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# The structure and evolution of the Solar System comet cloud

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**Summary**. The structure and evolution of a hypothetical cloud of comets surrounding the Solar System is investigated, with particular reference to showing how the derived results depend on the assumed cometary energy and velocity distribution functions. The mean energy transfer rate by stars and giant molecular clouds is calculated and it is shown that for reasonable values of the parameters the cumulative effect of the clouds is dominant. Thus it is unlikely that the loosely-bound Oort comet cloud could survive for the age of the Solar System.

The equation describing the evolution of the comet cloud between close encounters with nebulae is solved in a good approximation for an initial condition where the energy spectrum is a power law. Formulae are also given which relate the energy spectrum to the velocity distribution function and density distribution, and it is shown how the current energy spectrum can be inferred from observations. Analytic results for the standard Oort model are presented and we show how this model is just one of a family of hypothetical comet clouds with power-law energy spectra. The spectral index of this standard model does not agree well with that obtained from observations, these indicating a model with flatter spectrum and higher degree of central condensation. More centrally condensed models may be easier to understand as a by-product of Solar System formation, and are more stable against disruption by encounters with nebulae.

## 1 Introduction

The question of whether comets owe their ultimate origin to processes associated intimately with stellar and planet formation, or whether they are formed in the denser regions of interstellar space ranks highly amongst the oldest of unsolved astronomical problems. Quite apart from the possibly significant implications for Galactic chemistry and evolution (Tinsley & Cameron 1974) the problem of cometary origin also has important links with Solar System studies and questions of the long time-scale evolution of the Earth (e.g. Clube 1978, Section 12; Napier & Clube 1979; McCrea 1981; Clube & Napier 1982a). Even a cursory study of cometary astronomy (e.g. Bailey 1975) shows that the debate 'Solar System versus interstellar' has flourished for at least four hundred years, with a mean interval of order fifty years separating periods of major advance or when one idea or the other was dominant [e.g. Kant, Laplace, Schiaparelli, Fabry (see Newton 1878; Richter 1963), Bobrovnikoff 1929; Öpik 1932; Van Woerkom 1948; Lyttleton 1948; Oort 1950]. During the past thirty years or so Oort's (1950) hypothesis, that the Solar System is surrounded by a huge primordial swarm of some $10^{11}$ comets extending half-way to the nearest star, seems generally to have been adopted, while the interstellar hypothesis has attracted only relatively few supporters.

In recent papers, however, Clube & Napier (1982b) and Napier & Staniucha (1982) have argued strongly for the demise of this so-called Oort Cloud. Briefly, in order to work successfully the standard Oort theory requires external stellar perturbations to be large enough that at large distances from the Sun the comet velocities are always distributed more or less isotropically, thus maintaining a steady observable influx of comets with small perihelion distances. If the perturbations are too weak, too few comets will be seen; if they are too large the majority of those initially present will by now have escaped from the Solar System. These authors argue that inclusion of perturbations by the recently discovered giant molecular clouds would ensure that an occasional close encounter will efficiently remove all comets with orbits lying beyond $\approx 2 \times 10^4$ AU – just the region where the Oort theory could be applied. In this way it was concluded that adoption of an interstellar origin for comets was probable and it was proposed specifically that those we currently see are simply the remnants of the most recent capture event (*cf.* Bobrovnikoff 1929).

In this paper we re-examine the evolution of a primordial cloud of Solar System comets, assuming (e.g. Kuiper 1951; Öpik 1973; Cameron 1973) that such a comet cloud *can* be formed as a by-product of the formation of the Sun and planets. Section 2 gives a brief overview of the standard Oort (1950) model and we present previously unpublished analytic results. In Section 3 we give a new calculation of the mean energy transfer rate to comets in the cloud as a result of passing encounters with stars and nebulae. At the end of this section we estimate the expected number of passages of the Solar System through a giant molecular cloud. Section 4 demonstrates how the observable properties of the comet cloud depend on the assumed cometary energy spectrum and velocity distribution function. The partial differential equation describing the evolution of the energy spectrum is solved analytically for a power-law initial condition. Section 5 discusses the implications of this work for theories of cometary origin and compares the empirical $(1/a)$-distribution with the theory. Finally the main conclusions from this study are summarized in Section 6.

## 2 Summary of Oort's (1950) model

The standard model considers a spherically symmetric swarm of comets with outer radius $R_0$ moving with the Sun in a gravitational field dominated by the Sun, at centre. Random external stellar perturbations lead to the development of an iostropic velocity distribution at all radii $r \geqslant R_{is}$ (say). Within $R_{is}$ results from the standard model must be used with care, as there is usually no guarantee of isotropy of the velocity distribution. Oort (1950) took the inner and outer radii of validity of his theory to be $4 \times 10^4$ and $2 \times 10^5$ AU respectively; more recent estimates of the outer radius (e.g. Bailey 1977) suggest values closer to $10^5$ AU.

The standard model is defined by the chosen velocity distribution and the outer radius $R_0$. For simplicity Oort assumed that velocity space at radius $r$ was filled uniformly out to a maximum velocity

$$v_{\max}(r) = \left[2GM_\odot\left(\frac{1}{r} - \frac{1}{R_0}\right)\right]^{1/2} \tag{1}$$

corresponding to the speed of free-fall from $R_0$ to $r$. Thus the mean-square velocity dispersion at radius $r$ is

$$\overline{v^2(r)} = \int_0^{v_{\max}(r)} v^2 f(r,v)\, 4\pi v^2\, dv \Big/ \int_0^{v_{\max}(r)} f(r,v)\, 4\pi v^2\, dv = {}^3\!/_5\, v_{\max}^2(r)$$

(since $f$ is independent of $v$ for the assumed velocity distribution) and

$$\overline{v^2(r)} \equiv 3\,\sigma_{\rm r}^2(r) = {}^6\!/_5\, GM_\odot \left(\frac{1}{r} - \frac{1}{R_0}\right) \tag{2}$$

where $\sigma_{\rm r}(r)$ is the radial component of velocity dispersion at $r$.

The number density distribution $n(r)$ can then be obtained by solving the equation of hydrostatic equilibrium, given (e.g. Binney 1980) by

$$\frac{d}{dr}\,[n(r)\,\sigma_{\rm r}^2(r)] = -\frac{GM_\odot n(r)}{r^2} - \frac{2\beta(r)\,n(r)\,\sigma_{\rm r}^2(r)}{r}. \tag{3}$$

Here we have kept the assumption that points in velocity space are distributed uniformly, but for completeness have allowed the distribution function $f$ to be anisotropic. The velocity dispersion anisotropy parameter $\beta(r)$ is defined by $\beta^2 = 1 - \sigma_{\rm t}^2/\sigma_{\rm r}^2$, where $\sigma_{\rm t}(r)$ is the velocity dispersion in each transverse direction. Notice that $\beta = 0$ corresponds to velocity dispersion isotropy; $\beta = 1$ to purely radial motions. Equation (3) can be integrated for $\beta$ = constant to give

$$n(z) = A_1 \left(\frac{1}{z} - 1\right)^{3/2} \left(\frac{z_1}{z}\right)^{2\beta} \tag{4}$$

where the constant $A_1$ is given in terms of an arbitrary $z_1$ by

$$A_1 = n(z_1)\left(\frac{z_1}{1 - z_1}\right)^{3/2} \tag{5}$$

and $z = r/R_0$. In the standard model, $\beta = 0$.

The total number of comets in the standard model between $z_{\rm is}$ and $z$ is then

$$N_{\rm c}(z, z_{\rm is}) = 4\pi A_1 R_0^3 \int_{z_{\rm is}}^{z} z^{1/2}(1-z)^{3/2}\, dz$$

$$= 4\pi A_1 R_0^3 \left[\frac{\sin^{-1}(z^{1/2})}{8} - \frac{z^{1/2}(1-z)^{1/2}}{24}(3 - 14z + 8z^2)\right]_{z_{\rm is}}^{z}. \tag{6}$$

Extrapolating the model to small radii ($z_{\rm is} \ll 1$), the total number of comets is found to be

$$N_{\rm c}(1, 0) = \frac{\pi^2}{4} A_1 R_0^3. \tag{7}$$

The normalizing factor $A_1$ in (4) and (6) is determined empirically by comparing the observed distribution of semi-major axes, $a$, with that predicted by theory. We define the predicted $a$-distribution to be $F_{\rm 1c}(a, D)$, so that the number of comets passing perihelion per unit time with semi-major axes $(a, a + da)$ and perihelia $q \leqslant D$ is $F_{\rm 1c}(a, D)\,da$. Our notation $F_{\rm 1c}(a, D)$ is designed to make explicit the analogy with so-called 'loss-cone' consumption of stars in galactic nuclei by a massive black hole (see Section 4). Following Oort's

(1950) calculation we obtain

$$F_{1c}(a,D)\,da = \frac{3\pi A_1}{4}(2GM_\odot)^{1/2} D R_0^{3/2} a^{-2}\,da \qquad (a \geqslant \tfrac{1}{2}R_{is}). \tag{8}$$

We note (*cf.* Bailey 1977) that $F_{1c}(a,D)$, proportional to $a^{-2}$, has a quite different functional form from either $n(r=2a)$ or $N_c(r=2a, z_{is})$: the density structure of the comet cloud is not related *directly* to observations of the $a$-distribution. On the Oort theory the $a$-distribution peaks close to $a_{is} \simeq \tfrac{1}{2}R_{is}$ because within $R_{is}$ the loss-cone orbits are not replenished sufficiently rapidly by stellar perturbations.

Integrating (8) out to $a_{max} \simeq \tfrac{1}{2}R_0$ gives the total rate of passage of comets with $q \leqslant D$ and semi-major axes greater than some value $a$, say. The observed rate, for $D \simeq 1.5$ AU and $a \simeq 25\,000$ AU, is roughly one per year (Oort 1950); i.e. $F_{obs} \approx 3\times10^{-8}\,s^{-1}$, with an uncertainty of about a factor of 2 (*cf.* Fernandez 1981). Thus

$$A_1 \simeq F_{obs} \Big/ \left[\frac{3\pi}{4}(2GM_\odot)^{1/2} D R_0^{3/2}\left(\frac{1}{a}-\frac{2}{R_0}\right)\right] \tag{9}$$

or

$$A_1 \approx 5\times10^{-5}(10^5\,\text{AU}/R_0)^{3/2} \qquad \text{comets (AU)}^{-3}. \tag{10}$$

The total number of comets in the standard model is then of order $N_c(1,0) \simeq 10^{11}(R_0/10^5\,\text{AU})^{3/2}$.

These equations completely determine the standard Oort (1950) model of the cometary cloud. Apart from the assumed velocity distribution [$f$ = constant, uniform and isotropic within $v_{max}(r)$], the only crucial free parameter in the theory is $R_0$, the cloud's outer radius. This, and the value of $R_{is}$, depend principally on the magnitude of the energy transfer rate by stellar perturbations.

## 3 Mean energy transfer rate by stars and nebulae

### 3.1 OUTLINE OF THE MODEL

We consider a simplified model of the encounter process in which it is assumed that the Sun and its associated swarm of comets move with velocity **V** through a static distribution of field objects having masses $M$ and mean number density $n$. We define (see Fig. 1) a spherical coordinate system centred on the Sun ($S$) with $z$-axis parallel to the direction of **V**. In this coordinate system incoming field objects have impact parameter $b$, which in the model is also the distance of closest approach to $S$, and azimuthal angle $\phi_*$. They pass a typical comet, located at $(r,\theta,\phi)$, at a distance $d$ of closest approach given by

$$d^2 = b^2 + r^2\sin^2\theta - 2rb\sin\theta\cos(\phi_* - \phi). \tag{11}$$

In the impulse approximation the changes of velocity of the Sun and comet, $\mathbf{\Delta v}_\odot$ and $\mathbf{\Delta v}_c$ respectively, are

$$\mathbf{\Delta v}_\odot = \frac{2GM}{bV}\frac{\mathbf{b}}{b}$$

and

$$\mathbf{\Delta v}_c = \frac{2GM}{dV}\frac{\mathbf{d}}{d}.$$

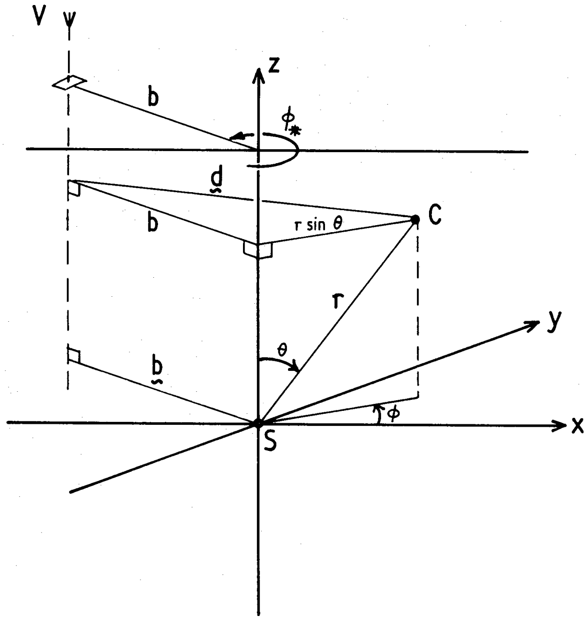


**Figure 1.** Geometry of the encounter model. A field object passes the Sun ($S$) with velocity $V$ and impact parameter $b$. It passes the comet ($C$), at position ($r$, $\theta$, $\phi$), at a distance of closest approach $d$. The origin for $\phi_*$ is the same as that for $\phi$.

