of this long-standing open problem in geometry. Our computer-assisted proof is based on an enumeration of irreducible graphs.

1 Introduction

1.1 The Thirteen-Sphere Problem

The *kissing number* $k(n)$ is the highest number of equal non-overlapping spheres in $\mathbb{R}^n$ that touch another sphere of the same size. In three dimensions, the kissing number problem is how many white billiard balls can *kiss* (touch) a black ball.

The most symmetrical configuration, 12 balls around one ball, is achieved if the 12 balls are placed at positions corresponding to the vertices of a regular icosahedron

O.R. Musin
Department of Mathematics, University of Texas at Brownsville, 80 Fort Brown, Brownsville, TX 78520, USA
e-mail: oleg.musin@utb.edu

A.S. Tarasov (✉)
Institute for System Analysis, Russian Academy of Science, 9 Pr. 60-letiya Oktyabrya, Moscow, Russia
e-mail: tarasov.alexey@gmail.com

concentric with the central ball. However, these 12 outer balls do not kiss each other and may all be moved freely. The space between the balls leads to a question: If you moved all of them to one side, would a 13th ball fit?

This problem was the subject of the famous discussion between Isaac Newton and David Gregory in 1694 (May 4, 1694; see [29] for details of this discussion). Most reports say that Newton believed the answer was 12 balls, while Gregory thought that 13 might be possible. However, Casselman [10] found some puzzling features in this story.

This problem is often called the *thirteen-sphere problem*. Hoppe [15] thought he had solved the problem (1874). But he made a mistake, and an analysis of this mistake was published by Hales in 1994 [14] (see also [29]). The problem was finally solved by Schütte and van der Waerden in 1953 [28]. A subsequent two-page sketch of an elegant proof was given by Leech [17] in 1956. Leech's proof was presented in the first edition of the well-known book by Aigner and Ziegler [1]; the authors removed this chapter from the second edition because a complete proof would have to include much spherical trigonometry.

The thirteen-sphere problem continues to be of interest, and new proofs have been published in the last several years by Hsiang [16], Maehara [19, 20] (this proof is based on Leech's proof), Böröczky [5], Anstreicher [2], and Musin [21].

Note that, for $n > 3$, the kissing number problem is solved only for $n = 8, 24$ [18, 23], and for $n = 4$ [22] (see [25] for a beautiful exposition of this problem).

1.2 The Tammes Problem

If N unit spheres kiss the unit sphere in $\mathbb{R}^n$, then the set of kissing points is an arrangement on the central sphere such that the (Euclidean) distance between any two points is at least 1. This allows us to state the kissing number problem in another way: How many points can be placed on the surface of $\mathbb{S}^{n-1}$ so that the angular separation between any two points is at least 60°?

This leads to an important generalization: a finite subset X of $\mathbb{S}^{n-1}$ is called a *spherical ψ-code* if for every pair (x, y) of X with $x \neq y$ its angular distance $\operatorname{dist}(x, y)$ is at least ψ.

Let X be a finite subset of $\mathbb{S}^2$. Denote

$$\psi(X) := \min_{x,y\in X} \{\operatorname{dist}(x, y)\}, \quad \text{where } x \neq y.$$

Then X is a spherical $\psi(X)$-code.

Denote by d_N the largest angular separation $\psi(X)$ with $|X| = N$ that can be attained in $\mathbb{S}^2$, i.e.

$$d_N := \max_{X\subset\mathbb{S}^2} \{\psi(X)\}, \quad \text{where } |X| = N.$$

In other words, we ask: *how are N congruent, non-overlapping circles distributed on the sphere when the common radius of the circles has to be as large as possible?*

This question, also known as the problem of the "inimical dictators", get be put this way *where should N dictators build their palaces on a planet so as to be as*

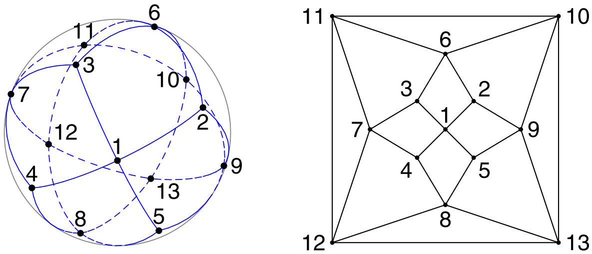

Fig. 1 An arrangement of 13 points P_{13} and its contact graph Γ_{13} with $\psi(P_{13}) \approx 57.1367°$

far away from each other as possible? The problem was first asked by the Dutch botanist Tammes [30] (see [8, Sect. 1.6: Problem 6]), who was led to this problem by examining the distribution of openings on the pollen grains of different flowers.

The Tammes problem is presently solved only for several values of N: for $N = 3, 4, 6, 12$ by L. Fejes Tóth [12]; for $N = 5, 7, 8, 9$ by Schütte and van der Waerden [27]; for $N = 10, 11$ by Danzer [11] (for $N = 11$ see also Böröczky [4]); and for $N = 24$ by Robinson [26].

1.3 The Tammes Problem for $N = 13$

The first unsolved case of the Tammes problem is $N = 13$, which is particularly interesting because of its relation to the kissing problem and the Kepler conjecture [6, 13, 29].

Actually, this problem is equivalent to *the strong thirteen-sphere problem*, which seeks to find the maximum radius of and an arrangement for 13 equal-size non-overlapping spheres in $\mathbb{R}^3$ touching the unit sphere.

It is clear that the equality $k(3) = 12$ implies $d_{13} < 60°$. Böröczky and Szabó [6] proved that $d_{13} < 58.7°$. Recently Bachoc and Vallentin [3] have shown that $d_{13} < 58.5°$.

