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THE STRUCTURE MAPPINGS ON A REGULAR SEMIGROUP

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In (5) the author showed how to construct all inverse semigroups from their trace and semilattice of idempotents: the construction is by means of a family of mappings between $\mathcal{R}$-classes of the semigroup which we refer to as the *structure mappings* of the semigroup. In (7) (see also (8) and (9)) K. S. S. Nambooripad has adopted a similar approach to the structure of regular semigroups: he shows how to construct regular semigroups from their trace and biordered set of idempotents by means of a family of mappings between $\mathcal{R}$-classes and between $\mathcal{L}$-classes of the semigroup which we again refer to as the *structure mappings* of the semigroup. In the present paper we aim to provide a simpler set of axioms characterising the structure mappings on a regular semigroup than the axioms (R1)–(R7) of Nambooripad (9). Two major differences occur between Nambooripad's approach (9) and the approach adopted here: first, we consider the set of idempotents of our semigroups to be equipped with a *partial regular band* structure (in the sense of Clifford (3)) rather than a *biorder* structure, and second, we shall enlarge the set of structure mappings used by Nambooripad.

1. Basic notions and notation

The standard notation of Clifford and Preston (4) will be used throughout. We denote the set of inverses of a regular element x in a semigroup by $V(x)$.

We shall assume familiarity with Nambooripad's papers (8) and (9). In particular we use the notation $(E, \omega^r, \omega^l, \tau)$ for a biordered set: we shall denote $\omega^r \cap (\omega^r)^{-1}$ by $\mathcal{R}$ rather than $\overset{r}{\approx}$ and $\omega^l \cap (\omega^l)^{-1}$ by $\mathcal{L}$—no confusion with Green's $\mathcal{R}$ and $\mathcal{L}$ relations on a semigroup should occur. Thus a biordered set is a set E together with quasi-orders ω^r and ω^l and family τ of the projections which satisfy Nambooripad's axioms (B1)–(B5) and their duals: Nambooripad shows in (8) that these axioms characterise the system of idempotents of a regular semigroup as an ordered structure.

The problem of characterising the system of idempotents of a regular semigroup as a partial groupoid structure seems to have been first posed in the literature by Baird (1). The set E of idempotents of a regular semigroup S forms a partial groupoid $(E, \cdot)$ with partial binary operation "$\cdot$" defined by $e \cdot f = ef$ if $ef \in E$ and $e \cdot f$ is undefined otherwise: a partial groupoid which arises this way as the partial groupoid of idempotents of a regular semigroup is called a *partial regular band*. In (2) and (3) Clifford attacked the problem of characterising partial regular bands axiomatically. His major step is to introduce the concept of a *regular warp* which is a partial groupoid E satisfying his conditions (W1)–(W5) and (R1)–(R2). He has shown that the partial groupoid of idempotents of a regular semigroup is a regular warp but that the converse is false: thus axioms (W1)–(W5) and (R1)–(R2) do not completely characterise partial regular bands. However, Clifford goes on to show how to extend the partial product in a regular warp E in a minimal fashion so as to make E a partial regular band: this depends on his characterisation of $B(E)$, the universal regular idempotent-generated semigroup on a regular warp. Thus, by starting with the class of all regular warps, we may construct all partial regular bands. We adopt Clifford's notation: in particular if E is a partial groupoid "$\exists ef$" means that the product ef is defined in E: an equation "$ef = g$" means "$\exists ef$ and $ef = g$". A product efg is interpreted as in (W1).