The relative change of velocity of the comet with respect to the Sun is $\mathbf{\Delta v} = \mathbf{\Delta v}_c - \mathbf{\Delta v}_\odot$, giving a net relative energy gain per unit mass of comet $\Delta\epsilon = \frac{1}{2}(\mathbf{\Delta v}_c - \mathbf{\Delta v}_\odot)^2$. This reduces to

$$2\Delta\epsilon = \left(\frac{2GM}{bV}\right)^2 \left(\frac{r\sin\theta}{d}\right)^2. \tag{12}$$

The impulse approximation gives a good estimate of the systematic energy increase provided that the comet's orbital angular velocity $\omega$ is small compared to that, $\omega_f \simeq V/b$, of the passing field object. A rough estimate of the maximum impact parameter $b_{max}$, beyond which energy transfer can be neglected, is given (*cf.* Spitzer 1958; Jackson 1962; Knobloch 1976) by

$$b_{max} \simeq \frac{V}{\omega} \approx V(r^3/GM_\odot)^{1/2}. \tag{13}$$

Here for $\omega$ we have used the angular frequency $\omega_0$ corresponding to that of a circular orbit at radius $r$; the exact value of $b_{max}$ is not too important because of the rapid fall-off in $\Delta\epsilon$ for large impact parameters (equation 12).

In addition to $b_{max}(r)$ several other length-scales enter the problem. First we define $b_{min}$, which is the minimum impact parameter expected to occur during the lifetime $T$ of the Solar System. If $\overline{nV}$ is the mean value of $nV$ averaged over time $T$, the expected number of encounters with impact parameters less than some value $b$ is $N_{enc}(b) = \pi b^2 \overline{nV}T$. Setting this equal to a half (say) gives an estimate of $b_{min}$; i.e.

$$b_{min} \simeq (2\pi\, \overline{nV}T)^{-1/2}. \tag{14}$$

Next we introduce $b'_{min}(r)$, which is the minimum impact parameter which might be expected to occur during a typical orbital period $P_{orb}(r) \approx 2\pi/\omega_0 = 2\pi r^{3/2}/(GM_\odot)^{1/2}$. This is

$$b'_{min}(r) = b_{min}\,[T/P_{orb}(r)]^{1/2} \tag{15}$$

which gives a rough estimate of the impact parameter beyond which the energy transfer may be regarded as approximately continuous.

We now define the distance $d_{\rm min}$ within which a single encounter would cause immediate ejection of the comet from the Solar System. The energy/mass needed to escape from the radius $r$ is or order $GM_\odot/2r$ (assuming a circular orbit), so

$$\left(\frac{2GM}{bV}\right)^2 \left(\frac{r\sin\theta}{d_{\rm min}}\right)^2 \simeq \frac{GM_\odot}{r}. \qquad (16)$$

For encounters with stars, $d_{{\rm min},*} < b_{{\rm min},*} \ll r$, so these close star–comet encounters all have $b \approx r\sin\theta$. Thus we obtain

$$d_{{\rm min},*} \simeq 2\left(\frac{GM_\odot r}{V^2}\right)^{1/2} \frac{M_*}{M_\odot}. \qquad (17)$$

For encounters with giant molecular clouds, $d \simeq b \gg r$. (16) then implies (averaging over $\theta$, setting $<\sin^{1/2}\theta> = 0.874$)

$$d_{\rm min,GMC} \approx \left(\frac{G}{M_\odot}\right)^{1/4} \left(\frac{M_{\rm GMC}}{V}\right)^{1/2} r^{3/4}. \qquad (18)$$

In this approximation all comets beyond $r$ are eliminated from the cloud whenever a giant molecular cloud passes closer to the Sun than $d_{\rm min,GMC}$. The mean interval between such elimination of comets is

$$\tau_{\rm elim}(r) \simeq 2T\,(b_{\rm min,GMC}/d_{\rm min,GMC})^2. \qquad (19)$$

Setting $\tau_{\rm elim}(r) = T$ defines a radius, $r_{\rm s}$ say, within which comets are 'safe' from direct ejection by close encounters with nebulae. We should note, however, that (18) and (19) depend on a model where the giant molecular cloud is treated as a point mass. Very close enounters will involve passage of the Solar System *through* the cloud, in which case the net $\Delta\epsilon$ per encounter will depend on the cloud's internal structure.

Stellar encounters may cause direct removal of comets from the cometary cloud in two ways. First those that pass very close to the Sun may perturb the Sun away from its comets. In this case the limiting radius for survival of comets is obtained from (16) by setting $b \simeq b_{{\rm min},*}$ and $d_{\rm min} \simeq r\sin\theta$. This gives an outer limiting radius defined by the condition that $d_{{\rm min},*}(r) = b_{{\rm min},*}$. The second way that stars directly remove comets is by ejecting all those that happen to lie within a distance $d_{{\rm min},*}(r)$ of their paths. This 'sweeping' of the cloud is a less dramatic evolutionary effect than a close nebular encounter or a close star–Sun encounter, but it could nevertheless still significantly reduce the cometary numbers at large radii where $d_{{\rm min},*}(r)$ is larger. It can be shown, however, that neglecting all other evolutionary processes the fraction of the original number density of comets at $r$ which survive this dissipation mechanism is $\exp(-d^2_{{\rm min},*}/b^2_{{\rm min},*})$ (*cf.* Nezhinski 1972). Thus, since $d_{{\rm min},*}(r) < b_{{\rm min},*}$ (since otherwise the Sun would have been perturbed away from the comets beyond $r$), 'sweeping' of the cloud can be neglected at the factor of 2 level. For typical stellar parameters (see below) the radius where $d_{{\rm min},*}(r) = b_{{\rm min},*}$ is of order $2.5\times10^5$ AU, somewhat larger than the limit already imposed by the Galactic tidal field ($\approx 10^5$ AU; *cf.* Wyatt & Faintich 1971).

Lastly we calculate the limiting radius $r_{\rm elim}$, beyond which comets are removed by single encounters with giant molecular clouds on an orbital time-scale. We define this by setting

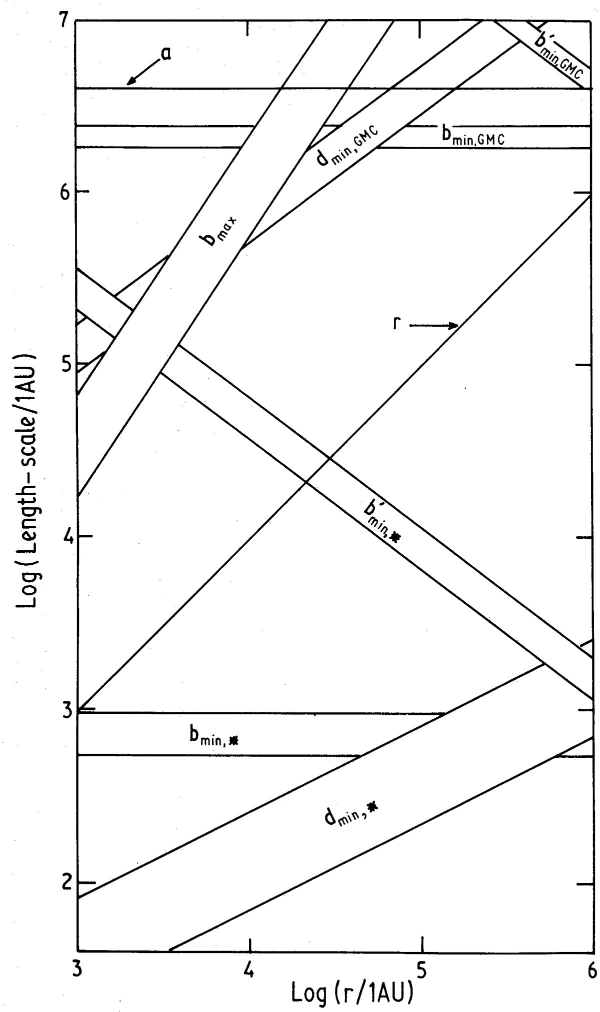


**Figure 2.** The length-scales defined in Section 3.1 are shown plotted versus log ($r$). The width of each band illustrates the range of uncertainty introduced by variations of the parameters as discussed in Section 3.4.

$\tau_{\text{elim}}(r) = P_{\text{orb}}(r)$, which implies

$$r_{\text{elim}} \simeq 0.37 \left(\frac{n_{\text{GMC}} M_{\text{GMC}}}{M_\odot}\right)^{-1/3}. \tag{20}$$

For typical giant molecular cloud parameters this radius too is larger than the Galactic tidal field limit.

These length-scales are shown in Fig. 2, plotted versus log ($r$) as narrow bands. The width of each band illustrates the range of uncertainty in each length-scale caused by the uncertainty in basic parameters discussed below in Section 3.4.

### 3.2 QUASI-STEADY RATE OF INCREASE OF COMET ENERGIES BY SMALL PERTURBATIONS

We define the mean 'quasi-steady' rate of increase of a comet's energy/mass to be $\bar{\dot{\epsilon}}_{\text{qs}}(r)$. This is obtained by integrating (12) over all impact parameters $b'_{\min}(r) \leqslant b \leqslant b_{\max}(r)$ satisfying the constraint $d \geqslant d_{\min}$, and then averaging the result over the angular coordinates $(\theta, \phi)$.

The number of encounters per unit time with impact parameters $(b, b+db)$, $(\phi_*, \phi_*+d\phi_*)$ is $nVb\, d\phi_*\, db$, and each such encounter gives a mean-square velocity increase $2\Delta\epsilon$ given by (12). Thus

$$d(2\dot{\epsilon}_{\rm qs}) = n\left(\frac{2GM}{bV}\right)^2 \; bV\frac{r^2\sin^2\theta}{d^2}\; d\phi_*\, db. \tag{21}$$

We first evaluate the $\phi_*$-integral. This is most easily effected by changing variables from $\phi_*$ to $d$ using (11); i.e.

$$d(2\dot{\epsilon}_{\rm qs}) = \frac{4G^2M^2n}{V}\;\frac{r^2\sin^2\theta}{b}\; I_{\rm d}(b,\theta)\, db \tag{22}$$

where the $d$-integral $I_{\rm d}$ is given by

$$I_{\rm d} = 2\int_{d_{\rm L}}^{d_{\rm U}} \frac{1}{d^2}\frac{d\phi_*}{dd}\, dd \tag{23}$$

and the lower and upper limits to the range of integration, $d_{\rm L}$ and $d_{\rm U}$ respectively, are

$$\left.\begin{aligned} d_{\rm L} &= \max\{d_{\rm min},\quad |b - r\sin\theta|\} \\ d_{\rm U} &= b + r\sin\theta \end{aligned}\right\}. \tag{24}$$

If $d_{\rm U} \leqslant d_{\rm L}$ (e.g. if $d_{\rm min}$ should be greater than $b_{\rm min}$), $I_{\rm d} = 0$. Otherwise (23) reduces to

$$I_{\rm d} = \begin{cases} 2\pi/|b^2 - r^2\sin^2\theta| & |b - r\sin\theta| > d_{\rm min} \\ \dfrac{2}{b\, d_{\rm min}}\left(1 - \dfrac{d_{\rm min}^2}{4b^2}\right)^{1/2} & b = r\sin\theta \\ \dfrac{2}{|b^2 - r^2\sin^2\theta|}\left\{\dfrac{\pi}{2} - \sin^{-1}\left[\dfrac{(b^2 + r^2\sin^2\theta)\, d_{\rm min}^2 - (b^2 - r^2\sin^2\theta)^2}{2br\sin\theta\, d_{\rm min}^2}\right]\right\} & |b - r\sin\theta| < d_{\rm min}. \end{cases} \tag{25}$$

The majority of encounters satisfy the condition $|b - r\sin\theta| > d_{\rm min}$. For the others ($|b - r\sin\theta| < d_{\rm min}$) it is a great simplification to work with an approximation to the rather complicated expression above. We define

$$\left.\begin{aligned} x &= (b - r\sin\theta)/d_{\rm min} \\ \alpha &= d_{\rm min}/b \end{aligned}\right\} \tag{26}$$

and rewrite (25), after some algebra, in the form

$$I_{\rm d}(b,\theta) = \frac{1}{b\, d_{\rm min}}\; f(x,\alpha) \tag{27}$$

where

$$f(x,\alpha) = \frac{1}{(1-\alpha x/2)}\;\frac{1}{|x|}\;\cos^{-1}\left[\frac{2(1-\alpha x/2)^2(1-x^2)}{(1-\alpha x)} - 1\right]. \tag{28}$$

The mean value of $f$ over the $x$-range of interest ($-1 \leqslant x \leqslant 1$) is

$$\bar{f}(\alpha) = \int_0^1 f(x,\alpha)\, dx$$

which for the special case $\alpha = 0$ (i.e. $b \gg d_{\min}$) reduces to $\pi \ln(2)$. For other values of $\alpha \leqslant 1$ it is within 20 per cent of this value. We therefore approximate (25) as

$$I_d(b, \theta) = \begin{cases} \dfrac{2\pi}{|b^2 - r^2 \sin^2\theta|} & |b - r\sin\theta| > d_{\min} \\ \dfrac{\pi \ln(2)}{b\, d_{\min}} & |b - r\sin\theta| \leqslant d_{\min}. \end{cases} \tag{29}$$

We next integrate (22) over all relevant impact parameters. This gives

$$2\dot{\epsilon}_{qs} = \frac{4G^2M^2n}{V} I_b(r, \theta) \tag{30}$$

where the $b$-integral $I_b$ is defined by

$$I_b = \int_{b'_{\min}(r)}^{b_{\max}(r)} \frac{r^2 \sin^2\theta}{b} I_d(b, \theta)\, db. \tag{31}$$

We evaluate this integral separately for the three cases

(A) $b'_{\min} > r\sin\theta + d_{\min}$,
(B) $r\sin\theta - d_{\min} < b'_{\min} < r\sin\theta + d_{\min}$, and
(C) $b'_{\min} < r\sin\theta - d_{\min}$.

