TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 341, Number 2, February 1994

THE STRUCTURE OF A RANDOM GRAPH AT THE POINT OF THE PHASE TRANSITION

TOMASZ ŁUCZAK, BORIS PITTEL, AND JOHN C. WIERMAN

ABSTRACT. Consider the random graph models $G(n, \#\text{edges} = M)$ and $G(n, \text{Prob(edge)} = p)$ with $M = M(n) = (1 + \lambda n^{-1/3})n/2$ and $p = p(n) = (1 + \lambda n^{-1/3})/n$. For $l \geq -1$ define an l-component of a random graph as a component which has exactly l more edges than vertices. Call an l-component with $l \geq 1$ a complex component. For both models, we show that when λ is constant, the expected number of complex components is bounded, almost surely (a.s.) each of these components (if any exist) has size of order $n^{2/3}$, and the maximum value of l is bounded in probability. We prove that a.s. the largest suspended tree in each complex component has size of order $n^{2/3}$, and deletion of all suspended trees results in a "smoothed" graph of size of order $n^{1/3}$, with the maximum vertex degree 3. The total number of branching vertices, i.e., of degree 3, is bounded in probability. Thus, each complex component is almost surely topologically equivalent to a 3-regular multigraph of a uniformly bounded size. Lengths of the shortest cycle and of the shortest path between two branching vertices of a smoothed graph are each of order $n^{1/3}$. We find a relatively simple integral formula for the limit distribution of the numbers of complex components, which implies, in particular, that all values of the "complexity spectrum" have positive limiting probabilities. We also answer questions raised by Erdös and Rényi back in 1960. It is proven that there exists $p(\lambda)$, the limiting planarity probability, with $0 < p(\lambda) < 1$, $p(-\infty) = 1$, $p(\infty) = 0$. In particular, $G(n, M)$ ($G(n, p)$, resp.) is almost surely nonplanar iff $(M - n/2)n^{-2/3} \to \infty$ ($(np - 1)n^{-1/3}) \to \infty$, resp.).

1. INTRODUCTION

Since the seminal work by Erdös and Rényi (1960), there has been considerable interest in the evolution of random graphs. A principal object of study is the random graph model $G(n, M)$, constructed on n labeled vertices with a set of M edges chosen randomly, equally likely among all possible

$$\binom{\binom{n}{2}}{M}$$

sets of edges. The number of edges, M, typically varies as a function of n,

Received by the editors September 24, 1990 and, in revised form, December 6, 1991.
1991 *Mathematics Subject Classification*. Primary 05C80; Secondary 05C10, 05C15, 05C30.
Key words and phrases. Random graph, planar graph, threshold function, limit distribution.
Research of the second author supported by the National Science Foundation through grant DMS-9002347.
Research of the third author supported by the National Science Foundation through grant DMS-8801209.

with $M(n) \to \infty$ as $n \to \infty$. A property is said to hold *almost surely* if the probability that the property is true converges to one as $N \to \infty$.

We may view $G(n, M)$ as the $(M+1)$th stage of the *random graph process* $G = (G(n, M))_{M=0, \ldots, \binom{n}{2}}$, which is a Markov chain whose states are graphs on n vertices. $G(n, 0)$ is the empty graph, and for each $M = 1, 2, \ldots, \binom{n}{2}$, the graph $G(n, M)$ is obtained from $G(n, M-1)$ by adding a new edge, where all possible choices for the new edge are equiprobable.

In a well-known alternative model, the random graph is constructed on n labeled vertices with each of the $\binom{n}{2}$ possible edges present independently with probability p, $0 \le p \le 1$. The $G(n, M)$ and $G(n, p)$ models are expected to have a similar asymptotic behavior provided that M is near $p\binom{n}{2}$. $G(n, p = 1 - e^{-t})$ may be viewed as a tth state of another random graph process $G^* = (G_n(\tau))_{\tau \ge 0}$ which is a continuous time Markov process, such that $G_n(0)$ is the empty graph, and the birth times of the $\binom{n}{2}$ edges are independent, exponentially (with parameter 1) distributed random variables (see, e.g., Stepanov (1970b)).

Most important results on the evolution of random graphs determine a threshold function $f(n)$ for $M(n)$ ($p(n)$, respectively) such that the almost sure behavior of the random graph changes dramatically when we switch from $M(n) \ll f(n)$ to $M(n) \gg f(n)$ (from $p(n) \ll f(n)$ to $p(n) \gg f(n)$, respectively).

A spectacular phenomenon in the evolution of random graphs is the "phase transition" or "double-jump threshold" which occurs at $M = n/2$ and $p = 1/n$ respectively. If $M(n) = cn/2$ (and $p = c/n$) where $c < 1$ then almost surely $G(n, M)$ ($G(n, p)$ respectively) has no components which contain more than one cycle, and the size of the largest component is of order $\log n$. When $c > 1$, this size jumps to the order of magnitude n (Erdös and Rényi (1960), Stepanov (1970a), (1970b)). However, when $c = 1 + dn^{-1/3}(\log n)^{1/2}$, it is of order $n^{2/3}(\log n)^{1/2}$ (Bollobás (1984a)).

We consider the case when $c = 1 + \lambda n^{-1/3}$ where λ is a constant, identified by the work of Bollobás (1984b). At this threshold, one expects the first appearance of component which are neither trees nor unicyclic. We define an *l-component* of a graph G as a component of G which has exactly l more edges than vertices. Since a component must be connected, we must have $l \ge -1$. The cases $l = -1$ and $l = 0$ correspond to tree components and unicyclic components, respectively. Since trees and unicyclic components are fairly abundant at this stage of the evolution, we focus on the presence of l-components with $l \ge 1$, which we call *complex components.* Let $X(n; l)$ be the total number of l-components, and call $\{X(n; l)\}_{l \ge 1}$ the *complexity spectrum* of the random graph.

In this paper, we find an unexpectedly simple formula for the limiting distribution of the complexity spectrum. It involves the limiting probability $h(x)$ that the random graph with $M(n) = (1 + xn^{-1/3})n/2$ ($p(n) = (1 + xn^{-1/3})/n$ resp.) does not have complex components. (The integral formula for $h(\cdot)$ was obtained in Flajolet, Knuth, and Pittel (1989).) This formula implies, in particular, that the complexity spectrum assumes every possible value with positive limiting probability for each λ.

The limiting distribution is obtained in §2, using a rough almost sure description of the complex components established by means of Cayley's formula for the number of labeled trees, plus asymptotic estimates of Wright (1980) and bounds of Bollobás (1984a) for the number of connected graphs with k labeled vertices and $k+l$ edges. We show in §2 that $E(\sum_{l\geq 1} X(n; l))$ is bounded, that $\max\{l \geq 1: X(n; l) > 0\}$ is bounded in probability, and that almost surely (a.s.) each one of the complex components (if any exist) has size of order $n^{2/3}$. Using this information, in §3 we show that a.s. the largest suspended tree in each complex component has size of order $n^{2/3}$, and deletion of all suspended trees results in a smoothed graph of size of order $n^{1/3}$, with maximum vertex degree 3. Furthermore, the total number of branching vertices (of degree 3) in the smoothed complex components is bounded in probability. Thus, almost surely every complex component is topologically equivalent to a 3-regular multigraph of a bounded size. Almost surely the lengths of the shortest cycle and of the shortest path between any two branching vertices in a smoothed complex component are each of order $n^{1/3}$.

An additional motivation for this work was to clarify results about the planarity threshold of random graphs. Erdös and Rényi (1960) claimed that $G(n, n/2)$ contains a cycle with three diagonals with positive limiting probability. Since approximately $1/15$ of all such cycles contain a topological copy of $K_{3,3}$, their claim implies that $G(n, n/2)$ is nonplanar with positive limiting probability. However, Luczak and Wierman (1989) found a flaw in the proof of Erdös and Rényi, and showed that almost surely there are no cycles with diagonals in $G(n, n/2)$. The planarity of random graphs was also considered by Stepanov (1987), who proved that when $\lambda \leq 0$, the probability that $G(n, p = (1 + \lambda n^{-1/3})/n)$ is nonplanar is larger than some positive constant asymptotically. Notice that $1 - h(\lambda)$, the limiting probability that the random graph has a complex component, is an obvious upper bound for the nonplanarity probability. $(1 - h(0) = 1 - \sqrt{2/3} \approx 0.18)$. In this paper, we show existence of $p(\lambda) = \lim_{n\to\infty} \text{Prob}(G(\lambda)$ is planar), such that $p(\lambda) \in (0, 1)$ and $p(-\infty) = 1$, $p(\infty) = 0$. (Here and below $G(\lambda)$ stands for both of the random graph models.) Consequently, $\lim_{n\to\infty} \text{Prob}(G(n, M)$ is nonplanar$) = 1$ iff $(M - n/2)n^{-2/3} \to \infty$, and $\lim_{n\to\infty} \text{Prob}(G(n, p)$ is nonplanar$) = 1$ iff $(np - 1)n^{1/3} \to \infty$.

A parallel study of the two models is indispensable since it allows us to establish some of the necessary estimates for the (arguably) more important and difficult model $G(n, M)$ by proving first their versions for $G(n, p)$.

Note that some aspects of the near critical behavior of $G(n, p)$ were earlier studied by Stepanov (1987) (see also Kolchin (1986)). In particular, Stepanov proved that, when $\lambda n^{1/24} \to -\infty$, a.s. the maximum vertex degree of each complex component is 3. If $\lambda < 0$ then, for every *fixed* m, a.s. there is no complex component with at most m branching vertices in its smooth part whose maximum vertex degree is at least 4. Our results show that both statements hold when λ is simply $O(1)$, *unconditionally* with respect to m. Furthermore, Stepanov suspected that closer to $\lambda = 0$ the random graph would a.s. contain more complex components, so that $\max\{l \geq 1 : X(l; n) > 0\}$ would be unbounded in probability and that the most complex components would have branching vertices of degree at least 4. We know now that neither of these conjectures is true.

For more comprehensive discussions of the evolution of random graphs, we refer the reader to the review papers by Grimmett (1980) and Karoński (1982), the introductory volume by Palmer (1985), and the research monograph by Bollobás (1985).

2. The number and size of l-components

For integers $l \geq -1$, and $n \geq k \geq 1$, define the random variable $X(n; k, l)$ as the number of l-components of size k in the graph $G(\lambda)$. (Recall that $G(\lambda)$ is the shorthand for the two models of the random graph $G(n, p = (1+\lambda n^{-1/3})/n)$ and $G(n, M = (1+\lambda n^{-1/3})n/2)$, and λ is fixed.) Note that $X(n; k, l) = 0$ if $l > \binom{n}{2} - k$.

Clearly,

$$X(n; l) := \sum_{k=1}^{n} X(n; k, l),$$

and

$$Y(n; l) := \sum_{k=1}^{n} kX(n; k, l),$$

are respectively the total number and the total size of l-components.

The exact expression for the expectation of $X(n; k, l)$ is

$$E_M[X(n; k, l)] = \binom{n}{k} C(k, k+l) \binom{\binom{n-k}{2}}{M-k-l} \binom{\binom{n}{2}}{M}^{-1} \tag{2.1}$$

for $G(n, M)$, and

$$E_p[X(n; k, l)] = \binom{n}{k} C(k, k+l) p^{k+l} q^{\binom{k}{2}-(k+l)+k(n-k)} \tag{2.2}$$

for $G(n, p)$ $(q = 1-p)$. In these expressions, $C(k, k+l)$ denotes the number of connected graphs with k labeled vertices and $k+l$ edges. A well-known formula due to Cayley states that $C(k, k-1) = k^{k-2}$, and Rényi (1959) established that

$$C(k, k) = \frac{1}{2}\sum_{j=3}^{k} [k]_j k^{k-j-1} \sim \left(\frac{\pi}{8}\right)^{1/2} k^{k-1/2} \tag{2.3}$$

where $[k]_j = k(k-1)\cdots(k-j+1)$.