We note that there is an arrangement of 13 points on $\mathbb{S}^2$ such that the distance between any two points of the arrangement is at least 57.1367° (see [13, Chap. VI, Sect. 4]). This arrangement is shown in Fig. 1.

Remark Denote the constant $\psi(P_{13})$ by δ_{13}. The value $d = \delta_{13}$ can be found analytically. Indeed, we have (see for notation and functions Fig. 9 and Sect. 3) $u_0 + 2u_{13} + u_2 = 2\pi$, where $u_2 = \pi/2$, $a := u_0 = \alpha(d)$, $u_{13} = \rho(u_9, d)$, $u_9 = 2\pi - 2u_5$, $u_5 = \rho(u_2, d)$. This yields

$$2\tan\left(\frac{3\pi}{8} - \frac{a}{4}\right) = \frac{1 - 2\cos a}{\cos^2 a}, \qquad \cos d = \frac{\cos a}{1 - \cos a}.$$

Thus, we have $a_{13} := \alpha(\delta_{13}) \approx 69.4051°$ and $\delta_{13} \approx 57.1367°$.

2 Main Theorem

In this paper, we present a solution of the Tammes problem for $N = 13$.

Theorem 1 *The arrangement of* 13 *points in* $\mathbb{S}^2$ *which is shown in Fig.* 1 *is the best possible*; *the maximal arrangement is unique up to isometry*, *and* $d_{13} = \delta_{13}$.

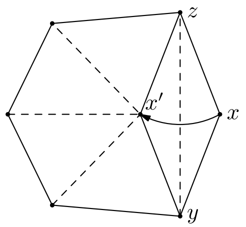

Fig. 2 Danzer's flip

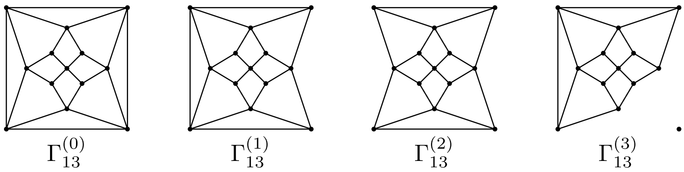

Fig. 3 Graphs $\Gamma_{13}^{(i)}$

2.1 Basic Definitions

Contact graphs. Let X be a finite set in $\mathbb{S}^2$. The *contact graph* $\mathrm{CG}(X)$ is the graph with vertices in X and edges (x, y), $x, y \in X$ such that $\mathrm{dist}(x, y) = \psi(X)$.

Shift of a single vertex. Let X be a finite set in $\mathbb{S}^2$. Let $x \in X$ be a vertex of $\mathrm{CG}(X)$ with $\deg(x) > 0$, i.e. there is $y \in X$ such that $\mathrm{dist}(x, y) = \psi(X)$. We say that there exists a shift of x if x can be slightly shifted to x' such that $\mathrm{dist}(x', X \setminus \{x\}) > \psi(X)$.

Danzer's flip. Danzer [11, Sect. 1] defined the following flip. Let x, y, z be vertices of $\mathrm{CG}(X)$ with $\mathrm{dist}(x, y) = \mathrm{dist}(x, z) = \psi(X)$. We say that x is flipped over yz if x is replaced by its mirror image x' relative to the great circle yz (see Fig. 2). We say that this flip is *Danzer's flip* if $\mathrm{dist}(x', X \setminus \{x, y, z\}) > \psi(X)$.

Irreducible graphs. We say that the graph $\mathrm{CG}(X)$ is *irreducible*[1] (or *jammed*) if there are neither Danzer's flips nor shifts of vertices.

P_{13} and Γ_{13}. Denote by P_{13} the arrangement of 13 points in Fig. 1. Let $\Gamma_{13} := \mathrm{CG}(P_{13})$. It is not hard to see that the graph Γ_{13} is irreducible.

Maximal graphs G_{13}. Let X be a subset of $\mathbb{S}^2$ with $|X| = 13$ and $\psi(X) = d_{13}$. Denote by G_{13} the graph $\mathrm{CG}(X)$. Actually, this definition does not assume that G_{13} is unique. We use this designation for some $\mathrm{CG}(X)$ with $\psi(X) = d_{13}$.

Graphs $\Gamma_{13}^{(i)}$. Let us define four planar graphs $\Gamma_{13}^{(i)}$ (see Fig. 3), where $i = 0, 1, 2, 3$, and $\Gamma_{13}^{(0)} := \Gamma_{13}$. Note that $\Gamma_{13}^{(i)}$, $i > 0$, is obtained from Γ_{13} by removing certain edges.

[1] This terminology was used by Schütte–van der Waerden [27, 28], Fejes Tóth [13], and Danzer [11].

2.2 Main Lemmas

Lemma 1 *G_{13} is isomorphic to $\Gamma_{13}^{(i)}$ with $i = 0, 1, 2,$ or 3.*

Lemma 2 *G_{13} is isomorphic to $\Gamma_{13}^{(0)}$ and $d_{13} = \delta_{13} \approx 57.1367°$.*

It is clear that Lemma 2 yields Theorem 1. Now our goal is to prove these lemmas.

3 Properties of G_{13}

3.1 Combinatorial Properties of G_{13}

Proposition 3.1 *Let X be a finite set in $\mathbb{S}^2$. Then $\mathrm{CG}(X)$ is a planar graph.*

Proof Let $a, b, x, y \in X$ with $\mathrm{dist}(a, b) = \mathrm{dist}(x, y) = \psi(X)$. Then the shortest arcs ab and xy do not intersect. Otherwise, the length of at least one of the arcs ax, ay, bx, by has to be less than $\psi(X)$. This leads to planarity of $\mathrm{CG}(X)$. □

The following three propositions are proved in [11] (also see [13, Chap. VI, 6, 7]).