In Case (A), which applies principally to encounters with nebulae, we obtain

$$I_{b,A} = \int_{b'_{\min}(r)}^{b_{\max}(r)} \frac{2\pi r^2 \sin^2\theta}{b(b^2 - r^2\sin^2\theta)}\, db$$

$$I_{b,A}(r, \theta) = \pi \ln\left(\frac{b_{\max}^2 - r^2\sin^2\theta}{b_{\max}^2} \quad \frac{b'^2_{\min}}{b'^2_{\min} - r^2\sin^2\theta}\right). \tag{32}$$

In the limit $b'_{\min} \gg r$ this reduces approximately to

$$I_{b,A} \simeq \pi r^2 \sin^2\theta \left(\frac{1}{b'^2_{\min}(r)} - \frac{1}{b^2_{\max}(r)}\right). \tag{33}$$

Note that this could have been obtained more simply from (21) using the approximation $b \simeq d \gg r$.

If $b'_{\min} - d_{\min} < r$, as may be the case for stellar encounters, (32) should be used for angles $\theta < \theta_1$, where

$$\theta_1 \simeq \sin^{-1}\left(\frac{b'_{\min} - d_{\min}}{r}\right). \tag{34}$$

The other two cases, (B) and (C), apply only to stellar encounters. In Case (B), covering the $\theta$-range $\theta_1 \leqslant \theta \leqslant \theta_2$, where

$$\theta_2 \simeq \sin^{-1}\left(\frac{b'_{\min} + d_{\min}}{r}\right) \tag{35}$$

we obtain

$$I_{b,B}(r,\theta) \simeq \pi \ln(2) \frac{r^2 \sin^2\theta}{d_{min}} \left[\frac{1}{b'_{min}} - \frac{1}{(d_{min} + r\sin\theta)}\right]$$
$$+ \pi \ln\left[\frac{(b_{max}^2 - r^2\sin^2\theta)}{b_{max}^2} \cdot \frac{(d_{min} + r\sin\theta)^2}{d_{min}(2r\sin\theta + d_{min})}\right]. \quad (36)$$

And in Case (C), covering $\theta_2 \leqslant \theta \leqslant \pi/2$, we have

$$I_{b,C}(r,\theta) \simeq \pi \ln\left[\frac{(r\sin\theta - d_{min})^2}{d_{min}(2r\sin\theta - d_{min})} \cdot \frac{(r^2\sin^2\theta - b'^2_{min})}{b'^2_{min}}\right] + \pi\ln(2)\frac{2r^2\sin^2\theta}{(r^2\sin^2\theta - d_{min}^2)}$$
$$+ \pi \ln\left[\frac{(b_{max}^2 - r^2\sin^2\theta)}{b_{max}^2} \cdot \frac{(r\sin\theta + d_{min})^2}{d_{min}(2r\sin\theta + d_{min})}\right]. \quad (37)$$

The required mean energy increase rate, $\bar{\dot{\epsilon}}_{qs}(r)$, is now obtained by averaging (30) over all $\theta$, with $I_b(r,\theta)$ given by (32), (36) and (37). Thus

$$2\bar{\dot{\epsilon}}_{qs}(r) = \frac{8\pi G^2 M^2 n}{V} \cdot I_{qs}(r) \quad (38)$$

where the dimensionless function $I_{qs}$ is

$$I_{qs}(r) = \frac{1}{2\pi}\int_0^{\pi/2} \sin\theta \, I_b(r,\theta)\, d\theta. \quad (39)$$

In Case (A) [i.e. $b'_{min}(r) > r + d_{min}(r)$] we obtain

$$I_{qs} = \sqrt{\beta^2 - 1}\,\sin^{-1}\left(\frac{1}{\beta}\right) - \sqrt{\alpha^2 - 1}\,\sin^{-1}\left(\frac{1}{\alpha}\right) \quad (40)$$

where $\beta = b_{max}(r)/r$ and $\alpha = b'_{min}(r)/r$. This can be approximated by

$$I_{qs}(r) \simeq \frac{1}{3} r^2 \left(\frac{1}{b'^2_{min}} - \frac{1}{b^2_{max}}\right) \qquad (b'_{min} \gtrsim \text{few } r) \quad (41)$$

which also follows from the approximation (33). Equation (40), or the approximation (41), applies to all encounters with giant molecular clouds, and also at small radii ($r \lesssim 3 \times 10^4$ AU) to encounters with stars.

At large radii ($r \gtrsim 3 \times 10^4$ AU), $b'_{min,*}(r)$ becomes rapidly smaller than $r$ (see Fig. 2). Then, both $\theta_1$ and $\theta_2$ are small and $I_{qs}$ may be approximated by neglecting the small contributions to the integral resulting from $\theta < \theta_2$ (i.e. Cases A and B). In this limit ($d_{min} < b'_{min} \ll r \ll b_{max}$), $I_{b,C} \simeq 2\pi \ln[r^2\sin^2\theta/d_{min}(r)\, b_{min}(r)]$, which implies

$$I_{qs} \simeq \ln\left[\frac{r^2}{d_{min}(r)\, b'_{min}(r)}\right] - 2[1 - \ln(2)]. \quad (42)$$

Since $r^2 \gg b'_{min,*}(r)\, d_{min,*}(r)$, the logarithmic term usually dominates the small numerical factor, giving the rough approximation

$$\overline{2\dot{\epsilon}_{qs,*}}(r) \approx \frac{8\pi G^2 M_*^2 n_*}{V} \cdot \ln\left(\frac{r^2}{d_{min,*}(r)\, b'_{min,*}(r)}\right). \quad (43)$$

For encounters with nebulae equation (41) can be used. This gives

$$\overline{2\dot{\epsilon}_{\mathrm{qs,GMC}}}(r) \simeq \frac{8\pi G^2 M_{\mathrm{GMC}}^2 n_{\mathrm{GMC}}}{V} \cdot \frac{1}{3}\left[\frac{r^2}{b'^2_{\mathrm{min,GMC}}(r)} - \frac{r^2}{b^2_{\mathrm{max}}(r)}\right] \quad [b_{\mathrm{max}}(r) > b'_{\mathrm{min,GMC}}(r)]. \tag{43}$$

If $b_{\max}(r) < b'_{\min,\mathrm{GMC}}(r)$ (as it is, typically, for $r \lesssim 10^{4.8}$ AU; see Fig. 2), $\overline{2\dot{\epsilon}_{\mathrm{qs,GMC}}} = 0$ and the energy transfer by giant molecular clouds is entirely intermittent in character.

### 3.3 TOTAL AND INTERMITTENT ENERGY INCREASE RATE INCLUDING CLOSE ENCOUNTERS

The mean total energy increase rate can be obtained from the above formulae by replacing $b'_{\min}(r)$ with $b_{\min}$. For stars, equation (42) applies for $r \gtrsim 10^3$ AU and to a very good approximation we have

$$\overline{2\dot{\epsilon}_{\mathrm{tot},*}}(r) \simeq \frac{8\pi G^2 M_*^2 n_*}{V} \left\{\ln\left[\frac{r^2}{b_{\min,*}\, d_{\min,*}(r)}\right] - 2\,[1 - \ln(2)]\right\}. \tag{45}$$

For giant molecular clouds, provided $b_{\max}(r) > b_{\min,\mathrm{GMC}}$, we have

$$\overline{2\dot{\epsilon}_{\mathrm{tot,GMC}}}(r) \simeq \frac{8\pi G^2 M_{\mathrm{GMC}}^2 n_{\mathrm{GMC}}}{V} \cdot \frac{1}{3}\left[\frac{r^2}{b^2_{\mathrm{min,GMC}}} - \frac{r^2}{b^2_{\mathrm{max}}(r)}\right]. \tag{46}$$

We define the intermittent energy increase rate, due to encounters with $b < b'_{\min}(r)$, to be the total minus the quasi-steady contribution. Such encounters occur on average about once every two orbital periods $P_{\mathrm{orb}}(r)$, and depending on how close was the encounter, or how recently, the instantaneous value of this contribution can vary widely about the mean. These random fluctuations in the actual energy transfer rate have important implications for detailed models of the comet cloud and its evolution (e.g. Hills 1981), but are outside the scope of the present investigation.

The mean intermittent energy increase rate is simply the mean total energy increase rate minus the quasi-steady contribution. For stars, at large radii ($r \gtrsim 3 \times 10^4$ AU) we therefore have approximately

$$\overline{2\dot{\epsilon}_{\mathrm{int},*}}(r) \simeq \frac{8\pi G M_*^2 n_*}{V} \cdot \ln\left[\frac{b'_{\min,*}(r)}{b_{\min,*}}\right] \qquad (r \gtrsim 3 \times 10^4\ \mathrm{AU}) \tag{47}$$

which in order of magnitude is generally comparable with the quasi-steady contribution. At smaller radii [$r \lesssim 3 \times 10^4$ AU; where $b'_{\min,*}(r) > r$] the quasi-steady contribution (equation 40) falls with decreasing radius very much more rapidly than (45). At small radii, therefore, the stellar energy transfer is almost entirely due to intermittent encounters with $b < b'_{\min,*}(r)$, giving $\overline{2\dot{\epsilon}_{\mathrm{int},*}}(r) \simeq \overline{2\dot{\epsilon}_{\mathrm{tot},*}}(r)$.

For nebulae, (44) and (46) show that the energy transfer rate at all radii is dominated by the intermittent contribution, giving $\overline{2\dot{\epsilon}_{\mathrm{int,GMC}}} \simeq \overline{2\dot{\epsilon}_{\mathrm{tot,GMC}}}$. We note, however, that (46) is strictly applicable only provided the Solar System always remains outside the giant molecular cloud. In fact (see Fig. 2) for reasonable values of the parameters this condition is usually not met and it is therefore necessary to consider in more detail the effect of penetrating encounters. This question is treated in the Appendix, where it is shown that the total mean energy transfer rate due to giant molecular cloud encounters with $b_{\min,\mathrm{GMC}} < a$ (the cloud radius) can be written in the form

$$\overline{2\dot{\epsilon}_{\mathrm{tot,GMC}}}(r) \simeq \frac{8\pi G^2 M_{\mathrm{GMC}}^2 n_{\mathrm{GMC}}}{V} \cdot \frac{1}{3} \frac{r^2}{b_{\min,\mathrm{GMC}}^2} \cdot g_{\mathrm{tot}} \tag{48}$$

where $g_{\mathrm{tot}}$ is a dimensionless constant or order unity which depends mainly on details of the cloud's internal structure. Equation (48) breaks down only at small radii where $b_{\max}(r)$ becomes small compared with the dimensions of the giant molecular cloud; this may enable a small region of the comet cloud ($r \lesssim 10^{3.7\pm0.2}$; see Appendix) to remain relatively unaffected by giant molecular clouds despite their inhomogeneous structures.

Although the instantaneous value of the giant molecular cloud energy transfer rate will fluctuate widely about its mean, we may still use (48) to get an estimate of the total energy transferred to comets at $r$ integrated over the lifetime $T$ of the Solar System. Neglecting evolution of the various parameters, this is

$$2\Delta\epsilon_{\mathrm{GMC}}(r) \simeq \frac{1}{3} g_{\mathrm{tot}} (4\pi G\, n_{\mathrm{GMC}} M_{\mathrm{GMC}}\, T r)^2 \tag{49}$$

which, with $T = 4.5 \times 10^9$ yr, implies

$$[2\Delta\epsilon_{\mathrm{GMC}}(r)]^{1/2} \simeq 35\, g_{\mathrm{tot}}^{1/2} \left(\frac{n_{\mathrm{GMC}}}{10^{-8}\,\mathrm{pc}^{-3}}\right)\left(\frac{M_{\mathrm{GMC}}}{5\times10^5 M_\odot}\right)\left(\frac{r}{10^4\,\mathrm{AU}}\right)\ \mathrm{m\,s^{-1}}. \tag{50}$$

This exceeds the energy needed to escape from $r$ at a radius, $r_0$, given by

$$r_0 \simeq 4.2\times10^4 g_{\mathrm{tot}}^{-1/3} \left(\frac{10^{-8}\,\mathrm{pc}^{-3}}{n_{\mathrm{GMC}}}\right)^{2/3} \left(\frac{5\times10^5 M_\odot}{M_{\mathrm{GMC}}}\right)^{2/3}\ \mathrm{AU} \tag{51}$$

which is comparable in order of magnitude to the 'safe' radius introduced earlier (equation 19).