As for $l \geq 1$, Wright (1980) proved that for $1 \leq l = o(k^{1/3})$ as $k \to \infty$,

$$C(k, k+l) = \gamma_l k^{k+3l/2-1/2}(1 + O(l^{3/2}k^{-1/2})). \tag{2.4}$$

Note that $C(k, k-1)$ and $C(k, k)$ also obey (2.4), with $\gamma_{-1} = 1$ and $\gamma_0 = (\pi/8)^{1/2}$. Bollobás (1984a) was able to show that the principal factor in (2.4) limits the growth of $C(k, k+l)$ for larger values of l. Namely, he proved existence of absolute constants c_1, $c_2 > 0$ such that

$$C(k, k+l) \leq (c_1/l)^{1/2} k^{k+(3l-1)/2}, \qquad 1 \leq l \leq k, \tag{2.5}$$

and

$$C(k, k+l) \leq (c_2 k)^{k+l}, \qquad 1 \leq l \leq \binom{k}{2} - k. \tag{2.6}$$

The key estimate is (2.5), while (2.6) directly follows from $C(k, k+l) \le \binom{N}{k+l}$, where $N = \binom{k}{2}$.

We shall estimate the expectation of $X(n; k; l)$ sharply for $k = o(n^{3/4})$, and more crudely for the remaining k's. In the latter case, we first bound $E_p[X(n; k; l)]$ and then $E_M[X(n; k; l)]$ by means of a general inequality (Angluin and Valiant (1979), Pittel (1982), Bollobás (1985)):

$$(2.7)\quad E_M[Z] \le \left(\binom{n}{2} + 1\right) E_{p'}[Z], \qquad p' = M \Big/ \binom{n}{2} \quad (Z = Z(G) \ge 0)\ .$$

This bound is instrumental on several other occasions, as well.

Note. If $M = O(n)$, the quadratic factor can be replaced by $\text{const}\, n^{1/2}$.

We shall frequently use the "exponential approximation"

$$(2.8)\qquad [b]_a/b^a = [1 + O(ab^{-1}) + O(a^4b^{-3})]\exp(-a^2/2b - a^3/6b^2)$$

for $a = o(b^{3/4})$, and the uniform bound

$$(2.9)\qquad [b]_a/b^a = O(\exp(-a^2/2b - a^3/6b^2 - ca^4/b^3)), \qquad c > 0.$$

Lemma 2.1. *For both models of* $G(\lambda)$, *if* $l \ge -1$ *is fixed then*

$$(2.10)\quad \begin{aligned} E[X(n; k; l)] = {} & [1 + \lambda l n^{-1/3} + O(n^{-2/3}) + O(k/n) + O(k^4/n^3)] \\ & \cdot n^{-l}(C(k, k+l)e^{-k}/k!)\exp(-F(x_k)), \end{aligned}$$

provided that $k = o(n^{3/4})$, *and for all* k

$$(2.11)\quad E[X(n; k, l)] = O(n^{2-l}(C(k, k+l)e^{-k}/k!)\exp(F(x_k) - ck^4/n^3)).$$

Here $x_k = k/n^{2/3}$ *and*

$$(2.12)\qquad F(x) = (x^3 - 3x^2\lambda + 3x\lambda^2)/6.$$

Proof. (a) Using the exponential approximation (2.8),

$$\binom{n}{k} = [1 + O(kn^{-1}) + O(k^4n^{-3})]\frac{n^k}{k!}\exp\left(-\frac{k^2}{2n} - \frac{k^3}{6n^2}\right)$$

if $k = o(n^{3/4})$, and from the uniform bound (2.9), for all $1 \le k \le n$,

$$(2.13)\qquad \binom{n}{k} = O\left(\frac{n^k}{k!}\exp\left(-\frac{k^2}{2n} - \frac{k^3}{6n^2} - c\frac{k^4}{n^3}\right)\right).$$

The exponential approximation also yields

$$\begin{aligned} & \binom{\binom{n-k}{2}}{M-k-l}\binom{\binom{n}{2}}{M}^{-1} \\ & = \left[1 + O\left(\frac{1}{n}\right)\right]\left[\frac{\binom{n-k}{2}^{M-k-l}}{(M-k-l)!}\exp\left(-\frac{(M-k-l)^2}{(n-k)(n-k-1)}\right)\right] \\ & \quad \cdot \left[\frac{\binom{n}{2}^M}{M!}\exp\left(-\frac{M^2}{n(n-1)}\right)\right]^{-1}, \end{aligned}$$

since

$$O\left(\frac{M}{\binom{n}{2}}\right) + O\left(\frac{M^4}{\binom{n}{2}^3}\right) + O\left(\frac{M^3}{\binom{n}{2}^2}\right) = O(n^{-1}),$$

and similarly for the other error term. The product of the exponential factors is $1+O(kn^{-1})$, since

$$\frac{M^2}{n(n-1)}-\frac{(M-k-l)^2}{(n-k)(n-k-1)} \\ = M^2\left(\frac{1}{n(n-1)}-\frac{1}{(n-k)(n-k-1)}\right)+O(kn^{-1})=O(kn^{-1}).$$

Thus, by rearrangement,

(2.14)
$$\binom{\binom{n-k}{2}}{M-k-l}\binom{\binom{n}{2}}{M}^{-1}=[1+O(kn^{-1})][M]_{k+l}\binom{n-k}{2}^{M-k-l}\binom{n}{2}^{-M}.$$

By the exponential approximation, then eliminating l in the exponential, we may replace $[M]_{k+l}$ in (2.14) by

$$(2.15)\quad \begin{aligned}&\left[1+O\left(\frac{k+l}{M}\right)+O\left(\frac{(k+l)^4}{M^3}\right)\right]M^{k+l}\exp\left(-\frac{(k+l)^2}{2M}-\frac{(k+l)^3}{6M^2}\right)\\ &=\left[1+O(kn^{-1})+O(k^4n^{-3})\right]M^{k+l}\exp\left(-\frac{k^2}{2M}-\frac{k^3}{6M^2}\right).\end{aligned}$$

We also compute

$$(2.16)\quad \begin{aligned}\binom{n-k}{2}^{M}\binom{n}{2}^{-M}&=\left(\frac{n-k}{n}\right)^{2M}\left[1-\frac{k}{n^2-nk-n+k}\right]^{M}\\ &=\left(1-\frac{k}{n}\right)^{2M}[1+O(kn^{-1})],\\ \binom{n-k}{2}^{-k-l}&=\binom{n}{2}^{-k-l}\left(1-\frac{k}{n}\right)^{-2k}[1+O(kn^{-1})].\end{aligned}$$

Combining these approximations, we find that

$$\binom{n}{k}\binom{\binom{n-k}{2}}{M-k-l}\binom{\binom{n}{2}}{M}^{-1}=[1+O(kn^{-1})+O(k^3n^{-4})] \\ \times\frac{n^{-l}}{k!}\left(\frac{Mn}{\binom{n}{2}}\right)^{k+l}\left(1-\frac{k}{n}\right)^{2(M-k)}\exp\left\{-\frac{k^2}{2}\left(\frac{1}{n}+\frac{1}{M}\right)-\frac{k^3}{6}\left(\frac{1}{n^2}+\frac{1}{M^2}\right)\right\}.$$

A straightforward calculation shows that

$$Mn\Big/\binom{n}{2}=1+\lambda n^{-1/3}+O(n^{-1}).$$

Using the definition of M throughout, a two-term Taylor expression for $\log(1+\lambda n^{-1/3})$ and a three-term expansion for $\log(1-k/n)^{2(M-k)}$, then combining exponents simplifies the above expression to

$$\left[1+\lambda l n^{-1/3}+O(n^{-2/3})+O\left(\frac{k}{n}\right)+O\left(\frac{k^4}{n^3}\right)\right]\frac{n^{-l}e^{-k}}{k!}\exp(-F(x_k)).$$

(A conscientious referee has checked our calculations and noticed, correctly, that besides the error terms $O(k/n)$ and $O(k^4/n^3)$ there appears an error term $O(k^3/n^{7/3})$. There is, in fact, yet another error term, $O(k^2/n^{5/3})$. Both of these terms were dropped because $k^2/n^{5/3}$ and $k^3/n^{7/3}$ are each of the order $O(k/n)+O(k^4/n^3)$ uniformly over all $k \geq 1$.) So, the formula (2.10) is proved for $G(n, M)$.

(b) The case of $G(n, p)$ is much simpler. First of all,

$$\begin{aligned} p^{k+l} &= n^{-k-l}(1+\lambda n^{-1/3})^{k+l} \\ &= [1+\lambda l n^{-1/3} + O(n^{-2/3})]n^{-k-l}(1+\lambda n^{-1/3})^k. \end{aligned}$$

Also

$$q^{\binom{k}{2}-(k+l)+k(n-k)} = [1+O(k/n)]q^{kn-k^2/2}.$$

So, using the two-term expansion for $\log(1+\lambda n^{-1/3})$, and the one-term expansion for $\log(1-p)$, we have [see (2.2)]

(2.17)
$$\begin{aligned} \binom{n}{k} p^{k+l} q^{\binom{k}{2}-(k+l)+k(n-k)} = &\left[1+\lambda l n^{-1/3} + O(n^{-2/3}) + O\left(\frac{k}{n}\right) + O\left(\frac{k^4}{n^3}\right)\right] \\ &\cdot (n^{-l}e^{-k}/k!)\exp(-F(x_k)). \end{aligned}$$

(c) Using the uniform bound (2.9), we get similarly

$$\binom{n}{k} p^{k+l} q^{\binom{k}{2}-(k+l)+k(n-k)} = O\left(\frac{n^{-l}e^{-k}}{k!}\exp\left(-F(x_k) - \frac{ck^4}{n^3}\right)\right).$$

This and (2.7) imply the bound (2.11). □

Using this lemma, we now establish the asymptotic formula for the expectation of $Y(n; \geq 1) = \sum_{l\geq 1}\sum_{k=1}^{n} kX(n; k, l)$, which is the total size of all complex components.

Lemma 2.2. *For both models,*

$$E[Y(n;-1)] = n - n^{2/3}[f_{-1}(\lambda) + O(n^{-1/3})], \tag{2.18}$$

$$E[Y(n;0)] = n^{2/3}[f_0(\lambda) + O(n^{-1/3})]. \tag{2.19}$$

So, since $Y(n;-1) + Y(n;0) + Y(n;\geq 1) = n$,

$$E[Y(n;\geq 1)] = n^{2/3}[f_{\geq 1}(\lambda) + O(n^{-1/3})], \tag{2.20}$$

where $f_{\geq 1}(\lambda) = f_{-1}(\lambda) - f_0(\lambda)$. *Here*

$$f_{-1}(\lambda) = (2\pi)^{-1/2}\int_0^\infty x^{-3/2}[1-\exp(-F(x))]\,dx + \lambda,$$

$$f_0(\lambda) = 4^{-1}\int_0^\infty \exp(-F(x))\,dx.$$

In addition, if $k_0 = \omega(n)n^{2/3}$ *and* $\omega(n)\to\infty$ *arbitrarily slowly,*

$$E\left[\sum_{k\geq k_0} X(n;k,-1)\right] = o(1). \tag{2.21}$$

Note. (1) In particular,

$$f_0(0) = 4^{-1}\int_0^\infty e^{-x^3/6}\,dx = \frac{6^{1/3}\Gamma(1/3)}{12}$$

(Erdös and Rényi (1960), Bollobás (1985, Theorem V.23)) and

$$f_{-1}(0) = (2\pi)^{-1/2} \int_0^\infty x^{-3/2}(1 - e^{-x^3/6})\, dx,$$

so, integrating by parts,

$$f_{-1}(0) = (2\pi)^{-1/2} 6^{5/6} \Gamma(5/6)/3.$$

(2) The relation (2.21) was proved by Bollobás (1985) for $G(n, p)$ and arbitrary $p = p(n)$.