If E is a regular warp, then we can associate a biordered set $(E, \omega^r, \omega^l, \tau)$ with it in the following fashion: just define $e\omega^r f$ iff $fe = e$, $e\omega^l f$ iff $ef = e$, and for each $e \in E$ define $\tau^r(e)\colon \omega^r(e) \to E$ and $\tau^l(e)\colon \omega^l(e) \to E$ by $x\tau^r(e) = xe \forall x \in \omega^r(e)$, $x\tau^l(e) = ex\ \forall x \in \omega^l(e)$: one checks that $(E, \omega^r, \omega^l, \tau)$ is a biordered set, called the *biordered set determined by the regular warp E*. Since $(E, \omega^r, \omega^l, \tau)$ is a biordered set it satisfies Nambooripad's axiom (B5) and so the regular warp E must satisfy the additional axiom (W) below:

(W) let $e,f \in E$, $h \in S(e,f)$ and $g \in \omega^l(e) \cap \omega^r(f)$: then there exist $g',g'' \in E$ such that $g'\mathscr{L}g$, $g''\mathscr{R}g$, $eg' = eg$, $g''f = gf$, $g'\omega^r h$, $g''\omega^l h$ and $g'h = hg'' = g'g''$.

Actually, axiom (B5) guarantees the existence of $g',g'' \in E$ satisfying all of (W) except the last statement $g'h = g'g''$. To see that $g'h = g'g''$ we argue as follows. Since $ge = g$ and $he = h$ and $\exists(eg)(eh)$ it follows by (W4) that $\exists gh$ and so

$$\begin{aligned} g'h &= (g'g)h = g'(gh) \quad \text{by (W1)} \\ &= g'(g(fh)) = g'((gf)h) \quad \text{by (W1)} \\ &= g'((g''f)h) = g'(g''(fh)) \quad \text{by (W1)} \\ &= g'(g''h) = g'g''. \end{aligned}$$

To every biordered set there corresponds at least one regular warp, but in general more than one: two regular semigroups may have sets of idempotents which are isomorphic as biordered sets but not as partial groupoids. Thus the partial groupoid approach gives a finer classification of regular semigroups than does the biordered set approach. In (3) Clifford shows how to construct all regular warps which correspond to a given biordered set.

2. The structure mappings

Let S be a regular semigroup whose partial regular band of idempotents is E and whose biordered set of idempotents is $(E, \omega^r, \omega^l, \tau)$. Define $\kappa = \omega^r \cup \omega^l$, so that $e \kappa f$ iff $e\omega^r f$ or $e\omega^l f$. It is an easy matter to check that if $f \kappa e$ $(e, f \in E)$ and if $x \in R_e$ then $fx \in R_f$, and if $y \in L_e$ then $yf \in L_f$. Define mappings $\phi_{e,f}\colon R_e \to R_f$ and $\psi_{e,f}\colon L_e \to L_f$ (for $f \kappa e$) by

$$x\phi_{e,f} = fx, \quad y\psi_{e,f} = yf \quad \forall x \in R_e, \quad y \in L_e \tag{1}$$

We refer to the mappings $\phi_{e,f}$ and $\psi_{e,f}$ defined by (1) as the *structure mappings* on S.

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As Nambooripad showed (**9**), these mappings may be used to reduce all products in S to products in the trace of S: in fact if $a,b \in S$, $e^2 = e \in L_a$, $f^2 = f \in R_b$ and if $h \in S(e,f)$ then one sees that

$$a \cdot b = (a\psi_{e,h})(b\phi_{f,h}). \tag{2}$$

(This is independent of the choice of e and f: also $a\psi_{e,h}\mathcal{L}h$ and $h\mathcal{R}b\phi_{f,h}$, so the product on the right-hand side of (2) is a product in the trace of S.)

We remark that the structure mappings introduced in (1) differ from those used by Nambooripad. Nambooripad used only the mappings $\phi_{e,f}: R_e \to R_f$ for $f\omega^r e$ and $\psi_{e,f}: L_e \to L_f$ for $f\omega^l e$, and we are in addition going to use the mappings $\phi_{e,f}$ for $f\omega^l e$ and $\psi_{e,f}$ for $f\omega^r e$. We now examine conditions which the structure mappings (1) must satisfy.