In Fig. 3 we show the quasi-steady and total stellar energy transfer rates calculated directly from (39) for the parameters $M_* = 0.7\, M_\odot$, $n_* = 0.1\,\mathrm{pc}^{-3}$ and two values of the velocity: $V = 16$ and $60\ \mathrm{km\,s^{-1}}$. Also shown, as dashed lines, are the analytic approximations (equations 41 and 43) to the quasi-steady contribution; the approximation (45) to the total stellar energy transfer rate would be indistinguishable on the figure from the directly calculated function. At large radii the quasi-steady stellar energy transfer rate can be roughly approximated by a power law

$$\overline{\dot{\epsilon}_{\mathrm{qs},*}} \simeq C r^{1/2} \quad \mathrm{m^2\,s^{-3}} \quad (r \gtrsim 3\times10^4\,\mathrm{AU}) \tag{52}$$

where $C \approx C_0 = 10^{-21}\,\mathrm{m^{3/2}\,s^{-3}}$ has an uncertainty of order 2, depending mainly on $V$. Thus at $10^5$ AU the stellar energy transfer rate, dominated by the quasi-steady contribution, is of order $10^{-13}\,\mathrm{m^2\,s^{-3}}$, larger than the corresponding values quoted by Öpik (1973, equation 9.4) and Weissman (1980a, equation 8) by factors of order 10 and 2 respectively. The cumulative velocity perturbation at $10^5$ AU due to encounters with stars integrated over the lifetime of the Solar System is thus of order

$$[\overline{2\dot{\epsilon}_{\mathrm{tot},*}}(r = 10^5\,\mathrm{AU})\, T]^{1/2} \simeq 170(\overset{\times}{\div} \sqrt{2})\ \mathrm{m\,s^{-1}}. \tag{53}$$

We note that the lower limit of this quantity (corresponding to a predominance of high-velocity stellar encounters) is in agreement with Weissman's (1980a) estimate. However, the

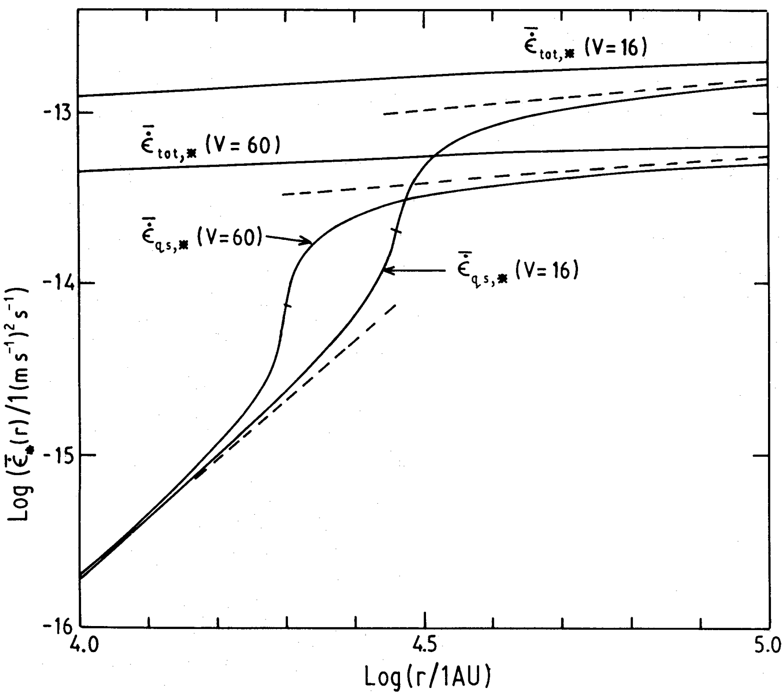


**Figure 3.** The mean total stellar energy transfer rate and the quasi-steady contribution are shown for two values of the velocity ($V = 16$ and $60\ \mathrm{km\ s^{-1}}$). Each curve assumes $M_* = 0.7\ M_\odot$ and $n_* = 0.1\ \mathrm{pc^{-3}}$. The dashed lines show the analytic approximations (equations 41 and 43) to the quasi-steady contribution. The tick on each $\bar{\dot{\epsilon}}_{qs,*}$ curve indicates the radius where $b'_{min,*}(r) = r$.

important point to be emphasized is that this is smaller than (50) for all but an extreme combination of giant molecular cloud parameters. This shows that, at large radii, the mean energy increase due to giant molecular clouds is greater than that due to stars. However, this dominant contribution is basically intermittent in character, itself dominated by occasional close or penetrating encounters with $b \lesssim$ few $a$. These occur at intervals separated by periods of order $10^8$ yr during which the total energy transfer rate is dominated by stars.

We thus arrive at a picture in which the total energy transfer rate at large radii is dominated for long periods by the stellar contribution, comprising a significant quasi-steady component together with occasional fluctuations due to intermittent close encounters. At intervals of order $10^8$ yr the Solar System will pass within a few tens of parsecs of a giant molecular cloud complex, and for a short time this contribution will dominate the energy transfer rate. The evolution of the comet cloud during these times may be quite dramatic, as the outer layers will almost certainly be stripped off and replaced (presumably) by comets from more tightly bound orbits within. We note, however, that this kind of evolution does *not* lead to a progressive shrinking of the comet cloud as suggested by Weissman (1980a) and Napier & Staniucha (1982). Athough the total number of comets in the cloud declines, its size does not.

### 3.4 DISCUSSION OF PARAMETERS AND VALIDITY OF MODEL

For the stellar encounters we follow previous authors (e.g. Oort 1950; Rickman 1976) in adopting $\langle M_*^2 n_*(0)\rangle \simeq 0.05\ M_\odot^2\ \mathrm{pc^{-3}}$, corresponding to a mid-plane space density $n_*(0) \simeq 0.1\ \mathrm{pc^{-3}}$ and a typical stellar mass of order $0.7\ M_\odot$. For the giant molecular clouds, observations (e.g. Solomon, Sanders & Scoville 1979 and references therein) indicate a total number of order 4000 in the 'molecular ring' of the Galaxy ($4 \lesssim R \lesssim 8$ kpc). Each cloud has a typical mass $M_{\rm GMC} \approx 5 \times 10^5 M_\odot$ and radius $a \approx 20$ pc. In the Solar neighbourhood (assumed here to lie at about 10 kpc from the Galactic Centre), the giant molecular cloud space density is about 1/5 the mean within the ring. Perpendicular to the plane the density distribution is Gaussian,

$$n_{\rm GMC}(z) = n_{\rm GMC}(0) \exp(-z^2/2h^2) \tag{54}$$

with a scale-height (Gordon & Burton 1979) $h \simeq 50$ pc. With these parameters we obtain $n_{\rm GMC}(0) \simeq 4 \times 10^{-8}\,{\rm pc}^{-3}$ and $M_{\rm GMC}^2\, n_{\rm GMC}(0) \simeq 10^4 M_\odot^2\,{\rm pc}^{-3}$.

In our simplified encounter model we have neglected the random velocities of the field objects and taken $V$ simply to be the velocity of the Solar System through a static distribution of field particles. In fact, although $V$ is known quite well at the present time (the space motion, $V$, is $\simeq 16\,{\rm km\,s^{-1}}$, while that, $V_\perp(0)$, perpendicular to the plane is $\simeq 7\,{\rm km\,s^{-1}}$; Woolley 1971), it is not clear what value should be chosen as representative of the solar motion over the past $4.5 \times 10^9$ yr. On the one hand, since the Sun was presumably formed close to the Galactic plane with a low peculiar motion, the present observation of a low velocity might suggest that its velocity has always been small. However, the relaxation time-scale (dominated by massive complexes in the plane such as giant molecular clouds) is only of order $5 \times 10^8$ yr (Wielen 1977), so the Sun's present small velocity ought not to persist. It might therefore be argued that it would be more reasonable to choose a value of $V$ more typical of stars of the same age as the Sun. Following Wielen (1977) such parameters at the present time would be $V \simeq 60\,{\rm km\,s^{-1}}$ and $V_\perp(0) \simeq 35\,{\rm km\,s^{-1}}$.

Clearly, without a detailed evolutionary calculation of the Solar motion it is not possible to choose reliably between the possibilities. We have decided therefore to accept that $V$ is uncertain, and have considered the above two cases: Case (1) $V = 16\,{\rm km\,s^{-1}}$ and $V_\perp(0) = 7\,{\rm km\,s^{-1}}$; and Case (2) $V = 60\,{\rm km\,s^{-1}}$ and $V_\perp(0) = 35\,{\rm km\,s^{-1}}$, as extremes.

This uncertainty in the mean Solar motion introduces an associated uncertainty in the mean density $\bar{n}$ of the field objects encountered by the Sun. Assuming that the Solar motion perpendicular to the Galactic plane is simple harmonic with amplitude $A = V_\perp(0)/\sqrt{C_{\rm gal}}$, where the Galactic constant $C_{\rm gal} \approx 10^{-29}\,{\rm s}^{-2}$ (Wielen 1973), the fraction of the total time spent between heights $(z, z+dz)$ is $dz/(\pi A\sqrt{1 - z^2/A^2})$. The mean density of field objects encountered by the Solar System is then approximately

$$\bar{n} = n(0)\frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-x^2}} \exp(-\alpha^2 x^2)\, dx \tag{55}$$

where $x = z/A$, $\alpha = A/\sqrt{2}h = V_\perp(0)/\sqrt{2}\sigma_z$ and $\sigma_z$ is the $z$-velocity dispersion of the field objects. For stars we find $\bar{n}_*/n_*(0) \simeq 1$ (Case 1) and 0.8 (Case 2), not significantly different from one another. For giant molecular clouds, however, $\sqrt{3}\sigma_z \simeq 8\,{\rm km\,s^{-1}}$ (Blitz 1979), and we find $\bar{n}_{\rm GMC}/n_{\rm GMC}(0) \simeq 0.6$ (Case 1) and 0.1 (Case 2). If we now introduce a further factor $\gamma \geqslant 1$ to allow for probable secular evolution of the Galactic gas content [i.e. evolution in either $M_{\rm GMC}$ or $n_{\rm GMC}(0)$] we obtain $M_{\rm GMC}^2\, n_{\rm GMC} \approx (4 \pm 3) \times 10^3 \gamma M_\odot^2\,{\rm pc}^{-3}$. The uncertain solar motion thus leads nearly to an order of magnitude uncertainty in the importance of giant molecular clouds; the uncertain Galactic evolution, represented by $\gamma$, probably introduces a further uncertainty of a factor of 2. Napier & Staniucha (1982) took $\gamma = 1.5$.

These values of the parameters and their ranges were used to calculate the various length-scales shown in Fig. 2. For clarity of presentation the effect of varying $\gamma$ was not included in the bands representing $b_{\rm min,GMC}$ and $b'_{\rm min,GMC}$.

Given these values of the parameters it is worthwhile estimating, for comparison with Napier & Staniucha (1982), the expected number of penetrative encounters with giant molecular clouds. Allowing for gravitational focusing, this is

$$N_{\rm pen} \simeq n_{\rm GMC}\,\pi a^2\, V\, T\left(1 + \frac{2GM_{\rm GMC}}{aV^2}\right) \tag{56}$$

where $n_{\rm GMC} \approx (1.6 \pm 1.2) \times 10^{-8}\,\gamma\ {\rm pc}^{-3}$ depending on the adopted mean solar motion. Case (1) then gives $N_{\rm pen} \simeq 5\gamma$, while Case (2) yields $N_{\rm pen} \simeq 1.5\gamma$. The expected number of penetrating encounters is thus of order $3\gamma(\overset{\times}{\div}\,2)$, a result somewhat smaller than that ($\approx 20$) found by Napier & Staniucha (1982). Most of the difference can be attributed to our improved treatment of the average density of giant molecular clouds encountered by the Sun, taking account of the $z$-distribution. Although the estimate of $N_{\rm pen}$ is still quite uncertain, we note that if the number of penetrating encounters is typically only of order a few during the lifetime of the Solar System, the argument made by Napier & Staniucha (1982) against survival of a primordial comet cloud becomes less compelling. This is reflected in our estimate of $r_0$ (equation 51), which even for $n_{\rm GMC} = 2.8 \times 10^{-8}\ {\rm pc}^{-3}$, $\gamma = 1.5$ and $g_{\rm tot} = 2$ is still of order $10^4$ AU. This suggests that comets formed with orbits initially within this radius could have survived to the present day. We cannot of course rule out the possibility that an exceptionally close encounter with an exceptionally massive cloud might have very seriously depleted a primordial comet cloud. An assessment of the likelihood of such an event lies outside the scope of the present study.

Finally we comment briefly on the validity of our simplified encounter model. An obvious alternative would be to assume that the Solar System is at rest within a system of field objects moving with random velocities. In this case, following Oort's (1950) calculation, the mean value of $2\Delta\epsilon$ averaged over angles is

$$\overline{2\Delta\epsilon} = \left(\frac{2GM}{DV}\right)^2 \frac{r}{2D} \ln\left[\frac{1 + r/D}{\pm(1 - r/D)}\right] \tag{57}$$

where the $\pm$ refers to cases where the impact parameter $D$ to the comet is $\lessgtr$ the distance $r$ of the comet from the Sun. We now define $n(M)\,dM$ to be the number of particles/volume with masses in the range $(M, M + dM)$ and $f(V)\,dV$ to be the fraction with speeds $V$ in the range $(V, V + dV)$. The mean energy transfer rate then becomes (integrating over all $D$)

$$\overline{2\dot\epsilon} = 8\pi G^2 \int_0^\infty n(M)\, M^2\, dM \int_0^\infty V^{-1} f(V)\, dV \cdot I_{\rm D} \tag{58}$$

where

$$I_{\rm D} = \frac{1}{2}\int_{D_{\rm min}}^{D_{\rm max}} \frac{r}{D^2} \ln\left[\frac{1 + r/D}{\pm(1 - r/D)}\right] dD. \tag{59}$$

This integral can be solved by substituting $x = r/D$ to give

$$I_{\rm D} = 1/2\ \{(1 + x)\ln(1 + x) + (1 - x)\ln[\pm(1 - x)]\}_{r/D_{\rm max}}^{r/D_{\rm min}}. \tag{60}$$

When both $D_{\rm min}$ and $D_{\rm max}$ are $\gg r$, as is the case for encounters with nebulae, (60) reduces to

$$I_{\rm D} \simeq \frac{1}{2}\left(\frac{r^2}{D_{\rm min}^2} - \frac{r^2}{D_{\rm max}^2}\right) \tag{61}$$

which may be compared with (41). In the other extreme ($D_{min} \ll r, D_{max} \gg r$) the analysis leading to (57) is not entirely justified (see Oort 1950; Rickman 1976 for discussion). Ignoring this detail, however, in this limit (60) becomes

$$I_D \simeq \ln\left(\frac{r}{D_{min}}\right) + 1 \tag{62}$$

which may be compared with (45). Taking account of the note added in proof in Oort's (1950) article, we conclude that the detailed encounter geometry affects the energy transfer rate only by a small numerical factor whose deviation from unity is negligible compared with the ranges of the other model parameters discussed. The general problem (both Sun and field objects moving) still, however, remains an important problem for future investigation.