Proof. We prove only (2.18), which is a genuinely new result even for $\lambda = 0$. We write

$$E[Y(n; -1)] = \sum_{k=1}^{n} kE[X(n; k, -1)] = \sum_{k \le n^\alpha} + \sum_{k > n^\alpha},$$

where $\alpha \in (2/3, 3/4)$. By Lemma 2.1 (2.11),

$$\begin{aligned}\sum_{k>n^\alpha} &= O\left(n^3 \exp(-cn^{4\alpha-3}) \sum_{k\ge 1} \frac{k^{k-1}e^{-k}}{k!} \exp(-F(x_k))\right) \\ &= O(n^3 \exp(-cn^{4\alpha-3})),\end{aligned}$$

since $F(x) = [(x-\lambda)^3 + \lambda^3]/6 \ge 0$, for all $x \ge 0$, and

$$\sum_{k \ge 1} \frac{k^{k-1}e^{-k}}{k!} = 1. \tag{2.22}$$

Using (2.10),

$$\sum_{k \le n^\alpha} = (1 - \lambda n^{-1/3} + O(n^{-2/3}))n \sum_{k \le n^\alpha} \frac{k^{k-1}e^{-k}}{k!} \exp(-F(x_k)) + R_n,$$

where

$$R_n = O\left(\sum_{k \ge 1} \frac{k^k e^{-k}}{k!} e^{-F(x_k)} + n^{-2} \sum_{k\ge 1} \frac{k^{k+3}e^{-k}}{k!} e^{-F(x_k)}\right).$$

By Stirling's formula,

$$\begin{aligned}\sum_{k\ge 1} \frac{k^k e^{-k}}{k!} e^{-F(x_k)} &= O\left(\sum_{k \ge 1} k^{-1/2} e^{-F(x_k)}\right) \\ &= O\left(n^{1/3} \sum_{k\ge 1} x_k^{-1/2} e^{-F(x_k)} \Delta x_k\right) \\ &= O\left(n^{1/3} \int_0^\infty x^{-1/2} e^{-F(x)}\, dx\right) = O\left(n^{1/3}\right).\end{aligned}$$

Similarly,

$$n^{-2} \sum_{k \ge 1} \frac{k^{k+3}e^{-k}}{k!} e^{-F(x_k)} = O\left(n^{1/3} \int_0^\infty x^{5/2} e^{-F(x)}\, dx\right) = O(n^{1/3}),$$

so that $R_n = O(n^{1/3})$. Also,

$$n \sum_{k>n^\alpha} \frac{k^{k-1}e^{-k}}{k!} e^{-F(x_k)} \le n \sum_{k>n^\alpha} \frac{k^{k-1}e^{-k}}{k!} \exp\left(-\frac{x_k^3}{7}\right)$$
$$= O\left(n \exp\left(-\frac{n^{3\alpha-2}}{7}\right)\right),$$

since $x_k \ge n^{\alpha-2/3} \to \infty$ for $k > n^\alpha$, $\alpha > 2/3$, and $F(x) > x^3/7$ for x sufficiently large. Collecting the estimates, we get

$$E[Y(n;-1)] = (1 - \lambda n^{-1/3} + O(n^{-2/3})) \cdot n \sum_{k\ge 1} \frac{k^{k-1}e^{-k}}{k!} e^{-F(x_k)} + O(n^{1/3}). \tag{2.23}$$

Next,

$$\sum_{k\ge 1} \frac{k^{k-1}e^{-k}}{k!} e^{-F(x_k)} = 1 - \sum_{k\ge 1} \frac{k^{k-1}e^{-k}}{k!} (1 - e^{-F(x_k)}),$$

by (2.22). Since the kth term in the last series is

$$(2\pi)^{-1/2} k^{-3/2} (1 - e^{-F(x_k)})[1 + O(k^{-1})],$$

a simple argument shows that its sum equals

$$n^{-1/3}(2\pi)^{-1/2} \int_0^\infty x^{-3/2}(1 - e^{-F(x)})\, dx + O(n^{-2/3}). \tag{2.24}$$

Combining (2.23) and (2.24) yields (2.18). □

Corollary 1. *The size of the largest component of $G(\lambda)$ is $O_p(n^{2/3})$, i.e., for every $\omega(n) \to \infty$, a.s. all the components are smaller than $\omega(n)n^{2/3}$.*

Note. According to Bollobás (1985), for λ of order $(\log n)^{1/2}$, the largest component has size $O_p(n^{2/3}(\log n)^{1/2})$.

Define $\mathscr{L}_n = \max\{l \ge 1 : X(n;l) > 0\}$, which is the maximum excess of the number of edges over the number of vertices, taken among all the components, in each of the two versions of the random graph $G(\lambda)$.

Theorem 1. *$\mathscr{L}_n$ is bounded in probability, i.e., $\mathscr{L}_n = O_p(1)$ as $n \to \infty$.*

Notice that the total number of simple cycles in an l-component cannot exceed 3^l. (For $l \ge 1$, an l-component with c cycles can be obtained by inserting an additional edge into an $(l-1)$-component with c' cycles, and this insertion may create (quite crudely) at most $2c'$ new cycles; thus $c \le 3c'$, and the bound 3^l follows.) So, we immediately get the following result.

Corollary 2. *The maximum number of cycles of a component is bounded in probability.*

Proof of Theorem 1. In view of Corollary 1, it suffices to show that $E(n)$, the expected number of l-components of size k satisfying conditions

$$l \ge l_0 := \omega(n), \quad k \le k_0 := \omega_1(n)n^{2/3} \qquad (\omega_1(n) := \omega(n)^{1/4}), \tag{2.25}$$

tends to 0, provided that $\omega(n) \to \infty$, and $\omega(n) = O(n^{4/3-\delta})$, $\delta > 0$.

I. Consider first $G(n, p)$. By definition,

$$E_p(n) = \sum_{\substack{k \le k_0 \\ l_0 \le l \le \binom{k}{2} - k}} \binom{n}{k} C(k, k+l) p^{k+l} q^{\binom{k}{2} - (k+l) + k(n-k)}. \tag{2.26}$$

Here (see (2.9))

$$\binom{n}{k} = O((n^k/k!)e^{-k^2/2n}), \tag{2.27}$$

and, denoting $\varepsilon = \lambda n^{-1/3}$,

$$p^{k+l} = ((1+\varepsilon)/n)^{k+l} \le p^l e^{\varepsilon k}/n^k,$$

and

$$\begin{aligned} q^{\binom{k}{2} - (k+l) + k(n-k)} &= O(q^{-l}(1-p)^{kn - k^2/2}) \\ &= O(q^{-l} \exp((k^2/2 - kn)(1+\varepsilon)/n)) \\ &= O(q^{-l} \exp(-(1+\varepsilon)k + (1+\varepsilon)k^2/2n)). \end{aligned}$$

Notice that, by the definitions of k_0 and $\varepsilon = \lambda n^{-1/3}$,

$$k^2|\varepsilon|/n \le k_0^2|\varepsilon|/n = |\lambda|\omega_1(n)^2 = |\lambda|\omega(n)^{1/2}.$$

So, from (2.26) we obtain

$$E_p(n) = O\left[e^{|\lambda|\omega(n)^{1/2}} \sum_{\substack{k \le k_0 \\ l_0 \le l \le \binom{k}{2} - k}} E_p(n; k, l) \right], \tag{2.28}$$

where

$$E_p(n; k, l) = \frac{C(k, k+l)e^{-k}}{k!} \left(\frac{p}{q}\right)^l.$$

Now, break the sum in (2.28) into $\sum_1$ and $\sum_2$ according to whether $k < l$ or $k \ge l$. By the bound (2.6), and the inequality $k! \ge (k/e)^k$,

$$\begin{aligned} \sum\nolimits_1 &\le \sum_{k=1}^{k_0} \frac{e^{-k}}{k!} \left(\sum_{l \ge \max(k, l_0)} (c_2 k)^{k+l} \left(\frac{p}{q}\right)^l \right) \\ &= O\left(\sum_{k=1}^{k_0} c_2^k \left(\sum_{l \ge \max(k, l_0)} \left(\frac{c_2 k p}{q}\right)^l \right) \right) \\ &= O\left(\sum_{k=1}^{k_0} c_2^k \left(\frac{c_2 k p}{q}\right)^{\max(k, l_0)} \right) \\ &= O\left(\sum_{k=1}^{k_0} \left(\frac{c_2^2 k p}{q}\right)^{\max(k, l_0)} \right) = O\left(l_0 \left(\frac{c_2^2 k_0 p}{q}\right)^{l_0} \right) \\ &= O((2c_2^2 \omega(n)^{1/4}/n^{1/3})^{\omega(n)}) = O((2c_2^2 n^{-\delta/4})^{\omega(n)}), \end{aligned} \tag{2.29}$$

because $\omega(n) = O(n^{4/3-\delta})$. (We may, and do, assume that $c_2 > 1$.)

Next, by the key bound (2.5), and Stirling's formula for $k!$,

$$\sum_2 \le \sum_{k=l_0}^{k_0} \frac{e^{-k}}{k!}\left(\sum_{l=l_0}^{k}\left(\frac{c_1}{l}\right)^{l/2} k^{k+(3l-1)/2}\left(\frac{p}{q}\right)^l\right)$$
$$= O\left(\sum_{k=l_0}^{k_0}\left(\sum_{l=l_0}^{k}\left(\frac{c_1}{l}\right)^{l/2} k^{(3l-2)/2}\left(\frac{p}{q}\right)^l\right)\right).$$

In the innermost sum, the ratio of the $(l+1)$th term and the lth term equals

$$\frac{(c_1/(l+1))^{(l+1)/2}}{(c_1/l)^{l/2}} k^{3/2}\frac{p}{q} \le \left(\frac{c_1}{l}\right)^{1/2} k^{3/2}\frac{p}{q} = O\left(\frac{k_0^{3/2}}{l_0^{1/2} n}\right) = O(\omega(n)^{-1/8}).$$

Therefore,

(2.30)

$$\sum_2 = O\left(\sum_{k=l_0}^{k_0}\left(\frac{c_1}{l_0}\right)^{l_0/2} k^{(3l_0-2)/2}\left(\frac{p}{q}\right)^{l_0}\right) = O\left(k_0\left(\frac{c_1}{l_0}\right)^{l_0/2} k_0^{(3l_0-2)/2}\left(\frac{p}{q}\right)^{l_0}\right)$$
$$= O\left(\left(\frac{c_1^{1/2}k_0^{3/2}}{l_0^{1/2}}\frac{p}{q}\right)^{l_0}\right) = O\left(\left(\frac{2c_1^{1/2}}{\omega(n)^{1/8}}\right)^{\omega(n)}\right).$$

Collecting the estimates (2.28), (2.29), and (2.30) we obtain

$$E_p(n) = O\left(e^{|\lambda|\omega(n)^{1/2}}\left(2c_2^2 n^{-\delta/4} + \frac{2c_1^{1/2}}{\omega(n)^{1/8}}\right)^{\omega(n)}\right) = o(1).$$

Note. Thus, the order of the expected total number of components satisfying the conditions (2.25) is determined by those components for which, in addition $k \ge l$.