Proposition 3.2 *Let X be a subset of $\mathbb{S}^2$ with $|X| = N$ and $\psi(X) = d_N$. Then for $N > 6$ the graph $\mathrm{CG}(X)$ is irreducible.*

Proposition 3.3 *Let $X \subset \mathbb{S}^2$. If the graph $\mathrm{CG}(X)$ is irreducible, then the degrees of its vertices can take only the values 0 (isolated vertices), 3, 4, or 5.*

Proposition 3.4 *Let $X \subset \mathbb{S}^2$ with $|X| = N$. If the graph $\mathrm{CG}(X)$ is irreducible, then its faces are polygons with at most $\lfloor 2\pi/d_N \rfloor$ vertices.*

Böröczky and Szabó [6, Lemmas 8 and 9(iii)] considered isolated vertices in irreducible graphs with 13 vertices.

Proposition 3.5 *Let $X \subset \mathbb{S}^2$ with $|X| = 13$. Let the graph $\mathrm{CG}(X)$ be irreducible. If $\mathrm{CG}(X)$ contains an isolated vertex, then it lies in the interior of a hexagon of $\mathrm{CG}(X)$, and this hexagon cannot contain other vertices of $\mathrm{CG}(X)$.*

Combining these propositions, we obtain the following combinatorial properties of G_{13}.

Corollary 3.1

1. *G_{13} is a planar graph;*
2. *any vertex of G_{13} is of degree 0, 3, 4, or 5;*
3. *any face of G_{13} is a polygon with three, four, five or six vertices;*
4. *if G_{13} contains an isolated vertex v, then v lies in a hexagonal face. Moreover, a hexagonal face of G_{13} cannot contain two or more isolated vertices.*

3.2 Geometric Properties of G_{13}

Let $X \subset \mathbb{S}^2$ with $|X| = 13$. Let the graph $\mathrm{CG}(X)$ be irreducible. Note that all faces of $\mathrm{CG}(X)$ are convex polygons. (Otherwise, a "concave" vertex of a polygon P can be shifted to the interior of P.) Then the faces of the graph $\mathrm{CG}(X)$ in $\mathbb{S}^2$ are regular triangles, rhombi, convex equilateral pentagons, and convex equilateral hexagons. Polygons with more than six vertices cannot occur. Note that the triangles, rhombi, or pentagons of $\mathrm{CG}(X)$ cannot contain isolated vertices in their interiors. The lengths of all edges of $\mathrm{CG}(X)$ equal $\psi(X)$.

Consider as parameters (variables) of $\mathrm{CG}(X)$ in $\mathbb{S}^2$ the set of all angles u_i of its faces, and set $d := \psi(X)$. Clearly, the graph $G = \mathrm{CG}(X)$, d, and the set $\{u_i\}$ uniquely (up to isometry) determine an embedding $X \setminus \{\text{isolated vertices}\}$ in $\mathbb{S}^2$.

We obviously have the following constraints for these parameters.

Proposition 3.6

1. $u_i < \pi$ *for all* u_i;
2. $u_i \geq \alpha(\psi(X))$ *for all* u_i, *where*

$$\alpha(d) := \cos^{-1}\left(\frac{\cos d}{1 + \cos d}\right)$$

   is the angle of a regular triangle in $\mathbb{S}^2$ *with sides of length* d;
3. $\sum_{k \in I(v)} u_k = 2\pi$ *for all vertices* v *of* G, *where* $I(v)$ *is the set of subscripts of angles that are adjacent to* v.

Let F be a face of G. Then F is a polygon with m vertices, where $m = 3, 4, 5$, or 6. Consider all possible cases.

1. $m = 3$*: triangle.* In this case, F is a regular triangle.

Proposition 3.7 *Let* F *be a triangular face of* G_{13} *with angles* u_1, u_2, u_3. *Then* $u_1 = u_2 = u_3 = \alpha_{13} := \alpha(d_{13})$.

2. $m = 4$*: quadrilateral.* In this case, $F = A_1A_2A_3A_4$ is a rhombus. Then we have $u_1 = u_3$ and $u_2 = u_4$. Using the spherical Pythagorean theorem, one can show that

$$\cot\frac{u_1}{2}\cot\frac{u_2}{2} = \cos d.$$

Then

$$u_2 = \rho(u_1, d) := 2\cot^{-1}\bigl(\tan(u_1/2)\cos d\bigr).$$

Since $u_2 \geq \alpha(d)$, we have $u_1 = \rho(u_2, d) \leq \rho(\alpha(d), d) = 2\alpha(d)$ (Fig. 4).

Proposition 3.8 *Let* F *be a quadrilateral of* G_{13} *with angles* u_1, u_2, u_3, *and* u_4. *Then* $u_3 = u_1$, $u_4 = u_2$, $u_2 = \rho(u_1, d_{13})$, $u_1 = \rho(u_2, d_{13})$, *and* $\alpha_{13} \leq u_i \leq 2\alpha_{13}$ *for all* $i = 1, 2, 3, 4$.