## 4 Structure and evolution of the comet cloud

### 4.1 PREDICTED $a$-DISTRIBUTION

We consider a spherically symmetric comet cloud with density distribution $n(r, t)$ corresponding to some specified distribution function $f(r, \mathbf{v}, t)$. Planetary perturbations remove all comets in the cloud which have perihelia $\leqslant R_p$, which we take here to be of the same order, $\approx 10$ AU, as the radii of the orbits of Jupiter and Saturn. Since comets from the cloud generally have semi-major axes $a \gg R_p$, the specific angular momentum $J$ corresponding to such a 'loss-cone' orbit satisfies $J \leqslant J_p$, where

$$J_p = (2GM_\odot R_p)^{1/2}. \tag{63}$$

The outer radius of the cloud, $R_0 \simeq 10^5$ AU, is determined principally by the overall effect of the Galaxy.

The distribution function $f$ is defined so that the number of comets per unit volume with velocities within $d^3\mathbf{v}$ about $\mathbf{v}$ in velocity space is $f(r, \mathbf{v}, t)\, d^3\mathbf{v}$. In the standard model for example, $f = 3n(r)/4\pi v_{max}^3(r) = (3A_1/4\pi)(R_0/2GM_\odot)^{3/2}$ = constant. The functional form of $f$, however, is not arbitrary: provided a comet's energy does not change significantly on a dynamical time-scale, Jeans's theorem applies and we know therefore that $f$ must be constant along a particle's orbit. Thus $f(r, \mathbf{v}, t) \equiv f(E, J, t)$, where the energy/mass $E$ is

$$E = \frac{1}{2} v_r^2 + \frac{1}{2} v_t^2 - \frac{GM_\odot}{r} \tag{64}$$

and $J = rv_t$. Here $v_r$ and $v_t$ are the radial and transverse components of velocity respectively.

We first check that between close encounters with giant molecular clouds 'collisions' are indeed negligible. This means that a comet's energy $E = -GM_\odot/2a$ must not change significantly on a dynamical time-scale $P(a) = 2\pi a^{3/2}/(GM_\odot)^{1/2}$. If we define $\mathbf{v}_0$ to be the orbital velocity of a comet at radius $r$, the net change of energy after receiving an impulse $\mathbf{\Delta v} = \mathbf{\Delta v}_c - \mathbf{\Delta v}_\odot$ relative to the Sun is

$$\Delta E = \mathbf{v}_0 \cdot \mathbf{\Delta v} + 1/2\, (\mathbf{\Delta v})^2. \tag{65}$$

The first term describes the randomizing or 'thermalizing' effect of stellar perturbations; the second the systematic energy increase calculated in the previous section.

We first consider whether $(\mathbf{\Delta v})^2$ is negligible compared with the mean-square orbital velocity $v_0^2 = GM_\odot/a$. From (12) the order of magnitude of $\Delta v$ in the two limits $b \ll r$ and

$b \gg r$ is readily shown to be

$$\Delta v \simeq \frac{2GM_*}{bV} \times \begin{cases} 1 & b < r \\ r/b & b > r \end{cases} \tag{66}$$

The smallest impact parameter expected to occur during one orbital period is of order $b'_{\min}$ $(r = a)$, given (equation 15) by

$$b'_{\min,*}(a) = \frac{1}{2\pi} (GM_\odot)^{1/4} (n_* V)^{-1/2} a^{-3/4}. \tag{67}$$

Thus for $a \gtrsim 10^{4.4}$ AU, where $b'_{\min}(a) < a$ (see Fig. 2), the maximum expected $\Delta v$ is of order

$$\Delta v_{\max} \simeq 4\pi (GM_\odot a)^{3/4} \left(\frac{n_*}{V}\right)^{1/2} \left(\frac{M_*}{M_\odot}\right). \tag{68}$$

Here we have used (66) and set $r = a$. This approximation is no less accurate than (66), since averaged over an elliptical orbit $\bar{r} = a(1 + \tfrac{1}{2}e^2)$ and for an isotropic distribution of orbits $\langle e^2 \rangle = \tfrac{1}{2}$. Thus $\langle \bar{r} \rangle = 5a/4 \simeq a$. $\Delta v$ is therefore negligible compared with $v_0$ provided

$$a \ll (4\pi)^{-4/5} (GM_\odot)^{-1/5} \left(\frac{V}{n_*}\right)^{2/5} \left(\frac{M_\odot}{M_*}\right)^{4/5} \simeq 3 \times 10^5 \left(\frac{V}{1 \text{ km s}^{-1}}\right) \text{AU} \tag{69}$$

where for the numerical factor we have taken $n_* = 0.1 \text{ pc}^{-3}$. A similar limit may be derived by considering encounters with comets of smaller $a$.

Thus the mean rate of increase of $\Delta E$ is

$$\langle \Delta\dot{E} \rangle = 1/2 \langle (\Delta\dot{\mathbf{v}})^2 \rangle = \bar{\dot{\epsilon}}_{\mathrm{qs},*} \tag{70}$$

and the mean rate of increase of $(\Delta E)^2$ is approximately

$$\langle (\Delta\dot{E})^2 \rangle \simeq v_0^2 \langle (\Delta\dot{\mathbf{v}})^2 \rangle = \frac{2GM_\odot}{a} \bar{\dot{\epsilon}}_{\mathrm{qs},*}. \tag{71}$$

Here $\bar{\dot{\epsilon}}_{\mathrm{qs},*}$ should be taken as the mean value of (52) averaged over a typical elliptical orbit, but because this average (i.e. $\overline{r^{1/2}}$ is within 20 per cent of $a^{1/2}$ (and equals $1.1\,a^{1/2}$ for the mean eccentricity $\langle e \rangle = \tfrac{2}{3}$), no great error is introduced by assuming simply that

$$\bar{\dot{\epsilon}}_{\mathrm{qs},*}(a) \simeq Ca^{1/2}. \tag{72}$$

We can now check both that the expected systematic energy increase over an orbital period is negligible compared with $|E|$, and that the expected 'random-walk' energy change is also negligible. The first is $\Delta E_{\mathrm{syst}} \simeq \bar{\dot{\epsilon}}_{\mathrm{qs},*}(a)P(a)$ and the second, $\Delta E_{\mathrm{rand}}$, is $[4|E|\,\bar{\dot{\epsilon}}_{\mathrm{qs},*}(a)P(a)]^{1/2}$. The ratio $\Delta E_{\mathrm{syst}}/\Delta E_{\mathrm{rand}}$ is less than unity for $a \lesssim 5 \times 10^5 (C_0/C)^{1/3}$ AU, so it is sufficient to consider the second. This is negligible compared with $|E|$ provided

$$a \ll (16\pi C)^{-1/3} (GM_\odot)^{1/2} \simeq 2 \times 10^5 \left(\frac{C_0}{C}\right)^{1/3} \text{AU}. \tag{73}$$

Since comets in the cloud generally have semi-major axes $a \lesssim 10^5$ AU this shows that $f$ should indeed be $f(E, J, t)$.

Although external perturbations do not usually cause significant energy changes on an orbital time-scale (excepting intermittent close encounters with stars or clouds), they can still cause appreciable changes in $J$. We define $J_D$ to be the rms change in $J$ expected for a nearly parabolic orbit during one orbital period. Then, since the transverse velocity impulse $\Delta v_t$ per revolution is $\Delta v_t^2 = \frac{2}{3}(\Delta v)^2 = 1.6\,\bar{\dot{\epsilon}}_{qs,*}(a)P(a)$, using $\overline{r^{1/2}} = 1.2\,a^{1/2}$ for $e=1$, we have

$$J_D \simeq 2a\,\Delta v_t \simeq 8\left(\frac{\pi}{5}\right)^{1/2}(GM_\odot)^{-1/4}C^{1/2}a^2. \tag{74}$$

Because the change in energy during one orbital period is negligible compared with $E$, the angular momentum vectors (represented by $J$) can be regarded in a first approximation as diffusing in $(E,J)$-space on spheres of constant $E$. The loss-cone is defined by $J \leqslant J_p$.

The problem of working out the distribution function when allowance is made for the removal of loss-cone orbits is analogous to that of the stellar distribution function in a galactic nucleus dominated by a massive black hole (Lightman & Shapiro 1977; Bahcall & Wolf 1976; Young 1977, and references therein). In spite of the loss-cone, the distribution function is found still to be very nearly isotropic (i.e. independent of $J$), varying only logarithmically with $J$ even in the 'diffusion' limit $J_D \ll J_p$. Following Young (1977) in this limit we have

$$f(E,J,t) \simeq \begin{cases} f[E,J_{max}(E),t]\left\{1 - \dfrac{\ln[J/J_{max}(E)]}{\ln[J_p/J_{max}(E)]}\right\} & J \geqslant J_p \\ 0 & J < J_p \end{cases} \tag{75}$$

whereas in the opposite extreme ($J_D \gg J_p$), when particles scatter rapidly in and out of the loss-cone,

$$f(E,J,t) \simeq f[E,J_{max}(E),t] \equiv f(E,t). \tag{76}$$

Here $J_{max}(E) \equiv GM_\odot/(-2E)^{1/2}$ is the specific angular momentum of the circular orbit with energy $E$.

Combining (63) and (74) we obtain

$$\frac{J_D}{J_p} \simeq 0.21\left(\frac{C}{C_0}\right)^{1/2}\left(\frac{10\ \text{AU}}{R_p}\right)^{1/2}\left(\frac{a}{10^4\ \text{AU}}\right)^2 \tag{77}$$

so that the distribution function is isotropic provided $a \gtrsim a_{is}$ given by

$$a_{is} \simeq 2.2\times10^4\left(\frac{C_0}{C}\right)^{1/4}\left(\frac{R_p}{10\ \text{AU}}\right)^{1/4}\ \text{AU}. \tag{78}$$

We note that this limit could be reduced temporarily if particular stellar encounters had combined to produce an unusually large $\delta J$ during one orbital period.

Given the distribution function, the flux into the loss-cone can be calculated straightforwardly. Here we follow Young (1977, equation II–13), who derives

$$F_{lc}(E)\,dE = \begin{cases} 4\pi f[E,J_{max}(E),t]\,\dfrac{\pi J_D^2(E)}{k^2(E)}\,dE & J_D < J_{trans} \\ 4\pi f[E,J_{max}(E),t]\,\pi J_p^2\,dE & J_D > J_{trans} \end{cases} \tag{79}$$

where $k(E) = \{2\ln[J_{max}(E)/J_p]\}^{1/2} = [\ln(a/2R_p)]^{1/2}$ and $J_{trans} = k(E)J_p$. The transition between the two forms of the expression occurs at an $a$-value or order $1.6\,a_{is}$, given by

$$a_{trans} = \left[\ln\left(\frac{a_{trans}}{2R_p}\right)\right]^{1/4} a_{is}. \tag{80}$$

To abbreviate the notation we write $f[E, J_{\max}(E), t] \equiv f(E, t)$. The $a$-distribution implied by (79) is then [using $F_{\text{lc}}(a)\,da = F_{\text{lc}}(E)\,|(dE/da)|\,da$]

$$F_{\text{lc}}(a)\,da = \begin{cases} \dfrac{128\pi^3}{3\sqrt{2}}\, f(E,t)\, \dfrac{(GM_\odot)^{1/2}}{k^2(a)}\, Ca^2\,da & a < a_{\text{trans}} \\ 4\pi^2 f(E,t)\,(GM_\odot)^2 R_{\text{p}}\, a^{-2}\,da & a > a_{\text{trans}}. \end{cases} \tag{81}$$

The flux $F_{\text{lc}}$ into the loss-cone defines the flux of comets which may become observable, although in practice we only detect those which pass rather closer to the Sun than $R_{\text{p}}$, having $J$ rather less than $J_{\text{p}}$. In the standard model (Section 2) $f = (3A_1/4\pi)(R_0/2GM_\odot)^{3/2}$ = constant. Substituting this into (81) for the case $a > a_{\text{trans}}$ gives an expression identical to (8) except that $R_{\text{p}}$ has replaced $D$. Because in this limit ($a > a_{\text{trans}}$) loss-cone depletion of the orbits is negligible, the change is not important; the flux of such orbits with perihelia less than some value, $q$ say, is proportional to $q$. In the other extreme ($a < a_{\text{trans}}$) comets coming from the cloud for the first time should only be found with $J$-values within an amount of order $J_{\text{D}}$ about $J_{\text{p}}$; i.e. perihelia within a small range about $R_{\text{p}}$. If comets with small $a$-values *are* observed, they must be explained either as comets entering the Solar System for the second or subsequent time (Bailey 1977), or as the result of a particular combination of stellar or molecular cloud encounters which led to a $\delta J$ larger than the rms value $J_{\text{D}}$ (e.g. Oort 1950; Hills 1981).