II. Turn now to $G(n, M)$. In this case,

$$E_M(n) = \left(\sum_{\substack{k\le k_0\\ l_0\le l\le\binom{k}{2}-k}}\binom{n}{k}C(k, k+l)\right)\binom{\binom{n-k}{2}}{M-k-l}\binom{\binom{n}{2}}{M}^{-1}.$$

Define

$$p' = M\Big/\binom{n}{2} = 1 + \lambda' n^{-1/3}, \qquad \lambda' = \lambda + O(n^{-2/3}).$$

Using (2.7) and (2.28), (2.29), we obtain that the total contribution of the terms with $k < l$ is

$$O(n^2 e^{|\lambda'|\omega(n)^{1/2}}(2c_2^2 n^{-\delta/4})^{\omega(n)}) = o(1).$$

So, it remains to estimate the sum of the terms with $k \ge l$. By (2.14)–(2.16),

(2.31)

$$\binom{\binom{n-k}{2}}{M-k-l}\binom{\binom{n}{2}}{M}^{-1} = O\left(\left(\frac{2M}{n^2}\right)^{k+l}\left(1-\frac{k}{n}\right)^{2(M-k-l)}\exp\left(-\frac{(k+l)^2}{2M}\right)\right).$$

Here

$$\left(\frac{2M}{n^2}\right)^{k+l} \le \frac{(1+\varepsilon)^l}{n^{k+l}} e^{\varepsilon k} \qquad (\varepsilon := \lambda n^{-1/3}), \tag{2.32}$$

and, since $k^3/n^2 \le \omega(n)^{3/4}$ (see (2.25)),

$$\begin{aligned} \left(1-\frac{k}{n}\right)^{2(M-k-l)} &\le \exp\left[2(M-k-l)\left(-\frac{k}{n}-\frac{k^2}{2n^2}\right)\right] \\ &= \exp\left[-\frac{2Mk}{n}+\frac{2(k+l)k}{n}-\frac{Mk^2}{n^2}+O(\omega(n)^{3/4})\right] \\ &= e^{-k}\exp\left[-\varepsilon k+\frac{2(k+l)k}{n}-\frac{k^2}{2n}+O(\omega(n)^{3/4})\right]. \end{aligned} \tag{2.33}$$

Besides,

$$(k+l)^2/2M = (k+l)^2/n + O(\omega(n)^{1/2}),$$

so, collecting the estimates (2.27), (2.31)–(2.33), we show that, for $k \ge l$, the generic term is at most

$$\begin{aligned} O&\left[\frac{C(k\,,\,k+l)e^{-k}}{k!}\left(\frac{1+\varepsilon}{n}\right)^l \exp\left(-\frac{l^2}{n}+O(\omega(n)^{3/4})\right)\right] \\ &= O\left[e^{c\omega(n)^{3/4}}\frac{C(k\,,\,k+l)e^{-k}}{k!}\left(\frac{1+\varepsilon}{n}\right)^l\right], \end{aligned} \tag{2.34}$$

$c > 0$ being an absolute constant. Therefore, as in part I(a), the sum of all these terms is bounded by

$$O[e^{c\omega(n)^{3/4}}(2c_1^{1/2}/\omega(n)^{1/8})^{\omega(n)}] = o(1), \qquad n \to \infty. \quad \square$$

Lemma 2.3. *For fixed* $l \ge 1$,

$$\lim_{n\to\infty} E[X(n\,;\,l)] = \frac{\gamma_l}{(2\pi)^{1/2}}\int_0^\infty x^{3l/2-1}e^{-F(x)}\,dx, \tag{2.35}$$

and

$$E\left[\sum_{k<k_1} X(n\,;\,k\,,\,l)\right] = O(\omega(n)^{-3l/2}), \tag{2.36}$$

if $k_1 = [n^{2/3}/\omega(n)]$ *and* $\omega(n) \to \infty$ *arbitrarily slowly.*

Proof. The first relation follows from (2.10), Lemma 2.1, and approximating the resulting sum by the integral. The second part is similar, via a bound

$$E\left[\sum_{k<k_1} X(n\,;\,k\,,\,l)\right] = O\left(\int_0^{\omega(n)^{-1}} x^{3l/2-1}e^{-F(x)}\,dx\right).$$

(The integrand is monotone increasing for all sufficiently small $x \ge 0$.) $\square$

In combination, Lemma 2.2 (2.20), Theorem 1, and Lemma 2.3 (2.36) mean that a.s. the total size of all complex components is at most $n^{2/3}\omega(n)$, while the smallest complex component has size at least $n^{2/3}/\omega(n)$, if $\omega(n) \to \infty$ arbitrarily slowly.

Corollary 3. *$\mathscr{C}(n)$, the total number of complex components, is bounded in probability.*

However, we can prove a stronger result.

Theorem 2. *Let*

$$g(x) = (2\pi)^{-1/2} \sum_{l \geq 1} \gamma_l x^{3l/2-1}.$$

Then (for both versions of $G(\lambda)$)

$$\lim E[\mathscr{C}(n)] = I(\lambda), \tag{2.37}$$

where

$$I(\lambda) = \int_0^\infty g(x) e^{-F(x)}\, dx < \infty, \qquad \left(F(x) = \frac{x^3 - 3x^2\lambda + 3x\lambda^2}{6} \right).$$

Proof. Fix $d > 1$ and write

$$\mathscr{C}(n) = \mathscr{C}_1(n) + \mathscr{C}_2(n) + \mathscr{C}_3(n),$$

where $\mathscr{C}_1(n)$, $\mathscr{C}_2(n)$, and $\mathscr{C}_3(n)$ are the number of complex components of size at most $d^{-1}n^{2/3}$, between $d^{-1}n^{2/3}$ and $dn^{2/3}$, and at least $dn^{2/3}$, respectively.

(a) By Lemma 2.2 (2.20),

$$E[\mathscr{C}_3(n)] \leq (dn^{2/3})^{-1} E[Y(n; \geq 1)] = O(d^{-1}). \tag{2.38}$$

(b) Let us bound $E[\mathscr{C}_1(n)]$. Consider $G(n, p)$ first. Similarly to (2.28) (*but with $k^2|\varepsilon|/n \leq d^{-2}|\lambda| = O(1)$ uniformly for $d > 1$, this time*), we have

$$E_p[\mathscr{C}_1(n)] = O\left(\sum_{\substack{4 \leq k \leq d^{-1}n^{2/3} \\ 1 \leq l \leq \binom{k}{2} - k}} E_p(n; k, l) \right).$$

$\sum_1$, the sum of $E_p(n; k, l)$ with $k < l$, is bounded (crudely) by

$$\begin{aligned} \sum_{k=4}^{d^{-1}n^{2/3}} \frac{e^{-k}}{k!} \left(\sum_{l>k} (c_2 k)^{k+l} \left(\frac{p}{q}\right)^l \right) &= O\left[\sum_{k=4}^{d^{-1}n^{2/3}} \left(\sum_{l>k} \left(\frac{c_2^2 k}{n} \right)^l \right) \right] \\ &= O\left[\sum_{k \geq 4} (c_2^2 d^{-1} n^{-1/3})^{k+1} \right] = O(n^{-5/3}). \end{aligned} \tag{2.39}$$

$\sum_2$, the sum of $E_p(n; k, l)$ with $k \geq l$, is bounded (according to (2.5)) by

$$\begin{aligned} \sum_{1 \leq l \leq k \leq d^{-1}n^{2/3}} \left(\frac{c_1}{l}\right)^{l/2} k^{(3l-2)/2} \left(\frac{p}{q}\right)^l &\leq \sum_{l \geq 1} \left(\frac{c_1}{l}\right)^{1/2} \left(\frac{p}{q}\right)^l (d^{-1}n^{2/3})^{(3l/2)} \\ &\leq \sum_{l \geq 1} (2c_1 d^{-3})^{l/2} = O(d^{-3/2}), \end{aligned}$$

provided $d > 1$ is such that $2c_1 d^{-3} < 1$, which we may, and do assume. Thus,

$$E_p[\mathscr{C}_1(n)] = O(d^{-3/2}). \tag{2.40}$$

Turn to $E_M[\mathscr{C}_1(n)]$. The overall contribution of the components with $k < l$ is bounded by $O(n^{1/2}n^{-5/3}) = O(n^{-7/6})$ (see (2.39) and the note following (2.7)). The bound (2.34) implies that for $k \le d^{-1}n^{2/3}$ $(d > 1)$ the expected number of components with k vertices and l edges is of the order

$$O\left(\frac{C(k, k+l)e^{-k}}{k!}\left(\frac{1+\varepsilon}{n}\right)^l\right).$$

So the total expected number of all those components with $k \ge l$ is at most

$$\begin{aligned}
O&\left(\sum_{1\le l\le k\le d^{-1}n^{2/3}} \frac{C(k, k+l)e^{-k}}{k!}\left(\frac{1+\varepsilon}{n}\right)^l\right)\\
&= O\left(\sum_{1\le l\le k\le d^{-1}n^{2/3}} \left(\frac{c_1}{l}\right)^{l/2} k^{(3l-2)/2}\left(\frac{1+\varepsilon}{n}\right)^l\right)\\
&= O(d^{-3/2}),
\end{aligned}$$

so that

$$E_M[\mathscr{C}_1(n)] = O(d^{-3/2}). \tag{2.41}$$

(c) Now consider $E[\mathscr{C}_2(n)]$. Define $\mathscr{C}_2^*(n)$ as the total number of components with sizes between $d^{-1}n^{2/3}$ and $dn^{2/3}$ such that $l < d^4$, and let $\Delta\mathscr{C}_2(n) = \mathscr{C}_2(n) - \mathscr{C}_2^*(n)$. While proving Theorem 1, we actually demonstrated that the expected number of components of size $\le dn^{2/3}$ such that $l \ge d^4$ is $O[e^{|\lambda|d^2}(2c_1^{1/2}/d^{1/2})^{d^4}]$ provided that d is sufficiently large (see (2.25), (2.29) and (2.30)). Hence,

$$E[\Delta\mathscr{C}_2(n)] = O[e^{|\lambda|d^2}(2c_1^{1/2}/d^{1/2})^{d^4}]. \tag{2.42}$$

As for $\mathscr{C}_2^*(n)$, by Lemma 2.1, Wright's formula (2.4) and Stirling's formula for factorials,

$$\begin{aligned}
E[\mathscr{C}_2^*(n)] = [1 + O(n^{-1/3})]&\\
\cdot \sum_{1\le l\le d^4} \frac{\gamma_l}{(2\pi)^{1/2}}&\left(n^{-l}\sum_{d^{-1}n^{2/3}\le k<dn^{2/3}} k^{3l/2-1}e^{-F(x_k)}\right).
\end{aligned}$$

Consequently

$$\lim_{n\to\infty} E[\mathscr{C}_2^*(n)] = \sum_{1\le l\le d^4}\frac{\gamma_l}{(2\pi)^{1/2}}\int_{d^{-1}}^{d} x^{3l/2-1}e^{-F(x)}\,dx. \tag{2.43}$$

The relations (2.38)–(2.43) lead directly to

$$\overline{\lim_{n\to\infty}}\, E[\mathscr{C}(n)] = O(d^{-1}) + \sum_{1\le l\le d^4}\frac{\gamma_l}{(2\pi)^{1/2}}\int_{d^{-1}}^{d} x^{3l/2-1}e^{-F(x)}\,dx,$$

$$\underline{\lim_{n\to\infty}}\, E[\mathscr{C}(n)] = O(d^{-1}) + \sum_{1\le l\le d^4}\frac{\gamma_l}{(2\pi)^{1/2}}\int_{d^{-1}}^{d} x^{3l/2-1}e^{-F(x)}\,dx,$$

for all d sufficiently large. Letting $d \uparrow \infty$ completes the proof of (2.37).

Let us show that the integral $\int_0^\infty g(x)e^{-F(x)}\,dx$ converges. According to Wright (1980),

$$(2.44)\qquad \gamma_l = \frac{\sqrt{2\pi}3^l(l-1)!\delta_l}{2^{5l/2}\Gamma(3l/2)}$$

where $\delta_l \uparrow \delta = .159155\ldots$ (Voblyi (1987) identified δ as $(2\pi)^{-1}$. Bender, Canfield, and McKay (1989) found a three term asymptotic expansion for δ_l using a method suggested by Meertens. We are aware of at least two other independent proofs of $\delta = (2\pi)^{-1}$, one by the first author and another by Knuth (private communication).)

