The Strongest Possible Lewisian Triviality Result

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April 23, 2015

1 Setting The Stage

Once upon a time, various philosophers (*e.g.*, [1, 15]) defended the idea that the probability of an indicative conditional ($P \to Q$) is the conditional probability of its consequent (Q), given its antecedent (P). More precisely, the following principle has been proposed and defended by various authors.[2]

The Equation. $\Pr(P \to Q) = \Pr(Q \mid P)$, provided $\Pr(P) > 0$.

David Lewis [9, 8] published several triviality results involving **The Equation**. Since then, several other authors have published similar triviality results (see, *e.g.*, [6, 13]). In section two, I will explain the basic ideas behind these Lewisian triviality results. In section three, I will prove a new Lewisian triviality result. In fact, I will prove the *strongest possible* result of its kind. All other (published) Lewisian triviality results are strictly weaker than ours, and there can be no stronger result along these lines (in a sense to be made precise in section four).

2 Given The Equation, Lewisian Triviality is *Equivalent* to Import-Export

There's nothing trivial about **The Equation** *per se*. But, if we combine **The Equation** with another (seemingly plausible) assumption about the probabilities of *nested* conditionals, then Lewisian trivialities ensue. That assumption is the so-called *Import-Export Law*, which (probabilistically) is expressed as follows.

Import-Export. $\Pr(P \to (Q \to R)) = \Pr((P \,\&\, Q) \to R)$, provided $\Pr(P \,\&\, Q) > 0$.

In the presence of **The Equation**, **Import-Export** is *equivalent* to the following "resilient" equation.[3]

The Resilient Equation. $\Pr(P \to Q \mid X) = \Pr(Q \mid P \,\&\, X)$, provided $\Pr(P \,\&\, X) > 0$.

It is actually **The Resilient Equation** that is the true target of Lewisian triviality arguments.[4] In the next section, I will present a new Lewisian triviality result, which subsumes all existing results of its kind.

3 The Strongest Lewisian Triviality Result

In this section, I will prove the the following triviality result.

Triviality. Provided that $\Pr(P \,\&\, Q) > 0$ and $\Pr(P \,\&\, {\sim}Q) > 0$,

$$\textbf{The Resilient Equation} \iff \Pr\left(P \,\&\, (Q \equiv (P \to Q))\right) = 1.$$

[1]Thanks to Andrew Bacon, Fabrizio Cariani, Thony Gillies, Simon Goldstein, Alan Hájek, and an anonymous referee for useful feedback on various versions of this paper.

[2]See [6] for a nice survey article on **The Equation** and its history.

[3]See the Appendix for a proof of this equivalence. I use the term "resilient" here, because it is reminiscent of the Skyrmsian [14] notion of resiliency. More recently, Hannes Leitgeb has endorsed a resilient version of the Lockean Thesis [7], which also has various unintuitive consequences [11, 2, 10]. I think the ultimate source of Lewisian triviality is this requirement of resiliency (and not **The Equation** *per se*). Moreover, the Import-Export Law is implicated in various other "triviality" results for the indicative conditional [5, 12, 3]. As such, I'd be inclined to reject **Import-Export** here, rather than **The Equation**. But, I'll have to leave the proper treatment of that question for another investigation.

[4]I am describing Lewisian triviality in terms of *resiliency* of **The Equation**, relative to a *single* probability function $\Pr(\cdot)$. Lewis's original arguments traded on the assumption that **The Equation** holds *throughout a class* of probability functions (including Pr) that is *closed under conditionalization*. But, from the point of view of classical Bayesianism (which assumes that all updating goes *via* conditionalization), these are (for all intents and purposes) equivalent ways of running Lewisian triviality arguments.

What follows is an algebraic proof of **Triviality**. The generic stochastic truth-table representation of the class of probability functions $\Pr(\cdot)$ over the eight states determined by $P, Q, P \to Q$ is as follows.[5]

P	Q	$P \to Q$	$\Pr(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	b
T	**F**	**T**	c
T	**F**	**F**	d
F	**T**	**T**	e
F	**T**	**F**	f
F	**F**	**T**	g
F	**F**	**F**	h

It turns out that one does not need the full strength of **The Resilient Equation** in order two show that it implies the right-hand side of **Triviality**. That is, one does not need to conditionalize on *all* X's such that $\Pr(P \,\&\, X) > 0$ in order to derive this (strongest) triviality result from **The Resilient Equation**. In fact, all we need are *three instances* of **The Resilient Equation**. I will now work my way up to **Triviality**, in three stages.

3.1 Stage 1: The $\sim Q$–instance of The Resilient Equation

Consider the following instance of **The Resilient Equation**, where $X := \sim Q$.[6]

The Resilient Equation$_{\sim Q}$. $\Pr(P \to Q \mid \sim Q) = \Pr(Q \mid P \,\&\, \sim Q)$, provided $\Pr(P \,\&\, \sim Q) > 0$.