### 4.2 THE COMETARY ENERGY SPECTRUM

Between major encounters with giant molecular clouds, when the outer layers of the comet cloud may be removed completely, evolution of the comet cloud is governed by two kinds of process: the removal of comets with energy $E$ into the loss-cone at a rate $F_{\text{lc}}(E)$, and the random and systematic energy changes caused by external stellar perturbations. It is therefore natural to introduce a function $N(E, J, t)$, defined so that the total number of comets in the cloud with specific energies $(E, E + dE)$ and angular momenta $(J, J + dJ)$ is $N(E, J, t)\,dE\,dJ$. The energy spectrum is then

$$N(E, t) = \int_0^{J_{\max}(E)} N(E, J, t)\,dJ. \tag{82}$$

The function $N(E, J, t)$ is related to the distribution function $f$ by (Lightman & Shapiro 1977; Young 1977)

$$N(E, J, t) = 4\sqrt{2}\pi^3 (GM_\odot)\, J\, (-E)^{-3/2} f(E, J, t). \tag{83}$$

In the present application the distribution function is very nearly isotropic (except under some initial conditions at small $a$-values), so (82) and (83) combine to give (*cf.* Bahcall & Wolf 1976, equation 42)

$$N(E, t) = \sqrt{2}\pi^3 (GM_\odot)^3 (-E)^{-5/2} f(E, t). \tag{84}$$

The evolution equation is then (*cf.* Lightman & Shapiro 1977, equation 45)

$$\frac{\partial N}{\partial t} = -\frac{\partial}{\partial E}(N\langle\Delta\dot{E}\rangle) + \frac{1}{2}\frac{\partial^2}{\partial E^2}\,[N\langle(\Delta\dot{E})^2\rangle] - F_{\text{lc}}(E) \tag{85}$$

where $\langle\Delta\dot{E}\rangle$ and $\langle(\Delta\dot{E})^2\rangle$ are given approximately by (70) and (71). We introduce the dimensionless variables $\tau = t/T$ and $x = E/\hat{E} = R_0/a$, where $R_0$ is the outer radius of the cloud and $\hat{E}$ is the specific energy of the circular orbit with radius $R_0$ (i.e. $\hat{E} = -GM_\odot/2R_0$), and assume (*cf.* 70 and 71)

$$\left.\begin{aligned}\langle \Delta\dot{E}\rangle &= Ax^\alpha\\ \langle(\Delta\dot{E})^2\rangle &= -4E\langle\Delta\dot{E}\rangle = \frac{2GM_\odot A}{R_0}x^{\alpha+1}\end{aligned}\right\}. \tag{86}$$

Then (85) reduces to

$$\frac{\partial N}{\partial\tau} = \frac{2AR_0T}{GM_\odot}\left\{2x^{\alpha+1}\frac{\partial^2 N}{\partial x^2} + (4\alpha+5)x^\alpha\frac{\partial N}{\partial x} + \alpha(2\alpha+3)x^{\alpha-1}N\right\} - NF^*_{\rm lc}(x) \tag{87}$$

where

$$F^*_{\rm lc}(x) = \begin{cases}\dfrac{32AR_0T}{5GM_\odot}\,\dfrac{x^{-3/2}}{k^2(x)} & x > x_{\rm trans}\\[2ex] \dfrac{1}{\pi}(GM_\odot)^{1/2}TR_{\rm p}R_0^{-5/2}x^{5/2} & x < x_{\rm trans}\end{cases} \tag{88}$$

Here $k^2(x) = \ln(R_0/2R_{\rm p}x)$ is typically of order 7 for $x$-values of interest. The transition between the two forms of (88) occurs at $x_{\rm trans}$ given (*cf.* 79) by

$$x_{\rm trans}k^{1/2}(x_{\rm trans}) = 2\left(\frac{2\pi}{5}\right)^{1/4}(GM_\odot)^{-3/8}C^{1/4}R_0R_{\rm p}^{-1/4}. \tag{89}$$

For representative values of the parameters $x_{\rm trans} \approx 3$ and $a_{\rm trans} \simeq 3\times10^4$ AU.

Equation (87) can be solved numerically once a suitable initial condition $N(x, 0)$ and boundary conditions are specified. It should be remarked, however, that the equation is strictly only applicable between close encounters with giant molecular clouds. Having obtained the present-day energy spectrum $N(x, 1)$, it is then straightforward to obtain the cloud's distribution function from (84) and thereby deduce other cloud properties such as $n(r)$ and $F_{\rm lc}(a)\,da$ for comparison with observations. For example, in the standard model we should have

$$N(x) = \frac{3\pi^2}{\sqrt{2}}A_1R_0^4(GM_\odot)^{-1}x^{-5/2} \tag{90}$$

$$n(r) = \int_{v=0}^{v_{\max}} 4\pi v^2 f(r,v)\,dv = 4\pi\int_{E(v=0)}^{E(v=v_{\max})} vf(E)\,dE$$

i.e.

$$n(r) = 4\pi\sqrt{2}f\int_{E(v=0)}^{E(v=v_{\max})}\left(E+\frac{GM_\odot}{r}\right)^{1/2}dE. \tag{91}$$

The upper and lower limits on the integral are $-GM_\odot/R_0$ and $-GM_\odot/r$ respectively, so (90) reduces finally to

$$n(r) = A_1\left(\frac{R_0}{r}-1\right)^{3/2}$$

in agreement with (4). It can also be verified that the total number of comets given by (7) agrees with that obtained alternatively by integrating (90) over all energies.

### 4.3 APPROXIMATE SOLUTIONS OF THE EVOLUTION EQUATION

We consider first the effect of neglecting secular and random energy changes. Evolution of $N$ is then due solely to the finite loss-cone in which comets either evolve into short-period comets or are ejected from the Solar System into hyperbolic orbits. Setting $A \equiv 0$ in the first factor of (87) we thus obtain

$$N(x,\tau) = N(x,0)\exp\left[-F_{\mathrm{lc}}^{*}(x)\,\tau\right]. \tag{92}$$

The $e$-folding time-scale at $x$, $\tau_{1/2}(x) \equiv F_{\mathrm{lc}}^{*}(x)^{-1}$, is shortest at $x = x_{\mathrm{trans}}$ corresponding to the maximum of $F_{\mathrm{lc}}(E)\,dE$ (*cf.* Lightman & Shapiro 1977, fig 2). If we assume (72) for the energy transfer rate, $A = CR_0^{1/2}$, and the minimum value of $\tau_{1/2}(x)$ becomes

$$\tau_{1/2,\mathrm{min}} \simeq 0.78\left(\frac{C_0}{C}\right)^{5/8}\left(\frac{10\,\mathrm{AU}}{R_{\mathrm{p}}}\right)^{3/8} k^{5/4}(x_{\mathrm{trans}}).$$

For typical values of the parameters this is of order 2. Thus on time-scales shorter than the age of the Solar System, neglect of the loss-cone would introduce an error of at most a factor $\lesssim 2$. This conclusion might have been anticipated, because in the standard model the total flux into the loss-cone is only on the order of 10 per year, too small to make a significant impact on the total number of comets during the age of the Solar System.

We are therefore justified, in a first approximation, in ignoring loss-cone losses. The equation to be solved is then

$$\frac{\partial N}{\partial \tau} \simeq 2D_0\left\{2x^{\alpha+1}\frac{\partial^2 N}{\partial x^2} + (4\alpha+5)\,x^{\alpha}\frac{\partial N}{\partial x} + \alpha(2\alpha+3)\,x^{\alpha-1}N\right\} \tag{93}$$

where $D_0 \equiv AR_0T/GM_\odot$. For typical values of the parameters (e.g. $R_0 \simeq 10^5\,\mathrm{AU}$, $A \approx 10^{-13}\,\mathrm{m^2\,s^{-3}}$) $D_0$ is of order unity. We note that the outward flux of comets, $F$, measured positive if the net flow is towards higher-energy orbits, is given from conservation of comet number by

$$\frac{\partial N}{\partial t} = -\frac{\partial F}{\partial E}. \tag{94}$$

Thus (93) implies

$$F(x) = A\left[(2\alpha+3)\,Nx^{\alpha} + 2x^{\alpha+1}\frac{\partial N}{\partial x}\right]. \tag{95}$$

We first investigate steady-state solutions of (93), assuming that conditions are such that a steady-state energy spectrum can be achieved. The evolution time can be estimated crudely from (93) to be of order

$$\tau_{\mathrm{ev}} \approx x^{1-\alpha}/\{2D_0\,[2\alpha^2 + 7(\alpha+1)]\}$$

i.e.

$$t_{\mathrm{ev}} \approx \frac{GM_\odot}{AR_0}\left(\frac{R_0}{a}\right)^{1-\alpha}\frac{1}{2\,[2\alpha^2 + 7(\alpha+1)]}. \tag{96}$$

For typical values of the parameters (e.g. $A \approx 10^{-13}\,\mathrm{m^2\,s^{-3}}$, $\alpha \simeq -1/2$, $a \simeq R_0$) this is of order $10^9$ yr, indicating that there may not be enough time between major encounters with giant molecular clouds for a quasi-steady state to be reached. However, possible steady-state solutions are still of interest, and setting $\partial/\partial\tau \equiv 0$ in (93) or $F$ = constant in (95) it is readily shown that these solutions are power laws of the form $N_{ss}(x) = Kx^{-q}$, with $q = \alpha$ or $\alpha + 3/2$. The $q = \alpha$ solution tends to zero at small binding energies ($x \to 0$), because (see Fig. 3) $\alpha \approx -1/2$ is generally negative. This solution, in which there is no external source of comets, has positive net flux $F = 3AK$. By contrast the $q = \alpha + 3/2$ solution diverges as $x \to 0$ and has zero net flux: the systematic outward drift of comets is just balanced by inward diffusion. It is possible that this kind of solution might become relevant during an extended period of capture of comets from an interstellar population. In the application considered below we shall be interested in the evolution of a comet cloud with no external sources, so it is the $q = \alpha$ solution that we expect to apply.

We now consider a more general solution, using the technique of separation of variables to solve (93). We define the separation constant to be $-X^2$, and resolve (93) into modes each of the form $A_X(x)\,B_X(\tau)$. It is then readily verified that

$$B_X(\tau) = \exp(-X^2\tau) \tag{97}$$

and

$$x^2\frac{d^2A_X}{dx^2} + \frac{(4\alpha+5)}{2}\,x\,\frac{dA_X}{dx} + \left[\frac{\alpha(2\alpha+3)}{2} + \frac{X^2}{4D}\,x^{1-\alpha}\right]A_X = 0. \tag{98}$$

This differential equation leads to Bessel's equation (e.g. Gradshteyn & Ryzhik 1980; equation 8.491.12), with the general solution

$$A_X(x) = a(X)\,x^{-(3/4+\alpha)}\,J_\nu[\gamma x^{(1-\alpha)/2}] + b(X)\,x^{-(3/4+\alpha)}\,J_{-\nu}[\gamma x^{(1-\alpha)/2}] \quad (\nu \neq \text{integer}). \tag{99}$$

If $\nu$ is an integer, the $J_{-\nu}$ component must be replaced by the Bessel function $N_\nu(\gamma x^{(1-\alpha)/2})$ of the second kind. The constants $\nu$ and $\gamma$ are given by

$$\left.\begin{aligned} \nu &= 3/[2(1-\alpha)] \\ \gamma &= X/[D_0^{1/2}(1-\alpha)] \end{aligned}\right\}. \tag{100}$$

Here we shall investigate only those solutions that remain finite as $x \to 0$. (This seems a reasonable restriction on cloud models, although any finite comet cloud will have its outer boundary defined at $x > 0$.) Then $b(X) \equiv 0$, and the general solution to (93) becomes

$$N(x,\tau) = 2x^{-(3/4+\alpha)}\int_{X=0}^{\infty} Xa(X)\,J_\nu[\beta(x)X]\exp(-\tau X^2)\,dX \tag{101}$$

where the weight function $a(X)$ is determined by the initial condition

$$N(x,0) = 2x^{-(3/4+\alpha)}\int_{X=0}^{\infty} Xa(X)\,J_\nu[\beta(x)X]\,dX. \tag{102}$$

Here we have defined

$$\beta(x) = x^{(1-\alpha)/2}/[D_0^{1/2}(1-\alpha)] = \left(\frac{GM_\odot}{AR_0T}\right)^{1/2}\frac{1}{(1-\alpha)}\,x^{(1-\alpha)/2}. \tag{103}$$

Using the Fourier–Bessel theorem, (102) can be inverted to give the weight function $a(X)$ explicitly; i.e.

$$a(X) = \frac{GM_\odot}{AR_0T}\,\frac{1}{4(1-\alpha)}\int_{x=0}^{\infty} x^{3/4}N(x,0)\,J_\nu[\beta(x)X]\,dx. \tag{104}$$

To illustrate the behaviour of these time-dependent solutions, we now consider the evolution of power-law initial conditions of the form $N(x,0) = Kx^{-q}$. After some straightforward algebra and use of formulae (6.561.14) and (6.631.1) of Gradshteyn & Ryzhik (1980) we find (subject to $\alpha < q < 5/2$) that

$$N(x,\tau) = K\left(\frac{GM_\odot}{AR_0T}\right)^a [2(1-\alpha)]^{-2a} x^{-\alpha} \tau^{-a} \frac{\Gamma(b-a)}{\Gamma(b)} \, {}_1F_1\left(a;b;\frac{-\beta^2(x)}{4\tau}\right) \tag{105}$$

where $\Gamma(z)$ is the gamma function, $a = (q-\alpha)/(1-\alpha)$ and $b = (5-2\alpha)/2(1-\alpha)$. Here ${}_1F_1(a;b;z)$ is the degenerate hypergeometric function defined by

$${}_1F_1(a;b;z) = 1 + \frac{az}{b1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \dots \tag{106}$$

For small values of $|z|$ ${}_1F_1(a;b;z) \to 1$, while if $z$ is large and negative it tends to the value $\Gamma(b)/\Gamma(b-a)(-z)^a$. Using these relations it can be verified that (105) tends to the assumed initial condition, $Kx^{-q}$, as $\tau \to 0$; and that as $\tau \to \infty$ the solution tends to the '$q = \alpha$' steady-state form.