Proposition 2.1. *Let S be a regular semigroup with partial regular band of idempotents E and associated biordered set $(E, \omega^r, \omega^l, \tau)$. Let $\Phi = \{\phi_{e,f}: R_e \to R_f | f \kappa e\}$ and $\Psi = \{\psi_{e,f}: L_e \to L_f | f \kappa e\}$ be the structure mappings on S defined by* (1). *Then $\Phi \cup \Psi$ satisfies the following conditions and their duals.* (*Here $e,f,g \in E$.*)

(K1) *If $g \kappa f \kappa e$ and $\exists gfe$ then $\phi_{e,f}\phi_{f,g} = \phi_{e,gfe}$.*

(K2) *If $f \kappa e$ then $e\phi_{e,f} = fe$.*

(K3) *If $e\mathcal{R}f$ then $\phi_{e,f}$ is the identity map on R_e.*

(K4) *Let $a \in L_e$, $b \in R_e$, $a' \in V(a) \cap R_e$ and $f \kappa e$. Then*

(i) *$a'\phi_{e,f} \in V(a\psi_{e,f})$ and $f_1 = (a\psi_{e,f})(a'\phi_{e,f})\ \omega aa'$;*

(ii) *the map $f \to f_1$ restricts to a (partial groupoid) isomorphism of $\omega(e)$ onto $\omega(aa')$;*

(iii) *$(ab)\phi_{aa',f_1} = (a\psi_{e,f})(b\phi_{e,f})$.*

Remarks. If $\exists gfe$ then clearly $gfe \kappa e$ and condition (K1) makes sense: also (K2) is meaningful since $\exists fe$ if $f \kappa e$. Note also that all products which occur in (K4) are products in the trace of S. Conditions (K1)–(K4) clearly reduce to the author's conditions (C1)–(C4) in the inverse case (see (**5**)).

Proof of Proposition 2.1. It is straightforward to check that $\Phi \cup \Psi$ satisfies (K1)–(K3) and (K4)(i) so we shall only check (K4)(ii) and (iii) here. By (K4)(i), the map $f \to f_1 = afa'$ maps $\omega(e)$ into $\omega(aa')$: it is again easy to check that this map is (1,1) and onto. If $f,g \in \omega(e)$ and $\exists fg$ then $fg \in \omega(e)$ and $\exists a(fg)a' = (afa')(aga')$: on the other hand if $\exists (afa')(aga') = a(fg)a'$, then $\exists fg = a'(afga')a$ and it follows that $f \to f_1$ is a partial groupoid isomorphism as required.

To check (K4)(iii) note that $(ab)\phi_{aa',f_1} = f_1ab = afa'ab = afeb = afb = (af)(fb) = (a\psi_{e,f})(b\phi_{e,f})$.

3. Partial regular groupoids

The partial groupoid of non-zero elements of a completely 0-simple semigroup is called a Rees groupoid: clearly the trace of a regular semigroup is a disjoint union of Rees groupoids. We denote the set of idempotents of a partial groupoid S by $E(S)$. If

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T is a subset of a partial groupoid $(S, \cdot)$ then by $(T, \cdot)$ we mean the partial groupoid T with partial operation induced by "$\cdot$", i.e. for $a,b \in T$, $a \cdot b$ is defined in $(T, \cdot)$ iff $a \cdot b$ is defined in $(S, \cdot)$ and $a \cdot b \in T$. A groupoid $(S, \cdot)$ satisfying the following conditions is called a *partial regular groupoid*:

(G1) $S = \dot{\cup}\{S_\alpha | \alpha \in J\}$ where each $(S_\alpha, \cdot)$ is a Rees groupoid;

(G2) $(E(S), \cdot)$ is a regular warp;

(G3) the product $x \cdot y$ of $x,y \in S$ is defined iff it is defined either in some $(S_\alpha, \cdot)$ or in $(E(S), \cdot)$.