Thus, using Stirling formula for the gamma function we can estimate γ_l from above by

$$(2.45)\qquad \begin{aligned} \gamma_l &= \delta_l(3\pi)^{1/2}(e/12l)^{l/2}(1+O(l^{-1})) \\ &\le \delta(3\pi)^{1/2}(e/12l)^{l/2}(1+O(l^{-1})). \end{aligned}$$

Consequently, there exists a constant $a > 0$ such that for all $x > 0$

(2.46)

$$\begin{aligned} \sum_{l\le 1}\gamma_l x^{3l/2-1} &\le \frac{a}{x}\sum_{l\ge 1}\left(\frac{e}{12l}\right)^{l/2}x^{3l/2} \\ &\le \frac{a}{x}(1+x^{3/2})\sum_{m\ge 1}\left(\frac{e}{24m}\right)^m x^{3m} \le \frac{a}{x}(1+x^{3/2})(e^{x^3/16}-1), \end{aligned}$$

since $m! \le (m/2)^m$ and $16 < 48/e$. It remains to notice that $F(x) = (x^3 - 3x^2\lambda + 3x\lambda^2)/6 \ge x^3/7$ for all sufficiently large x. □

Notes. (1) Extending the above argument, we can prove the existence of a finite $\lim_{n\to\infty} E[\mathscr{C}(n)^k]$ for each $k > 1$, but the explicit formulae are more complicated, cf. Theorems 3 and 5.

(2) Only minor changes of the above argument are necessary to show that

$$E[Y(n;\ge 1)] = n^{2/3}\int_0^\infty xg(x)e^{-F(x)}\,dx + O(n^{1/3});$$

here $Y(n;\ge 1)$ is the total size of all complex components. But we have already estimated this expected value, see Lemma 2.2 (2.20). Comparing the two estimates we obtain a remarkable integral identity

$$(*)\qquad \int_0^\infty xg_0(x)e^{-F(x)}\,dx = \lambda + (2\pi)^{-1/2}\int_0^\infty x^{-3/2}[1-e^{-F(x)}]\,dx,$$

where

$$g_0(x) = (2\pi)^{-1/2}\sum_{l\ge 0}\gamma_l x^{3l/2-1},$$

(recall that $\gamma_0 = (\pi/8)^{1/2}$). What happens if we let $\lambda\to\infty$?

Let $c > 0$ be fixed. The series $\sum_{k=1}^\infty (c\xi/k)^{k/2}$ converges for all $\xi > 0$. Furthermore, $(c\xi/k)^{k/2}$ achieves its maximum at $k = \overline{k} = c\xi/e$. For $\xi\to\infty$, we have $\overline{k}\to\infty$ and if $k - \overline{k} = o(\overline{k}^{2/3})$ then

$$\left(\frac{c\xi}{k}\right)^{k/2} = (1+o(1))\exp\left(\frac{c\xi}{2e} - \frac{(k-\overline{k})^2}{4c\xi e^{-1}}\right).$$

The overall contribution of these leading terms to the sum is therefore asymptotic (as $\xi \to \infty$) to

$$\exp\left(\frac{c\xi}{2e}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{4c\xi e^{-1}}\right)dx = 2\exp\left(\frac{c\xi}{2e}\right)\left(\frac{\pi c\xi}{e}\right)^{1/2}.$$

This expression provides a sharp asymptotic formula for $\sum_{k=1}^{\infty}(c\xi/k)^{k/2}$ since the remaining terms contribute negligibly little to its value. (The reader has probably identified the above computations as an application of Laplace's method for asymptotic estimation of sums and integrals.) So, according to the definition of $g_0(x)$ and (2.45),

$$g_0(x) = (1+o(1))\delta(\pi/2)^{1/2}e^{x^3/24}x^{1/2}, \qquad x \to \infty. \tag{2.47}$$

Getting back to the integral identity $(*)$ break the integral on the left-hand side into $I_1(\lambda)$ and $I_2(\lambda)$, over $x \in [0, \lambda^{1/2}]$ and $x \in [\lambda^{1/2}, \infty)$ respectively. Clearly (see 2.46)),

$$I_1(\lambda) = O\left(\int_0^{\lambda^{1/2}}\exp(-\tfrac{1}{2}x\lambda^2)\,dx\right) = O(\lambda^{-2}).$$

Next,

$$\begin{aligned} I_2(\lambda) &= (1+o(1))\delta\left(\frac{\pi}{2}\right)^{1/2}\int_{\lambda^{1/2}}^{\infty}x^{3/2}\exp\left[\frac{x^3}{24} - \frac{x^3 - 3x^2\lambda + 3x\lambda^2}{6}\right]dx \\ &= (1+o(1))\delta\left(\frac{\pi}{2}\right)^{1/2}\int_{\lambda^{1/2}}^{\infty}x^{3/2}\exp\left[-\frac{x(x-2\lambda)^2}{8}\right]dx, \end{aligned}$$

so, applying Laplace's method to the last integral obtains

$$\begin{aligned} I_2(\lambda) &= (1+o(1))\delta\left(\frac{\pi}{2}\right)^{1/2}(2\lambda)^{3/2}\int_{-\infty}^{\infty}\exp\left(-\frac{\lambda z^2}{4}\right)dz \\ &= (1+o(1))4\pi\lambda\delta. \end{aligned}$$

Therefore, as $\lambda \to \infty$,

$$\int_0^{\infty}xg_0(x)e^{-F(x)}\,dx = (1+o(1))4\pi\lambda\delta.$$

Integrating by parts the integral on the right-hand side of the integral identity $(*)$, we get (omitting some simple intermediate estimates)

$$\begin{aligned} &(2\pi)^{-1/2}\int_0^{\infty}x^{-3/2}[1-e^{-F(x)}]\,dx \\ &= (2\pi)^{-1/2}\int_0^{\infty}(x-\lambda)^2x^{-1/2}e^{-F(x)}\,dx \\ &= (1+o(1))(2\pi)^{-1/2}\lambda^2\int_0^{\infty}x^{-1/2}e^{-x\lambda^2/2}\,dx \\ &= (1+o(1))\lambda\pi^{-1/2}\Gamma(1/2) = (1+o(1))\lambda. \end{aligned}$$

Comparing the estimates of both integrals, and using the integral identity $(*)$, we obtain $4\pi\delta = 2$, so that $\delta = (2\pi)^{-1}$. (!) Thus, we have obtained an entirely new proof of Voblyi's result, a proof that is based on the structural properties of the random graph at the critical stage.

(3) The estimate (2.46) and the dominated convergence theorem imply that $\lim_{\lambda\to-\infty} I(\lambda) = 0$. Intuitively, we should have expected this, since for $n^{-2/3}(M-n/2)\to-\infty$ the graph $G(n, M)$ a.s. does not have complex components (Kolchin (1986)). We also know that for $n^{-2/3}(M-n/2)\to\infty$ the graph $G(n, M)$ a.s. has exactly one complex component (Bollobás (1985), Łuczak (1989)). So, we should be able to prove that $\lim_{\lambda\to+\infty} I(\lambda) = 1$. It is easy! Analogously to the estimate in the preceding argument,

$$\begin{aligned} I(\lambda) &= \int_0^\infty g(x)e^{-F(x)}\,dx \\ &= (1+o(1))\delta\left(\frac{\pi}{2}\right)^{1/2}(2\lambda)^{1/2}\int_{-\infty}^{\infty}\exp\left(-\frac{\lambda z^2}{4}\right)dz \\ &= (1+o(1))2\pi\delta = 1+o(1), \end{aligned}$$

as $\lambda\to\infty$, since $2\pi\delta = 1$.

We conjecture that $I(\lambda)$ is monotone increasing. *If this is so, then, for each* λ, *the limiting expected number of complex components is less than* $I(\infty)=1$.

Theorems 1 and 2 make it plausible that there must exist a limiting distribution of the complexity spectrum $X(n) = \{X(n; l)\}_{l\geq1}$. This is indeed the case.

Let $h_n(x)$ stand for the probability that the random graph $G(x)$ (i.e., $G(n, M = (1+x^{-1/3})n/2)$ or $G(n, p = (1+xn^{-1/3})/n)$) does not have a complex component.

Flajolet, Knuth, and Pittel (1989) proved existence of $h(x) = \lim_{n\to\infty} h_n(x)$, given by

$$h(x) = \frac{1}{(2\pi)^{1/2}i}\int_{\mathscr{L}} z^{1/2}e^{(2z-x)(z+x)^2}\,dz, \tag{2.48}$$

where $\mathscr{L}$ is a certain contour in the half-plane $\mathrm{Re}(z)>0$.

It is possible to obtain a series-type formula for $h(x)$, namely

$$h(x) = e^{-x^3/6}\left(\frac{2}{3\pi}\right)^{1/2}\sum_{j\geq0}\frac{1}{j!}\left(-\frac{x}{2}3^{2/3}\right)^j\cos\left(\frac{\pi j}{4}\right)\Gamma\left(\frac{2}{3}j+\frac{1}{2}\right).$$

The derivation is based on an identity for the graph $G(n, p)$:

$$h_n(x) = n!q^{n^2/2}(pq^{-3/2})^n[x^n]\exp\left(\frac{q}{p}W_{-1}(x)+W_0(x)\right),$$

(cf. Flajolet, Knuth, and Pittel (1989)). Here $W_{-1}(x)$, $W_0(x)$ are the exponential generating functions for the (unrooted) trees and the unicyclic graphs, that is

$$W_{-1}(x) = T(x) - T^2(x)/2,$$

and

$$W_0(x) = \frac{1}{2}\left[\log\frac{1}{1-T(x)} - T(x) - \frac{T^2(x)}{2}\right]$$

($T(x) = \sum_{j\geq1} j^{j-1}x^j/j!$) (Moon (1970), Wright (1977)). To estimate the coefficient $[x^n]\exp(\cdot)$ we use the Cauchy integral formula for a contour $x = e^{-1+i\theta}$, $-\pi<\theta\leq\pi$. (Recall that $T(e^{-1}) = 1$.) The dominant part of the integral corresponds to the values $\theta = O(n^{-2/3+\delta})$, $\delta \in (0, 2/3)$, since (see Britikov (1988), for instance)

$$W_{-1}(e^{-1+i\theta}) = 1/2 + i\theta - (4/3\sqrt{2})e^{i\pi/4}\theta^{3/2} + O(\theta^2) \qquad (\theta \geq 0).$$

Rotating the segment $[0, n^{-2/3+\delta}]$ in the complex plane θ counterclockwise at the angle $\pi/6$, changing the variable of integration in order to linearize the leading exponent in the integrand, and then Taylor-expanding the remaining exponent factor, we arrive at the above series for $h(x)$.

The function $h(x)$ is strictly decreasing, $h(-\infty) = 1$, $h(\infty) = 0$ and $h(0) = (2/3)^{1/2}$. It comes in handy in the next theorem concerning the limiting distribution of the complexity spectrum.

Theorem 3. *Let $r = \{r_l\}_{l\geq 1}$ be a sequence of nonnegative integers with only finitely many nonzero terms, a finitary sequence in short. Then, for both models of the random graph $G(\lambda)$, there exists $p(r) = \lim_{n\to\infty} P(X(n) = r)$, where*

$$p(\vec{0}) = h(\lambda),$$

and, for $r \neq \vec{0}$,

(2.49)

$$p(r) = \frac{1}{\Gamma(3L/2)}\left(\prod_{l\geq 1}\frac{(\gamma_l\Gamma(3l/2)/(2\pi)^{1/2})^{r_l}}{r_l!}\right)\int_0^\infty e^{-F(x)}x^{3L/2-1}h(\lambda - x)\,dx$$

$(L := \sum_l lr_l)$. Furthermore, $\sum_r p(r) = 1$; so, in terms of the finite-dimensional distributions, $X(n)$ converges to a random finitary sequence $X = \{X(l)\}$ such that $P(X = r) = p(r)$.

Notes. (1) $p(r) > 0$ for all r. Thus, the limiting distribution of the complexity spectrum is supported by the whole set R of finitary sequences of nonnegative integers !