### 4.4 RESULTS FOR CLOUDS WITH POWER-LAW ENERGY SPECTRA

In this section we give some basic properties of comet clouds with power-law energy spectra, the evolution of which is given by (105). Here we ignore the time-dependence, and assume simply that the number of comets with energies in the range $(E, E+dE)$ is

$$N(E)\,dE = Kx^{-q}\,dE \tag{107}$$

where, as before, $x = E/\hat{E} = R_0/a$ is the normalized binding energy. The distribution function $f$ is then given (equation 84) by

$$f(x) = \frac{K}{8\pi^3}(GM_\odot)^{-1/2}R_0^{-5/2}x^{5/2-q} \tag{108}$$

and the density distribution (*cf.* 91) by

$$n(r) = 2\pi\left(\frac{GM_\odot}{R_0}\right)^{3/2}\int_2^{2R_0/r} f(x)\left(\frac{2R_0}{r} - x\right)^{1/2} dx. \tag{109}$$

The density distribution can be obtained analytically for distribution functions of the form (107) when $q$ is half-integral. Denoting this $q$-dependence of $n(r)$ by $n_q(r)$, we have for example

$$n_{5/2}(r) = \frac{\sqrt{2}}{3\pi^2}KGM_\odot R_0^{-4}\left(\frac{R_0}{r} - 1\right)^{3/2},$$

$$n_{3/2}(r) = \frac{4\sqrt{2}}{15\pi^2}KGM_\odot R_0^{-4}\left(\frac{R_0}{r} - 1\right)^{3/2}\left[\left(\frac{R_0}{r}\right) + \frac{3}{2}\right],$$

$$n_{1/2}(r) \frac{32\sqrt{2}}{105\pi^2}KGM_\odot R_0^{-4}\left(\frac{R_0}{r} - 1\right)^{3/2}\left[\left(\frac{R_0}{r}\right)^2 + \frac{3}{2}\left(\frac{R_0}{r}\right) + \frac{15}{8}\right],$$

$$n_{-1/2}(r) = \frac{128\sqrt{2}}{315\pi^2}KGM_\odot R_0^{-4}\left(\frac{R_0}{r} - 1\right)^{3/2}\left[\left(\frac{R_0}{r}\right)^3 + \frac{3}{2}\left(\frac{R_0}{r}\right) + \frac{15}{8}\left(\frac{R_0}{r}\right) + \frac{35}{16}\right]. \tag{110}$$

These density profiles illustrate the kind of variation which might occur between models differing only in respect of the index $q$ of the energy spectrum. They show too, as noted by Oort (1950), that the standard model has only a relatively weak density gradient towards the centre; most of its comets are to be found in weakly-bound orbits far from the Sun. In general there is no need for $q$ to take the standard value $q = 5/2$; and it might even be the case, particularly in models where comets originate in much more tightly-bound orbits (*cf.* Öpik 1973), that $q$ is small or negative. Viewed from this angle the standard model might appear to be implausible on physical grounds (*cf.* Hills 1981).

Until a reliable theory of the origin of comets has been developed, it seems best in the present context to regard the spectral index $q$ as a possible parameter to be determined from observations. With this in mind we give below the predicted loss-cone flux, both as a function of $a$ and as a function of $y = 1/a$. From (79) and (81) we obtain

$$F_{1c}(a)\,da = \begin{cases} \dfrac{16}{3\sqrt{2}}\,KCR_0^{-q}\,\dfrac{1}{\ln(a/2R_p}\,a^{q-1/2}\,da & a < a_{\rm trans}, \\[2ex] \dfrac{1}{2\pi}\,K\,(GM_\odot)^{3/2}R_pR_0^{-q}\,a^{q-9/2}\,da & a > a_{\rm trans}, \end{cases} \tag{111}$$

and

$$F_{1c}(y)\,dy = \begin{cases} \dfrac{16}{3\sqrt{2}}\,KCR_0^{-q}\,\dfrac{1}{\ln(1/2R_py)}\,y^{-(q+3/2)}\,dy & y > y_{\rm trans}, \\[2ex] \dfrac{1}{2\pi}K\,(GM_\odot)^{3/2}R_pR_0^{-q}\,y^{5/2-q}\,dy & y < y_{\rm trans}. \end{cases} \tag{112}$$

Here $C$ is defined in equation (52) and the transition $a$-value, $a_{\rm trans}$, is of order $3\times10^4$ AU (equation 80). The normalizing constant $K$ can be obtained in the same way as for the standard model (Section 2, equation 9). Integrating (111) over $a_{\rm trans} < a < R_0/2$, setting $R_p$ equal to $D \simeq 1.5$ AU and equating the result to $F_{\rm obs} \approx 1\ {\rm yr}^{-1}$, we obtain

$$K = \frac{(7-2q)\,\pi\,(GM_\odot)^{-3/2}D^{-1}R_0^q}{[a_{\rm trans}^{q-7/2} - (R_0/2)^{q-7/2}]}\,F_{\rm obs}. \tag{113}$$

Finally the total number of comets in the cloud can be obtained by integrating (107) over all energies; i.e.

$$N_{\rm tot} = \int_{E_{\min}}^{E_{\max}} N(E)\,dE = K\cdot\frac{GM_\odot}{R_0}\,\frac{1}{2(q-1)}\,[2^{1-q} - x(E_{\min})^{1-q}] \qquad (q \neq 1) \tag{114}$$

For completeness we note that our spectral index $q$ corresponds to Hills' (1981) $n = 2 - q$; the standard model then has $n = -1/2$.

## 5 Discussion and comparison with observations

Setting aside those theories (e.g. van Flandern 1978; Yabushita 1979) in which comets are a short-lived transient phenomenon attributed to a recent event which occurred $(3\text{–}6)\times10^6$ yr ago, we expect that most of the development of the previous section should be applicable irrespective of the detailed theory of comet origin under discussion. Thus it is possible in principle to use the observed orbital data to determine the present distribution function (equation 81). This can then be combined with a theory of the spectral evolution (e.g. equation 105) to determine the initial energy spectrum. Given this, it might be possible to provide significant constraints on theories of cometary origin. This is in contrast to Weissman's (1980a) rather pessimistic statement, although the chain of argument is weakened by the possibility that successive encounters with giant molecular clouds could have obliterated all trace of the initial energy spectrum. It is possible, however, that some spectral structure might survive, and in any case the theory should still be applicable to models (e.g. Clube & Napier 1982b) in which comets have been captured from interstellar space during a recent giant molecular cloud encounter.

Since, to the writer's knowledge, no theory of cometary origins yet proposed has predicted the initial energy spectrum, it is not yet possible to use the machinery of Section 4 to try to discriminate between theories. However, it is still of some interest to attempt to determine the observed energy spectrum from present orbital data. For example, the discussion of Section 4.4 shows (equation 112) that a logarithmic plot of the $(1/a)$-distribution, $\log [N_{\rm obs}(1/a)]$ versus $\log (1/a)$, should show a break of about four in slope at $y = y_{\rm trans} \simeq 1/(3 \times 10^4\ \text{AU})$. This prediction can be checked, and, if the data were good enough, the current value of $q$ in the neighbourhood of $y_{\rm trans}$ could be estimated.

The most recent compilation of original $(1/a)$-values of new comets (those coming from the Oort Cloud for the first time) is that of Marsden, Sekanina & Everhart (1978). We have taken from their sample of 200 comets the 111 best-determined 'Class I' orbits and represented each by a unit Gaussian:

$$N_{\rm i} = \frac{1}{\sqrt{2\pi}\,\sigma_{\rm i}} \exp\,[-(y - y_{\rm i})^2/2\sigma_{\rm i}^2] \tag{115}$$

where $y_{\rm i}$ is the original $(1/a)$-value of the orbit and $\sigma_{\rm i}$ is its quoted mean error. This seemed, at least for the comets of longest period ($a \gtrsim 10^4$ AU, say) which interest us here, to be a preferable way of combining data of widely-varying $\sigma_{\rm i}$ than the alternative of simply constructing a $(1/a)$-histrogram. Adding the 111 such Gaussians together we obtained the upper (broken) curve of Fig. 4. Because it is probable that many of these comets have been affected by non-gravitational forces (e.g. due to out-gassing) we have also plotted in Fig. 4 the curve (solid line) obtained from the 37 comets with perihelia greater than 2 AU.

The data are still too patchy to allow completely firm conclusions to be drawn, but the graph *does* show a sharp break at $\log (y) \simeq 1.55$, corresponding to $a \simeq 2.8 \times 10^4$ AU. The size of the break is difficult to determine accurately, but eye-fitting two straight lines to the curve for the 37 orbits least affected by non-gravitational perturbations gives values for the slopes either side of $y_{\rm trans}$ of $0.9 \pm 0.2$ and $-4.5 \pm 1.5$. The break in slope is thus consistent with the theoretically expected value of 4, and the spectral index for the long-period comets ($a \gtrsim a_{\rm trans}$) is of order $1.6 \pm 0.2$, somewhat smaller than the values (5/2) for the standard model.

We thus interpret Fig. 4 as indicating that the considerations of Section 4 *are* appropriate to the present cloud of comets surrounding the Solar System. This does not prove that the cloud is primordial, and until competing theories have been developed sufficiently we prefer not to pre-judge the issue. For example, although the current mean spectral index $q$ determinded above is quite uncertain, even should a power-law approximation be the case (*cf.* 105), the value $q \approx 1.6$ admits of at least two possible interpretations within the framework of Section 4. In the first it might be speculated that the effect of a close encounter with a giant molecular cloud might be to produce an inverted energy spectrum of the kind observed or assumed in the standard model (e.g. $q = 2$, say). Then it could be argued that the observed spectrum represents evolution of this initial condition towards the positive-flux '$q = \alpha$'

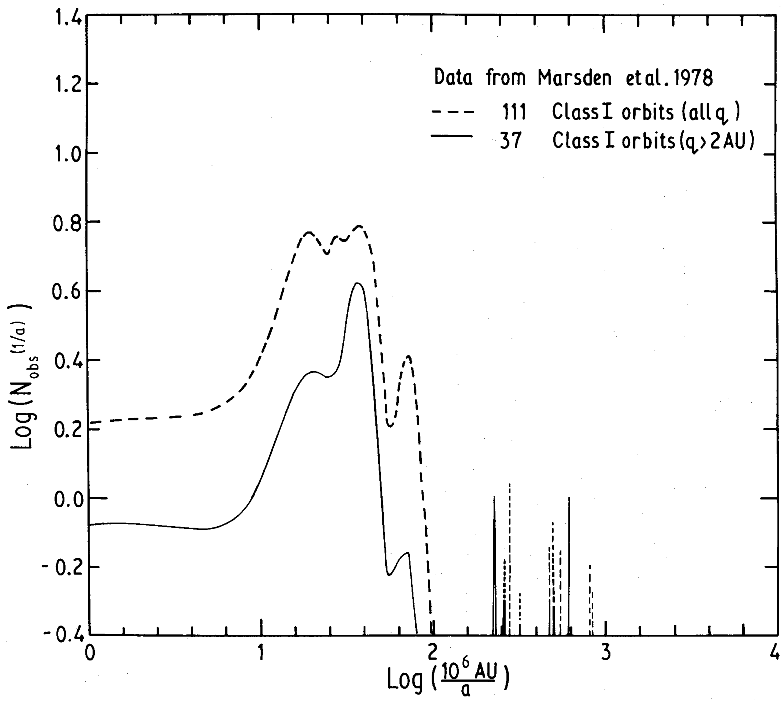


**Figure 4.** The Gaussian-smoothed $(1/a)$-distribution is shown. Data from Marsden, Sekanina & Everhart (1978).

steady-state form along an evolutionary track described by (105). In this case one would have a primordial hypothesis. Alternatively it might be argued that the energy spectrum had evolved to a steady state. Then, since $\alpha \approx 0$, one could perhaps argue that the cloud had relaxed to the zero-flux '$q = \alpha + 3/2$' solution, indicating an external interstellar source of comets. Clearly, neither of these interpretations in their current form can be regarded seriously as an explanation of the observations – but they do serve to illustrate the difficulty of shooting ghosts! It is most important that theories of cometary origin should attempt to predict the energy spectrum, because this is one of the very few ways that different theories can be tested.