Remarks. (1) A product of e and f in S may be defined in both $(E(S), \cdot)$ (in which case we write $\exists e \cdot f$ in $E(S)$) and in some $(S_\alpha, \cdot)$ (in which case we write $\exists e \cdot f$ in S_α): both of these products are of course the product $e \cdot f$ in $(S, \cdot)$. Thus if $(E(S), \omega^r, \omega^l, \tau)$ is the biordered set associated with $(E(S), \cdot)$ then we see that for $e,f \in E(S)$, $e\mathcal{R}f$ iff $e \cdot f = f$, $f \cdot e = e$ and $e\mathcal{L}f$ iff $e \cdot f = e$, $f \cdot e = f$. Thus $(S, \cdot)$ is a regular groupoid in the sense of Nambooripad **(9)** and the relations ω^r, ω^l, κ, $\mathcal{R}$ and $\mathcal{L}$ all have the usual meaning: the restriction of the $\mathcal{R}$-relation on $(S, \cdot)$ to $(E(S), \cdot)$ is the $\mathcal{R}$-relation on the biordered set $E(S)$ so no confusion can arise.

(2) If S is a regular semigroup define the partial groupoid $(S, \cdot)$ as follows: $a \cdot b$ is defined (and equal to ab) iff either $\exists e^2 = e \in L_a \cap R_b$ or a,b and ab are idempotents. Then $(S, \cdot)$ is a partial regular groupoid.

The main theorem of the paper is the following.

Theorem 3.1. *Let $S = \dot{\cup}\{S_\alpha | \alpha \in J\}$ be a partial regular groupoid (with partial operation denoted by juxtaposition) and let $\Phi \cup \Psi$ be a family of mappings $\Phi = \{\phi_{e,f}: R_e \to R_f | f \kappa e\}$ and $\Psi = \{\psi_{e,f}: L_e \to L_f | f \kappa e\}$ which satisfy* (K1)–(K4) *and their duals.* Define a binary operation "$\cdot$" on S by* (2) *and denote the resulting groupoid $(S, \cdot)$ by $S(\Phi, \Psi)$. Then $S(\Phi, \Psi)$ is a regular semigroup with trace $S = \dot{\cup}\{S_\alpha | \alpha \in J\}$, partial regular band $E(S)$ and set $\Phi \cup \Psi$ of structure mappings. Conversely if $(S, \cdot)$ is a regular semigroup with set $\Phi \cup \Psi$ of structure mappings and associated partial regular groupoid S, then $\Phi \cup \Psi$ satisfies* (K1)–(K4) *and their duals, and $S = S(\Phi, \Psi)$.*

Proof. The "converse" part of the theorem has already been established (Proposition 2.1), so we assume that S is a partial regular groupoid admitting a family $\Phi \cup \Psi$ of mappings which satisfy (K1)–(K4) and their duals. We show that $\Phi \cup \Psi$ satisfies Nambooripad's conditions (R1)–(R7) of **(9)** (and their duals): this will show that $S(\Phi, \Psi)$ is a regular semigroup with trace $\dot{\cup}\{S_\alpha | \alpha \in J\}$ and biordered set $(E(S), \omega^r, \omega^l, \tau)$ associated with the regular warp $E(S)$. The proof will be completed by checking that $E(S)$ is actually the partial regular band of $S(\Phi, \Psi)$. We break the proof into several lemmas: in each of these lemmas we assume that S is a partial regular groupoid admitting a family $\Phi \cup \Psi$ of mappings which satisfy (K1)–(K4) and their duals.

*A partial regular groupoid S does not necessarily admit a family $\Phi \cup \Psi$ of mappings which satisfy (K1)–(K4), even if each S_α is a Brandt groupoid and the biordered set $E(S)$ is a semilattice: see **(6)** for a discussion of this situation.