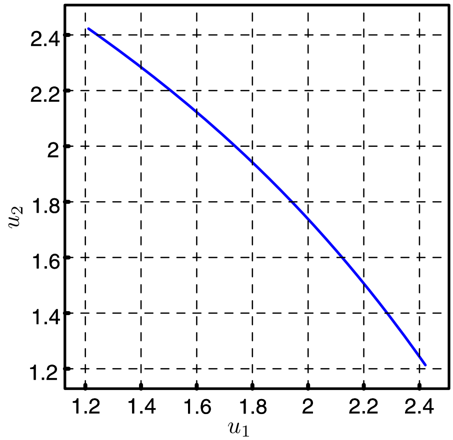

Fig. 4 The graph of the function $u_2 = \rho(u_1, d)$, where $d = 57.1367°$

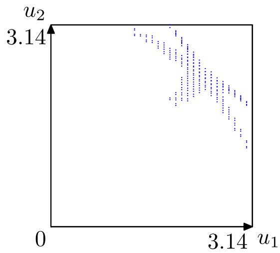

Fig. 5 The set of admissible pairs (u_1, u_2) for a pentagon with $d = 57.1367°$

3. $m = 5$: pentagon. In this case, F is a convex equilateral pentagon $A_1A_2A_3A_4A_5$. Let u_1, u_2, u_3, u_4, and u_5 be its angles. Then F is uniquely determined by d and any pair of these angles, for instance, by (u_1, u_2) (Fig. 5).

It is not hard, for given parameters $x = u_1$, $y = u_2$, and d, to find u_3, u_4, and u_5 as functions of x, y, d, i.e. $u_i = f_i(x, y, d)$, where $i = 3, 4, 5$. Let $f_1(x, y, d) = x$ and $f_2(x, y, d) = y$. Then we have $u_i = f_i(x, y, d)$ for all $i = 1, \dots, 5$. We find that all $f_i(x, y, d) \geq \alpha(d)$.

Denote by A_i' the image of A_i after a Danzer flip. Let $\xi_i(x, y, d)$ denote the minimum distance between A_i' and A_j, where $j \neq i$. If F is a face of $\mathrm{CG}(X)$ and $\mathrm{CG}(X)$ is irreducible, then F does not admit a Danzer flips. Therefore, $\xi_i(x, y, d) < d$ for all i. Thus we have the following proposition.

Proposition 3.9 *Let F be a pentagonal face of G_{13} with angles $u_1, \dots, u_5$. Then $f_i(u_1, u_2, d_{13}) \geq \alpha_{13}$ and $\xi_i(u_1, u_2, d_{13}) < d_{13}$ for all $i = 1, \dots, 5$.*

4. $m = 6$: hexagon. In this case, $F = A_1A_2A_3A_4A_5A_6$ is a convex equilateral hexagon with angles $u_1, \dots, u_6$. Clearly, F is uniquely defined by any three angles and d.

Let $u_i = g_i(u_1, u_2, u_3, d)$ for $i = 4, 5, 6$. Let $g_i(u_1, u_2, u_3, d) = u_i$ for $i = 1, 2, 3$. Then we have $u_i = g_i(u_1, u_2, u_3, d)$ for all $i = 1, \dots, 6$.

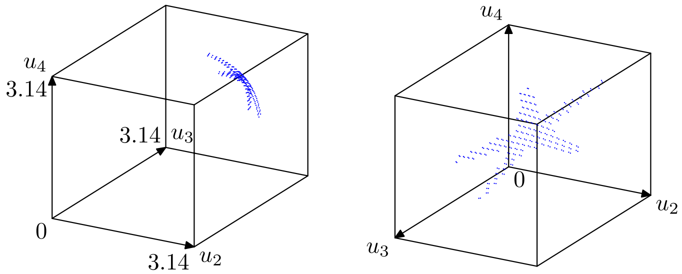

Fig. 6 Admissible angles $(u_1, u_2, u_3, u_4, u_5)$ of a pentagon projected into (u_2, u_3, u_4)

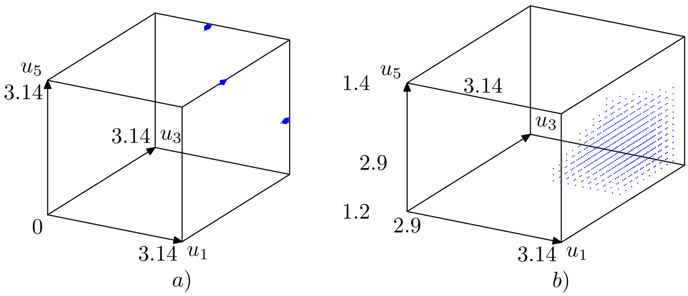

Fig. 7 (**a**) The set of admissible triplets for (u_1, u_2, u_3) for empty hexagon with $d = 57.1367°$. (**b**) A component with zoom

In fact, for the case $m = 6$ we have two subcases: (a) F has no isolated vertices, and (b) F has an isolated vertex.

It is easy to see that for case 4(a) there exists an analog of Proposition 3.9. Let $\zeta_i(u_1, u_2, u_3, d)$ denote the minimum distance between A'_i and A_j, where $j \neq i$.

Proposition 3.10 *Let F be a hexagonal face of G_{13} with angles $u_1, \dots, u_6$. Suppose that the face F has no isolated vertices in its interior. Then $g_i(u_1, u_2, u_3, d_{13}) \geq \alpha_{13}$ and $\zeta_i(u_1, u_2, u_3, d_{13}) < d_{13}$ for all $i = 1, \dots, 6$ (Fig. 7).*

Now consider case 4(b). Denote by Π the set of all points p in the interior of F such that there is a pair (i, j), $1 \leq i, j \leq 6$, $i \neq j$, with $\operatorname{dist}(p, A_i) = \operatorname{dist}(p, A_j) = d$. Clearly, $|\Pi| \leq 18$.