Algebraically, **The Resilient Equation**$_{\sim Q}$ is equivalent to the following [4], provided $\Pr(P \,\&\, \sim Q) > 0$.

$$\Pr(P \to Q \mid \sim Q) = \frac{\Pr((P \to Q) \,\&\, \sim Q)}{\Pr(\sim Q)} = \frac{c + g}{c + d + g + h} = 0 = \Pr(Q \mid P \,\&\, \sim Q)$$

This equation will be true iff $c + g = 0$, which implies that c and g *must both be equal to zero*. The effect of **The Resilient Equation**$_{\sim Q}$ is therefore reflected in the following revised stochastic truth-table.

P	Q	$P \to Q$	$\Pr(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	b
T	**F**	**T**	0
T	**F**	**F**	d
F	**T**	**T**	e
F	**T**	**F**	f
F	**F**	**T**	0
F	**F**	**F**	h

3.2 Stage 2: The $P \supset Q$–instance of The Resilient Equation

Consider the following instance of **The Resilient Equation**, where $X := P \supset Q$.[7]

The Resilient Equation$_{P \supset Q}$. $\Pr(P \to Q \mid P \supset Q) = \Pr(Q \mid P \,\&\, (P \supset Q))$, provided $\Pr(P \,\&\, (P \supset Q)) > 0$.

[5]Here, I'm using the terminology and setup of [4], which provides a general technique for reasoning algebraically about the probability calculus. Moreover, I will be assuming (without loss of generality) that P, Q, and $P \to Q$ are *logically independent* of each other. If there were logical dependencies between them, then this would only serve to *strengthen* our triviality result.

[6]This was one of the instances used by Lewis [9] to derive his original triviality results. The other instance he used was $X := Q$. It can be shown that Lewis's pair of constraints is *strictly weaker* than our (maximally strong) set of three constraints. For instance, Lewis's pair of instances do not jointly entail $\Pr(P) = 1$. See the companion *Mathematica* notebook (*fn.* 9) for a proof of this.

[7]This is the instance used by Milne [13] to derive his triviality result. Milne's instance is strictly weaker than our (maximally strong) set of three constraints. For instance, Milne's instance does not entail *either* $\Pr(P) = 1$ *or* $\Pr(Q) = \Pr(P \to Q)$. See the companion *Mathematica* notebook (*fn.* 9) for a proof of this.

Algebraically, **The Resilient Equation**$_{P \supset Q}$ is equivalent to the following, provided $\Pr(P \,\&\, Q) > 0$.

$$\Pr(P \to Q \mid P \supset Q) = \frac{\Pr((P \to Q) \,\&\, (P \supset Q))}{\Pr(P \supset Q)} = \frac{a + e}{a + b + e + f + h} = 1 = \Pr(Q \mid P \,\&\, (P \supset Q))$$

Cross-multiplying (and expanding and simplifying) this equation yields

$$0 = b + f + h$$

This equation will be true iff b, f and h *are all equal to zero*. The effects of **The Resilient Equation**$_{\sim Q}$ + **The Resilient Equation**$_{P \supset Q}$ are reflected in the following revised stochastic truth-table.

P	Q	$P \to Q$	$\Pr(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	0
T	**F**	**T**	0
T	**F**	**F**	d
F	**T**	**T**	e
F	**T**	**F**	0
F	**F**	**T**	0
F	**F**	**F**	0

3.3 Stage 3: The $\top$–instance of The Resilient Equation — *i.e.*, The Equation *Itself*

Consider the following instance of **The Resilient Equation**, where $X := \top$.

The Resilient Equation$_{\top}$. $\Pr(P \to Q \mid \top) = \Pr(Q \mid P \,\&\, \top)$, provided $\Pr(P \,\&\, \top) > 0$.

Of course, **The Resilient Equation**$_{\top}$ *is just* **The Equation** *itself*. Algebraically, **The Equation** is now

$$\Pr(P \to Q) = a + e = \frac{a}{a + d} = \Pr(Q \mid P)$$

Cross-multiplying (and expanding and simplifying) this equation yields the following quadratic equation

$$a^2 + ad + ae + de - a = 0$$

Recall, we are assuming (from Stage 1) that $\Pr(P \,\&\, \sim Q) > 0$. That is, we are assuming that $d > 0$. As it happens, when $d > 0$ (and the background probabilistic constraints on a, d, e hold [4]), the quadratic equation above is satisfied *iff* $e = 0$, $d = 1 - a$, and $a, d \in (0, 1)$.

The effects of **The Resilient Equation**$_{\sim Q}$ + **The Resilient Equation**$_{P \supset Q}$ + **The Equation** are reflected in the following (final) *single-parameter* stochastic truth-table, where $a \in (0, 1)$.