Lemma 3.2. *The mappings* $\Phi \cup \Psi$ *satisfy* (R1)–(R3) *and their duals.*

Proof. Axiom (R1) follows immediately from (K3), and (R2) follows from (K2). To check (R3), suppose that $e\mathcal{R}e_1$ and $e'\omega^r e$. By Clifford's axioms (W3) and (W1), $\exists e'e = e'e'e$, so $\phi_{e,e'}\phi_{e',e'} = \phi_{e,e'e}$ by (K1): thus by (K3), $\phi_{e,e'} = \phi_{e,e'e}$ (and also $\phi_{e_1,e'} = \phi_{e_1,e'e_1}$). Again, by (K3), (K1), (W1) and (W3) we have

$$\phi_{e_1,e'} = \phi_{e,e_1}\phi_{e_1,e'} = \phi_{e,e'e_1e} = \phi_{e,e'e} = \phi_{e,e'}.$$

The axiom (R3) now follows.

Lemma 3.3. *If* $e,f \in E$ *and* $e\mathcal{L}f$ *then* $x\phi_{e,f} = fx\ \forall x \in R_e$ *(and dually, if* $e\mathcal{R}f$ *then* $y\psi_{e,f} = yf\ \forall y \in L_e$*).*

Proof. Note that $\exists fx$ in S since $e^2 = e \in L_f \cap R_x$. We apply axiom (K4) with $a = a' = e$ and $b = x$: it follows by (K2) and its dual that $f_1 = (e\psi_{e,f})(e\phi_{e,f}) = (ef)(fe) = ef = e$ and that $x = x\phi_{e,e} = (ex)\phi_{e,e} = (e\psi_{e,f})(x\phi_{e,f}) = e(x\phi_{e,f})$. Hence, liberally using associativity within the Rees groupoid containing e, $fx = fe(x\phi_{e,f}) = f(x\phi_{e,f}) = x\phi_{e,f}$ as required. The dual result follows in a dual fashion of course.

Lemma 3.4. *The mappings* $\Phi \cup \Psi$ *satisfy* (R4) *and its dual.*

Proof. Let $e',e'' \in \omega(e)$ and $e'\mathcal{R}e''$, $x \in R_e$, $x' \in V(x) \cap L_e$. We already know from (K4)(i) that $x'\psi_{e,e'} \in V(x\phi_{e,e'})$. By (W1), the dual of (K1) and lemma 3.3 we have $x'\psi_{e,e''} = x'\psi_{e,e'}\psi_{e',e''} = (x'\psi_{e,e'})e''$. Thus (by using associativity in the Rees groupoid containing e') we have

$$\begin{aligned}(x'\psi_{e,e''})(x\phi_{e,e'})(x'\psi_{e,e''}) &= (x'\psi_{e,e'})(e''(x\phi_{e,e'}))(x'\psi_{e,e'})e'' \\ &= (x'\psi_{e,e'})e'' = x'\psi_{e,e''}.\end{aligned}$$

Similarly $(x\phi_{e,e'})(x'\psi_{e,e''})(x\phi_{e,e'}) = x\phi_{e,e'}$.

Lemma 3.5. *The mappings* $\Phi \cup \Psi$ *satisfy* (R5) *and* (R6) *and their duals.*

Proof. In the notation of (K4) we know that the mapping $f \to f_1 = (a\psi_{e,f})(a'\phi_{e,f})$ $\forall f \in \omega(e)$ is a partial groupoid isomorphism of $\omega(e)$ onto $\omega(aa')$. Clifford has shown in (2) that an isomorphism of warps is a biorder isomorphism (but not in general conversely) and so the map $f \to f_1$ above is an ω-isomorphism (in the sense of Nambooripad (8)) from $\omega(e)$ onto $\omega(aa')$: thus Nambooripad's axiom (R5) is satisfied.