Let $p \in \Pi$ be defined by a pair (i, j). Denote by $K(p)$ the set of all $k = 1, \dots, 6$ such that $k \neq i$ and $k \neq j$. Let

$$\lambda(u_1, u_2, u_3, d) = \tilde{\lambda}(F) := \max_{p \in \Pi} \min_{i \in K(p)} \{\operatorname{dist}(p, A_i)\}.$$

Since F contains an isolated vertex, we have $\tilde{\lambda}(F) \geq d$.

Proposition 3.11 *Let F be a hexagonal face of G_{13} with angles $u_1, \dots, u_6$. Suppose that the face F has an isolated vertex in its interior.*

Then $g_i(u_1, u_2, u_3, d_{13}) \geq \alpha_{13}$ for all $i = 1, \dots, 6$ and $\lambda(u_1, u_2, u_3, d_{13}) \geq d_{13}$.

4 Proof of Lemma 1

Here we give a sketch of our computer proof. For more details see http://dcs.isa.ru/taras/tammes13/.

The proof consists of two parts:
(I) create the list L_{13} of all graphs with 13 vertices that satisfy Corollary 3.1;
(II) using linear approximations and linear programming, remove from the list L_{13} all graphs that do not satisfy the geometric properties of G_{13} (see Propositions 3.6–3.11).

(I). To create L_{13} we use the program *plantri* (see [24]).[2] This program is the isomorph-free generator of planar graphs, including triangulations, quadrangulations, and convex polytopes. (Reference [9] describes plantri's principles of operation, the basis for its efficiency, and recursive algorithms behind many of its capabilities.)

The program plantri generates 94,754,965 graphs in L_{13}, i.e. graphs that satisfy Corollary 3.1. Namely, L_{13} contains 30,829,972 graphs with triangular and quadrilateral faces; 49,665,852 with at least one pentagonal face, and with triangular and quadrilaterals; 13,489,261 with at least one hexagonal face which do not contain isolated vertices; 769,375[3] graphs with one isolated vertex, 505[3] with two isolated vertices, and no graphs with three or more isolated vertices.

(II). Let us consider a graph G from L_{13}. We start from the level of approximation $\ell = 1$. Now using Propositions 3.6–3.11 we write linear equalities and inequalities for the parameters (angles) $\{u_i\}$ of this graph.

For $\ell = 1$ we use the following linear equalities and inequalities:
(i) 13 linear equalities $\sum_{k\in I(v)} u_k = 2\pi$ in Proposition 3.6(3);
(ii) since $57.1367° = 0.9972 \le d_{13} < 1.021 = 58.5°$, we have $1.2113 \le \alpha_{13} < 1.2205$;
(iii) for a quadrilateral from Proposition 3.8 we have the equalities $u_3 = u_1$, $u_4 = u_2$, and the inequalities $\alpha_{13} \le u_i \le 2\alpha_{13}$, $i = 1, 2$;
(iv) for a quadrilateral, (ii) and $u_2 = \rho(u_1, d_{13})$ yield $3.6339 \le u_1 + u_2 \le 3.779657$;
(v) let F be a pentagonal face. Consider all vectors $U_5 := \{(u_1, \dots, u_5)\}$ that satisfy Proposition 3.9 (see Fig. 6). We use a convex polytope P_5 in $\mathbb{R}^5$ which contains U_5. Actually, P_5 is defined by certain linear inequalities. For instance, $2.96 \le u_1 + u_2 - 0.63u_4 \le 3.26$, $u_1 + u_3 + 1.8u_2 \le 9.05$, etc.
(vi) For a hexagonal face F that contains no isolated vertices, using Proposition 3.10, we find a set of three polytopes P_6^k, $U_6 \subset \bigcup_{k=1}^3 P_6^k$, which are defined by the inequalities $1.2 \le u_k, u_{k+3} \le 1.34$ and $2.9 \le u_{k+1}, u_{k+2}, u_{k+4}, u_{k+5}$;
(vii) for a hexagonal face with an isolated vertex, Proposition 3.11 yields $\sum_{i=1}^6 u_i \ge 15.936$.

Using this set of linear inequalities, we find the minimal and maximal value of each variable by linear programming. This gives us a convex region in the space of possible solutions that contains all possible solutions for given graph (if they exist).

[2] The authors of this program are Gunnar Brinkmann and Brendan McKay.

[3] This figure may include isomorphic graphs.

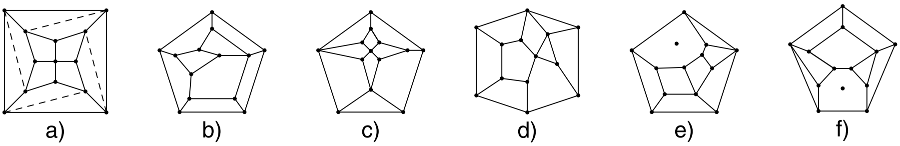

Fig. 8 Strongest eliminated graphs

If the region becomes empty, this means that we can eliminate the graph considered. This step "kills" almost all graphs. After this step, there remain 2013 graphs without hexagons, 40910 graphs with hexagons and without isolated vertices, 9073 graphs with one isolated vertex, and 272 graphs with two isolated vertices.

We use the following idea for $\ell = 2$. This region is smaller than the original region, so we can adjust linear estimates for nonlinear equalities and inequalities. For quadrilaterals we adjust inequalities using (iv). For pentagons we use an additional set of inequalities. Namely, using the functions $f_3(u_1, u_5, d)$, $f_3(u_2, u_4, d)$, and bounds for u_1, u_2, u_4, u_5, d, minimal and maximal linear bounds for u_3.