(2) Consider a Banach space l_1 of all finitary sequences $r = \{r_l\}$ of integers with the norm $\|r\| = \sum_{l\geq 1}|r_l|$. From the proof of Theorem 2, and Theorem 3, it follows that $X \in l_1$ almost certainly, and that $X(n)$ converges weakly to X in l_1. In particular, $\mathscr{C}(n) = \sum_{l\geq 1} X(n; l) \Rightarrow \sum_{l\geq 1} X(l)$. (In fact, $E[\mathscr{C}(n)] \to E[\sum_{l\geq 1} X(l)]$, too; so, by Theorem 2,

$$\sum_r p(r)\sum_{l\geq 1} r_l = I(\lambda).)$$

Proof of Theorem 3. For each $l \geq 1$, let k_{lm} $(1 \leq m \leq r_l)$ stand for possible sizes of all l-components. Set $\vec{k} = \{k_{lm} : 1 \leq m \leq r_l,\ l \geq 1\}$, and $k = \sum_{l,m} k_{lm}$. In the case of $G(n, p)$, we have

$$\begin{aligned}P(X(n; l) &= r_l, l \geq 1)\\ &= \sum_{\vec{k}}\frac{n!}{(n-k)!}\left(\prod_{l\geq 1}\frac{1}{r_l!}\right)\left(\prod_{l,m}\frac{C(k_{lm}, k_{lm}+l)}{k_{lm}!}p^{k_{lm}+l}q^{\binom{k_{lm}}{2}-(k_{lm}+l)}\right)\\ &\quad\cdot q^{(1/2)\sum_{(l,m)\neq(l',m')}k_{lm}k_{l'm'}+k(n-k)}\mathrm{H}_{n-k}(p).\end{aligned}$$

Here, moving backward, $\mathrm{H}_{n-k}(p)$ is the probability that the graph $G(n-k, p)$ does not have complex components; the next factor is the probability that there are no edges joining the designated subsets of vertices either to each other or to the remaining $n-k$ vertices;

$$C(k_{lm}, k_{lm}+l)p^{k_{lm}+l}q^{\binom{k_{lm}}{2}-(k_{lm}+l)}$$

is the probability that the set of k_{lm} vertices induces a connected subgraph with $k_{lm}+l$ edges; the remaining factors account for the total number of partitions with the parameter $\vec{k}$. Notice at once that

$$\frac{1}{2}\sum_{(l,m)\neq(l',m')} k_{lm}k_{l'm'} = \frac{1}{2}k^2 - \frac{1}{2}\sum_{(l,m)} k_{lm}^2 .$$

So, rearranging factors,

$$P(X(n;l)=r_l;\, l\geq 1) = \sum_{\vec{k}} P(\vec{k}), \tag{2.50}$$

$$\begin{aligned} P(\vec{k}) &= \frac{n!}{(n-k)!}q^{kn-k^2/2}\left(\prod_{l\geq 1}\frac{1}{r_l!}\right) \\ &\cdot \left(\prod_{l,m}\frac{C(k_{lm}, k_{lm}+l)}{k_{lm}!}p^{k_{lm}+l}q^{(-3k_{lm}/2-l)}\right) H_{n-k}(p). \end{aligned} \tag{2.51}$$

The summands $P(\vec{k})$, which satisfy the conditions $1\leq l\leq d^4$, $d^{-1}n^{2/3}\leq k_{lm}\leq dn^{2/3}$, account for almost all the value of the probability in question, except for a remainder of order $O(d^{-1})$. (See the proof of Theorem 2.) Let us determine a sharp asymptotic formula for a leading generic term $P(\vec{k})$. First,

$$\frac{n!}{(n-k)!}q^{kn-k^2/2} = n^k e^{-k}\exp\left[-k\lambda n^{-1/3} - \frac{x_k^3}{6} + \frac{\lambda x_k^2}{2} + O(n^{-1/3})\right] \tag{2.52}$$

($x_k := k/n^{-2/3}$). The double-product factor in (2.51) is asymptotic, via Wright's formula (1980), to

$$\begin{aligned} &[1+O(n^{-1/3})]\prod_{l,m}\frac{\gamma_l k_{lm}^{k_{lm}+3l/2-1/2}}{k_{lm}!}p^{k_{lm}+l} \\ &= [1+O(n^{-1/3})]e^k\prod_{l,m}\frac{\gamma_l k_{lm}^{3l/2-1}}{(2\pi)^{1/2}}\left(\frac{1+\lambda n^{-1/3}}{n}\right)^{k_{lm}+l} \\ &= [1+O(n^{-1/3})]n^{-k}e^k\left(\prod_{l,m}\frac{n^{-l}\gamma_l k_{lm}^{3l/2-1}}{(2\pi)^{1/2}}\right)(1+\lambda n^{-1/3})^k \\ &= [1+O(n^{-1/3})]n^{-k}e^k\exp(k\lambda n^{-1/3}-\lambda^2 x_{k_{lm}}/2)\prod_{l,m}\frac{\gamma_l x_{k_{lm}}^{3l/2-1}}{(2\pi)^{1/2}}\Delta x_{k_{lm}} \end{aligned}$$

($x_{k_{lm}} := k_{lm}/n^{2/3}$, $\Delta x_{k_{lm}} = n^{-2/3}$). Furthermore, since

$$p = (1+\lambda n^{-1/3})/n = [1+(\lambda - x_k)(n-k)^{-1/3}+O(n^{-2/3})]/(n-k),$$

we have

(2.53) $\mathrm{H}_{n-k}(p) = h_{n-k}(\lambda - x_k + O(n^{-1/3})) = h(\lambda - x_k) + o(1) \qquad (n \to \infty).$

Collecting (2.50)–(2.53) yields

$$P(X(n;l) = r_l;\, l \geq 1) = [1 + O(n^{-1/3})] \sum_{\substack{d^{-1}n^{2/3} \leq k_{lm} \leq dn^{2/3} \\ l \leq m \leq r_l}} \left(\prod_{l \geq 1} \frac{1}{r_l!} \right)$$

$$\cdot \left(\prod_{l,m} \frac{\gamma_l x_{k_{lm}}^{3l/2-1}}{(2\pi)^{1/2}} \Delta x_{k_{lm}} \right) \exp\left[-\frac{(x_k^3 - 3x_k^2\lambda + 3x_k\lambda^2)}{6} \right] (h(\lambda - x_k) + o(1)) \tag{2.54}$$

$$+ O(d^{-1}),$$

So, letting $n \to \infty$ and then $d \to \infty$,

$$\lim_{n \to \infty} P(X(n;l) = r_l;\, l \geq 1)$$

$$= \int_{0 \leq x_{lm} < \infty} \cdots \int \left(\prod_{l \geq 1} \frac{1}{r_l!} \right) \left(\prod_{l,m} \frac{\gamma_l x_{k_{lm}}^{3l/2-1}}{(2\pi)^{1/2}} \right) e^{-F(x)} h(\lambda - x) \prod_{l,m} dx_{lm}, \tag{2.55}$$

$$x := \sum_{l,m} x_{lm}.$$

In view of the identity

$$\int_{\substack{0 \leq x_\mu < \infty \\ 1 \leq \mu \leq \nu}} \cdots \int \psi\left(\sum_{\mu=1}^{\nu} x_\mu \right) \left(\prod_{\mu=1}^{\nu} x_\mu^{\alpha_\mu} \right) dx_1 \cdots dx_\nu$$

$$= \frac{\prod_\mu \Gamma(\alpha_\mu + 1)}{\Gamma\left(\sum_\mu (\alpha_\mu + 1) \right)} \int_0^\infty \psi(x) x^{\sum_\mu (\alpha_\mu + 1) - 1}\, dx, \tag{2.56}$$

the relation (2.55) immediately leads to (2.49).

In the case of $G(n, M)$, the analogous formula for $P(\vec{k})$ is

$$P(\vec{k}) = \frac{n!}{(n-k)!} \frac{\binom{\binom{n-k}{2}}{M-k-L}}{\binom{\binom{n}{2}}{M}} \prod_{l \geq 1} \frac{1}{r_l!}$$

$$\cdot \prod_{l,m} \frac{C(k_{lm}, k_{lm} + l)}{(k_{lm})!} \mathscr{H}_{n-k}(M - k - L).$$

Here $L = \sum_l l r_l$, and $\mathscr{H}_\nu(\mu)$ is the probability that $G(\nu, \mu)$ does not have complex components. As in the part (a) of the proof of Lemma 2.1, for the dominating $\vec{k}$'s,

$$\frac{n!}{(n-k)!} \frac{\binom{\binom{n-k}{2}}{M-k-L}}{\binom{\binom{n}{2}}{M}} = [1 + O(n^{-1/3})] n^{-L} e^{-k} e^{-F(x_k)}.$$

Also,

$$\mathscr{H}_{n-k}(M-k-L) = h(\lambda - x) + o(1).$$

Using Wright's formula for $C(k_{lm}, k_{lm}+l)$ and Stirling's formula for $k_{lm}!$, we obtain (2.49).

To finish the proof, it remains to notice that

$$\sum_r p(r) = \lim_{n\to\infty} P(\mathscr{C}(n) < \infty) = 1,$$

since $E(\mathscr{C}(n)) = O(1)$. ($\mathscr{C}(n)$ is the total number of complex components.) □

Note. The identity (2.56) can be established as follows. Its right-hand side equals $\int_0^\infty \psi(x)\phi(x)\,dx$, where

$$\phi(x) = \int \left(\prod_{\mu=1}^{\nu} x_\mu^{\alpha_\mu}\right)(dx_1 \cdots dx_{\nu-1}) \qquad \left(x_\nu = x - \sum_{\mu=1}^{\nu-1} x_\mu\right)$$

and $x_1, \dots, x_{\nu-1}$ are subject to restrictions $x_1 \ge 0, \dots, x_{\nu-1} \ge 0$, $\sum_{\mu=1}^{\nu-1} x_\mu \le x$ (that is $x_\nu \ge 0$, too). Clearly,

$$\phi(x) = x^{\sum_{\mu=1}^{\nu}\alpha_\mu + \nu - 1}\psi(1). \tag{2.57}$$

Applying this to $\psi(x) = e^{-x}$, we have then

$$\begin{aligned}
&\int_{\substack{0\le x_\mu<\infty\\ 1\le\mu\le\nu}} \exp\left(-\sum_{\mu=1}^{\nu} x_\mu\right)\left(\prod_{\mu=1}^{\nu} x_\mu^{\alpha_\mu}\right) dx_1\cdots dx_\nu \\
&\quad = \phi(1)\int_0^\infty e^{-x} x^{\sum_{\mu=1}^{\nu}(\alpha_\mu+1)-1}\,dx \\
&\quad = \phi(1)\Gamma\left(\sum_{\mu=1}^{\nu}(\alpha_\mu+1)\right).
\end{aligned} \tag{2.58}$$

But the integral (2.58) also equals

$$\prod_{\mu=1}^{\nu}\left(\int_0^\infty e^{-x_\mu} x_\mu^{\alpha_\mu}\,dx_\mu\right) = \prod_{\mu=1}^{\nu}\Gamma(\alpha_\mu+1),$$

hence

$$\phi(1) = \frac{\prod_{\mu=1}^{\nu}\Gamma(\alpha_\mu+1)}{\Gamma(\sum_{\mu=1}^{\nu}(\alpha_\mu+1))}. \tag{2.59}$$

The relations (2.57), (2.59) imply (2.56).

Note. Introduce C_n the total number of cycles. Since the total number of complex components and the maximal number of cycles in a component are both bounded in probability, $C_n - C_n^* = O_p(1)$, where C_n^* is the total number of unicyclic components. This can be used to show that $(C_n - 6^{-1}\log n)/\sqrt{6^{-1}\log n}$ is asymptotically standard Gaussian, regardless of the actual value of λ. We suggest the reader check, however, that $E(C_n) \sim 6^{-1}\log n$ for $\lambda = 0$, and $E(C_n) \sim 4^{-1}\log n$ for $\lambda = 0$, and $E(C_n) \sim c(\lambda)n^{-1/6}\exp(\lambda^2 n^{1/3}/2)$ for $\lambda > 0$.