We now check the dual of (R6). Let $e'\omega e$, $x \in R_e$, $x' \in L_e \cap V(x)$, $f = x'x$ and $f_1 = (x'\psi_{e,e'})(x\phi_{e,e'})$. Then $f_1\omega f$ (by (K4)) and by (K2) and (K4),

$$x'\phi_{f,f_1} = (x'e)\phi_{f,f_1} = (x'\psi_{e,e'})(e\phi_{e,e'}) = (x'\psi_{e,e'})e' = x'\psi_{e,e'}.$$

Dually, (R6) and the dual of (R5) are satisfied.

Lemma 3.6. *The mappings* $\Phi \cup \Psi$ *satisfy* (R7) *and its dual.*

Proof. Let $x \in L_e$, $y \in R_f$, $h \in S(e,f)$, $x' \in V(x) \cap R_e$, $g \in \omega^l(e) \cap \omega^r(f)$, $h_1 =$

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$(x\psi_{e,eh})(x'\phi_{e,eh})$ and $g_1=(x\psi_{e,eg})(x'\phi_{e,eg})$. By definition of $S(e,f)$, $eg\omega^r eh$ and so by (R5), $g_1\omega^r h_1$: we aim to show that

$$[(x\psi_{e,h})(y\phi_{f,h})]\phi_{h_1,g_1} = (x\psi_{e,g})(y\phi_{f,g}). \tag{3}$$

Let $a = x\psi_{e,h}$, $b = y\phi_{f,h}$ and $a' = x'\phi_{e,h}$: $a' \in V(a)$ by (K4). Also $a = x\psi_{e,eh}$ and $b = y\phi_{f,hf}$ by (R3) and its dual. In addition, (W1), (K1) and Lemma 3.3 imply that $x'\phi_{e,eh} = x'\phi_{e,h}\phi_{h,eh} = (eh)(x'\phi_{e,h})$, and thus by associativity in the Rees groupoid containing h, we see that $h_1=(x\psi_{e,eh})(x'\phi_{e,eh})=(x\psi_{e,h})(x'\phi_{e,h})$. Again, $E(S)$ is a regular warp so $\exists g',g'' \in E(S)$ satisfying the conclusions of axiom (W) (Section 1). In particular $g' \kappa h$, so let $c = a\psi_{h,g'}$ and $d = b\phi_{h,g'}$. By (K4), $(ab)\phi_{h_1,g_2} = cd$ where $g_2=(a\psi_{h,g'})(a'\phi_{h,g'})$. The proof will be completed by showing that

$$\phi_{h_1,g_1} = \phi_{h_1,g_2} \tag{4}$$

and

$$cd = (x\psi_{e,g})(y\phi_{f,g}). \tag{5}$$

To prove (4) we proceed as follows. Note first that by using (W) and Clifford's axioms for a regular warp we see that $\exists hg'$, ehg', $g'h$ and $g'he$ in $E(S)$ and so by (K1) and its dual,

$$\begin{aligned} g_2 &= (x\psi_{e,h}\psi_{h,g'})(x'\phi_{e,h}\phi_{h,g'}) = (x\psi_{e,ehg'})(x'\phi_{e,g'he}) \\ &= (x\psi_{e,eg'})(x'\phi_{e,g'h}) = (x\psi_{e,eg})(x'\phi_{e,g'h}). \end{aligned}$$

Again, by the axioms for a regular warp $\exists (eg)(eh)=(eg)h$ in $E(S)$ and since $eg\mathscr{L}g'$ and $\exists g'h$, it follows from the dual of Clifford's Proposition 2.3 (2) that $egeh\mathscr{L}g'h$. Hence by Lemma 3.3,

$$\begin{aligned} g_2 &= (x\psi_{e,eg})(x'\phi_{e,eg'h}) = (x\psi_{e,eg})((g'h)(x'\phi_{e,egeh})) \\ &= ((x\psi_{e,egeh})(eg))((g'h)(x'\phi_{e,egeh})) \qquad \text{(since } egeh\mathscr{R}eg) \\ &= (x\psi_{e,egeh})((eg)(g'h)(x'\phi_{e,egeh})) \end{aligned}$$