Repeating this procedure, we obtain a chain of nested convex regions, which contain all possible solutions. This chain converges to an empty or a non-empty region. If this result is empty, the graph is eliminated. After this step, only 260 graphs remain in the main group, 9991 graphs remain in the second group, 126 graphs remain in the third group, and no graphs remain in the fourth group.

For the level of approximation $\ell = 3$, we split the region into two smaller regions and repeat the same procedure as for $\ell = 2$ independently. For graphs with empty hexagons, we make a specific split by taking different values of k from item (vi) (see above).

Repeating the splitting procedure, we "kill" all graphs except $\Gamma_{13}^{(i)}$.

This result leads to two surprises. We expected that subgraphs were to remain, because they can be infinitesimally close to Γ_{13}, and so they cannot be eliminated by a computer program. But we did not expect that all other graphs would be killed. Also, we manually found two subgraphs which could be contact graphs: $\Gamma_{13}^{(1)}$ and $\Gamma_{13}^{(2)}$. But we missed the graph $\Gamma_{13}^{(3)}$ with one isolated vertex, which was found by the computer program.

Remark In Fig. 8 are presented examples of graphs which are not isomorphic to $\Gamma_{13}^{(i)}$ and have been eliminated only after many iterations. The strongest of the graphs in Fig. 8 is (a). This graph is also a subgraph of $\Gamma_{13}^{(0)}$. After eliminating four edges, the graph contains four pentagons. The reason why it was eliminated is that there are angles u_i which are slightly larger than π, so that the pentagons are not convex. Therefore, this graph is not irreducible. Most other surviving graphs were "strong" because they have several pentagons and hexagons. Note that here we use weak bounds for pentagons and hexagons given by (v), (vi), (vii). Our elimination procedure works very fast when we have sufficiently many triangles and quadrilaterals, and it works worse (slowly) when we have several pentagons and hexagons.

5 Proof of Lemma 2

Proof This proof is based on geometric properties of G_{13}. In Sect. 4 we substitute all nonlinear equations by certain linear inequalities. Note that the statement $d_{13} \approx \delta_{13}$ is a by-product of this approximation. Here we prove that $d_{13} = \delta_{13}$ based on the original equations.

Lemma 1 says that $G_{13} = \Gamma_{13}^{(i)}$, where $i = 0, 1, 2,$ or 3. We are going to prove that if $\mathrm{CG}(X) = \Gamma_{13}^{(i)}$ with $i > 0$, then $\psi(X) < \delta_{13} = \psi(P_{13})$.

5.0. Angles of $\Gamma_{13}^{(2)}$ Let $u_0 := \alpha(d)$. For $G_{13} = \Gamma_{13}^{(2)}$ we have (see Fig. 9):

$$u_5 = \rho(u_1, d) \qquad u_6 = \rho(u_2, d) \qquad u_9 = 2\pi - u_5 - u_6$$
$$u_{13} = \rho(u_9, d) \qquad u_{14} = 2\pi - u_0 - u_{13} - u_2 \qquad u_{10} = \rho(u_{14})$$
$$u_7 = 2\pi - u_6 - u_{10} \qquad u_3 = \rho(u_7, d) \qquad u_4 = 2\pi - u_1 - u_2 - u_3$$
$$u_8 = \rho(u_4, d) \qquad u_{11} = 2\pi - u_7 - u_8 \qquad u_{12} = 2\pi - u_8 - u_5$$
$$u_{15} = \rho(u_{11}, d) \qquad u_{16} = \rho(u_{12}, d)$$

Therefore, for $3 \le i \le 16$ the value u_i are functions of the variables u_1, u_2, and d. Since we have also an additional equation for the vertex v_8 (see Figs. 1 and 9):

$$u_0 + u_{15} + u_4 + u_{16} = 2\pi,$$

the value d is a function of u_1, u_2; also, u_2 is a function of the variables u_1 and d. Thus, all u_i and d are functions of u_1, u_2 or of u_1, d.

Now we consider three cases $G_{13} = \Gamma_{13}^{(i)}$, where $i = 1, 2, 3$.

5.1. The case $G_{13} = \Gamma_{13}^{(1)}$ In this case $u_{17} = u_0$. Then for vertex v_7 we have the equation

$$u_1 + u_{13} + u_0 + u_{16} = 2\pi.$$

From this it follows that u_1 and, therefore, all u_i are functions of d. Note that