3. The inner structure of complex components

Let G be a connected graph with k labeled vertices and $k+l$ edges, where $l \geq 1$. Let $C(G)$ denote the maximal subgraph of G with minimal vertex degree at least two. $C(G)$ is well defined since the union of such subgraphs also satisfies the condition. (To get G from $C(G)$, one has to grow a number of disjoint trees each sprouting from its own vertex of $C(G)$.) A *basic path* of G is a maximal path in $C(G)$ in which all vertices except the endpoints have degree two in $C(G)$. Let $S(G)$ be the multigraph obtained from $C(G)$ by replacing each basic path with a single edge which joins its endpoints. (The edge induced by a basic path is a loop if the path starts and ends at a vertex of degree at least three in $C(G)$.) We shall call $S(G)$ the support of G.

Using the previous results, we can obtain a rather complete rough description of the likely complex components in the random graph.

Theorem 4. *Let $\omega(n) \to \infty$ as $n \to \infty$. Then a.s.:*

(I) *Supports of all complex components of $G(\lambda)$ are 3-regular multigraphs, each with at most $\omega(n)$ vertices, and at most $\omega(n)$ edges and loops.*
(II) *Each basic path of every complex component of $G(\lambda)$ has at least $n^{1/3}/\omega(n)$ vertices, and at most $n^{1/3}\omega(n)$ vertices; thus all cycles in the complex components have lengths of order $n^{1/3}$.*
(III) *For each complex component G, the largest tree sprouting from a vertex of $C(G)$ has size at least $n^{2/3}/\omega(n)$, and at most $n^{2/3}\omega(n)$.*

Proof. (I) Let G be an l-component, $l \geq 1$, with a support $S = (V, E)$. Denote $|V| = a$, $|E| = b$. Notice that $b - a = l$. Indeed, $S(G)$ can be obtained from G by consecutive deletions of all vertices of degree 1 (together with the correspondent edges), followed by shrinking every basic path in the resulting graph $C(G)$ to an edge. For each of these operations, the difference between the number of edges and the number of vertices remains unchanged. Furthermore, the degree of each vertex in S is at least 3, since we have assumed that a loop at a vertex contributes two to this vertex degree. Thus, $2b \geq 3a$ and consequently $a \leq 2l$, $b \leq 3l$. But by Theorem 1, $\mathscr{L}_n = \max_G(|E(G)| - |V(G)|)$ is bounded in probability, hence a.s.

$$\max_G |E(G)| \leq \omega(n), \qquad \max_G |V(G)| \leq \omega(n), \tag{3.1}$$

if $\omega(n) \to \infty$ arbitrarily slowly.

It remains to show that a.s. all the supports are 3-regular. For this, we recall that a.s. each complex component has size between $n^{2/3}/\omega(n)$ and $n^{2/3}\omega(n)$. So (see (3.1)), it will suffice to prove the following statement.

Lemma 3.1. *Let $d > 1$ and positive integers a, b such that $2b > 3a$ be given. Let S be a multigraph with vertex set $\{1, \ldots, a\}$ and edges $u_1, \ldots, u_b$. Then E_n, the expected number of complex components of $G(\lambda)$ with support isomorphic to S and with size between $d^{-1}n^{2/3}$ and $dn^{2/3}$, tends to 0 as $n \to \infty$.*

Proof. Let $(i_1, \dots, i_b)$ and k be nonnegative integers which satisfy the conditions

$$d^{-1}n^{2/3} \le k \le dn^{2/3}, \qquad i := a + \sum_{t=1}^{b} i_t \le k. \tag{3.2}$$

Define $E(n; \vec{i}, k)$ as the expected number of complex components G of size k, subject to the conditions:

(1) the support $S(G)$ is isomorphic to S, and

(2) if a basic path in $C(G)$ stripped of its intermediate vertices (an edge (loop) of $S(G)$, that is) is the image of an edge (loop) u_t of the multigraph S, then the total number of those vertices is i_t $(1 \le t \le b)$.

A little reflection shows (cf. Stepanov (1987), for instance) that the total number of the subgraphs G of the complete graph K_n, which meet the conditions (1) and (2), is at most

$$\binom{n}{k}\binom{k}{a, i_1, \dots, i_b, k-i} a! \left(\prod_{t=1}^{b} i_t!\right) ik^{k-i-1} = \frac{[n]_k ik^{k-i-1}}{(k-i)!}.$$

Here the factor ik^{k-i-1} counts the total number of forests of i trees on k vertices $v_1, \dots, v_k$ such that the jth tree contains v_j $(1 \le j \le i)$. ("At most" is because in the underlying counting scheme a subgraph in question arises, in general, more than once.)

Consequently, for $G(n, p)$,

$$E(n; \vec{i}, k) \le E_*(n; \vec{i}, k) := \frac{[n]_k ik^{k-i-1}}{(k-i)!} p^{k+b-a} p^{\binom{k}{2}-(k+b-a)+k(n-k)}. \tag{3.3}$$

We notice that $E_*(n; \vec{i}, k)$ depends only on i and k. Since the number of nonnegative solutions of $\sum_{t=1}^{b} i_t = i - a$ (i fixed) equals $\binom{i-a+b-1}{b-1}$, we obtain

$$E(n; i, k) := \sum_{\vec{i}} E(n; \vec{i}, k) \le E_*(n; i, k), \tag{3.4}$$

where

$$\begin{aligned} E_*(n; i, k) &= \sum_{\vec{i}} E_*(n; \vec{i}, k) \\ &= [n]_k ik^{k-i-1} p^{k+b-a} q^{\binom{k}{2}-(k+b-a)+k(n-k)} \binom{i-a+b-1}{b-1} / (k-i)!. \end{aligned} \tag{3.5}$$

($E(n; i, k)$ is the expected number of complex components G in question such that the sizes of G and $C(G)$ are k and i respectively.) Using the bounds

$$\begin{aligned} &[n]_k = O(n^k e^{-k^2/2n}), \qquad \binom{i-a+b-1}{b-1} \le i^{b-1}(b-a)^{b-1}, \\ &(k-i)! \ge \text{const}(k-i+1)^{1/2}\left(\frac{k-i}{e}\right)^{k-i}, \end{aligned} \tag{3.6}$$

and one term expansions for $\log(1+\lambda n^{-1/3})$ and $\log(1-q)$, we transform (3.4) and (3.5) into

$$E(n; i, k) = O\left[\frac{1}{n^{b-a}} \frac{1}{k(k-i+1)^{1/2}} \exp(H(i, k))\right], \tag{3.7}$$

with

$$H(i, k) = (k - i)\log\frac{k}{k-i} - i + b\log i. \tag{3.8}$$

A bit of calculus is in order. We have

$$\frac{\partial H}{\partial i} = -\log\frac{k}{k-i} + \frac{b}{i}, \qquad \frac{\partial^2 H}{\partial i^2} = -\frac{1}{k-i} - \frac{b}{i^2} < -\frac{1}{k}.$$

It follows then that, for each k, $H(i, k)$ achieves its maximum at

$$i_* = i_*(k) = (bk)^{1/2} + O(1),$$

and

$$H(i_{*,}, k) = O(1) + b\log i_* = (b/2)\log k + O(1).$$

For $i \neq i_*$, the bound for $\partial^2 H/\partial i^2$ implies that

$$H(i, k) \leq H(i_*, k) - (i - i_*)^2/2k.$$

Therefore,

$$\begin{aligned}\sum_i E(n; i, k) &= O\left[\frac{k^{b/2-1}}{n^{b-a}}\sum_i (k-i+1)^{-1/2}\exp\left(-\frac{(i-i_*(k))^2}{2k}\right)\right]\\ &= O\left[\frac{k^{b/2-1}}{n^{b-a}}\sum_i k^{-1/2}\exp\left(-\frac{(i-i_*(k))^2}{2k}\right)\right]\\ &= O\left(\frac{k^{b/2-1}}{n^{b-a}}\right) = O(n^{-2/3}n^{-(2b-3a)/3}).\end{aligned} \tag{3.9}$$

(The dominant terms in $\sum_i (k-i)^{-1/2}\exp[-(i-i_*(k))^2/2k]$ correspond to i relatively close to $i_*(k)$ and, for those i, $(k-i)^{-1/2}$ is asymptotic to $k^{-1/2}$. Besides, the function $e^{-x^2/2}$ is Reimann integrable on $(-\infty, \infty)$.) So,

$$E(n) = \sum_k\sum_i E(n; i, k) = O(n^{-(2b-3a)/3}) \to 0, \qquad n \to \infty,$$

since $2b - 3a \geq 1$.

The same estimate can be obtained for $G(n, M)$.

(II) Define $E'(n)$ and $E''(n)$ like $E(n)$ in Lemma 3.1, except that for $E'(n)$ we consider only those complex components G for which $|V(G(C))| \geq n^{1/3}\omega(n)$, and for $E''(n)$—the complex components for which

$$\min\{i_t : 1 \leq t \leq b\} \leq n^{1/3}/\omega(n).$$

We need to show that $E'(n) \to 0$, $E''(n) \to 0$, as $n \to \infty$. Consider $G(n, p)$.

(1) Set $i' = n^{1/3}\omega(n)$. By the definition of $E'(n)$ and the numbers $E(n; i, k)$ (see the above proof of Lemma 3.1),

$$E'(n) = \sum_k \sum_{i' \leq i \leq k} E(n; i, k).$$

So, arguing as in (3.6)–(3.9), and using

$$i - i_*(k) \geq i' - i_*(k) \geq n^{1/3}\omega(n)/2,$$

we have

$$E'(n) = O\left(n^{-(2b-3a)/3}\int_{\delta\omega(n)}^{\infty} e^{-z^2/2}\,dz\right)$$
$$= O\left(\int_{\delta\omega(n)}^{\infty} e^{-z^2/2}\,dz\right) = o(1), \qquad n\to\infty$$

$(\delta = 2^{-1}d^{-1/2})$.

(2) Set $i'' = n^{1/3}/\omega(n)$. The number of nonnegative solutions of $\sum_{t=1}^{b} i_t = i-a$ such that $\min\{i_t : 1\le t\le b\}\le i''$ is at most

$$bi''\binom{i-a+b-2}{b-2} \le b(b-a-1)^{b-2}i''i^{b-2}.$$

So (cf. the second inequality in (2.6) and (3.9))

$$E''(n) = O\left[\sum_k \frac{k^{b/2-1}}{n^{b-a}}i''\sum_i i^{-1}(k-i)^{-1/2}\exp\left(-\frac{(i-i_*(k))^2}{2k}\right)\right]$$
$$= O[n^{-(2b-3a)/3}\omega(n)^{-1}] = O[\omega(n)^{-1}].$$

The case of $G(n, M)$ is similar.

(III) It remains to show that a.s. the largest pendant tree in each complex component has size at least $n^{2/3}/\omega(n)$. To this end, we apparently would have to show that the expected number of complex components without such a large pendant tree tends to 0 as $n\to\infty$. However, the previous results allow us to consider only $E'''(n)$ the expected number of "bad" components of size $k\in[d^{-1}n^{2/3}, dn^{2/3}]$, which also satisfy a condition

$$d_1^{-1}n^{1/3}\le i\le d_1 n^{1/3}, \tag{3.10}$$

where $d_1 > 1$ and is otherwise arbitrary.