Napier & Staniucha (1982) argued the case against a primordial comet cloud principally on the ground that encounters with nebulae would have removed all but a negligible fraction of the initial number of comets. However, although giant molecular clouds most probably *do* dominate the cumulative energy transfer (Section 3 and equations 50 and 53), it does not necessarily follow that a primordial comet cloud would be dissipated. Comets within $r \lesssim 10^4$ AU are relatively safe from giant molecular cloud perturbations, and if the initial index of the energy spectrum was smaller than unity, these would be in the majority (equation 110). In fact, on a primordial hypothesis it seems not unreasonable to expect more comets initially in more tightly-bound orbits (i.e. $q < 0$; *cf.* Hills 1981), so such a hypothesis might lead rather naturally to the possibility of a massive centrally-condensed reservoir of comets from which to draw during the subsequent evolution. This possible solution to the survival problem of the Oort Cloud would admittedly increase the total number (and mass) of comets required initially, but if comets formed in the outer parts of the early Solar nebulae (e.g. Hills 1982) this need not be a serious difficulty. (It *would*, however, be a problem for theories where comets formed initially in the planetary system.) Reducing $q$ might also alleviate the well-known difficulty of placement into the observable part of the cloud (e.g. Öpik 1973, section 9; Biermann & Michel 1978; Dermott & Gold 1978; Hills 1981).

Finally it may be possible to bring various indirect arguments to bear on the problem. For example, a well-established comet cloud would normally be expected to show the observed preponderence of retrograde-to-direct orbits amongst the apparently new comets (Bailey 1977; Fernandez 1981); and additionally (Bailey 1977) should show a weak concentration towards lines on the celestial sphere close to the Galactic Equator. Fernandez (1981) has also emphasized that on some, but not all, capture hypotheses a preponderance of *direct* orbits would be predicted, contrary to observation. A second example is the recent work by Clube & Napier (e.g. 1982c) in which the mean interval between major cometary impacts on the Earth is linked with that between close encounters with nebulae or passages through spiral arms. This is just the sort of connection which might be expected on an interstellar hypothesis, but it should also be noted that the mean interval between impacts of long-period comets, alone, on the Earth is of order $[F_{\rm obs}\,\tfrac{1}{2}R_\oplus^2/(1\,{\rm AU})^2]^{-1} \approx 10^9\,{\rm yr}$, so such correlations could also arise even if comets had a purely Solar System origin (see also Hills 1981; Weissman 1980b). Lastly one could use the fact that some theories of Solar System origin (e.g. Kuiper 1951) predict the existence of a comet 'belt', analogous to the asteroid belt, lying some distance beyond the orbits of the major planets. The existence of such a belt could help to resolve some difficulties with the observed numbers of short-period comets (e.g. Fernandez 1980); and if it could successfully be detected (Bailey 1976), a Solar System origin for comets would appear more probable.

## 6 Conclusions

The principal conclusions to be drawn from this work are the following:

(1) The number of passages of the Solar System through giant molecular clouds during its lifetime is in the range 1–10, depending on parameters; rather less than that worked out by Napier & Staniucha (1982). This makes survival of the loosely-bound primordial Oort (1950) comet cloud rather unlikely for reasonable values of the parameters.

(2) Analytic results derived for the standard Oort (1950) model show that this is just one of a family of hypothetical clouds with power-law energy spectra. The energy spectral index, $q = 5/2$ (equation 107) gives relatively poor agreement with the observed $(1/a)$-distribution. Observations (Fig. 4) suggest that better agreement would be provided by models with smaller $q$ and a higher degree of central concentration (equation 110).

(3) Loss-cone removal of comets and direct 'sweeping' of the cloud by stars are negligible at the factor of 2 level. The dominant evolutionary processes are the occasional close encounter with a giant molecular cloud, and the random and systematic energy changes caused by stellar perturbations. The secular evolution of the energy spectrum, neglecting the loss-cone and 'sweeping', is solved exactly for a power-law initial condition (equation 105).

(4) Provided that the velocity distribution function at large radii is isotropic, the present work should apply irrespective of the precise assumption about cometary origins (e.g. Solar System versus interstellar). We emphasize, however, that observations in principle can discriminate between different theories, provided that the theories in question predict the energy spectrum.

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## Appendix: Energy transfer rate allowing for penetrative encounters with nebulae

Here we estimate the mean energy transfer rate by giant molecular clouds for cases when the minimum impact parameter $b_{\min,\mathrm{GMC}}$ is smaller than the cloud radius $a$. In order to bracket the likely possibilities we consider two kinds of cloud model, each spherically symmetric with total mass $M_{\mathrm{GMC}}$ and space density $n_{\mathrm{GMC}}$. In the first we assume the cloud is simply a uniform sphere of radius $a$, while in the second the basic cloud itself is assumed to contain substructure comprising on average $N_s$ subclouds with masses $M_s$ distributed according to some prescribed mass function $N(M_s)\,dM_s$. For simplicity we assume the subclouds can be treated as point masses.

The total energy transfer rate due to giant molecular clouds when $b_{\min,\mathrm{GMC}} < a$ can then be written as the sum of three terms: that due to non-penetrating encounters with $b > a$, that due to the effect of the cloud-as-a-whole in cases with $b < a$, and that due to the cloud's substructure (if any) in cases with $b < a$. The first is easily written down. This is (*cf.* 46)

$$2\bar{\dot{\epsilon}}_{\mathrm{np}}(r) \simeq \frac{8\pi G^2 M_{\mathrm{GMC}}^2\, n_{\mathrm{GMC}}}{V}\,\frac{1}{3}\left(\frac{r^2}{a^2} - \frac{r^2}{b_{\max}^2(r)}\right) \qquad [b_{\max}(r) > a]. \tag{A1}$$

If $b_{\max}(r) < a$, as it is when $r \lesssim 10^{4.4\pm0.2}$ AU (Fig. 2), this contribution is zero: at small radii only penetrating encounters contribute to the giant molecular cloud energy transfer rate.

The second two terms, due to penetrating encounters, clearly depend in detail on the adopted cloud model. For a uniform cloud it is readily shown (e.g. Biermann 1978) that in the impulse approximation the velocity increment $\mathbf{\Delta v}$ given to a particle passing through the cloud with impact parameter $b < a$ is

$$\mathbf{\Delta v} = \frac{2GM_{\mathrm{GMC}}}{bV}\,\frac{\mathbf{b}}{b}\left[1 - \left(1 - \frac{b^2}{a^2}\right)^{3/2}\right]. \tag{A2}$$

Using the approximation $b \simeq d$ and averaging over $\theta$, the mean relative energy transfer is found to be

$$2\overline{\Delta\epsilon} \simeq \left(\frac{2GM_{\mathrm{GMC}}}{V}\right)^2 \frac{2}{3}\,\frac{r^2}{b^4}\left[1 - \left(1 - \frac{b^2}{a^2}\right)^{3/2}\right]^2. \tag{A3}$$

(Notice that this result, for a uniform cloud, can easily be extended to homogeneous clouds of arbitrary known density distributions.) The contribution to $\mathbf{\Delta v}$ resulting from that part of the motion while the particle lies outside the cloud is

$$\frac{2GM_{\mathrm{GMC}}}{bV}\,\frac{\mathbf{b}}{b}\left[1 - \left(1 - \frac{b^2}{a^2}\right)^{1/2}\right] \tag{A4}$$

so the net contribution to $\mathbf{\Delta v}$, treating the cloud as a whole, should lie between that given by (A3) and that obtained similarly from (A4); i.e.

$$2\overline{\Delta\epsilon}_{\mathrm{wh}} = \left(\frac{2GM_{\mathrm{GMC}}}{V}\right)^2 \frac{2}{3}\,\frac{r^2}{b^4}\,f_{\mathrm{wh}}(x) \tag{A5}$$

where $x = b/a$ and

$$[1-(1-x^2)^{1/2}]^2 \lesssim f_{\rm wh}(x) \lesssim [1-(1-x^2)^{3/2}]^2. \tag{A6}$$

In order not to underestimate the mean energy transfer rate by penetrating encounters with $b < a$, we assume here that $f_{\rm wh}(x) = [1-(1-x^2)^{3/2}]^2$. This is tantamount to assuming that, whatever the cloud's substructure, the effect of the cloud-as-a-whole can be modelled by treating the cloud as a uniform sphere. This contribution to the mean energy transfer rate is therefore of order

$$2\bar{\dot{\epsilon}}_{\rm wh}(r) \simeq \frac{8\pi G^2 M_{\rm GMC}^2 n_{\rm GMC}}{V} \frac{1}{3} \frac{r^2}{a^2} g_{\rm wh}(x_{\rm min}) \tag{A7}$$

where $x_{\rm min} = b_{\rm min,GMC}/a$ and

$$g_{\rm wh}(x_{\rm min}) = \left[3x^2 - \frac{1}{2}x^4 + 6(1-x^2)^{1/2} + \frac{2}{x^2}[(1-x^2)^{3/2}-1] + 6\ln\left[\frac{1-(1-x^2)^{1/2}}{x^2}\right]\right]_{x_{\rm min}}^{1}. \tag{A8}$$

This function is typically of order unity, showing that $\bar{\dot{\epsilon}}_{\rm wh}$ is comparable on average to that of all non-penetrative encounters put together.

We now determine the third contribution: that due to the cloud's assumed substructure. Treating the subclouds as point masses, the mean relative energy per encounter with each subcloud is

$$2\overline{\Delta\epsilon}_{\rm s} \simeq \left(\frac{2GM_{\rm s}}{V}\right)^2 \frac{2}{3} \frac{r^2}{b_{\rm s}^4} \tag{A9}$$

where $b_{\rm s}$ is the impact parameter for the subcloud. The number of subclouds with masses in the range $(M_{\rm s}, M_{\rm s}+dM_{\rm s})$ is $N(M_{\rm s})\,dM_{\rm s}$, so the total number of subclouds per giant molecular cloud is

$$N_{\rm s} = \int_{M_{\rm L}}^{M_{\rm U}} N(M_{\rm s})\,dM_{\rm s} \tag{A10}$$

where $m_{\rm L}$ and $m_{\rm U}$ are the assumed lower and upper mass limits for the substructure. Integrating (A9) over all impact parameters and summing over all subclouds thus gives for the mean energy transfer rate due to substructure

$$2\bar{\dot{\epsilon}}_{\rm s}(r) \simeq \frac{8\pi G^2 n_{\rm GMC}}{V} \int_{M_{\rm L}}^{M_{\rm U}} N(M_{\rm s}) M_{\rm s}^2\,dM_{\rm s} \frac{1}{3} r^2 \left(\frac{1}{b_{\rm min,s}^2} - \frac{1}{b_{\rm max,s}^2(r)}\right). \tag{A11}$$

We normally expect $b_{\rm min,s} \approx N_{\rm s}^{-1/2} b_{\rm min,GMC} \ll b_{\rm max,s} \approx a$, so

$$2\bar{\dot{\epsilon}}_{\rm s} \simeq \frac{8\pi G^2 M_{\rm GMC}^2 n_{\rm GMC}}{V} \frac{1}{3} g_{\rm s} \frac{r^2}{b_{\rm min,GMC}^2} \tag{A12}$$

where the dimensionless parameter $g_{\rm s} = N_{\rm s}^2 \langle M_{\rm s}^2\rangle / M_{\rm GMC}^2$ depends only on the mass-distribution of the substructure, i.e.

$$g_{\rm s} = \frac{N_{\rm s}}{M_{\rm GMC}^2} \int_{M_{\rm L}}^{M_{\rm U}} N(M_{\rm s}) M_{\rm s}^2\,dM_{\rm s}. \tag{A13}$$

Typically $g_s \approx 1$. For example, if all subclouds have the same mass [i.e. $N(M_s) = N_s \delta(M - M_s)$], $g_s = 1$; and if $N(M_s) = k M_s^{-2}$ with $N_s = 25$ and $m_U/M_{GMC} = 1/5$ we would have $g_s \simeq 1.75$.

The mean total energy transfer rate due to encounters with nebulae in which $b_{min,GMC} < a$ can therefore be written in the form

$$2\bar{\dot{\epsilon}}_{GMC}(r) \simeq \frac{8\pi G^2 M_{GMC}^2 n_{GMC}}{V} \frac{1}{3} \frac{r^2}{b_{min,GMC}^2} g_{tot} \tag{A14}$$

where

$$g_{tot} = g_s + \frac{b_{min,GMC}^2}{a^2} \left[ g_{wh}(x_{min}) + 1 - \frac{a^2}{b_{max}^2} \right]. \tag{A15}$$

In this expression we have assumed $b_{max}(r) > a$. If this is not the case, the last two terms of (A15) are zero, and the approximation leading to (A12) must be checked. At small radii $b_{max}(r)$ may become comparable with $b_{min,s}$, in which case even the existence of substructure may not necessarily cause significant energy transfer. For typical values of $N_s$ (e.g. $N_s \approx 25$, say), $b_{min,s} \approx \frac{1}{5}a \approx 4$ pc and comets with $r \lesssim 10^{3.7 \pm 0.2}$ AU will remain relatively unaffected by the giant molecular cloud (see Fig. 2).

For typical values of the parameters (e.g. $b_{min,GMC} \lesssim a/2$ and $g_s \simeq 1$) equation (A15) is dominated by $g_s$, the term due to substructure. Thus if substructure is present (as seems likely), it can dominate the mean energy transfer rate due to penetrating encounters; a point made previously by Napier & Staniucha (1982) and Clube & Napier (1982b). We have shown here that provided $r$ is not too small, the mean energy transfer rate including substructure is comparable to that obtained alternatively simply by treating the basic clouds as point masses.