$$u_{18} = 2\pi - u_{14} - u_3 - u_{15}.$$

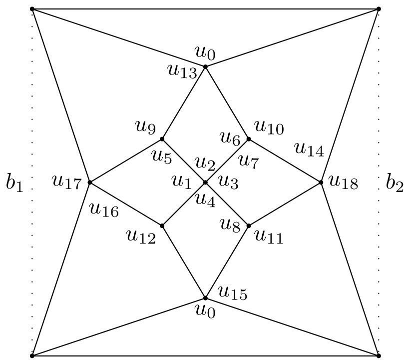

Fig. 9 Angles of $\Gamma_{13}^{(2)}$

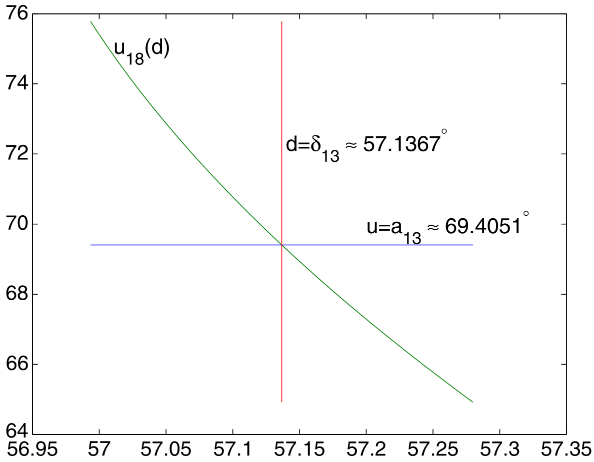

Fig. 10 The graph of the function $u_{18}(d)$

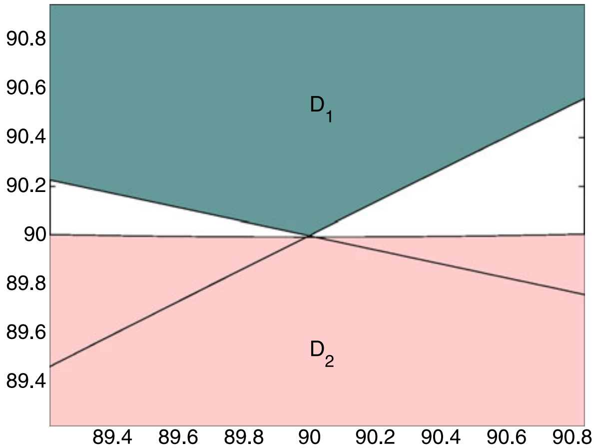

Fig. 11 D_1 and D_2

Thus, u_{18} is a function of d (see Fig. 10).

If $G_{13} = \Gamma_{13}^{(1)}$, then $u_{18} > u_0 \geq a_{13}$. Since the function $u_{18}(d)$ is monotonous decreasing, we have $u_{18}(d) > a_{13}$ only if $d < \delta_{13}$. Thus, $G_{13} \neq \Gamma_{13}^{(1)}$.

5.2. The case $G_{13} = \Gamma_{13}^{(2)}$ It is already shown that d and all u_i are functions of u_1, u_2. Let

$$D_1 := \{(u_1, u_2) : u_{17} \geq u_0,\ u_{18} \geq u_0\} \quad \text{and}$$
$$D_2 := \{(u_1, u_2) : u_0 = \alpha(d) \geq a_{13}\}.$$

We can see from Fig. 11 that the intersection $I := D_1 \cap D_2 \subset \mathbb{R}^2$ consists of one point with $u_1 = u_2 = 90°$. It is not hard to prove this fact. Indeed, conversely, $d_{13} > \delta_{13}$, and there is a point (u_1, u_2) on the boundary of I such that $u_{17} = u_0$ or $u_{18} = u_0$. Therefore, we have the same case as in case 5.1, a contradiction. Thus, $G_{13} \neq \Gamma_{13}^{(2)}$.

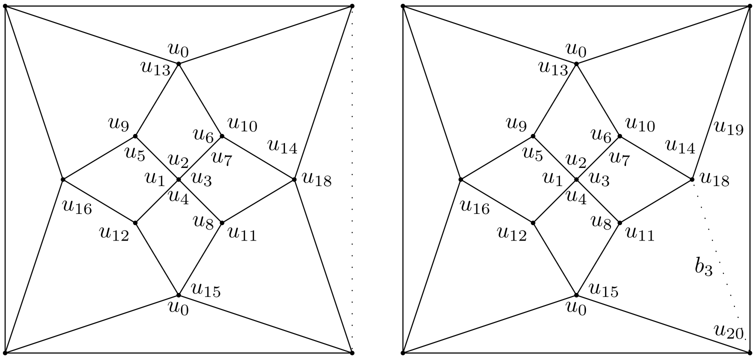

Fig. 12 Two subcases for the case $G_{13} = \Gamma_{13}^{(3)}$

5.3. The case $G_{13} = \Gamma_{13}^{(3)}$ This case can be considered by the same method as the case $G_{13} = \Gamma_{13}^{(2)}$. Actually, for given u_1, u_2, and d, all angles u_i, $3 \le i \le 16, i \ne 15$ can be found by the same formulas as in case 5.0. On the other hand,

$$u_{15} = 2\pi - u_4 - u_{16} - u_0.$$

Then all u_i are functions of the variables u_1, u_2, and d. Since $u_{17} = u_0$ (or equivalently $b_1 = d$), we have the equation

$$u_1 + u_{13} + u_0 + u_{16} = 2\pi.$$

We find that all u_i depend on two parameters.

The vertex v_{13} is isolated. In fact, we can shift this point in such a way that at least two edges $v_{13}v_k$, where $k = 8, 9, 10, 12$, have lengths d. Then for two other edges we have the inequalities $\operatorname{dist}(v_{13}, v_i) \ge d$ and $\operatorname{dist}(v_{13}, v_j) \ge d$.

Arguing as in case 5.2, we can show that there are parameters u_1, u_2 such that $u_0 > a_{13}$ and at least one of the inequalities $\operatorname{dist}(v_{13}, v_k) \ge d,\ k = i, j$, becomes an equality. It is not hard to see that there are exactly two geometrically nonequivalent cases with exactly one edge $v_{13}v_k, k = 8, 9, 10,$ or 12, such that $\operatorname{dist}(v_{13}v_k) > d$. These cases are shown in Fig. 12.