P	Q	$P \to Q$	$\Pr(\cdot)$
T	**T**	**T**	a
T	**T**	**F**	0
T	**F**	**T**	0
T	**F**	**F**	$1 - a$
F	**T**	**T**	0
F	**T**	**F**	0
F	**F**	**T**	0
F	**F**	**F**	0

In other words, **The Resilient Equation**$_{\sim Q}$ + **The Resilient Equation**$_{P \supset Q}$ + **The Equation** jointly entail that *the only two states which can be assigned non-zero probability* are $P \,\&\, Q \,\&\, (P \to Q)$ and $P \,\&\, \sim Q \,\&\, \sim(P \to Q)$. This is equivalent to saying that the proposition $P \,\&\, (Q \equiv (P \to Q))$ *must receive maximal probability*. *QED*

Triviality is *very* strong.[8] It implies that, for *every* P and Q that feature as the antecedent and consequent of some indicative conditional $P \to Q$ (and which are such that $\Pr(P \,\&\, Q) > 0$ and $\Pr(P \,\&\, \sim Q) > 0$), *both* P *and* the material biconditional $Q \equiv (P \to Q)$ must receive maximal probability (and, as a result, we must also have $\Pr(Q) = \Pr(P \to Q)$). All of the existing Lewisian triviality results are strictly weaker than this one. In fact, *there can be no stronger* Lewisian triviality result.

[8] Here's one interpretation of $\to$ and $\Pr(\cdot)$ which satisfies **Triviality**. Let $\Pr(\cdot)$ be an *indicator function*, and let $p \to q \overset{\text{def}}{=} q$.

4 Why Triviality is *The Strongest* (Lewisian) Triviality Result

Triviality is *the strongest* triviality result of its kind. Here's what I mean. If one assumes *all* of the instances of **The Resilient Equation**, then this *still* (*only*) implies **Triviality**. That is, adding further instances of **The Resilient Equation** to the three we used above *does not add any additional constraints to* $\Pr(\cdot)$. This can be shown algebraically by proving that the conjunction of *all* instances of **The Resilient Equation** (where X ranges over the 256 propositions in the Boolean algebra generated by $P, Q, P \to Q$) *is equivalent to* the conjunction of the *three* instances of **The Resilient Equation** that we used above (and this also secures the $\Longleftarrow$ direction of **Triviality**).[9]

Appendix: Proof of the Equivalence of Import-Export and The Resilient Equation, Given The Equation

Theorem. Given **The Equation**, **The Resilient Equation** is *equivalent* to ($\Leftrightarrow$) **Import-Export**.

Proof. Here is a proof of the $\Rightarrow$ direction of this theorem.

1.	$\Pr(P \to (Q \to R)) = \Pr(Q \to R \mid P)$, if $\Pr(P \,\&\, Q) > 0$	**The Equation**
2.	$\Pr(Q \to R \mid P) = \Pr(R \mid P \,\&\, Q)$, if $\Pr(P \,\&\, Q) > 0$	**The Resilient Equation**
3.	$\Pr(R \mid P \,\&\, Q) = \Pr((P \,\&\, Q) \to R)$, if $\Pr(P \,\&\, Q) > 0$	**The Equation**
	$\therefore \Pr(P \to (Q \to R)) = \Pr((P \,\&\, Q) \to R)$, if $\Pr(P \,\&\, Q) > 0$	(1), (2), (3) □

Here is a proof of the $\Leftarrow$ direction of this theorem.

1.	$\Pr(X \to (P \to Q)) = \Pr(P \to Q \mid X)$, if $\Pr(P \,\&\, X) > 0$	**The Equation**
2.	$\Pr(X \to (P \to Q)) = \Pr((P \,\&\, X) \to Q)$, if $\Pr(P \,\&\, X) > 0$	**Import-Export**
3.	$\Pr((P \,\&\, X) \to Q) = \Pr(Q \mid P \,\&\, X)$, if $\Pr(P \,\&\, X) > 0$	**The Equation**
	$\therefore \Pr(P \to Q \mid X) = \Pr(Q \mid P \,\&\, X)$, if $\Pr(P \,\&\, X) > 0$	(1), (2), (3) □

References

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[7] H. Leitgeb, *The stability theory of belief*, *Philosophical Review* (2014): 131-171.

[9] This can easily be verified using *Mathematica*. I have created a *Mathematica* (version 10) notebook which verifies that **Triviality** is *the strongest* (Lewisian) triviality result for the indicative conditional. It also shows (a) that the results of Lewis and Milne are strictly weaker than ours; and, (b) there are some (very complex) *pairs* of instances of **The Resilient Equation** that suffice to establish **Triviality**. This *Mathematica* notebook can be downloaded from the following URL: `http://fitelson.org/triviality.nb`.

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