Introduce $\mathscr{F}(i, k)$ a uniform random forest of i trees on k vertices $1, 2, \dots, k$ such that the jth tree contains the vertex j. Consider $M(i, k)$ the size of the largest tree in $\mathscr{F}(i, k)$. It should be clear that

$$E'''(n) = \sum_k\sum_i E(n; \vec{i}, k)P\left(M(i, k)\le \frac{n^{2/3}}{\omega(n)}\right)$$

where k, i satisfy (3.2), (3.10); see (3.3) for $E(n; \vec{i}, k)$. So, all we need to do is to show that $P(M(i, k)\le n^{2/3}/\omega(n))\to 0$ uniformly over i, k in this range.

As it happens, Pavlov (1977, 1979) studied the behavior of $M(i, k)$ under various assumptions regarding the relation of i and k. In particular, when i is of order precisely $k^{1/2}$, $M(i, k)$ is of order k, that is $M(i, k)/k$ has a nondegenerate limiting distribution. But in the case of our i, k, i is of order $k^{1/2}$, and k is of order $n^{2/3}$. Hence

$$P(M(i, k)\le n^{2/3}/\omega(n)) = P\left(\mathscr{M}(i, k)\le \frac{d}{\omega(n)}k\right)\to 0,$$

as $n\to\infty$. □

4. Planarity of $G(\lambda)$

The last problem we shall study in this paper is the determination of the asymptotic probability of planarity for $G(n, M)$ and $G(n, p)$.

Introduce $C^{(0)}(k, k+l)$ ($C^{(1)}(k, k+l)$, resp.) the total number of planar (nonplanar, resp.) connected graphs on k labeled vertices, with $k+l$ edges. Of course, $C(k, k+l) = C^{(0)}(k, k+l) + C^{(1)}(k, k+l)$. Mimicking Wright's derivation of the asymptotic formula (2.4) for $C(k, k+l)$, we obtain: for every $l \geq 1$, and $i = 0, 1$,

$$C^{(i)}(k, k+l) = (1+o())\gamma_l^{(i)} k^{k+3l/2-1/2}, \qquad k \to \infty.$$

Clearly, $\gamma_l^{(0)} + \gamma_l^{(1)} = \gamma_l$. Recall that, by Kuratowski's theorem, a graph is planar iff it does not contain a topological copy of $K_{3,3}$ or K_5. Since the number of edges of $K_{3,3}$ exceeds the number of vertices by 3, we can assert that $\gamma_3^{(1)} > 0$. Likewise, considering K_5, $\gamma_5^{(1)} > 1$. (It is worth mentioning at once that almost surely $G(\lambda)$ does not contain a topological copy of K_5. So, the only likely obstacle to planarity is presence of a copy of $K_{3,3}$.) Needless to say, Bollobás' inequalities (2.5), (2.6) hold for $C^{(i)}(k, k+l)$ as well.

Let $X^{(0)}(n; l)$, $X^{(1)}(n; l)$ denote the total number of planar l-components and nonplanar l-components, respectively. Repeating almost verbatim the proof of Theorem 3, we have:

Let $r^{(0)} = \{r_l^{(0)}\}_{l\geq 1}$, $r^{(1)} = \{r_l^{(1)}\}_{l \geq 1}$ be two finitary sequences and $r^{(0)} + r^{(1)} \neq \vec{0}$. Then

$$\lim_{n\to\infty} P(X^{(0)}(n) = r^{(0)},\ X^{(1)}(n) = r^{(1)}) = p(r^{(0)}, r^{(1)}), \tag{3.11}$$

where

$$p(r', r'') = \frac{1}{\Gamma(3L/2)} \left(\prod_{l \geq 1} \frac{(\gamma_l^{(0)} \Gamma(3l/2)/(2\pi)^{1/2})^{r_l^{(0)}}}{r_l^{(0)}!} \right) \cdot \left(\prod_{l\geq 1} \frac{(\gamma_l^{(1)} \Gamma(3l/2)/(2\pi)^{1/2})^{r_l^{(1)}}}{r_l^{(1)}!} \right) \cdot \int_0^\infty e^{-F(x)} x^{3L/2-1} h(\lambda - x)\, dx,$$

$(L := \sum_l l(\gamma_l^{(0)} + \gamma_l^{(1)}))$. So, we have established the following.

Theorem 5. *Denote by $P_n(\lambda)$ the probability that $G(\lambda)$ is planar. There exists $\lim_{n\to\infty} P_n(\lambda) = p(\lambda)$, given by*

$$p(\lambda) = h(\lambda) + \sum_r p^{(0)}(r), \tag{3.12}$$

where

$$p^{(0)}(r) = \frac{1}{\Gamma(3L/2)} \left(\prod_{l\geq 1} \frac{(\gamma_l^{(0)} \Gamma(3l/2)/(2\pi)^{1/2})^{r_l^{(0)}}}{r_l^{(0)}!} \right) \cdot \int_0^\infty e^{-F(x)} x^{3L/2-1} h(\lambda - x)\, dx,$$

and $L = \sum_l l r_l^{(0)}$. (Recall that $h(\lambda)$ is the limiting probability that $G(\lambda)$ does *not have complex components.) Furthermore, we have* $0 < p(\lambda) < 1$, *the lower bound following from* (3.10), *and the upper bound from an observation that, for instance,* $\gamma_3^{(1)} > 0$.

Note that $h(\lambda) \to 1$ as $\lambda \to -\infty$, and therefore $\lim_{\lambda\to-\infty} p(\lambda) = 1$.

There are also various ways to check that $\lim_{\lambda\to-\infty} p(\lambda) = 0$. We present here a simple and elegant argument proposed by Svante Janson.

If $\lambda = \lambda(n) \to +\infty$ then, with probability $1-o(1)$, the graph $G(n, n/2)$ contains at least five tree components $C_1, \dots, C_5$, each of size at least $\lambda^{-1/3}n^{2/3}$ (see Erdös and Rényi (1960)). The (conditional) probability that in the graph $G(n, (n+\lambda n^{2/3})/2)$, obtained by adding $s = \lambda n^{2/3}/2$ randomly chosen edges to $G(n, n/2)$, the components C_i, C_j are not joined by an edge for some $1 \le i < j \le 5$ is smaller than

$$\binom{5}{2}\binom{\binom{n}{2} - \frac{n}{2} - \lambda^{-2/3}n^{4/3}}{s}\Big/\binom{\binom{n}{2}-\frac{n}{2}}{s} \le 10\left(1 - \frac{\lambda^{-2/3}}{n^{2/3}}(1-o(1))\right)^s$$
$$\le \exp(-\lambda^{1/4}) \to 0.$$

Thus, almost surely $G(n, (n+\lambda n^{2/3})/2)$ contains a subgraph contractible to K_5, and is therefore nonplanar (Wagner's Theorem). We conclude that $\lim_{\lambda\to\infty} p(\lambda) = 0$.

Consequently, $G(n, M)$ ($G(n, p)$ resp.) is almost surely nonplanar iff $(M - n/2)n^{-2/3} \to \infty$ ($(np-1)n^{-1/3} \to \infty$, resp.)

Remark. The above argument works just as well for every K_r, $r \ge 5$. Thus, for $G(n, M)$ say, the contraction clique number (ccl) is bounded in probability if $(M - n/2)n^{-2/3}$ is bounded, and the ccl number goes to ∞ in probability, if $(M - n/2)n^{-2/3}$ goes to ∞ *however slowly.*

Acknowledgments

The research reported in this paper is a combination of two independent research programs. One investigation, by Tomasz Łuczak and John Wierman, began in 1985, during a National Academy of Sciences exchange visit by John Wierman to Poland, and continued during visits by both Łuczak and Wierman to the Institute for Mathematics and Its Applications at the University of Minnesota for the 1987–88 academic year. An independent investigation conducted by Boris Pittel was begun in 1986. He expresses his gratitude to Don Knuth for support, encouragement, and opportunity to participate in a study of the fine structure of random graphs during a visit to the Stanford University Computer Science Department in 1988. All three authors thank Michal Karoński for his matchmaking role in bringing about the combination of the two efforts, resulting in the present manuscript. We are very grateful to Alan Frieze and Svante Janson for stimulating discussions and valuable suggestions. Our sincere thanks go to Andrew Barbour, Béla Bollobás, and Trevor Fenner for giving us an opportunity to present the results of this study at the LMS Symposium (Durham, England), 1991. We gratefully acknowledge the financial support of the U.S. Academy of Sciences, the Institute for Mathematics and Its Applications, and the U. S. National Science Foundation through grants DMS-9002347 (Pittel) and DMS-8801209 (Wierman). Finally, we are very much indebted to a referee for his thorough reading of the paper and many constructive comments.

REFERENCES

[1979] D. Angluin and L. G. Valiant, *Fast probabilistic algorithms for Hamilton circuits and matchings*, J. Comput. System Sci. **18**, 155–193.

[1990] E. A. Bender, E. R. Canfield, and B. D. McKay, *The asymptotic number of labeled connected graphs with a given number of vertices and edges*, Random Structures & Algorithms **1**, 127–169.

[1984a] B. Bollobás, *The evolution of random graphs*, Trans. Amer. Math. Soc. **286**, 257–274.

[1984b] ——, *The evolution of sparse random graphs*, Graph Theory and Combinatorics (Proc. Cambridge Combinatorial Conf. in Honour of Paul Erdös), Academic Press, San Diego, CA, pp. 35–37.

[1985] ——, *Random graphs*, Academic Press, San Diego, CA.

[1988] V. E. Britikov, *Asymptotic number of forests from unrooted trees*, Math. Notes **43**, 387–394.

[1960] P. Erdös and A. Rényi, *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutato Int. Kozl. **5**, 17–61.

[1989] P. Flajolet, D. E. Knuth, and B. Pittel, *The first cycles in an evolving graph*, Discrete Math. **75**, 167–215.

[1980] G. Grimmett, *Random graphs*, Further Selected Topics in Graph Theory (L. Beineke and R. J. Wilson, eds.), Academic Press, San Diego, CA.

[1990] S. Janson, *Poisson approximation for large deviations*, Random Structures & Algorithms **1**, 221–229.

[1982] M. Karoński, *A review of random graphs*, J. Graph Theory **6**, 349–389.

[1988] D. E. Knuth, Private communication.

[1986] V. F. Kolchin, *On the behavior of a random graph near a critical point*, Theory Probab. Appl. **31**, 439–451.

[1989] T. Łuczak and J. C. Wierman, *The chromatic number of random graphs at the double-jump threshold*, Combinatorica **9**, 39–49.

[1990] T. Łuczak, *Component behavior near the critical point of the random graph process*, Random Structures & Algorithms **1**, 287–310.

[1970] J. W. Moon, *Counting labelled trees*, Canad. Math. Congress, Montreal.

[1985] E. M. Palmer, *Graphical evolution*, Wiley, New York.

[1977] Yu. L. Pavlov, *The asymptotic distribution of maximum tree size in a random forest*, Theory Probab. Appl. **22**, 509–520.

[1979] ——, *A case of limit distribution of the maximal volume of a tree in a random forest*, Math. Notes **25**, 387–392.

[1982] B. Pittel, *On the probable behavior of some algorithms for finding the stability number of a graph*, Math. Proc. Cambridge Philos. Soc. **92**, 511–526.

[1970a] V. E. Stepanov, *Phase transitions in random graphs*, Theory Probab. Appl. **15**, 187–203.

[1970b] ——, *On the probability of connectedness of a random graph* $G_m(t)$, Theory Probab. Appl. **15**, 55–67.

[1988] ——, *On some features of the structure of a random graph near a critical point*, Theory Probab. Appl. **32**, 573–594.

[1987] V. A. Voblyi, *Wright and Stepanov-Wright coefficients*, Math Notes **42**, 969–974.

[1977] E. M. Wright, *The number of connected sparsely edged graphs*, J. Graph Theory **1**, 317–330.

[1980] ——, *The number of connected sparsely edged graphs.* III: *Asymptotic results*, J. Graph Theory **4**, 393–407.

INSTITUTE OF MATHEMATICS, ADAM MICKIEWICZ UNIVERSITY, POZNAN, POLAND

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210

MATHEMATICAL SCIENCES DEPARTMENT, THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218