Actually, the first subcase is case 5.1. For the second subcase consider the pentagon $F := v_5v_8v_{12}v_{13}v_{10}$. All angles of F can be found as functions of u_1, d. Since d and any two angles of F define all other angles, we can use one of these equations to find u_1 as a function of d. Then u_{19} (see Fig. 12) is a function of d. In fact, the graph of the function $u_{19}(d)$ is very similar to the graph $u_{18}(d)$ in Fig. 10, and $u_{19}(d)$ is a monotonous decreasing function. Thus, $u_{19}(d)$ cannot be greater than a_{13}, and $G_{13} \ne \Gamma_{13}^{(3)}$.

We see that if $\mathrm{CG}(X) = G_{13}$, then $\mathrm{CG}(X)$ is isomorphic to Γ_{13}. Moreover, X is uniquely defined up to isometry and $\psi(X) = \delta_{13} \approx 57.1367°$. This completes the proof. □

Acknowledgements We wish to thank Robert Connelly, Alexey Glazyrin, Nikita Netsvetaev, and Günter Ziegler for helpful discussions and comments. Supported by the Russian government project 11.G34.31.0053.

O.R. Musin's research is supported in part by NSF grant DMS-0807640 and NSA grant MSPF-08G-201. A.S. Tarasov's research is supported by program P15 of presidium of RAS and RFBR grant 11-01-00633 and 11-01-00735.

References

1. Aigner, M., Ziegler, G.M.: Proofs from THE BOOK. Springer, Berlin (1998) 1st edn. and (2002) 2nd edn.
2. Anstreicher, K.: The thirteen spheres: A new proof. Discrete Comput. Geom. **31**, 613–625 (2004)
3. Bachoc, C., Vallentin, F.: New upper bounds for kissing numbers from semidefinite programming. J. Am. Math. Soc. **21**, 909–924 (2008)
4. Böröczky, K.: The problem of Tammes for $n = 11$. Studia Sci. Math. Hung. **18**, 165–171 (1983)
5. Böröczky, K.: The Newton-Gregory problem revisited. In: Bezdek, A. (ed.) Discrete Geometry, pp. 103–110. Dekker, New York (2003)
6. Böröczky, K., Szabó, L.: Arrangements of 13 points on a sphere. In: Bezdek, A. (ed.) Discrete Geometry, pp. 111–184. Dekker, New York (2003)
7. Böröczky, K., Szabó, L.: Arrangements of 14, 15, 16 and 17 points on a sphere. Studia Sci. Math. Hung. **40**, 407–421 (2003)
8. Brass, P., Moser, W.O.J., Pach, J.: Research Problems in Discrete Geometry. Springer, Berlin (2005)
9. Brinkmann, G., McKay, B.D.: Fast generation of planar graphs (expanded edition). http://cs.anu.edu.au/~bdm/papers/plantri-full.pdf
10. Casselman, B.: The difficulties of kissing in three dimensions. Not. Am. Math. Soc. **51**, 884–885 (2004)
11. Danzer, L.: Finite point-sets on $\mathbf{S}^2$ with minimum distance as large as possible. Discrete Math. **60**, 3–66 (1986)
12. Fejes Tóth, L.: Über die Abschätzung des kürzesten Abstandes zweier Punkte eines auf einer Kugelfläche liegenden Punktsystems. Jber. Deutch. Math. Verein. **53**, 66–68 (1943)
13. Fejes Tóth, L.: Lagerungen in der Ebene, auf der Kugel und in Raum. Springer, Berlin (1953). Russian translation, Moscow, 1958
14. Hales, T.: The status of the Kepler conjecture. Math. Intell. **16**, 47–58 (1994)
15. Hoppe, R.: Bemerkung der Redaktion. Archiv Math. Phys. (Grunet) **56**, 307–312 (1874)
16. Hsiang, W.-Y.: Least Action Principle of Crystal Formation of Dense Packing Type and Kepler's Conjecture. World Scientific, Singapore (2001)
17. Leech, J.: The problem of the thirteen spheres. Math. Gaz. **41**, 22–23 (1956)
18. Levenshtein, V.I.: On bounds for packing in n-dimensional Euclidean space. Sov. Math. Dokl. **20**(2), 417–421 (1979)
19. Maehara, H.: Isoperimetric theorem for spherical polygons and the problem of 13 spheres. Ryukyu Math. J. **14**, 41–57 (2001)
20. Maehara, H.: The problem of thirteen spheres—a proof for undergraduates. Eur. J. Comb. **28**, 1770–1778 (2007)
21. Musin, O.R.: The kissing problem in three dimensions. Discrete Comput. Geom. **35**, 375–384 (2006)
22. Musin, O.R.: The kissing number in four dimensions. Ann. Math. **168**, 1–32 (2008)
23. Odlyzko, A.M., Sloane, N.J.A.: New bounds on the number of unit spheres that can touch a unit sphere in n dimensions. J. Comb. Theory, Ser. A **26**, 210–214 (1979)
24. Plantri, Fullgen: http://cs.anu.edu.au/~bdm/plantri/
25. Pfender, F., Ziegler, G.M.: Kissing numbers, sphere packings, and some unexpected proofs. Not. Am. Math. Soc. **51**, 873–883 (2004)
26. Robinson, R.M.: Arrangement of 24 circles on a sphere. Math. Ann. **144**, 17–48 (1961)
27. Schütte, K., van der Waerden, B.L.: Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand 1 Platz? Math. Ann. **123**, 96–124 (1951)
28. Schütte, K., van der Waerden, B.L.: Das Problem der dreizehn Kugeln. Math. Ann. **125**, 325–334 (1953)
29. Szpiro, G.G.: Kepler's Conjecture. Wiley, New York (2002)
30. Tammes, R.M.L.: On the origin number and arrangement of the places of exits on the surface of pollengrains. Rec. Trv. Bot. Neerl. **27**, 1–84 (1930)