by associativity in the Rees groupoid containing h. Note that $(eg)(g'h)$ exists in the Rees groupoid containing h and also (from the axioms for a regular warp) in $E(S)$ and so by associativity in $E(S)$, $(eg)(g'h)=(egg')h = egh = egeh$. Hence

$$g_2 = (x\psi_{e,egeh})((egeh)(x'\phi_{e,egeh})) = (x\psi_{e,egeh})(x'\phi_{e,egeh}).$$

From the definition of g_1, from Nambooripad's definition of an ω-isomorphism and from (R5) it follows that $g_2 = g_1h_1$, and hence from (R3), $\phi_{h_1,g_1} = \phi_{h_1,g_2}$ as required. This establishes (4).

To prove (5) we proceed as follows. Note first that (again by (K1) and the axioms for a regular warp)

$$c = x\psi_{e,h}\psi_{h,g'} = x\psi_{e,eg'} \quad \text{and} \quad d = y\phi_{f,h}\phi_{h,g'} = y\phi_{f,hgf} = y\phi_{f,hfgf}.$$

Since $hfgh\mathscr{L}gf$ we have (by Lemma 3.3 and the fact that $eg = eg'$)

$$cd = (x\psi_{e,eg})((hfgf)(y\phi_{f,gf})) = (x\psi_{e,g})((hfgf)(y\phi_{f,g}))$$

(by (R3) and its dual). Again, using axioms for a regular warp we see that in $E(S)$, $\exists(g'h)(gf)$ and $g'hgf = hg''gf = hgf = hfgf$, so by associativity in the Rees groupoid containing g,

$$\begin{aligned} cd &= (x\psi_{e,g})(g'h)((gf)(y\phi_{f,g})) = (x\psi_{e,g})(g'h)(y\phi_{f,g}) \\ &= (x\psi_{e,g})(g'g'')(y\phi_{f,g}) = ((x\psi_{e,g})g')(g''(y\phi_{f,g})) \\ &= (x\psi_{e,g})(y\phi_{f,g}). \end{aligned}$$

This establishes (5) and hence the lemma.

Lemma 3.7. *$E(S)$ is the partial regular band of idempotents of the regular semigroup $S(\Phi, \Psi)$.*

Proof. We know from Nambooripad's theorem that $(E(S), \omega^r, \omega^l, \tau)$ is the biordered set of idempotents of $S(\Phi, \Psi)$. Let $e,f \in E(S)$. Then in $S(\Phi, \Psi)$ we have $e \cdot f = (e\psi_{e,h})(f\phi_{f,h})$ where $h \in S(e,f)$, so by (K2) and its dual, $e \cdot f = (eh)(hf)$ where the product on the right-hand side is a product in the Rees groupoid containing h. From Clifford's axiom (R2) (for regular warps) and from his Proposition 3.1 of (**2**) and from (W1) and (W5) we see that $\exists ef$ in $E(S)$ iff $\exists (eh)(hf)$ in $E(S)$ and in this case $ef = (eh)(hf) = e \cdot f$. Thus $\exists e \cdot f$ in $E(S)$ iff $\exists ef$ in $E(S)$ and in this case $e \cdot f = ef$ as required. The proof of Theorem 3.1 is now complete.

Remark. If S is a partial regular groupoid then Theorem 3.1 (and in particular Lemma 3.7) tells us that the regular warp $E(S)$ must in fact be a partial regular band. The multiplication in the Rees groupoids S_α induces a partial multiplication in $E(S)$ and a regular warp $E(S)$ which extends this partial multiplication (induced by the Rees groupoids S_α) must be a partial regular band. This gives us a reasonable feeling for the extent to which a regular warp approximates a partial regular band.

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